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Global well-posedness and nonlinear stability of a chemotaxis system modelling multiple sclerosis

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We consider a system of reaction–diffusion equations including chemotaxis terms and coming out of the modelling of multiple sclerosis. The global existence of strong solutions to this system in any dimension is proved, and it is also shown that the solution is bounded uniformly in time. Finally, a nonlinear stability result is obtained when the chemotaxis term is not too big. We also perform numerical simulations to show the appearance of Turing patterns when the chemotaxis term is large.

Keywords: Chemotaxis models; Global solutions; Uniform-in-time bounds; Nonlinear stability; Cross diffusion

2020 *Mathematics subject classification:* 35K51; 35K57; 35K92; 92C17; 35B40

1. Introduction and main results

In this paper, we study the global existence of strong solutions (and also some aspects of the large time behaviour) of the following reaction–diffusion system

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including chemotaxis terms:

$$\begin{cases} \partial_t m = \Delta m + m(1 - m^{a-1}) - \chi \nabla \cdot (f(m)\nabla c), & (x,t) \in Q_T, \\ \partial_t c = \varepsilon_0 \Delta c + \delta d - c + \beta m, & (x,t) \in Q_T, \\ \partial_t d = g(m)(1-d), & (x,t) \in Q_T, \\ \nabla m \cdot \nu = \nabla c \cdot \nu = 0, & (x,t) \in \partial\Omega \times (0,T), \\ m(x,0) = m_0(x), c(x,0) = c_0(x), d(x,0) = d_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $Q_T := \Omega \times (0,T)$, ν is the outward normal vector to $\partial\Omega$ at a point of $\partial\Omega$, the parameters satisfy $a > 1$, and $\chi, \varepsilon_0, \delta, \beta > 0$, and the initial data is nonnegative, bounded and sufficiently smooth. The nonlinearities f and g satisfy some conditions which will be specified later.

The system (1.1) with

$$a = 2, \quad f(m) = \frac{m}{1+m} \quad \text{and} \quad g(m) = \frac{rm^2}{1+m},\ r > 0, \tag{1.2}$$

was recently proposed [**12**] for the dynamics of multiple sclerosis. There, $m := m(x,t) \geqslant 0$ is the density of inflammatory immune cells (macrophages), $c := c(x,t) \geqslant 0$ is the density of a chemoattractant (cytokine), and $d := d(x,t) \geqslant 0$ is the density of destroyed oligodendrocytes. This model generalizes that of [**3**] and [**10**] to describe a rare and aggressive form of multiple sclerosis, namely Baló's sclerosis. It has been shown that by varying the parameter values within realistic ranges taken from the experimental literature, this model is able to reproduce different pathological scenarios typical of the disease ([**1**], [**2**]). The analysis of (1.1)–(1.2) has been carried out in two recent works: in [**5**], the global existence of very weak and classical solutions to (1.1)–(1.2) was shown in dimension one; global classical solutions in higher dimensions are shown in [**9**] under the smallness assumption of the chemoattractant coefficient χ (when $n > 3$). This latter work also shows that the solution is bounded uniformly in time, and the positive equilibrium is globally asymptotically stable if χ is small enough.

In the present work, we study (1.1) with a general logistic growth of order $a > 1$, and the nonlinearities f and g are assumed to satisfy

(F) $f \in C^2([0,\infty))$ with $f(0) = 0$, and there exist constants $\gamma, b, \ell \geqslant 0$ such that

$$|f(y)| \leqslant \gamma\, y^b \quad \text{and} \quad |f'(y)| \leqslant \gamma\, y^\ell \quad \text{for all} \quad y \geqslant 0, \tag{1.3}$$

(G) $g \in C^1([0,\infty))$ and $g(y) > 0$ for all $y > 0$.

It's easy to check that the special case (1.2) fulfills the assumptions (F) for f with $b = \ell = 0$, and the assumption (G) for g.

The first main result of this paper concerns the existence and uniqueness of (uniformly w.r.t. t) bounded strong solutions to system (1.1), when the parameter b appearing in assumption (1.3) is not too large.

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THEOREM 1.1. *Let Ω be a smooth ($C^{2+\alpha}$ for some $\alpha > 0$) bounded connected open subset of $\mathbb{R}^n$ (for $n \in \mathbb{N} - \{0\}$), and suppose $a > 1$, $\chi, \varepsilon_0, \delta, \beta > 0$, $\gamma, b, l \geqslant 0$, and the functions f and g satisfy (F) and (G). We further assume*

$$\max\left\{b-1, \frac{b}{2}\right\} < \frac{a}{n+2}, \tag{1.4}$$

and consider nonnegative initial data $(m_0, c_0, d_0) \in W^{(2-2/\tilde{p}),\tilde{p}}(\Omega) \times W^{(2-2/\tilde{p}),\tilde{p}}(\Omega) \times L^\infty(\Omega)$, for some $\tilde{p} > \max\{n+2, a\}$. Then, system (1.1) has a unique nonnegative (for each component) strong solution which is bounded uniformly in time. More precisely, $\partial_t m$, $\partial_t c$, $\partial_t d$, Δm, Δc, $\nabla \cdot (f(m)\nabla c)$ belong to $L^{\tilde{p}}(\Omega \times (0,T))$ for all $T > 0$, and there exists $C > 0$ (depending on Ω, n, a, b, ε_0, β, δ, m_0, c_0, d_0), and $\tilde{C} > 0$ (depending on Ω, n, a, b, ε_0, β, δ, d_0), such that

$$\sup_{t \geqslant 0} \left(\|m(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|d(\cdot,t)\|_{L^\infty(\Omega)} \right) \leqslant C, \tag{1.5}$$

and

$$\limsup_{t \to \infty} \left(\|m(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|d(\cdot,t)\|_{L^\infty(\Omega)} \right) \leqslant \tilde{C}. \tag{1.6}$$

Finally, if the initial data are smooth, i.e. $m_0, c_0, d_0 \in C^2(\overline{\Omega})$ and satisfy compatibility conditions $\nabla m_0 \cdot \nu = \nabla c_0 \cdot \nu = 0$ on $\partial\Omega$, then the solution is classical. That is

$$\partial_t m, \partial_t c, \partial_t d, \Delta m, \Delta c, \nabla \cdot (f(m)\nabla c) \in C^0(\overline{\Omega} \times [0,T])$$

for all $T > 0$.

REMARK 1.2. The difference between C and $\tilde{C}$ in (1.5) and (1.6) is that $\tilde{C}$ does not depend on m_0 and c_0. This means that in large time, the bound of solutions depends only on the size of d_0 but not m_0 and c_0.

For the special case (1.2), since $b = 0$ and $a = 2$, condition (1.4) is satisfied in all dimensions. Therefore, theorem 1.1 in particular shows that the system (1.1)–(1.2) has global unique strong solutions (for suitable initial data) in all dimensions. We also emphasize that theorem 1.1 improves the global existence in [**9**] by removing the requirement that χ is small in all cases.

Let us briefly describe the ideas underlying the proof of theorem 1.1. Using the comparison principle, it's easy to show that d is bounded uniformly in time and space. It follows straightforwardly from the logistic term that m is bounded in $L^a(\Omega \times (0,T))$. Using these bounds and the properties of the heat equation satisfied by c, thanks to its linearity, we obtain some $L^q(\Omega \times (0,T))$ bound for ∇c. In order to use this estimate in the equation satisfied by m, we exploit the so-called *duality method*, which proved its significant usefulness in the study of reaction-diffusion systems (cf. [**4**, **13**, **14**]), or cross diffusion systems ([**6**, **7**]). Under condition (1.4), the duality method yields an $L^p(\Omega \times (0,T))$ estimate for m for all $1 \leqslant p < \infty$. This information and a bootstrap argument allow us to conclude that c is bounded in $L^\infty(0,T; W^{1,\infty}(\Omega))$ and that m is bounded in $L^\infty(\Omega \times (0,T))$, hence

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the global existence of bounded solutions to (1.1). To show the uniform-in-time bound (1.5), we use a smooth truncation function in time to study (1.1) on each cylinder $\Omega \times (\tau, \tau+1)$, $\tau \in \mathbb{N}$. We repeat the previous arguments to obtain that m and c are bounded in $\Omega \times (\tau, \tau+1)$ *uniformly in* $\tau \in \mathbb{N}$, which consequently implies the desired bounds (1.5) and (1.6). It's worth noting that this uniform-in-time bound plays an important role in the nonlinear stability, which is discussed in the following paragraphs.

Our second main result of this paper is the nonlinear exponential stability of a positive (for all components) homogeneous equilibrium. A homogeneous equilibrium (m^*, c^*, d^*) to (1.1) solves $m^* = (m^*)^a$, $\delta d^* + \beta m^* = c^*$ and $g(m^*)(1-d^*) = 0$. It's straightforward that

$$(\bar{m}, \bar{c}, \bar{d}) = (1, \beta+\delta, 1)$$

is the unique *positive* (for all components) homogeneous equilibrium. Besides this, we also have other equilibria, namely:

$$(m^*, c^*, d^*) = \begin{cases} (0, \delta, 1) & \text{if } g(0) \neq 0, \\ (0, \delta\zeta, \zeta) \text{ with } \zeta \geqslant 0 \text{ arbitrary} & \text{if } g(0) = 0. \end{cases}$$

THEOREM 1.3 Exponential nonlinear stability of $(\bar{m}, \bar{c}, \bar{d})$. *Let the assumptions of theorem 1.1 hold and assume additionally that* $f(1) > 0$. *If*

$$\chi < \chi_{\text{subcrit}} := \frac{4\sqrt{\varepsilon_0(a-1)}}{\beta f(1)}, \tag{1.7}$$

then the constant steady state $(\bar{m}, \bar{c}, \bar{d})$ *is exponentially asymptotically stable. More precisely, there exists* $\varepsilon > 0$ *such that if* $(m_0, c_0, d_0) \in W^{(2-2/\tilde{p}),\tilde{p}}(\Omega) \times W^{(2-2/\tilde{p}),\tilde{p}}(\Omega) \times L^\infty(\Omega)$ *for* $\tilde{p} > \max\{n+2, a\}$, *and*

$$\|m_0 - \bar{m}\|_{L^2(\Omega)} + \|c_0 - \bar{c}\|_{L^2(\Omega)} + \|d_0 - \bar{d}\|_{L^2(\Omega)} \leqslant \varepsilon,$$

then for all $p \in [1, \infty[$ *and* $\delta \in]0, 1]$, *there exist* $C, \varrho > 0$ *depending on* p *and* δ *such that*

$$\|m(t) - \bar{m}\|_{L^p(\Omega)} + \|c(t) - \bar{c}\|_{W^{1-\delta,p}(\Omega)} + \left\|d(t) - \bar{d}\right\|_{L^p(\Omega)} \leqslant C\, e^{-\varrho t} \quad \text{for all } t \geqslant 0.$$

REMARK 1.4. We expect that the nonlinear stability in theorem 1.3 can be shown under the optimal condition $\chi < \chi_c$, where χ_c is defined in (3.2) using the eigenvalues of the Laplacian with Neumann boundary condition (see § 3 for more details). We remark that the threshold (1.7) becomes optimal when $\varepsilon_0^{-1} = a - 1$, which makes $\chi_{\text{subcrit}} = \chi_{c0}$ (see (3.3)).

To prove theorem 1.3, we first rewrite system (1.1) with a new variable $(\tilde{m}, \tilde{c}, \tilde{d}) = (m - \bar{m}, c - \bar{c}, d - \bar{d})$. Next, we show that under condition (1.7), the linear part, after a suitable scaling, has a spectral gap. This spectral gap and the uniform-in-time bound proved in theorem 1.1 allow us to show that the nonlinear part is dominated by the linear one, and consequently obtain the nonlinear stability of $(\bar{m}, \bar{c}, \bar{d})$.

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The organization of the paper is as follows: In § 2, we prove theorem 1.1 by first showing global existence, then uniqueness, and finally the uniform-in-time bound. In § 3.1, we prove the nonlinear stability in theorem 1.3. Then, in § 3.2 we perform numerical simulations on a 2D spatial domain to show Turing patterns which appear for sufficiently large chemoattractant coefficient χ. Finally, § 3.3 concludes the paper with a presentation of open problems relevant for the model under study.

Notation. Throughout this paper, we use the following notations

- The space $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, is equipped with the norm

$$\|v\|_{L^p(\Omega)} = \left(\int_\Omega |v(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \quad \text{when} \quad p < \infty,$$

and

$$\|v\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x \in \Omega} |v(x)|.$$

- When $1 \leqslant p \leqslant \infty$ and $r > 0$ is an integer, we define the space

$$W^{r,p}(\Omega) := \{ v \in L^p(\Omega) : \partial_x^s v \in L^p(Q_{\tau,T}) \text{ for } s \in \mathbb{N} \text{ with } s \leqslant r \},$$

and when r is not an integer, we use the definition in [**11**, chapter 2, section 2].

- For any $0 \leqslant \tau < T$, $Q_{\tau,T} := \Omega \times (\tau, T)$. When $\tau = 0$, we write Q_T instead of $Q_{0,T}$.

