Combinatorics, Probability and Computing (2008) 17, 437–470. © 2008 Cambridge University Press
doi:10.1017/S0963548307008681 Printed in the United Kingdom

Large-Deviation Approximations for General Occupancy Models

JIM X. ZHANG[†] and PAUL DUPUIS[‡]

Division of Applied Mathematics, Brown University, Providence, RI 02912, USA
(e-mail: jim.zhang@ubs.com)

Received 10 December 2004; revised 29 August 2005

We obtain large-deviation approximations for the empirical distribution for a general family of occupancy problems. In the general setting, balls are allowed to fall in a given urn depending on the urn's contents prior to the throw. We discuss a parametric family of statistical models that includes *Maxwell–Boltzmann*, *Bose–Einstein* and *Fermi–Dirac* statistics as special cases. A process-level large-deviation analysis is conducted and the rate function for the original problem is then characterized, via the contraction principle, by the solution to a calculus of variations problem. The solution to this variational problem is shown to coincide with that of a simple finite-dimensional minimization problem. As a consequence, the large-deviation approximations and related qualitative information are available in more-or-less explicit form.

1. Introduction

Occupancy problems centre on the distribution of r balls that have been thrown into n urns. In the simplest scenario each ball is equally likely to fall in any of the urns, *i.e.*, each ball is independently assigned to a given urn with probability $1/n$. In this case, we say that the urn model uses *Maxwell–Boltzmann* (MB) statistics. This model has been studied for decades and applied in diverse fields such as computer science, biology, and statistics. See [2, 8, 9] and the references therein. However, balls may also enter the urns in a non-uniform way. An important generalization is to allow the likelihood that the ball lands in a given urn to depend on its contents prior to the throw, as in *Bose–Einstein* (BE) and *Fermi–Dirac* (FD) statistics. See [5, 9, 11] and the references therein.

For MB statistics, many results have been obtained using 'exact' methods. For example, combinatorial methods are used in [6] and methods that use generating functions are discussed in [9]. Although they do not directly involve approximations, the implementation of these methods can be difficult. For example, in combinatorial methods one has to deal with the difference of

[†] Research supported in part by the National Science Foundation (NSF-DMS-0306070).
[‡] Research supported in part by the National Science Foundation (NSF-DMS-0306070 and NSF-DMS-0404806) and the Army Research Office (DAAD19-02-1-0425 and W911NF-05-1-0289).

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

events using the inclusion–exclusion formula and the resulting computations can involve large errors. In the moment-generating function approach in [9] similar difficulties occur.

Large-deviation approximations give an attractive alternative to both of these approaches. One reason is that they offer good approximations with just modest computation. A second, perhaps more important reason, is that qualitative insights can be obtained. In [4] the LDP for the MB model is obtained, and the rate function exhibited in more-or-less explicit form.

In the present paper, we discuss a parametric family of statistical models, of which the previously mentioned MB, BE and FD statistics are all special cases. We assume there are n urns and that $\lfloor Tn \rfloor$ balls are thrown into them (where $\lfloor s \rfloor$ denotes the integer part of s), and analyse the asymptotic properties as n goes to ∞. A typical problem of interest is to characterize the large-deviation asymptotics of the empirical distribution after all the balls are thrown. For example, one may wish to estimate the probability that at most half of the urns are empty after all the balls are thrown. A direct analysis of this problem is hard, and instead we lift the problem to the process level and analyse the large-deviation asymptotics at this process level.

Although process-level large deviations are by now quite standard, there are several interesting features, both qualitative and technical, which distinguish occupancy models and place them outside the range of existing theory. The most significant of these as far as the proof is concerned is that the singular transition rates occur in the (Markovian) process-level description of the model. We will use a weak convergence approach that is naturally suited to these problems and results in a nicely compact and self-contained proof, and one that can easily accommodate further generalization of the model. A second very interesting feature is that one can explicitly solve the variational problems that arise in the process-level approximations. Such explicit formulas are rare, and the solutions have many uses and add significantly to the practical value of the large-deviation approximations. Besides identifying exponential rates and the manner in which rare events are most likely to occur, they can be used to construct importance-sampling schemes that provide accurate non-asymptotic numerical approximations [12].

The outline of the paper is as follows. In Section 2 the parametric family of occupancy problems is described in detail. A dynamical system characterization of the random occupancy process is given, and a representation for certain exponential integrals is given in terms of a 'controlled' occupancy process. From this representation formula one can identify the large-deviation rate function immediately. In Section 3 we prove the lower bound for the Laplace principle, which corresponds to the large-deviation upper bound. In Section 4, the rate function $\mathscr{I}$ is studied more closely so as to deal with the technical difficulty of the singular transition rates. In Section 5, we prove the upper bound for the Laplace principle, which corresponds to the large-deviation lower bound. Finally, in Section 6 we give the explicit formula for the rate function of the process at a given fixed time.

2. Preliminaries and main result

In this section, we formulate the problem of interest and state the LDP. The proof is given in the following sections.