- The space-time space $L^p(Q_{\tau,T}) = L^p((\tau,T); L^p(\Omega))$, $1 \leqslant p \leqslant \infty$, is equipped with the norm

$$\|v\|_{L^p(Q_{\tau,T})} = \left(\int_\tau^T \int_\Omega |v(x,t)|^p \, \mathrm{d}x \, \mathrm{d}t \right)^{\frac{1}{p}} \quad \text{when} \quad p < \infty,$$

and

$$\|v\|_{L^\infty(Q_{\tau,T})} = \operatorname*{ess\,sup}_{(x,t) \in Q_{\tau,T}} |v(x,t)|.$$

- We denote the space-time space

$$W_p^{2,1}(Q_{\tau,T}) := \{ v \in L^p(Q_{\tau,T}) : \partial_t^r \partial_x^s v \in L^p(Q_{\tau,T}) \text{ for } \times r, s \in \mathbb{N} \text{ with } 2r + s \leqslant 2 \},$$

equipped with the norm

$$\|v\|_{W_p^{2,1}(Q_{\tau,T})} := \sum_{2r+s \leqslant 2} \|\partial_t^r \partial_x^s v\|_{L^p(Q_{\tau,T})}.$$

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2. Proof of theorem 1.1

2.1. Global existence of bounded solutions

In this subsection, we show the existence of solutions to system (1.1) which are bounded on Q_T for all $T > 0$.

The following lemma is useful in our analysis.

LEMMA 2.1 ([**11**, lemma 3.3] and [**15**, theorem 3]). *Assume $1 < p < \infty$. There exists a constant $C > 0$, only depending on $T - \tau, \Omega, p, n$, such that for all $v \in W_p^{2,1}(Q_{\tau,T})$,*

(*a*)

$$\|v\|_{L^q(Q_{\tau,T})} \leqslant C \ \|v\|_{W_p^{2,1}(Q_{\tau,T})}$$

where

$$q = \begin{cases} \dfrac{(n+2)p}{n+2-2p} & \text{if } p < (n+2)/2, \\ < +\infty \text{ arbitrary} & \text{if } p = (n+2)/2, \\ +\infty & \text{if } p > (n+2)/2. \end{cases}$$

(*b*)

$$\|\nabla v\|_{L^q(Q_{\tau,T})} \leqslant C \ \|v\|_{W_p^{2,1}(Q_{\tau,T})}$$

where

$$q = \begin{cases} \dfrac{(n+2)p}{n+2-p} & \text{if } p < n+2, \\ < +\infty \text{ arbitrary} & \text{if } p = n+2, \\ +\infty & \text{if } p > n+2. \end{cases}$$

Furthermore, when $p > n + 2$, the space $W_p^{2,1}(Q_{\tau,T})$ is compactly embedded in $C^{1,0}(\overline{\Omega} \times [0, T])$.

Next, we prove the

PROPOSITION 2.2. *Under the assumptions of theorem 1.1, and picking $T > 0$, system (1.1) has a nonnegative (for each component) strong solution which is bounded on Q_T. More precisely, there exists $C_T > 0$ (depending on Ω, T, n, a, b, ε_0, β, δ, $\|m_0\|_{L^\infty(\Omega)}$, $\|c_0\|_{W^{1,\infty}(\Omega)}$, $\|d_0\|_{L^\infty(\Omega)}$), such that*

$$\|m\|_{L^\infty(Q_T)} + \|c\|_{L^\infty((0,T);W^{1,\infty}(\Omega))} + \|d\|_{L^\infty(Q_T)} \leqslant C_T. \tag{2.1}$$

Proof of proposition 2.2. For a given $r > 0$, define $h_r \in C^\infty(\mathbb{R}_+, [0, 1])$ such that $h_r(y) = 1$ for $0 \leqslant y \leqslant r$, and $h_r(y) = 0$ for $y \geqslant 2r$. We also use the standard

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convention $m_+ := \max\{0; m\}$. Consider the system

$$\begin{cases} \partial_t m = \Delta m + m_+(1 - m_+^{a-1}) - \chi \nabla \cdot (f_r(m_+)\nabla c), & (x,t) \in Q_T, \\ \partial_t c = \varepsilon_0 \Delta c + \delta d - c + \beta m_+, & (x,t) \in Q_T, \\ \partial_t d = g(m_+)(1-d), & (x,t) \in Q_T, \\ \nabla m \cdot \nu = \nabla c \cdot \nu = 0, & (x,t) \in \partial\Omega \times (0,T), \\ m(x,0) = m_0(x),\ c(x,0) = c_0(x),\ d(x,0) = d_0(x), & x \in \Omega, \end{cases} \tag{2.2}$$

where the function f_r is defined as

$$f_r(y) := h_r(f(y)). \qquad \square$$

We will prove, thanks to Leray-Schauder's theorem (cf. [8, theorem 10.3]), that (2.2) has a nonnegative (for each component) solution which is bounded in sup-norm *independent of* $r > 0$, and consequently prove the existence of a solution to the original system (1.1) on Q_T. To this end, we define

$$\mathfrak{Y} := C^{1,0}(\overline{\Omega} \times [0,T]) \quad \text{ and } \mathfrak{X} := \mathfrak{Y} \times \mathfrak{Y},$$

where $C^{1,0}$ is the space of continuous functions such that their gradient (with respect to space) is also continuous, associated with the natural norm (making it a Banach space)

$$\|u\|_{\mathfrak{Y}} = \sup_{t\in[0,T]} \|u(t)\|_{L^\infty(\Omega)} + \sup_{t\in[0,T]} \|\nabla u(t)\|_{L^\infty(\Omega)},$$

and

$$\mathcal{F}_r : \mathfrak{X} \times [0,1] \to \mathfrak{X} \quad \text{ via } \quad \mathcal{F}_r(m,c,\lambda) := (\lambda\,\mathfrak{M}, \lambda\,\mathfrak{C}),$$

where $(\mathfrak{M}, \mathfrak{C})$ is given (in a unique way) by solving first an ODE (for d), and then successively (for $\mathfrak{C}$ and $\mathfrak{M}$, in that order) two Neumann problems for the heat equation with a source. More precisely, $(\mathfrak{M}, \mathfrak{C})$ is the (unique) solution on Q_T of the system

$$\begin{cases} \partial_t \mathfrak{M} = \Delta \mathfrak{M} + m_+ \left(1 - m_+^{a-1}\right) - \chi \nabla \cdot (f_r(m_+)\nabla \mathfrak{C}), & (x,t) \in Q_T, \\ \partial_t \mathfrak{C} = \varepsilon_0 \Delta \mathfrak{C} + \delta d - \mathfrak{C} + \beta m_+, & (x,t) \in Q_T, \\ \partial_t d = g(m_+)(1-d), & (x,t) \in Q_T, \\ \nabla \mathfrak{M} \cdot \nu = \nabla \mathfrak{C} \cdot \nu = 0, & (x,t) \in \partial\Omega \times (0,T), \\ \mathfrak{M}(x,0) = m_0(x), \mathfrak{C}(x,0) = c_0(x), d(x,0) = d_0(x), & x \in \Omega. \end{cases} \tag{2.3}$$

Note first that $\mathcal{F}_r(m,c,0) = 0$. Then, regardless of m, we always have that

$$0 \leqslant d(x,t) \leqslant \mu := \max\left\{1, \|d_0\|_{L^\infty(\Omega)}\right\}. \tag{2.4}$$

Also for any $1 < p \leqslant \tilde{p}$, there exists $C_p > 0$ depending on $\|m\|_{\mathfrak{Y}}$ and r (and Ω, T, m_0, c_0, μ) such that, thanks to maximal regularity results,

$$\|\mathfrak{C}\|_{W_p^{2,1}(Q_T)} \leqslant C_p, \tag{2.5}$$

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and

$$\|\mathfrak{M}\|_{W_p^{2,1}(Q_T)} \leqslant C_p. \tag{2.6}$$

Thanks to the embedding in lemma 2.1,

$$W_p^{2,1}(Q_T) \hookrightarrow \mathfrak{Y} \text{ compactly when } p > n+2. \tag{2.7}$$

So, we see that since $\tilde{p} > n+2$, the map $\mathcal{F}_r$ sends bounded sets in $\mathfrak{X}$ into compact sets of $\mathfrak{X}$. We now show that for any $\lambda \in [0,1]$, $\mathcal{F}_r(\cdot,\cdot,\lambda)$ is continuous from $\mathfrak{X}$ to $\mathfrak{X}$. Let $(m_1,c_1),(m_2,c_2) \in \mathfrak{X}$ and $(\lambda\mathfrak{M}_1,\lambda\mathfrak{C}_1) = \mathcal{F}_r(m_1,c_1,\lambda)$, $(\lambda\mathfrak{M}_2,\lambda\mathfrak{C}_2) = \mathcal{F}_r(m_2,c_2,\lambda)$. We have

$$\begin{cases} \partial_t(\mathfrak{M}_1-\mathfrak{M}_2) &= \Delta(\mathfrak{M}_1-\mathfrak{M}_2) + \left[m_{1+}(1-m_{1+}^{a-1}) - m_{2+}(1-m_{2+}^{a-1})\right] \\ &\quad +\chi\nabla\cdot\left[f_r(m_{1+})\nabla\mathfrak{C}_1 - f_r(m_{2+})\nabla\mathfrak{C}_2\right], \\ \partial_t(\mathfrak{C}_1-\mathfrak{C}_2) &= \varepsilon_0\Delta(\mathfrak{C}_1-\mathfrak{C}_2) + \delta(d_1-d_2) - (\mathfrak{C}_1-\mathfrak{C}_2) + \beta(m_{1+}-m_{2+}), \\ (d_1-d_2)_t &= g(m_{1+})(1-d_1) - g(m_{2+})(1-d_2). \end{cases} \tag{2.8}$$

By rewriting $g(m_{1+})(1-d_1) - g(m_{2+})(1-d_2) = g(m_{1+})(d_2-d_1) + (g(m_{1+}) - g(m_{2+}))(1-d_2)$, and using $g \in C^1([0,\infty))$, we obtain

$$d_1(x,t) - d_2(x,t) = \int_0^t e^{-\int_s^t g(m_{1+}(r))dr} g'(\theta(x,s))(m_{1+}(s)-m_{2+}(s))(1-d_2(s))\,\mathrm{d}s,$$

where $\theta(x,s)$ is between $m_{1+}(x,s)$ and $m_{2+}(x,s)$. Therefore, there exists $C_T > 0$ (which can depend on the norm $\|m_1\|_{\mathfrak{Y}}$) such that

$$\|d_1-d_2\|_{L^\infty(Q_T)} \leqslant C_T\|m_{1+}-m_{2+}\|_{L^\infty(Q_T)} \leqslant C_T\|m_1-m_2\|_{\mathfrak{Y}}.$$

By maximal regularity results and (2.7)

$$\begin{aligned} \|\mathfrak{C}_1-\mathfrak{C}_2\|_{\mathfrak{Y}} &\leqslant C_T\|\mathfrak{C}_1-\mathfrak{C}_2\|_{W_p^{2,1}(Q_T)} \\ &\leqslant C_T\left(\|d_1-d_2\|_{L^p(Q_T)} + \|m_{1+}-m_{2+}\|_{L^p(Q_T)}\right) \\ &\leqslant C_T\|m_1-m_2\|_{\mathfrak{Y}}. \end{aligned} \tag{2.9}$$

For the equation satisfied by $\mathfrak{M}_1-\mathfrak{M}_2$, we write

$$\left|m_{1+}(1-m_{1+}^{a-1}) - m_{2+}(1-m_{2+}^{a-1})\right| \leqslant |m_1-m_2| + |m_1-m_2|\,\max(m_{1+}^{a-1}, m_{2+}^{a-1}),$$

and

$$\begin{aligned} &|\nabla\cdot[f_r(m_{1+})\nabla\mathfrak{C}_1 - f_r(m_{2+})\nabla\mathfrak{C}_2]| \\ &\quad\leqslant |f_r(m_{1+})|\,|\Delta(\mathfrak{C}_1-\mathfrak{C}_2)| + |f_r(m_{1+})-f_r(m_{2+})|\,|\Delta\mathfrak{C}_2| \\ &\qquad + |f_r'(m_{1+})|\,|\nabla m_{1+}|\,|\nabla(\mathfrak{C}_1-\mathfrak{C}_2)| \\ &\qquad + |f_r'(m_{1+})|\,|\nabla(m_{1+}-m_{2+})|\,|\nabla\mathfrak{C}_2| + |f_r'(m_{1+})-f_r'(m_{2+})|\,|\nabla m_{2+}|\,|\nabla\mathfrak{C}_2|. \end{aligned}$$

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Using the fact that $f \in C^2([0,\infty))$, estimate (2.9), and maximal regularity results we get

$$\|\mathfrak{M}_1 - \mathfrak{M}_2\|_{\mathfrak{Y}} \leqslant C_T \|\mathfrak{M}_1 - \mathfrak{M}_2\|_{W^{2,1}_{\bar{p}}(Q_T)} \leqslant C_T \|m_1 - m_2\|_{\mathfrak{Y}}. \tag{2.10}$$

It follows from (2.9) and (2.10) that $\mathcal{F}_r(\cdot,\cdot,\lambda)$ is continuous from $\mathfrak{X}$ to $\mathfrak{X}$ for any $\lambda \in [0,1]$.

We now check the last assumption in Leray-Schauder's theorem. We consider therefore

$$\mathbf{Z} := \{(m,c) \in \mathfrak{X} \,:\, (m,c) = \mathcal{F}_r(m,c,\lambda) \ \text{ where } \ 0 < \lambda \leqslant 1\}, \tag{2.11}$$

and will show that $\mathbf{Z}$ is bounded in $\mathfrak{X}$.

Note that if $(m,c) \in \mathbf{Z}$, then $(m,c) = (\lambda\mathfrak{M}, \lambda\mathfrak{C})$, where $(\mathfrak{M},\mathfrak{C})$ solves (2.3). Therefore, by multiplying the equations in (2.3) by λ, we obtain (for $(m,c) \in \mathbf{Z}$)

$$\begin{cases} \partial_t m = \Delta m + \lambda \left[m_+(1 - m_+^{a-1}) - \chi \nabla \cdot (f_r(m_+)\nabla \mathfrak{C}) \right], & (x,t) \in Q_T, \\ \partial_t c = \varepsilon_0 \Delta c - c + \lambda(\delta d + \beta m_+), & (x,t) \in Q_T, \\ \partial_t d = g(m_+)(1-d), & (x,t) \in Q_T, \\ \nabla m \cdot \nu = \nabla c \cdot \nu = 0, & (x,t) \in \partial\Omega \times (0,T), \\ m(x,0) = \lambda m_0(x), c(x,0) = \lambda c_0(x), d(x,0) = d_0(x), & x \in \Omega. \end{cases} \tag{2.12}$$

Since $d \geqslant 0$, $m_+ \geqslant 0$, and $c_0 \geqslant 0$, we obtain immediately (for $(m,c) \in \mathbf{Z}$) that $c \geqslant 0$. We now show that m is also nonnegative. Indeed, denote by $m_- = \max\{0, -m\}$, and multiply the equation satisfied by m in (2.12) by $(m_-)^2$. Then, an integration by parts and the fact $m_- m_+ = 0$ give for all $t \in [0,T]$,

$$-\frac{1}{3}\int_\Omega m_-^3(x,t)\,\mathrm{d}x - 2\int_0^t \int_\Omega m_- |\nabla m_-|^2 \,\mathrm{d}x\,\mathrm{d}s \geqslant 0.$$

Therefore, $m_- = 0$ and thus $m \geqslant 0$. It follows from the nonnegativity of m, c and d (and system (2.12)) that (m,c,d) solves the system

$$\begin{cases} \partial_t m = \Delta m + \lambda m(1 - m^{a-1}) - \chi \nabla \cdot (f_r(m)\nabla c), & (x,t) \in Q_T, \\ \partial_t c = \varepsilon_0 \Delta c - c + \lambda(\delta d + \beta m), & (x,t) \in Q_T, \\ \partial_t d = g(m)(1-d), & (x,t) \in Q_T, \\ \nabla m \cdot \nu = \nabla c \cdot \nu = 0, & (x,t) \in \partial\Omega \times (0,T), \\ m(x,0) = \lambda m_0(x), c(x,0) = \lambda c_0(x), d(x,0) = d_0(x), & x \in \Omega. \end{cases} \tag{2.13}$$

In order to show that $\mathbf{Z}$ is bounded, we propose a series of lemmas. We start with the

Lemma 2.3. *Under the assumptions of theorem 1.1, and supposing that $\lambda \in]0,1]$, we consider (m,c,d) a strong, nonnegative (for each component) solution to (2.13).*

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Then there exists $K_1 > 0$ *(depending on* T, a, β, δ *and* $\|m_0\|_{L^1(\Omega)}$, $\|c_0\|_{L^1(\Omega)}$, $\|d_0\|_{L^\infty(\Omega)}$, *but not depending on* λ*) such that*

$$\sup_{t\in[0,T]} \left(\|m(\cdot,t)\|_{L^1(\Omega)} + \|c(\cdot,t)\|_{L^1(\Omega)}\right) \leqslant K_1, \tag{2.14}$$

and if $0 \leqslant \tau < T$ *then*

$$\|m\|_{L^a(Q_{\tau,T})} \leqslant \left[K_1(T-\tau+1)\right]^{1/a}.$$

Proof. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (m+c)\,\mathrm{d}x = \int_\Omega \left[\lambda(\delta d + (\beta+1)m - m^a) - c\right]\,\mathrm{d}x. \tag{2.15}$$

Since $a > 1$, there exists $k_a > 0$ such that

$$(\beta+1)m - m^a \leqslant k_a - m.$$

Applying this and the fact that $0 \leqslant d \leqslant \mu$ to (2.15), we obtain some $K_1 > 0$ such that (2.14) holds. In addition, integrating the equation satisfied by m in (2.13) gives

$$\int_\Omega m(x,T)\,\mathrm{d}x = \int_\tau^T \int_\Omega \lambda m(1-m^{a-1})\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega m(x,\tau)\,\mathrm{d}x.$$

Consequently,

$$\|m\|^a_{L^a(Q_{\tau,T})} \leqslant K_1(T-\tau+1). \tag{2.16}$$

□

We now turn to the

LEMMA 2.4. *Under the assumptions of theorem 1.1, and supposing that* $\lambda \in]0,1]$, *we consider* (m,c,d) *a strong, nonnegative (for each component) solution to* (2.13). *Then there exists* $K_2 > 0$ *(depending on* Ω, T, n, a, $\varepsilon_0, \beta, \delta$ *and* $\|c_0\|_{W^{(2-2/a),a}(\Omega)}$, $\|d_0\|_{L^\infty(\Omega)}$, *but not depending on* λ*) such that*

$$\|c\|_{W^{2,1}_a(Q_T)} + \|\nabla c\|_{L^{q_1}(Q_T)} \leqslant K_2\,K_1\,(T+1), \tag{2.17}$$

where

$$q_1 = \begin{cases} \dfrac{(n+2)a}{n+2-a} & \text{if } a < n+2, \\ < +\infty \text{ arbitrary} & \text{if } a = n+2, \\ = +\infty & \text{if } a > n+2 \end{cases} \tag{2.18}$$

Proof. Returning to the equation of c in (2.13), and applying lemma 2.1, we get (2.17). □

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We finally prove the

LEMMA 2.5. *Under the assumptions of theorem 1.1, and supposing that* $\lambda \in]0,1]$, *we consider* (m,c,d) *a strong, nonnegative (for each component) solution to* (2.13). *Then for any* $1 \leqslant p < \infty$, *there exists* $F_p^* > 0$ *(depending on* Ω, T, n, p, a, b, ε_0, β, δ, $\|m_0\|_{W^{(2-2/\tilde{p}),\tilde{p}}(\Omega)}$, $\|c_0\|_{W^{(2-2/\tilde{p}),\tilde{p}}(\Omega)}$, $\|d_0\|_{L^\infty(\Omega)}$, *but not depending on* λ*) such that*

$$\|m\|_{L^p(Q_T)} \leqslant F_p^*. \tag{2.19}$$

Proof. Note that from (1.4), it follows that there exists $s \in (0,1)$ (depending on a, b, n only, and with s close to 1 if the inequality in (1.4) is close to being an equality) satisfying

$$b - s < \frac{a}{n+2}, \quad \text{and} \quad b - s < \frac{2a}{n+2} - 1. \tag{2.20}$$

It follows that

$$b < s - 1 + \frac{2a}{n+2} \tag{2.21}$$

and

$$(n+2)(b-s) < a. \tag{2.22}$$

Note also that

$$1 < \frac{2}{n+2} - \frac{b-s}{a}.$$

Since it is sufficient to prove (2.19) for p large enough, we pick any p' such that

$$1 < p' < \min\left\{\frac{n+2}{2}, \frac{2}{n+2} - \frac{b-s}{a}\right\}. \tag{2.23}$$

Let $\theta \in L^{p'}(Q_T)$ with $\theta \geqslant 0$ and $\|\theta\|_{L^{p'}(Q_T)} \leqslant 1$. Let ϕ be the unique nonnegative solution to the equation

$$\begin{cases} \partial_t \phi + \Delta \phi = -\theta, & (x,t) \in Q_T, \\ \nabla \phi \cdot \nu = 0, & (x,t) \in \partial\Omega \times (0,T), \\ \phi(x,T) = 0, & x \in \Omega. \end{cases} \tag{2.24}$$

To avoid any possible confusion, we remark that (2.24) is a forward heat equation for the function $\psi(x,s) = \phi(x,T-s)$, with respect to the new time variable $s = T - t$. From maximal regularity in [**11**] and lemma 2.1, we know that there exists $C_{p'} > 0$

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(depending on Ω, T, n and p only) such that

$$\|\partial_t\phi\|_{L^{p'}(Q_T)} + \|\phi\|_{L^{q_2}(Q_T)} + \|\nabla\phi\|_{L^{q_3}(Q_T)} \leqslant C_{p'}, \tag{2.25}$$

where

$$q_2 = \frac{(n+2)p'}{n+2-2p'} \quad \text{and} \quad q_3 = \frac{(n+2)p'}{n+2-p'}. \tag{2.26}$$

Set $p = p'/(p'-1)$. Then, by using the equation satisfied by m in (2.13) and integrating by parts, we have

$$\begin{aligned}\int_0^T\int_\Omega m\theta\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega m(-\partial_t\phi - \Delta\phi)\,\mathrm{d}x\,\mathrm{d}t\\ &= \lambda\int_\Omega m_0\phi(0)\,\mathrm{d}x + \int_0^T\int_\Omega \phi(\partial_t m - \Delta m)\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant \int_\Omega m_0\phi(0)\,\mathrm{d}x + \lambda\int_0^T\int_\Omega \phi m\,\mathrm{d}x\,\mathrm{d}t + \chi\int_0^T\int_\Omega f_r(m)\nabla\phi\nabla c\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant \int_\Omega m_0\phi(0)\,\mathrm{d}x + (I) + (II),\end{aligned} \tag{2.27}$$

where

$$(I) = \int_0^T\int_\Omega \phi m\,\mathrm{d}x\,\mathrm{d}t \quad \text{and} \quad (II) = \chi\int_0^T\int_\Omega |f_r(m)||\nabla\phi||\nabla c|\,\mathrm{d}x\,\mathrm{d}t.$$

For the first term on the right-hand side of (2.27), we use

$$\|\phi(0)\|^{p'}_{L^{p'}(\Omega)} = \int_\Omega \left|\int_0^T \partial_t\phi\,\mathrm{d}t\right|^{p'}\mathrm{d}x \leqslant T^{\frac{1}{p-1}}\|\partial_t\phi\|^{p'}_{L^{p'}(Q_T)} \leqslant T^{\frac{1}{p-1}}C^{p'}_{p'}$$

thanks to (2.25), in order to estimate

$$\int_\Omega m_0\phi(0)\,\mathrm{d}x \leqslant \|m_0\|_{L^p(\Omega)}\|\phi(0)\|_{L^{p'}(\Omega)} \leqslant \|m_0\|_{L^p(\Omega)}T^{\frac{1}{p}}C_{p'}. \tag{2.28}$$

For the treatment of (I), we consider two cases.

- **Case 1.** $a \geqslant (n+2)/2$. Note that

$$a \geqslant \frac{n+2}{2} \Longrightarrow a > \frac{(n+2)p'}{(n+2)(p'-1)+2p'} \Longrightarrow \frac{a}{a-1} < \frac{(n+2)p'}{n+2-2p'} = q_2.$$

  Therefore, by using Hölder's inequality, we have (for some C_T depending only on Ω, T, a, n and p)

$$\begin{aligned}(I) &\leqslant \|\phi\|_{L^{\frac{a}{a-1}}(Q_T)}\|m\|_{L^a(Q_T)} \leqslant C_T\,\|\phi\|_{L^{q_2}(Q_T)}\|m\|_{L^a(Q_T)}\\ &\leqslant C_T\,C_{p'}\,[K_1\,(T+1)]^{1/a},\end{aligned} \tag{2.29}$$

  thanks to (2.16) and (2.25).

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- **Case 2.** $a < (n+2)/2$. Define $\vartheta = 1 - 2a/(n+2)$ and note that $0 < \vartheta < 1$. From Hölder's inequality, there exists $R_{T,|\Omega|} > 0$ (depending only on Ω, T, a, n and p) such that

$$\begin{aligned}(I) &= \int_0^T \int_\Omega \phi m^{1-\vartheta} m^\vartheta \,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant R_{T,|\Omega|} \left(\int_0^T \int_\Omega \phi^{p'} m^{p'(1-\vartheta)} \right)^{\frac{1}{p'}} \|m\|^\vartheta_{L^p(Q_T)} \\ &\leqslant R_{T,|\Omega|} \, \|\phi\|_{L^{\frac{(n+2)p'}{n+2-2p'}}(Q_T)} \left(\int_0^T \int_\Omega m^{\frac{(n+2)(1-\vartheta)}{2}} \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{2}{n+2}} \|m\|^\vartheta_{L^p(Q_T)} \\ &= R_{T,|\Omega|} \, \|\phi\|_{L^{q_2}(Q_T)} \|m\|^{(1-\vartheta)}_{L^a(Q_T)} \|m\|^\vartheta_{L^p(Q_T)},\end{aligned} \tag{2.30}$$

using

$$q_2 = \frac{(n+2)p'}{n+2-2p'} \quad \text{and} \quad \frac{(n+2)(1-\vartheta)}{2} = a. \tag{2.31}$$

Therefore,

$$(I) \leqslant R_{T,\Omega} \, C_{p'} \, [K_1(T+1)]^{\frac{1-\vartheta}{a}} \, \|m\|^\vartheta_{L^p(Q_T)}. \tag{2.32}$$

From these two cases, or more precisely, from (2.29) and (2.32), we have

$$(I) \leqslant C_T^* \left(1 + \|m\|^\vartheta_{L^p(Q_T)} \right) \tag{2.33}$$

for some $0 < \vartheta < 1$ (depending on a, n), and C_T^* (depending on Ω, T, n, p, a, β, δ, $\|m_0\|_{L^1(\Omega)}$, $\|c_0\|_{L^1(\Omega)}$, $\|d_0\|_{L^\infty(\Omega)}$, but not depending on λ).

In order to estimate (II), we write

$$\begin{aligned}(II) &= \chi \int_0^T \int_\Omega |f_r(m)||\nabla\phi||\nabla c| \,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant \chi\gamma \int_0^T \int_\Omega |m|^b |\nabla\phi||\nabla c| \,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant \chi\gamma \left(\int_0^T \int_\Omega m^{b-s} |\nabla\phi||\nabla c|^{p'} \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{p'}} \|m\|^s_{L^p(Q_T)} \\ &= \chi\gamma \left(\int_0^T \int_\Omega m^{p'(b-s)} |\nabla\phi|^{p'} |\nabla c|^{p'} \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{p'}} \|m\|^s_{L^p(Q_T)}.\end{aligned} \tag{2.34}$$

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Note that $1 < p' < n+2$ (see (2.23)). We can use Hölder's inequality to estimate further

$$(II) \leqslant \chi\gamma \, \|\nabla\phi\|_{L^{\frac{(n+2)p'}{n+2-p'}}(Q_T)} \left(\int_0^T \int_\Omega m^{(n+2)(b-s)} |\nabla c|^{n+2} \, \mathrm{d}x \, \mathrm{d}t \right)^{\frac{1}{n+2}} \|m\|^s_{L^p(Q_T)}$$
$$\leqslant \chi\gamma \, \|\nabla\phi\|_{L^{q_3}(Q_T)} \|m\|^{b-s}_{L^a(Q_T)} \|\nabla c\|_{L^{q_4}(Q_T)} \|m\|^s_{L^p(Q_T)},$$

where

$$q_4 = \frac{(n+2)a}{a-(n+2)(b-s)} \tag{2.35}$$

is well defined since $(n+2)(b-s) < a$ (see (2.20)). Therefore, we can apply (2.25), (2.16) in order to get

$$(II) \leqslant \chi\gamma \, C_{p'} \, [K_1(T+1)]^{\frac{b-s}{a}} \, \|\nabla c\|_{L^{q_4}(Q_T)} \|m\|^s_{L^p(Q_T)}. \tag{2.36}$$

Note that from (2.35) we see that when $a < n+2$,

$$q_4 = \frac{(n+2)a}{a-(n+2)(b-s)} < \frac{(n+2)a}{n+2-a} = q_1, \tag{2.37}$$

and that when $a \geqslant n+2$, $q_4 < q_1$ since $q_1 < +\infty$ can be chosen arbitrarily. Therefore (for some $C_T > 0$ depending only on Ω, T, a, b, n),

$$\|\nabla c\|_{L^{q_4}(Q_T)} \leqslant C_T \|\nabla c\|_{L^{q_1}(Q_T)} \leqslant C_T \, K_2 \, K_1 \, (T+1),$$

thanks to (2.17). It then follows from (2.36) that

$$(II) \leqslant D_T^* \, \|m\|^s_{L^p(Q_T)}, \tag{2.38}$$

where $D_T^* > 0$ only depends on Ω, T, n, p, a, b, β, δ, ε_0, χ, γ, $\|m_0\|_{L^1(\Omega)}$, $\|c_0\|_{W^{\tilde{p},(2-2/\tilde{p})}(\Omega)}$, $\|d_0\|_{L^\infty(\Omega)}$, but does not depend on λ.

From (2.27), (2.28), (2.33) and (2.38), we get (for $E^* := C_{p'} T^{1/p} + C_T^* + D_T^*$)

$$\int_0^T \int_\Omega m\theta \, \mathrm{d}x \, \mathrm{d}t \leqslant E^* \left[\|m_0\|_{L^p(\Omega)} + 1 + \|m\|^\vartheta_{L^p(Q_T)} + \|m\|^s_{L^p(Q_T)} \right]$$

for all $0 \leqslant \theta \in L^{p'}(Q_T)$ satisfying $\|\theta\|_{L^{p'}(Q_T)} = 1$. Therefore, by duality, we get

$$\|m\|_{L^p(Q_T)} \leqslant E^* \left[\|m_0\|_{L^p(\Omega)} + 1 + \|m\|^\vartheta_{L^p(Q_T)} + \|m\|^s_{L^p(Q_T)} \right].$$

Note that $0 < \vartheta, s < 1$. Thus we can use Young's inequality in order to get

$$\|m\|_{L^p(Q_T)} \leqslant F_p^* \, \left(1 + \|m_0\|_{L^p(\Omega)} \right), \tag{2.39}$$

where F_p^* depends on the same quantities as D_T^*.

From (2.23), we recall from $p = \frac{p'}{p'-1}$ that $1 < p < \infty$ can be chosen arbitrarily. We also insist that the bound in (2.39) does not depend on $\lambda \in (0,1]$. □

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We now use lemma 2.5 in order to conclude the proof of proposition 2.2.

Since $\tilde{p} > n+2$, lemma 2.1 guarantees $W^{2,1}_{\tilde{p}}(Q_T) \hookrightarrow \mathfrak{Y}$ compactly. Applying this to the equation satisfied by c in (2.13), we get (for some constant $C_T > 0$ depending only on Ω, T, n, p and ε_0)

$$\|c\|_{L^\infty(Q_T)} \leqslant C_T \left(\|\lambda(\delta d + \beta m)\|_{L^{\tilde{p}}(Q_T)} + \|c_0\|_{W^{\tilde{p},(2-2/\tilde{p})}(\Omega)} \right) \leqslant \S_T, \tag{2.40}$$

for some $\S_T > 0$ (depending only on the same quantities as D^*_T, except $\tilde{p}$).

Applying the heat semigroup property and maximal regularity, we can find $\S^*_T$ (depending on the same quantities as D^*_T) such that

$$\|\nabla c\|_{L^\infty(Q_T)} + \|c\|_{W^{2,1}_{\tilde{p}}(Q_T)} \leqslant \S^*_T, \tag{2.41}$$

and therefore

$$\|c\|_{\mathfrak{Y}} \leqslant S_T + \S^*_T. \tag{2.42}$$

We now consider the equation satisfied by m in (2.13):

$$\begin{aligned} \partial_t m &= \Delta m + \lambda m(1 - m^{a-1}) - \chi \nabla \cdot (f_r(m) \nabla c) \\ &= \Delta m - \chi f_r'(m) \nabla c \cdot \nabla m + \lambda m(1 - m^{a-1}) - \chi f_r(m) \Delta c. \end{aligned} \tag{2.43}$$

From (2.39) with $p = \tilde{p}$, (2.42), and assumption (1.3), we apply the properties of regularity for parabolic equations (cf [**11**]) to (2.43), and get

$$\begin{aligned} \|m\|_{W^{2,1}_{\tilde{p}}(Q_T)} &\leqslant C_T \left(\left\| \lambda m(1 - m^{a-1}) - \chi f_r(m) \Delta c \right\|_{L^{\tilde{p}}(Q_T)} + \|m_0\|_{W^{\tilde{p},(2-2/\tilde{p})}(\Omega)} \right) \\ &\leqslant \S^{**}_T, \end{aligned}$$

for some $\S^{**}_T > 0$ depending on the same parameters as $\S_T$. By the embedding (2.7)

$$\|m\|_{\mathfrak{Y}} \leqslant C_T \|m\|_{W^{2,1}_{\tilde{p}}(Q_T)} \leqslant C_T \, \S^{**}_T. \tag{2.44}$$

Note that the bounds (2.42) and (2.44) do not depend on $\lambda \in (0, 1]$ (or (m, c)). This means that the set $\mathbf{Z}$ defined in (2.11) is bounded in $\mathfrak{X} = \mathfrak{Y} \times \mathfrak{Y}$. This shows that the last assumption of Leray-Schauder fixed point theorem holds, and therefore that the mapping $\mathcal{F}_r$ has a fixed point, which satisfies as a consequence system (2.13), together with estimates (2.42), (2.44). Since moreover the upper bound $\S_T$ in (2.44) does not depend on r, by taking $r \geqslant \S_T$, we obtain a solution to the original system (1.1) (for all $T > 0$). This solution is strong in the sense that all terms appearing in the system are defined a.e., (as $L^1(Q_T)$ functions).

2.2. Uniqueness

In this subsection, we write down a stability result for strong solutions of system (1.1), which entails the uniqueness result in theorem 1.1. More precisely, we write the

PROPOSITION 2.6. *Let $T > 0$ and let Ω be a smooth ($C^{2+\alpha}$ for some $\alpha > 0$) bounded connected open set of $\mathbb{R}^n$ (for $n \in \mathbb{N} - \{0\}$). In addition, assume $a > 1$, $\chi, \varepsilon_0, \delta, \beta >$*

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0, $\gamma, b, l \geqslant 0$, *and* f *and* g *satisfying (F) and (G). We consider two sets of nonnegative (for each component), initial data* (m_{01}, c_{01}, d_{01}) *and* (m_{02}, c_{02}, d_{02}) *in* $W^{(2-2/\tilde{p}),\tilde{p}}(\Omega) \times W^{(2-2/\tilde{p}),\tilde{p}}(\Omega) \times L^{\infty}(\Omega)$ *for some* $\tilde{p} > \max\{n+2, a\}$*, and two sets of nonnegative (for each component) strong (in the sense of theorem 1.1) solutions* (m_1, c_1, d_1) *and* (m_2, c_2, d_2) *to system (1.1) (with corresponding initial data) on* Q_T*, satisfying estimate (2.1). We denote* $\mu := \|d_2\|_{L^{\infty}(Q_T)}$, $\mu_c := \|\nabla c_1\|_{L^{\infty}(Q_T)}$, $\mu_m := \max(\|m_1\|_{L^{\infty}(Q_T)}, \|m_2\|_{L^{\infty}(Q_T)})$, $\mu_f := \|f\|_{L^{\infty}([0,\mu_m])}$, $\mu_{f'} := \|f'\|_{L^{\infty}([0,\mu_m])}$, $\mu_{g'} := \|g'\|_{L^{\infty}([0,\mu_m])}$.

Then, for all $t \in [0, T]$,

$$\begin{aligned}
&\int_{\Omega} |m_1(\cdot,t) - m_2(\cdot,t)|^2 \, dx + \frac{\chi^2 \mu_f^2}{2\varepsilon_0} \int_{\Omega} |c_1(\cdot,t) - c_2(\cdot,t)|^2 \, dx \\
&\qquad + \int_{\Omega} |d_1(\cdot,t) - d_2(\cdot,t)|^2 \, dx \\
&\quad \leqslant e^{G\,t} \bigg(\int_{\Omega} |m_{10} - m_{20}|^2 \, dx + \frac{\chi^2 \mu_f^2}{2\varepsilon_0} \int_{\Omega} |c_{10} - c_{20}|^2 \, dx + \int_{\Omega} |d_{10} - d_{20}|^2 \, dx \bigg),
\end{aligned} \tag{2.45}$$

where

$$G := \max\left(2 + \chi^2 \mu_{f'}^2 \mu_c^2 + \frac{\chi^2 \beta \mu_f^2}{2\varepsilon_0} + \mu_{g'}(1+\mu); \beta + \delta; \frac{\chi^2 \delta \mu_f^2}{2\varepsilon_0} + (1+\mu)\right).$$

Proof. Substracting the equations satisfied by m_1 and m_2, c_1 and c_2, and finally d_1 and d_2, and then performing integrations by parts, we end up with the identities:

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} |m_1 - m_2|^2 \, \mathrm{d}x + \int_{\Omega} |\nabla(m_1 - m_2)|^2 \, \mathrm{d}x + \int_{\Omega} (m_1^a - m_2^a)(m_1 - m_2) \, \mathrm{d}x \\
&\quad = \int_{\Omega} |m_1 - m_2|^2 \, \mathrm{d}x + \chi \int_{\Omega} f(m_2) \nabla(m_1 - m_2) \cdot \nabla(c_1 - c_2) \, \mathrm{d}x \\
&\qquad + \chi \int_{\Omega} (f(m_1) - f(m_2)) \nabla(m_1 - m_2) \cdot \nabla c_1 \, \mathrm{d}x,
\end{aligned} \tag{2.46}$$

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} |c_1 - c_2|^2 \, \mathrm{d}x + \varepsilon_0 \int_{\Omega} |\nabla(c_1 - c_2)|^2 \, \mathrm{d}x + \int_{\Omega} |c_1 - c_2|^2 \, \mathrm{d}x \\
&\quad = \delta \int_{\Omega} (c_1 - c_2)(d_1 - d_2) \, \mathrm{d}x + \beta \int_{\Omega} (c_1 - c_2)(m_1 - m_2) \, \mathrm{d}x,
\end{aligned} \tag{2.47}$$

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} |d_1 - d_2|^2 \, \mathrm{d}x + \int_{\Omega} g(m_1) |d_1 - d_2|^2 \, \mathrm{d}x \\
&\quad = -\int_{\Omega} (1 - d_2)(d_1 - d_2)(g(m_1) - g(m_2)) \, \mathrm{d}x.
\end{aligned} \tag{2.48}$$

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Then we perform the following estimates (the last one uses Young's inequality):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |d_1-d_2|^2\,\mathrm{d}x \leqslant (1+\mu)\int_\Omega |d_1-d_2|^2\,\mathrm{d}x + (1+\mu)\,\mu_{g'}^2\int_\Omega |m_1-m_2|^2\,\mathrm{d}x, \tag{2.49}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |c_1-c_2|^2\,\mathrm{d}x + 2\varepsilon_0\int_\Omega |\nabla(c_1-c_2)|^2\,\mathrm{d}x \\ \leqslant \delta\int_\Omega |d_1-d_2|^2\,\mathrm{d}x + \beta\int_\Omega |m_1-m_2|^2\,\mathrm{d}x + (\beta+\delta)\int_\Omega |c_1-c_2|^2\,\mathrm{d}x,\end{aligned} \tag{2.50}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |m_1-m_2|^2\,\mathrm{d}x \leqslant (2+\chi^2\,\mu_{f'}^2\mu_c^2)\int_\Omega |m_1-m_2|^2\,\mathrm{d}x \\ + \chi^2\mu_f^2\int_\Omega |\nabla(c_1-c_2)|^2\,\mathrm{d}x.\end{aligned} \tag{2.51}$$

Estimate (2.45) is directly obtained from these estimates by an application of Gronwall's lemma to the quantity

$$\int_\Omega |m_1-m_2|^2\,\mathrm{d}x + \frac{\chi^2\,\mu_f^2}{2\varepsilon_0}\int_\Omega |c_1-c_2|^2\,\mathrm{d}x + \int_\Omega |d_1-d_2|^2\,\mathrm{d}x.$$

Note that uniqueness in theorem 1.1 is a direct consequence of proposition 2.6. Moreover, since T is arbitrary in propositions 2.2 and 2.6, one can build the unique solution of system (1.1) on $\mathbb{R}_+$ that appears in the conclusion of theorem 1.1 by patching together the solutions defined on finite time intervals. □

2.3. Uniform-in-time bounds

In this section, we will conclude the proof of theorem 1.1 by showing that the unique solution to (1.1) obtained in propositions 2.2 and 2.6 is globally (w.r.t. time) bounded. Thanks to proposition 2.2, it is sufficient to show that the sequences $\left\{\|m\|_{L^\infty(Q_{\tau,\tau+1})}\right\}_{\tau=1}^\infty$ and $\left\{\|c\|_{L^\infty(Q_{\tau,\tau+1})}\right\}_{\tau=1}^\infty$ are bounded. As a consequence, we will obtain estimates in this subsection in which all constants *do not depend on* τ. In addition, we will see that the lim sup of each sequence is independent of m_0 and c_0. We first recall the uniform in time bound

$$\sup_{t\in\mathbb{R}_+}\|d(\cdot,t)\|_{L^\infty(\Omega)} \leqslant \mu := \max\left\{1, \|d_0\|_{L^\infty(\Omega)}\right\}.$$

Coming back to the proof of lemma 2.3, and specializing it in the case when $\lambda = 1$, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (m+c) \leqslant |\Omega|\,(\delta\,\mu + k_a) - \int_\Omega (m+c),$$

so that

$$\sup_{t\in\mathbb{R}_+}\left(\|m(t)\|_{L^1(\Omega)} + \|c(t)\|_{L^1(\Omega)}\right) \leqslant K_1^*, \tag{2.52}$$

where $K_1^* := \max(|\Omega|\,(\delta\,\mu + k_a), \|m_0\|_{L^1(\Omega)} + \|c_0\|_{L^1(\Omega)})$, and

$$\limsup_{t\to\infty}\left(\|m(t)\|_{L^1(\Omega)} + \|c(t)\|_{L^1(\Omega)}\right) \leqslant |\Omega|\,(\delta\,\mu + k_a). \tag{2.53}$$

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We also recall that k_a and μ only depend on β, a and $\|d_0\|_{L^\infty(\Omega)}$.

Then, we prove the

LEMMA 2.7. *Under the assumptions of theorem 1.1, we consider (m, c, d), the unique solution to (1.1) on $\Omega \times \mathbb{R}_+$. Then*

$$\sup_{\tau\in\mathbb{N}} \|m\|_{L^a(Q_{\tau,\tau+2})} \leqslant (3\,K_1^*)^{\frac{1}{a}}, \tag{2.54}$$

and

$$\limsup_{\tau\to\infty} \|m\|_{L^a(Q_{\tau,\tau+2})} \leqslant (3\,|\Omega|\,(\delta\,\mu + k_a))^{\frac{1}{a}}. \tag{2.55}$$

Proof. By integrating the equation satisfied by m on $Q_{\tau,\tau+2}$, we have

$$\int_\Omega m(x,\tau+2)\,\mathrm{d}x = \int_\Omega m(x,\tau)\,\mathrm{d}x + \int_\tau^{\tau+2}\int_\Omega m(x,s)(1-m(x,s)^{a-1})\,\mathrm{d}x\,\mathrm{d}s.$$

Thus

$$\int_\tau^{\tau+2}\int_\Omega m(x,s)^a\,\mathrm{d}x\,\mathrm{d}s \leqslant \|m(\tau)\|_{L^1(\Omega)} + \int_\tau^{\tau+2}\int_\Omega m(x,s)\,\mathrm{d}x\,\mathrm{d}s,$$

which proves both (2.54) and (2.55). □

LEMMA 2.8. *Under the assumptions of theorem 1.1, we consider (m, c, d), the unique solution to (1.1) on $\Omega \times \mathbb{R}_+$. Let q_1 be defined as in (2.18). Then, there exists $K_3 > 0$ (depending on Ω, n, a, $\varepsilon_0, \beta, \delta$ and $\|m_0\|_{L^1(\Omega)}$, $\|c_0\|_{L^1(\Omega)}$, $\|d_0\|_{L^\infty(\Omega)}$, but not on τ) and $\tilde{K}_3 > 0$ (depending on Ω, n, a, $\varepsilon_0, \beta, \delta$ and $\|d_0\|_{L^\infty(\Omega)}$, but not on τ) such that*

$$\sup_{\tau\in\mathbb{N}-\{0\}} \|\nabla c\|_{L^{q_1}(Q_{\tau,\tau+1})} \leqslant K_3, \tag{2.56}$$

and

$$\limsup_{\tau\to\infty} \|\nabla c\|_{L^{q_1}(Q_{\tau,\tau+1})} \leqslant \tilde{K}_3. \tag{2.57}$$

Proof. We define a smooth cutoff function $\psi \in C^\infty(\mathbb{R}; [0,1])$ such that $\psi(s) = 0$ for $s \leqslant 0$, and $\psi(s) = 1$ for $s \geqslant 1$. Moreover, we assume that $|\psi'(s)| \leqslant 2$ for all $s \in \mathbb{R}$. For any $\tau \in \mathbb{N}$, the shifted cutoff function is defined by $\psi_\tau(\cdot) := \psi(\cdot - \tau)$. By multiplying the equation satisfied by c by ψ_τ, we get

$$\partial_t(\psi_\tau c) = \varepsilon_0\Delta(\psi_\tau c) + \psi_\tau' c - \psi_\tau c + \psi_\tau[\delta d + \beta m], \tag{2.58}$$

and $(\psi_\tau c)(x,\tau) = 0$. By using the semigroup properties of the heat equation (lemma 2.1), we see that (for some $C > 0$ depending only on Ω, n, a, ε_0, but

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not τ)

$$\|\psi_\tau c\|_{L^{q_5}(Q_{\tau,\tau+2})} \leqslant C(\|\psi'_\tau c\|_{L^a(Q_{\tau,\tau+2})} + \|\psi_\tau c\|_{L^a(Q_{\tau,\tau+2})} + \|\psi_\tau(\delta d + \beta m)\|_{L^a(Q_{\tau,\tau+2})}), \tag{2.59}$$

where

$$q_5 = \begin{cases} \dfrac{(n+2)a}{n+2-2a} & \text{if } a < \dfrac{n+2}{2}, \\ < +\infty \text{ arbitrary} & \text{if } a = \dfrac{n+2}{2}, \\ = +\infty & \text{if } a > \dfrac{n+2}{2}. \end{cases}$$

In any case, we have $a < q_5$. By using the boundedness of ψ'_τ and ψ_τ, and Hölder's inequality, we have, for $\alpha \in (0,1)$ satisfying $(1/a) = (1-\alpha/1) + (\alpha/q_5)$,

$$\begin{aligned} \|\psi'_\tau c\|_{L^a(Q_{\tau,\tau+2})} &\leqslant 2\|c\|_{L^a(Q_{\tau,\tau+2})} \leqslant 2\|c\|^{1-\alpha}_{L^1(Q_{\tau,\tau+2})}\|c\|^{\alpha}_{L^{q_5}(Q_{\tau,\tau+2})} \\ &\leqslant 2(2K_1^*)^{1-\alpha}\,\|c\|^{\alpha}_{L^{q_5}(Q_{\tau,\tau+2})}, \end{aligned} \tag{2.60}$$

and similarly

$$\|\psi_\tau c\|_{L^a(Q_{\tau,\tau+2})} \leqslant (2K_1^*)^{1-\alpha}\,\|c\|^{\alpha}_{L^{q_5}(Q_{\tau,\tau+2})}. \tag{2.61}$$

From the uniform boundedness of d and (2.54), we see that

$$\|\psi_\tau(\delta d + \beta m)\|_{L^a(Q_{\tau,\tau+2})} \leqslant (3K_1^*)^{\frac{1}{a}}\beta + \delta\,\mu\,(2|\Omega|)^{1/a}. \tag{2.62}$$

Inserting (2.60)–(2.62) into (2.59) and using $\psi_\tau \equiv 1$ on $(\tau+1, \tau+2)$ yields, for some constant $C_2 > 0$ (depending only on Ω, n, δ, β, a, ε_0, $\|m_0\|_{L^1(\Omega)}$, $\|c_0\|_{L^1(\Omega)}$, $\|d_0\|_{L^\infty(\Omega)}$, but not τ)

$$\|c\|_{L^{q_5}(Q_{\tau+1,\tau+2})} \leqslant C_2\,\|c\|^{\alpha}_{L^{q_5}(Q_{\tau,\tau+2})} + C_2. \tag{2.63}$$

Denote by $\Lambda := \left\{\tau + 1 \in \mathbb{N} \,:\, \|c\|_{L^{q_5}(Q_{\tau,\tau+1})} \leqslant \|c\|_{L^{q_5}(Q_{\tau+1,\tau+2})}\right\}$. Then for all $\tau \in \Lambda$, we get from (2.63) that

$$\|c\|_{L^{q_5}(Q_{\tau+1,\tau+2})} \leqslant 2C_2\,\|c\|^{\alpha}_{L^{q_5}(Q_{\tau+1,\tau+2})} + C_2 \leqslant \frac{1}{2}\|c\|_{L^{q_5}(Q_{\tau+1,\tau+2})} + C_3,$$

where $C_3 > 0$ only depends on the same parameters as C_2. Thus

$$\|c\|_{L^{q_5}(Q_{\tau+1,\tau+2})} \leqslant 2C_3 \quad \text{for all} \quad \tau \in \Lambda.$$

By the definition of Λ, this in fact implies that

$$\|c\|_{L^{q_5}(Q_{\tau+1,\tau+2})} \leqslant 2C_3 \quad \text{for all} \quad \tau \in \mathbb{N}.$$

Since $a < q_5$, we get with $\frac{1}{a} = \frac{1-\alpha}{1} + \frac{\alpha}{q_5}$,

$$\sup_{\tau\in\mathbb{N}-\{0\}} \|c\|_{L^a(Q_{\tau,\tau+1})} \leqslant \sup_{\tau\in\mathbb{N}-\{0\}} \|c\|^{1-\alpha}_{L^1(Q_{\tau,\tau+1})}\|c\|^{\alpha}_{L^{q_5}(Q_{\tau,\tau+1})} \leqslant (K_1^*)^{1-\alpha}(2C_3)^{\alpha}. \tag{2.64}$$

We can now use this estimate, the uniform boundedness of d, and (2.54), and apply the semigroup properties of the heat kernel to (2.58), and get (for some constant

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$C > 0$ only depending on Ω, n, ε_0 and a),

$$\begin{aligned}\|\psi_\tau|\nabla c|\|_{L^{q_1}(Q_{\tau,\tau+2})} &\leqslant C\,\|\psi'_\tau c - \psi_\tau c + \psi_\tau[\delta d + \beta m]\|_{L^a(Q_{\tau,\tau+2})}\\ &\leqslant C\,(6(K_1^*)^{1-\alpha}(2C_3)^\alpha + (3K_1^*)^{\frac{1}{a}}\beta + \delta\,\mu\,(2|\Omega|)^{1/a}).\end{aligned}$$

Finally, by using $\psi_\tau \equiv 1$ on $(\tau+1, \tau+2)$, we obtain (2.56). Similarly, using (2.55), we obtain (2.57). □

Using lemmas 2.7 and 2.8, we obtain the following result:

LEMMA 2.9. *Under the assumptions of theorem 1.1, we consider (m, c, d), the unique solution to (1.1) on $\Omega \times \mathbb{R}_+$. For any $1 \leqslant p < \infty$, there exist a constant $K_p > 0$ (depending on Ω, n, p, a, β, δ, ε_0 and $\|m_0\|_{L^1(\Omega)}$, $\|c_0\|_{L^1(\Omega)}$, $\|d_0\|_{L^\infty(\Omega)}$, but not on τ), and a constant $\tilde{K}_p > 0$ (depending on Ω, n, p, a, β, δ, ε_0 and $\|d_0\|_{L^\infty(\Omega)}$, but not on τ) such that*

$$\sup_{\tau\in\mathbb{N}-\{0\}} \|m\|_{L^p(Q_{\tau,\tau+1})} \leqslant K_p, \tag{2.65}$$

and

$$\limsup_{\tau\to\infty} \|m\|_{L^p(Q_{\tau,\tau+1})} \leqslant \tilde{K}_p. \tag{2.66}$$

Proof. The proof of this lemma is similar to that of lemma 2.5, except that we will consider the cylinder $Q_{\tau,\tau+2}$ instead of Q_T. More precisely, by multiplying the equation satisfied by m in (1.1) with ψ_τ, we get

$$\partial_t(\psi_\tau m) = \Delta(\psi_\tau m) + \psi'_\tau m + \psi_\tau m(1 - m^{a-1}) - \chi\psi_\tau\nabla\cdot(f(m)\nabla c), \tag{2.67}$$

and $(\psi_\tau m)(x, \tau) = 0$. Let $s \in (0, 1)$ satisfy (2.20), and choose p' as in (2.23). Consider $0 \leqslant \theta \in L^{p'}(Q_{\tau,\tau+2})$ with $\|\theta\|_{L^{p'}(Q_{\tau,\tau+2})} = 1$. Let ϕ be the unique nonnegative solution to the equation

$$\begin{cases}\partial_t\phi + \Delta\phi = -\theta, & (x,t) \in Q_{\tau,\tau+2},\\ \nabla\phi\cdot\nu = 0, & (x,t) \in \partial\Omega \times (\tau, \tau+2),\\ \phi(x, \tau+2) = 0, & x \in \Omega.\end{cases}$$

From the semigroup properties of the heat equation (lemma 2.1), we know that there exists a constant $L_{p'} > 0$ (depending on Ω, n, p but not τ) such that

$$\|\phi\|_{L^{q_2}(Q_{\tau,\tau+2})} + \|\nabla\phi\|_{L^{q_3}(Q_{\tau,\tau+2})} \leqslant L_{p'}, \tag{2.68}$$

with q_2 and q_3 defined in (2.26). By integration by parts, we estimate similarly to (2.27) and get

$$\begin{aligned}\int_\tau^{\tau+2}\int_\Omega (\psi_\tau m)\theta\,\mathrm{d}x\,\mathrm{d}t &\leqslant \int_\tau^{\tau+2}\int_\Omega (\psi'_\tau m + \psi_\tau m)\,\phi\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \chi\int_\tau^{\tau+2}\int_\Omega |f(m)||\nabla\phi||\nabla c|\,\mathrm{d}x\,\mathrm{d}t\\ &=: (III) + (IV).\end{aligned} \tag{2.69}$$

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We estimate (III) similarly to (I) in (2.27). More precisely, in the case when $a \geqslant (n+2)/2$, we have (for some $C_4 > 0$ depending only on Ω, n, p, a, β, δ, ε_0 and $\|m_0\|_{L^1(\Omega)}$, $\|c_0\|_{L^1(\Omega)}$, $\|d_0\|_{L^\infty(\Omega)}$)

$$(III) \leqslant 3\|\phi\|_{L^{\frac{a}{a-1}}(Q_{\tau,\tau+2})}\|m\|_{L^a(Q_{\tau,\tau+2})} \leqslant C_4, \tag{2.70}$$

and in the case when $a < (n+2)/2$, we have (for some $C_5 > 0$ depending only on Ω, and some $C_6 > 0$ depending on the same quantities as C_4)

$$\begin{aligned}(III) &\leqslant 3\int_\tau^{\tau+2}\int_\Omega \phi m^{1-\vartheta} m^\vartheta \,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant C_5\,\|\phi\|_{L^{q_2}(Q_{\tau,\tau+2})}\|m\|^{1-\vartheta}_{L^a(Q_{\tau,\tau+2})}\|m\|^{\vartheta}_{L^p(Q_{\tau,\tau+2})} \\ &\leqslant C_6\,\|m\|^{\vartheta}_{L^p(Q_{\tau,\tau+2})},\end{aligned} \tag{2.71}$$

where ϑ is defined in (2.31). It follows from (2.70) and (2.71) that, in any case, we have the estimate

$$(III) \leqslant \max(C_4, C_6)\,\left(1 + \|m\|^{\vartheta}_{L^p(Q_{\tau,\tau+2})}\right). \tag{2.72}$$

The term (IV) can be estimated similarly to (II) in (2.34) and (2.36). More precisely, it follows from Hölder's inequality that (for some $C_7, C_8 > 0$ depending only on Ω, and $C_9 > 0$ depending on the same quantities as C_4 and b, γ, χ)

$$\begin{aligned}(IV) &\leqslant \chi\gamma\int_\tau^{\tau+2}\int_\Omega |m|^b|\nabla\phi||\nabla c|\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant \chi\gamma\left(\int_\tau^{\tau+2}\int_\Omega m^{p'(b-s)}|\nabla\phi|^{p'}|\nabla c|^{p'}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p'}}\|m\|^s_{L^p(Q_{\tau,\tau+2})} \\ &\leqslant C_7\,\chi\gamma\|\nabla\phi\|_{L^{q_3}(Q_{\tau,\tau+2})}\|m\|^{b-s}_{L^a(Q_{\tau,\tau+2})}\|\nabla c\|_{L^{q_4}(Q_{\tau,\tau+2})}\|m\|^s_{L^p(Q_{\tau,\tau+2})} \\ &\qquad (q_4 \text{ is defined in (2.35)}) \\ &\leqslant C_8\,\chi\gamma L_{p'}\|m\|^{b-s}_{L^a(Q_{\tau,\tau+2})}\|\nabla c\|_{L^{q_1}(Q_{\tau,\tau+2})}\|m\|^s_{L^p(Q_{\tau,\tau+2})} \\ &\qquad (\text{since } q_4 < q_1, \text{see (2.37)}) \\ &\leqslant C_9\,\|m\|^s_{L^p(Q_{\tau,\tau+2})}.\end{aligned}$$

Inserting this and (2.72) into (2.69), we get

$$\int_\tau^{\tau+2}\int_\Omega (\psi_\tau m)\theta\,\mathrm{d}x\,\mathrm{d}t \leqslant \max(C_4, C_6, C_9)\,\left(1 + \|m\|^{\vartheta}_{L^p(Q_{\tau,\tau+2})} + \|m\|^s_{L^p(Q_{\tau,\tau+2})}\right)$$

for all $0 \leqslant \theta \in L^{p'}(Q_{\tau,\tau+2})$ satisfying $\|\theta\|_{L^{p'}(Q_{\tau,\tau+2})} = 1$. Therefore, by duality, we get (for $C_{10} := \max(C_4, C_6, C_9)$)

$$\|\psi_\tau m\|_{L^p(Q_{\tau,\tau+2})} \leqslant C_{10}\,\left(1 + \|m\|^{\vartheta}_{L^p(Q_{\tau,\tau+2})} + \|m\|^s_{L^p(Q_{\tau,\tau+2})}\right). \tag{2.73}$$

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Define $\Theta := \{\tau + 1 \in \mathbb{N} : \|m\|_{L^p(Q_{\tau,\tau+1})} \leqslant \|m\|_{L^p(Q_{\tau+1,\tau+2})}\}$. Then for any $\tau \in \Theta$, we have

$$\begin{aligned}\text{RHS of (2.73)} &\leqslant 2\,C_{10}\left(1 + \|m\|^{\vartheta}_{L^p(Q_{\tau+1,\tau+2})} + \|m\|^{s}_{L^p(Q_{\tau+1,\tau+2})}\right)\\ &\leqslant C_{11} + \frac{1}{2}\|m\|_{L^p(Q_{\tau+1,\tau+2})},\end{aligned}$$

(where C_{11} depends on the same quantities as C_9) thanks to the fact that $0 < \vartheta, s < 1$. From this, (2.73), and $\psi_\tau \equiv 1$ on $(\tau+1, \tau+2)$, we get

$$\|m\|_{L^p(Q_{\tau+1,\tau+2})} \leqslant C_{11} + \frac{1}{2}\|m\|_{L^p(Q_{\tau+1,\tau+2})} \quad \text{for all} \quad \tau \in \Theta,$$

and consequently

$$\|m\|_{L^p(Q_{\tau+1,\tau+2})} \leqslant 2\,C_{11} \quad \text{for all} \quad \tau \in \Theta.$$

Thanks to the definition of Θ, we finally obtain the desired estimate (2.65). In a similar manner, we obtain (2.66). □

We are now ready to show the uniform-in-time boundedness of the solution to (1.1).

The uniform-in-time boundedness of d is already shown. It remains to show it for m and c. We start by rewriting equation (2.58) as

$$\partial_t(\psi_\tau c) = \varepsilon_0\,\psi_\tau\,\Delta c + (\psi'_\tau - \psi_\tau)c + H, \quad \text{with } H = \psi_\tau[\delta d + \beta m].$$

Thanks to the lemma 2.9, we know that for any $1 \leqslant p < \infty$, $\sup_{\tau\in\mathbb{N}} \|H\|_{L^p(Q_{\tau,\tau+2})} \leqslant \beta\,K_p + \delta\,(2|\Omega|)^{1/p}$. Proceeding as in lemma 2.8, we see that for any $1 \leqslant p < \infty$, there exists $K^*_p > 0$ (depending on the same quantities as K_p) such that $\sup_{\tau\in\mathbb{N}} \|c\|_{L^p(Q_{\tau,\tau+2})} \leqslant K^*_p$. By maximal regularity, recalling that $(\psi_\tau c)(x,\tau) = 0$, we have (for some $C > 0$ depending on Ω, p, n, ε_0, but not τ)

$$\|\psi_\tau c\|_{W^{2,1}_p(Q_{\tau,\tau+2})} \leqslant C\|(\psi'_\tau - \psi_\tau)c + H\|_{L^p(Q_{\tau,\tau+2})},$$

so that picking $p > (n+2)/2$, we can use the semigroup properties of the heat kernel (lemma 2.1) in order to get

$$\|\psi_\tau c\|_{L^\infty(Q_{\tau,\tau+2})} \leqslant C\|\psi_\tau c\|_{W^{2,1}_p(Q_{\tau,\tau+2})} \leqslant C_{[\frac{n+2}{2}]+1}.$$

Due to $\psi_\tau \equiv 1$ on $(\tau+1, \tau+2)$, we finally obtain

$$\|c\|_{L^\infty(Q_{\tau+1,\tau+2})} \leqslant C_{[\frac{n+2}{2}]+1}.$$

Moreover, by picking $p > n+2$, we obtain also the uniform estimate for ∇c, i.e. (for some $C > 0$ depending on Ω, p, n, ε_0, but not on τ)

$$\|\nabla c\|_{L^\infty(Q_{\tau+1,\tau+2})} \leqslant \|\psi_\tau|\nabla c|\|_{L^\infty(Q_{\tau,\tau+2})} \leqslant C\|c\|_{W^{2,1}_p(Q_{\tau,\tau+2})} \leqslant C_{n+3}. \tag{2.74}$$

This concludes the proof of boundedness of c (and ∇c).

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For the uniform boundedness of m, we observe that from (2.67), one can write the Duhamel formula for all $t \in (\tau, \tau+2)$

$$\psi_\tau m(t) = \int_\tau^t S(t-\tau-s)\left[(\psi_\tau' m) + \psi_\tau m(1-m^{a-1}) - \chi\psi_\tau \nabla\cdot(f(m)\nabla c)\right](s)\,\mathrm{d}s$$

where $\{S(t)\}$ is the semigroup of the heat kernel with homogeneous Neumann boundary condition. We have the following estimates of $\{S(t)\}$, (cf [**16**]), for p and q well chosen,

$$\|S(t)f\|_{L^\infty(\Omega)} \leqslant C^*\left(1+t^{-\frac{n}{2p}}\right)\|f\|_{L^p(\Omega)} \quad \text{and}$$
$$\|S(t)\nabla f\|_{L^\infty(\Omega)} \leqslant C^*\left(1+t^{-\frac{1}{2}-\frac{n}{2q}}\right)\|f\|_{L^q(\Omega)},$$

where C^* only depends on Ω, n, p and q.

Using these estimates into the above Duhamel formula, we get

$$\begin{aligned}
&\sup_{t\in(\tau,\tau+2)} \|\psi_\tau m(t)\|_{L^\infty(\Omega)} \\
&\quad\leqslant 3\int_\tau^{\tau+2} \|[S(t-\tau-s)m](s)\|_{L^\infty(\Omega)}\,\mathrm{d}s \\
&\qquad + \int_\tau^{\tau+2} \|[S(t-\tau-s)m^a](s)\|_{L^\infty(\Omega)}\,\mathrm{d}s \\
&\qquad + \chi\int_\tau^{\tau+2} \|[S(t-\tau-s)\nabla\cdot(f(m)\nabla c)](s)\|_{L^\infty(\Omega)}\,\mathrm{d}s \\
&\quad\leqslant 3C^*\|m\|_{L^p(Q_{\tau,\tau+2})}\int_\tau^{\tau+2}\left(1+(t-\tau-s)^{-\frac{n}{2p}}\right)\mathrm{d}s \\
&\qquad + C^*\|m\|^a_{L^{ap}(Q_{\tau,\tau+2})}\int_\tau^{\tau+2}\left(1+(t-\tau-s)^{-\frac{n}{2p}}\right)\mathrm{d}s \\
&\qquad + C^*\gamma\chi\|\nabla c\|_{L^\infty(Q_{\tau,\tau+2})}\|m\|^b_{L^{bp}(Q_{\tau,\tau+2})}\int_\tau^{\tau+2}\left(1+(t-\tau-s)^{-\frac{1}{2}-\frac{n}{2p}}\right)\mathrm{d}s.
\end{aligned}$$

By applying lemma 2.9 and (2.74), we can choose $p > n$, which implies that the last three integrals converge and are bounded w.r.t τ. We finally obtain the estimate (for a constant $C > 0$ depending on the parameters stated in theorem 1.1)

$$\sup_{t\in(\tau,\tau+2)} \|\psi_\tau m\|_{L^\infty(\Omega)} \leqslant C.$$

That is, (1.5) is proved. The result in (1.6) is obtained in a similar manner, by applying the earlier lim sup estimates and parabolic regularity, concluding the proof of theorem 1.1.

3. Nonlinear stability and Turing instability

We study in this section the homogeneous equilibria of system (1.1). Those equilibria $(\bar m, \bar c, \bar d)$ satisfy $\bar m = \bar m^a$, $\delta\,\bar d + \beta\,\bar m = \bar c$, and $g(\bar m)\,(1-\bar d) = 0$. They are therefore given by $(1, \beta+\delta, 1)$, $(0, \delta, 1)$, and, if $g(0) = 0$, $(0, \delta\,\zeta, \zeta)$ for all $\zeta \geqslant 0$.

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It is easy to see that the equilibria $(0, \delta, 1)$, and (if $g(0) = 0$) $(0, \delta\,\zeta, \zeta)$ for all $\zeta \geqslant 0$, are linearly unstable by considering x-independent solutions to (1.1).

In this section, we assume that $f(1) > 0$.

In order to study the stability of the positive equilibrium $(\bar{m}, \bar{c}, \bar{d}) = (1, \beta + \delta, 1)$, we consider $\tilde{m} := m - \bar{m}$, $\tilde{c} := c - \bar{c}$, $\tilde{d} := d - \bar{d}$. The system (1.1) can be rewritten under the form

$$\partial_t \tilde{X} = L_1\,\tilde{X} + L_2\,\tilde{X} + R(\tilde{X}), \tag{3.1}$$

where

$$\tilde{X} := \begin{pmatrix} \tilde{m} \\ \tilde{c} \\ \tilde{d} \end{pmatrix}.$$

The linear part of the system is associated to the matrices

$$L_1 := \begin{pmatrix} 1-a & 0 & 0 \\ \beta & -1 & \delta \\ 0 & 0 & -g(1) \end{pmatrix}, \qquad L_2 := \begin{pmatrix} \Delta & -\chi\, f(1)\,\Delta & 0 \\ 0 & \varepsilon_0\,\Delta & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the nonlinear part of the system is

$$\begin{aligned} R(\tilde{X}) &:= \begin{pmatrix} R_1(\tilde{X}) \\ R_2(\tilde{X}) \\ R_3(\tilde{X}) \end{pmatrix} \\ &= \begin{pmatrix} -[(1+\tilde{m})^a - 1 - a\,\tilde{m}] - \chi \nabla \cdot \{[f(1+\tilde{m}) - f(1)]\,\nabla \tilde{c}\} \\ 0 - [g(1+\tilde{m}) - g(1)]\,\tilde{d} \end{pmatrix}. \end{aligned}$$

We will show that when χ is small enough, the steady state $(\bar{m}, \bar{c}, \bar{d})$ is nonlinearly asymptotically stable. This is obtained by first proving that the linear part possesses a spectral gap. This implies the exponential decay (for the linear part) of the solution. Thanks to the fact that the solutions are uniformly bounded in time, we then show that the nonlinear part is dominated by the linear one provided that the initial data are close enough to $(\bar{m}, \bar{c}, \bar{d})$, and thus finally obtain the nonlinear stability.

Denoting by $(\lambda_n)_{n\in\mathbb{N}}$ the eigenvalues of $-\Delta$ (with Neumann boundary condition) on Ω in such a way that $\lambda_0 = 0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \to +\infty$, we see that the matrix of the linearized system associated to (3.1) projected on the eigenspace $\mathbb{P}\,e_n$ of $L^2(\Omega)$ associated to λ_n, is given by

$$M_n := \begin{pmatrix} 1 - a - \lambda_n & \chi\, f(1)\,\lambda_n & 0 \\ \beta & -1 - \varepsilon_0\,\lambda_n & \delta \\ 0 & 0 & -g(1) \end{pmatrix}.$$

Recalling that $g(1) > 0$, we see that the spectral properties of M_n can be studied by computing the trace and the determinant of the extracted (from M_n) matrix

$$N_n := \begin{pmatrix} 1 - a - \lambda_n & \chi\, f(1)\,\lambda_n \\ \beta & -1 - \varepsilon_0\,\lambda_n \end{pmatrix}.$$

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The trace of N_n is

$$\text{Tr } N_n = -a - (1+\varepsilon_0)\,\lambda_n < 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

The determinant of N_n is

$$\text{Det } N_n = [a-1] + [1 + \varepsilon_0\,(a-1) - \chi\, f(1)\,\beta]\,\lambda_n + \varepsilon_0\,\lambda_n^2.$$

We define

$$\chi_c := \sup\{\chi \,|\, \forall n \in \mathbb{N}, \quad [a-1] + [1 + \varepsilon_0\,(a-1) - \chi\, f(1)\,\beta]\,\lambda_n + \varepsilon_0\,\lambda_n^2 > 0\}, \tag{3.2}$$

and observe that

$$\chi_c \geqslant \chi_{c0} := \frac{2\sqrt{\varepsilon_0(a-1)} + 1 + \varepsilon_0\,(a-1)}{f(1)\,\beta} > 0. \tag{3.3}$$

Note that χ_{c0} does not depend on Ω (and there exists Ω such that $\chi_c = \chi_{c0}$).

3.1. Nonlinear stability

Proof of theorem 1.3. Thanks to theorem 1.1, we recall that there exist $K_m, K_c, K_d > 0$ such that for all $t > 0$,

$$\|\tilde{m}(t)\|_{L^\infty(\Omega)} \leqslant K_m, \quad \|\tilde{c}(t)\|_{W^{1,\infty}(\Omega)} \leqslant K_c, \quad \left\|\tilde{d}(t)\right\|_{L^\infty(\Omega)} \leqslant K_d. \tag{3.4}$$

It follows from the condition (1.7) that there exist $\alpha \in (0,1)$ and $\vartheta_1 > 0$ such that

$$\vartheta_1 < \frac{4\alpha(a-1)}{\beta^2} \quad \text{and} \quad \chi < \frac{2\sqrt{\vartheta_1\varepsilon_0}}{f(1)}. \tag{3.5}$$

Now we choose

$$\vartheta_2 > \frac{\delta^2\vartheta_1}{4g(1)(1-\alpha)}, \tag{3.6}$$

and define a function

$$\varphi(t) = \frac{1}{2}\left(\|\tilde{m}\|^2_{L^2(\Omega)} + \vartheta_1\|\tilde{c}\|^2_{L^2(\Omega)} + \vartheta_2\|\tilde{d}\|^2_{L^2(\Omega)}\right).$$

Direct calculations give

$$\begin{aligned}\varphi'(t) = &- \|\nabla\tilde{m}\|^2_{L^2(\Omega)} + \chi f(1)\,\langle\nabla\tilde{m}, \nabla\tilde{c}\rangle - \vartheta_1\varepsilon_0\,\|\nabla\tilde{c}\|^2_{L^2(\Omega)} \\ &+ (1-a)\,\|\tilde{m}\|^2_{L^2(\Omega)} - \vartheta_1\,\|\tilde{c}\|^2_{L^2(\Omega)} - \vartheta_2 g(1)\|\tilde{d}\|^2_{L^2(\Omega)} + \beta\vartheta_1\,\langle\tilde{m}, \tilde{c}\rangle \\ &+ \vartheta_1\delta\left\langle\tilde{d}, \tilde{c}\right\rangle + \left\langle R_1(\tilde{X}), \tilde{m}\right\rangle + \left\langle R_3(\tilde{X}), \vartheta_2\tilde{d}\right\rangle. \end{aligned} \tag{3.7}$$

From the relation between χ and ϑ_1 in (3.5), there exists $\omega_1 > 0$ such that

$$\begin{aligned}&- \|\nabla\tilde{m}\|^2_{L^2(\Omega)} + \chi f(1)\,\langle\nabla\tilde{m}, \nabla\tilde{c}\rangle - \vartheta_1\varepsilon_0\,\|\nabla\tilde{c}\|^2_{L^2(\Omega)} \\ &\quad\leqslant -\omega_1\left(\|\nabla\tilde{m}\|^2_{L^2(\Omega)} + \|\nabla\tilde{c}\|^2_{L^2(\Omega)}\right).\end{aligned}$$

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It also follows from the condition of ϑ_1 in (3.5) that there exists $\omega_2 > 0$ satisfying

$$(1-a)\,\|\tilde{m}\|^2_{L^2(\Omega)} + \beta\vartheta_1\,\langle \tilde{m}, \tilde{c}\rangle - \alpha\vartheta_1\,\|\tilde{c}\|^2_{L^2(\Omega)} \leqslant -\omega_2\left(\|\tilde{m}\|^2_{L^2(\Omega)} + \|\tilde{c}\|^2_{L^2(\Omega)}\right),$$

where α is chosen in (3.5). Finally, from (3.6), there exists $\omega_3 > 0$ such that

$$-(1-\alpha)\vartheta_1\,\|\tilde{c}\|^2_{L^2(\Omega)} + \vartheta_1\delta\left\langle \tilde{d}, \tilde{c}\right\rangle - \vartheta_2 g(1)\|\tilde{d}\|^2_{L^2(\Omega)} \leqslant -\omega_3\left(\|\tilde{c}\|^2_{L^2(\Omega)} + \|\tilde{d}\|^2_{L^2(\Omega)}\right).$$

Note that these three estimates mean that the linear part of (3.1), after the rescaling by $\mathrm{diag}(1, \vartheta_1, \vartheta_2)$, has a negative spectral gap. Inserting all these estimates into (3.7), we obtain for some ω_4, $\omega_5 > 0$ that

$$\varphi'(t) \leqslant -\omega_4\left(\|\tilde{m}\|^2_{H^1(\Omega)} + \|\tilde{c}\|^2_{H^1(\Omega)}\right) - \omega_5\varphi(t) + \left\langle R_1(\tilde{X}), \tilde{m}\right\rangle + \left\langle R_3(\tilde{X}), \vartheta_2\tilde{d}\right\rangle. \tag{3.8}$$

To deal with the two last terms on the right-hand side of (3.8), we first use integration by parts to have

$$\begin{aligned}
&\left\langle R_1(\tilde{X}), \tilde{m}\right\rangle + \left\langle R_3(\tilde{X}), \vartheta_2\tilde{d}\right\rangle \\
&\quad = -\int_\Omega \tilde{m}\left[(1+\tilde{m})^a - 1 - a\tilde{m}\right]\mathrm{d}x + \chi\int_\Omega \left[f(1+\tilde{m}) - f(1)\right]\nabla\tilde{c}\cdot\nabla\tilde{m}\,\mathrm{d}x \\
&\qquad - \vartheta_2\int_\Omega \left[g(1+\tilde{m}) - g(1)\right]\tilde{d}^2\,\mathrm{d}x =: (A) + (B) + (C).
\end{aligned} \tag{3.9}$$

Now we fix a constant $\kappa = \kappa(n) \in (0, \min\{a-1, 1\})$ small enough such that $H^1(\Omega) \hookrightarrow L^{2+2\kappa}(\Omega)$. Thanks to this embedding and interpolation inequality, we have (for some $C(\Omega, \kappa)$ which only depends on Ω and κ)

$$\|\tilde{m}\|^{2+\kappa}_{L^{2+\kappa}(\Omega)} \leqslant \|\tilde{m}\|^2_{L^{2+2\kappa}(\Omega)}\,\|\tilde{m}\|^\kappa_{L^{1+\kappa}(\Omega)} \leqslant C(\Omega,\kappa)\,\|\tilde{m}\|^2_{H^1(\Omega)}\,\|\tilde{m}\|^\kappa_{L^2(\Omega)}. \tag{3.10}$$

To estimate (A), we use Taylor's expansion and the fact that $\tilde{m}$ is uniformly bounded in $L^\infty(\Omega)$ to get

$$\begin{aligned}
|(A)| &\leqslant \int_\Omega |\tilde{m}||(1+\tilde{m})^a - 1 - a\,\tilde{m}|\,\mathrm{d}x \\
&\leqslant C(a, K_m)\int_\Omega |\tilde{m}|^{1+a}\,\mathrm{d}x \\
&\leqslant C(a, K_m)K_m^{a-1-\kappa}\int_\Omega |\tilde{m}|^{2+\kappa}\,\mathrm{d}x \quad (\text{since } \kappa < a-1) \\
&\leqslant C(a, K_m, \kappa)\,\|\tilde{m}\|^{2+\kappa}_{L^{2+\kappa}(\Omega)} \\
&\leqslant C(a, K_m, \kappa)\,\|\tilde{m}\|^2_{H^1(\Omega)}\,\|\tilde{m}\|^\kappa_{L^2(\Omega)} \quad (\text{using (3.10)}).
\end{aligned} \tag{3.11}$$

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Similarly for (B) and (C)

$$\begin{aligned}
|(C)| &\leqslant \vartheta_2 \int_\Omega |g(1+\tilde{m}) - g(1)|\tilde{d}^2 \,\mathrm{d}x \leqslant C(K_m) \int_\Omega |\tilde{m}||\tilde{d}|^2 \,\mathrm{d}x \quad (\text{since } g \in C^{1,0}) \\
&\leqslant C(K_m, K_d) \int_\Omega |\tilde{m}||\tilde{d}|^{\frac{2(1+\kappa)}{2+\kappa}} \,\mathrm{d}x \\
&\leqslant C(K_m, K_d, \omega_5) \int_\Omega |\tilde{m}|^{2+\kappa} \,\mathrm{d}x + \frac{\omega_5 \vartheta_2}{4} \int_\Omega |\tilde{d}|^2 \,\mathrm{d}x \quad (\text{by Young's inequality}) \\
&\leqslant C(K_m, K_d, \omega_5, \kappa) \|\tilde{m}\|^2_{H^1(\Omega)} \|\tilde{m}\|^\kappa_{L^2(\Omega)} + \frac{\omega_5}{2}\varphi(t) \quad (\text{using (3.10)}),
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
|(B)| &\leqslant \chi \int_\Omega |f(1+\tilde{m}) - f(1)||\nabla\tilde{c}||\nabla\tilde{m}| \,\mathrm{d}x \\
&\leqslant C(K_m) \int_\Omega |\tilde{m}||\nabla\tilde{c}||\nabla\tilde{m}| \,\mathrm{d}x \quad (\text{since } f \in C^{1,0}) \\
&\leqslant C(K_m, K_c) \int_\Omega |\tilde{m}||\nabla\tilde{c}|^{\frac{1}{3}}|\nabla\tilde{m}| \,\mathrm{d}x \quad (\text{thanks to (3.4)}) \\
&\leqslant \frac{\omega_4}{2} \int_\Omega \left(|\nabla\tilde{m}|^2 + |\nabla\tilde{c}|^2\right) \mathrm{d}x + C(K_m, K_c, \omega_4) \int_\Omega |\tilde{m}|^3 \,\mathrm{d}x \\
&\qquad (\text{by Young's inequality}) \\
&\leqslant \frac{\omega_4}{2} \left(\|\tilde{m}\|^2_{H^1(\Omega)} + \|\tilde{c}\|^2_{H^1(\Omega)}\right) + C(K_m, K_c, \omega_4) K_m^{1-\kappa} \int_\Omega |\tilde{m}|^{2+\kappa} \,\mathrm{d}x \\
&\leqslant \frac{\omega_4}{2} \left(\|\tilde{m}\|^2_{H^1(\Omega)} + \|\tilde{c}\|^2_{H^1(\Omega)}\right) + C(K_m, K_c, \omega_4, \kappa) \|\tilde{m}\|^2_{H^1(\Omega)} \|\tilde{m}\|^\kappa_{L^2(\Omega)} \\
&\qquad (\text{using (3.10)}).
\end{aligned} \tag{3.13}$$

By inserting (3.11)–(3.12)–(3.13) into (3.8) we obtain

$$\begin{aligned}
\varphi'(t) &\leqslant -\frac{\omega_5}{2}\varphi(t) + \left(\|\tilde{m}\|^2_{H^1(\Omega)} + \|\tilde{c}\|^2_{H^1(\Omega)}\right) \left[C\|\tilde{m}\|^\kappa_{L^2(\Omega)} - \frac{\omega_4}{2}\right] \\
&\leqslant -\frac{\omega_5}{2}\varphi(t) + \left(\|\tilde{m}\|^2_{H^1(\Omega)} + \|\tilde{c}\|^2_{H^1(\Omega)}\right) \left[C\varphi(t)^{\kappa/2} - \frac{\omega_4}{2}\right].
\end{aligned}$$

Therefore, if $\varphi(0)$ is small enough, $t \mapsto \varphi(t)$ is decreasing and consequently

$$\varphi(t) \leqslant e^{-\frac{\omega_5}{2}t}\varphi(0)$$

which proves the nonlinear (exponential) stability in L^2 of $(\bar{m}, \bar{c}, \bar{d})$. Interpolating with the estimate (1.5), we conclude the proof of theorem 1.3. □

3.2. Turing instability and Turing patterns

When $\chi > \chi_c$, the equilibrium point $(1, \beta + \delta, 1)$ is linearly unstable, so that a Turing-type instability sets in. In this section, we display a numerical simulation showing the occurrence of this instability on the spatial domain $\Omega = [0; L_x] \times [0; L_y]$.

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First, we recall that on rectangular domains defined by $0 < x < L_x$ and $0 < y < L_y$, the Fourier transformation of solutions to the linearized problem with homogeneous Neumann boundary conditions has the following form:

$$\tilde{X} = \sum_{p,q\in\mathbb{N}} \mathbf{f}_{p,q} e^{\sigma(\lambda_n)t} \cos\left(\frac{p\pi}{L_x}x\right) \cos\left(\frac{q\pi}{L_y}y\right), \tag{3.14}$$

where $\mathbf{f}_{p,q}$ are the Fourier coefficients of the initial conditions, and λ_n are defined as follows:

$$\lambda_n = \left(\frac{p\pi}{L_x}\right)^2 + \left(\frac{q\pi}{L_y}\right)^2, \text{ with } p,\, q \in \mathbb{N}, \tag{3.15}$$

in such a way that they are ordered (that is $\lambda_n \leqslant \lambda_{n+1}$). The values $\sigma(\lambda_n)$ are derived from the dispersion relations $\sigma^2 - \mathrm{Tr}N_n\sigma + \mathrm{Det}N_n = 0$. When $\chi < \chi_c$, $\sigma(\lambda_n) < 0$, $\forall n \in \mathbb{N}$, and then the homogeneous state is linearly stable. For $\chi = \chi_c$, the homogeneous solution is marginally stable and $\exists\lambda_{n_c}$ such that $\min(\mathrm{Det}N_{n_c}) = 0$, or in other words $\sigma(\lambda_{n_c}) = 0$. For $\chi > \chi_c$, there exists a range $[n_1; n_2]$ such that $\sigma(\lambda_n) > 0$, $\forall n \in [n_1; n_2]$, and therefore a Turing instability occurs.

The numerical simulation has been performed on the square domain $[0; \pi] \times [0; \pi]$, i.e. $L_x = L_y = \pi$. For the discretization in space, we have adopted a Fourier spectral scheme with 64×64 modes. The integration in time has been realized by using the Crank-Nicholson method for the diffusive part and a second-order Runge-Kutta explicit method for the reaction terms.

We have adopted the functions

$$f(m) = \frac{m}{1+m} \quad \text{and} \quad g(m) = \frac{rm^2}{1+m},$$

which corresponds to the special case (1.2) (except for parameter a), and the following parameter values have been fixed: $r = 1$, $a = 3$, $\varepsilon_0 = 0.03125$, $\delta = 1$, $\beta = 1$. For this set of values, linear stability analysis predicts that $\lambda_{n_c} = 8$ and $\chi_c = 3.125$. In order to obtain a Turing pattern, we have fixed $\chi = 3.18$ (which is bigger than χ_c) and for this choice the only unstable λ_n allowed by the domain and the boundary conditions is λ_{n_c}, which has multiplicity one, in that $(p, q) = (2, 2)$ is the only pair which satisfies the condition (3.15).

We therefore expect that a random perturbation of the homogeneous state evolves towards a stationary Turing pattern of the form $\tilde{X} \propto \cos(2x)\cos(2y)$.

The temporal evolution of the numerical solution is shown in Fig. 1. As predicted by the previous analysis, a small random perturbation (see Fig. 1a) of the homogeneous state evolves towards a stationary square patten (Fig. 1d).

On the other hand, when $\chi < \chi_c$, the homogeneous steady state is stable and therefore perturbations of this equilibrium, as shown in Fig. 2, are dampened.

3.3. Global asymptotic stability

We expect that the convergence to the steady state $(\bar{m}, \bar{c}, \bar{d}) = (1, \beta + \delta, 1)$ is not only local, as proved in § 3.1 when $\chi < \chi_{\mathrm{subcrit}}$, but is in fact global and valid for $\chi < \chi_c$. This is left as an interesting open question. We performed numerical

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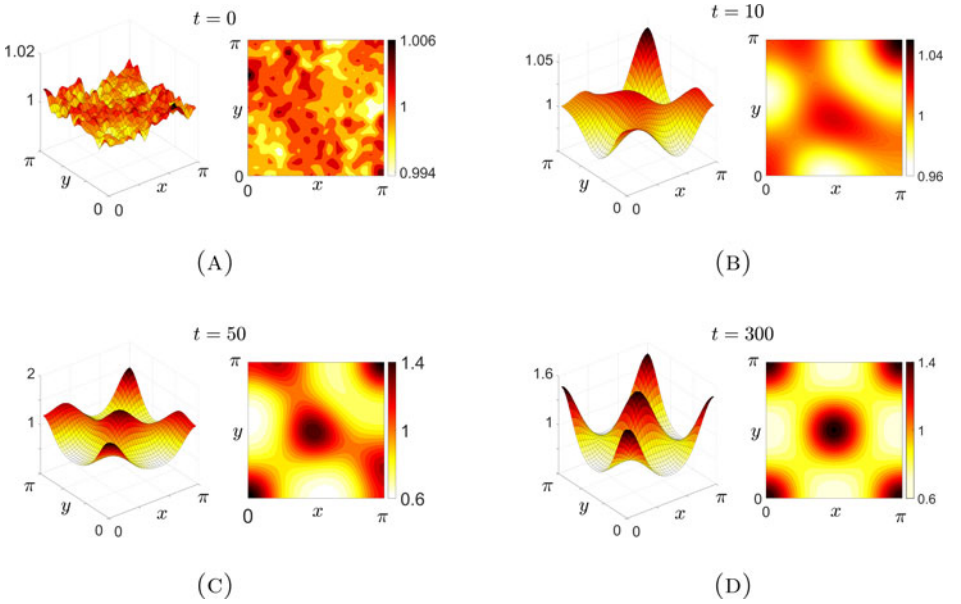


Figure 1. Spatio-temporal evolution of m on the spatial domain $\Omega = [0;\pi] \times [0;\pi]$ above the Turing threshold $\chi_c = 3.125$. The following parameter values have been fixed: $r = 1$, $a = 3$, $\varepsilon_0 = 0.03125$, $\delta = 1$, $\beta = 1$ and $\chi = 3.18$. The initial condition is a small random perturbation of the homogeneous equilibrium $\bar{m} = 1$.

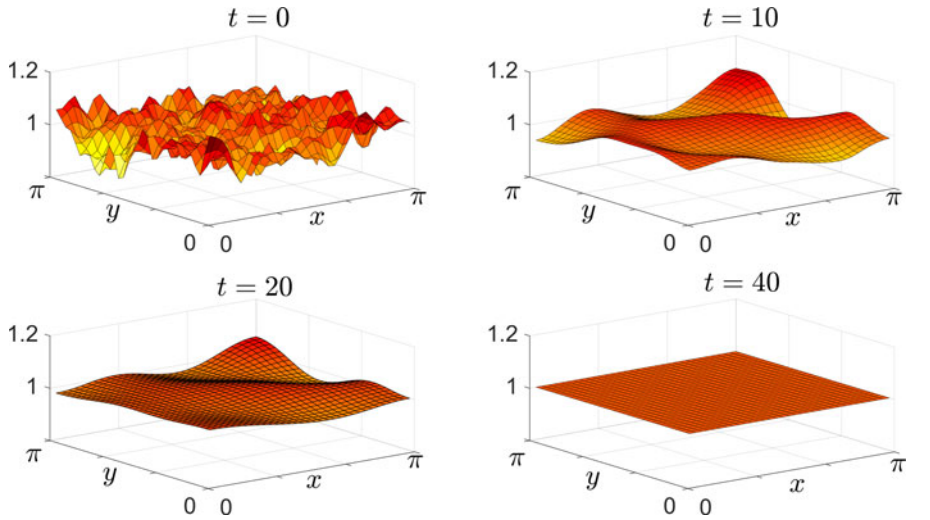


Figure 2. Spatio-temporal evolution of m on the spatial domain $\Omega = [0;\pi] \times [0;\pi]$ below the Turing threshold χ_c. All parameter values, except χ, are as in Fig. 1, and $\chi = 3$. The initial condition is a random perturbation of the homogeneous equilibrium $\bar{m} = 1$.

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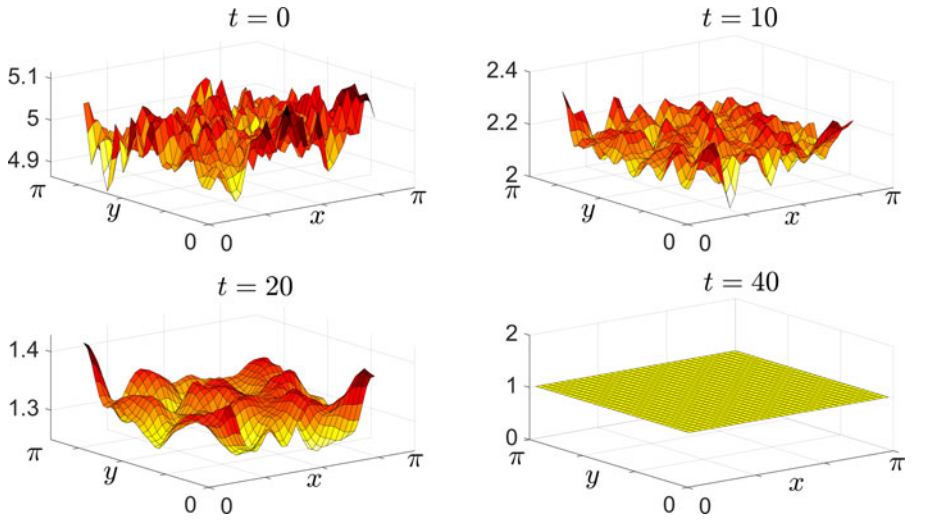


Figure 3. Spatio-temporal evolution of m on the spatial domain $\Omega = [0;\pi] \times [0;\pi]$ below the Turing threshold χ_c. All parameter values are as in Fig. 2. The initial condition is far from the homogeneous equilibrium $(\bar{m}, \bar{c}, \bar{d})$.

simulations which support this conjecture. We show one of these simulations in Fig. 3: below the Turing threshold χ_c, an initial datum far from the homogeneous equilibrium evolves towards the stable equilibrium point $(\bar{m}, \bar{c}, \bar{d})$.

Moreover we performed several other simulations adopting initial data further away from equilibrium and in any of these numerical experiments we observed solutions converging to the stable steady state.

Note that the large time behaviour of system (1.1) when $\chi > \chi_c$ is a much more complicated problem: it involves the research of inhomogeneous stable steady states, and the study of the basin of attraction.

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