The general occupancy problem has the same structure as the Maxwell–Boltzmann occupancy problem, except that in the general problem urns are distinguished according to the number of balls contained therein. The full collection of models will be indexed by a parameter a. This

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

parameter takes values in the set $(0,\infty]\cup\{-1,-2,\dots\}$, and its interpretation is as follows. Suppose that a ball is about to be thrown, and that any two urns (labelled, say, A and B) are selected. An urn is said to be of *category* i if it contains i balls. Suppose that urn A is of category i, while B is of category j. Then the probability that the ball is thrown into urn A, conditioned on the state of all the urns and that the ball is thrown into either urn A or B, is

$$\frac{a+i}{(a+i)+(a+j)}.$$

When $a=\infty$ we interpret this to mean that the two urns are equally likely. Also, when $a<0$ we use this ratio to define the probabilities only when $0\leqslant i\vee j\leqslant -a$ and $i<-a$ or $j<-a$, so the formula gives a well-defined probability. The probability that a ball is placed in an urn of category $-a$ is 0. Thus, under this model, urns can only be of category $0,1,\dots,-a$, and we only throw balls into categories $0,1,\dots,-a-1$.

In this set-up, certain special cases are distinguished. The cases $a=1$, $a=\infty$, $a\in-\mathbb{N}$ correspond to what are called *Bose–Einstein* statistics, *Maxwell–Boltzmann* statistics, and *Fermi–Dirac* statistics, respectively.

Suppose that before we throw a ball there are already tn balls in all the urns, and further suppose that the occupancy state is $(x_0,x_1,\dots,x_{I+})$. Here $x_i, i=0,1,\dots,I$ denotes the fraction of urns that contain i balls, and x_{I+} denotes the fraction containing more than I balls. Then the 'un-normalized' or 'relative' probability of throwing into a category i urn with $i\leqslant I$ is simply $(a+i)x_i$. Let us temporarily abuse notation, and let $x_{I+1},x_{I+2},\dots$ denote the exact fraction in each category i with $i>I$. Since there are tn balls in the urns before we throw, $\sum_{i=0}^{\infty} ix_i=t$. Thus the (normalized and true) probability that the ball is placed in an urn that contains exactly i balls, $i=0,1,\dots,I$, is $\frac{(a+i)x_i}{a+t}$, and the probability that the ball is placed in an urn that has more than I balls is $1-\sum_{j=0}^{I}\frac{a+j}{a+t}x_j$.

An explicit construction of this process is as follows. To simplify, we assume the empty initial condition, *i.e.*, all urns are empty. One can consider other initial conditions, with only simple notational changes in the results to be stated below. We introduce a time variable t that ranges from 0 to T. At a time t that is of the form l/n, with $0\leqslant l\leqslant\lfloor nT\rfloor$ an integer, l balls have been thrown. Let $X^n(t)=\{X_0^n(t),X_1^n(t),\dots,X_I^n(t),X_{I+}^n(t)\}$ be the *occupancy state* at that time. As noted previously, $X_i^n(t)$ denotes the fraction of urns that contain i balls at time t, $i=0,1,\dots,I$, and $X_{I+}^n(t)$ the fraction of urns that contain more than I balls. The definition of X^n is extended to all $t\in[0,T]$ not of the form l/n by piecewise linear interpolation. Note that $X^n(t)$ is indeed a probability vector in $\mathbb{R}^{I+2}$. If

$$\mathscr{S}_I\doteq\left\{x\in\mathbb{R}^{I+2}:x_i\geqslant 0, 0\leqslant i\leqslant I+1\quad\text{and}\quad\sum_{i=0}^{I+1}x_i=1\right\},$$

then for any $t\in[0,T]$, $X^n(t)\in\mathscr{S}_I$. Thus X^n takes values in $\mathscr{U}\doteq C([0,T],\mathscr{S}_I)$. We equip $\mathscr{U}$ with the usual supremum norm and on $\mathscr{S}_I$ we take the usual L_1-norm.

It will be convenient to work with the following 'dynamical system' representation. For $x\in\mathbb{R}^{I+2}$ and $t\in[0,-a1_{\{a<0\}}+\infty 1_{\{a>0\}})$ define the vector $\rho(t,x)\in\mathbb{R}^{I+2}$ by

$$\rho_k(t,x)=\frac{a+k}{a+t}x_k,\quad\text{for } k=0,1,\dots,I,\tag{2.1}$$

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

and

$$\rho_{I+1}(t,x) = 1 - \sum_{k=0}^{I} \frac{a+k}{a+t} x_k.$$

A direct calculation shows that if

$$x \in \mathscr{S}_I \quad \text{and} \quad \sum_{k=0}^{I+1} k x_k \leqslant t, \tag{2.2}$$

then $\rho(t,x)$ is indeed a probability vector in $\mathbb{R}^{I+2}$, *i.e.*, $\rho(t,x) \in \mathscr{S}_I$. We can then define a family of independent random vector fields

$$\{y^{i,n}(\cdot) : i = 0, 1, \ldots, \lfloor nT \rfloor - 1, \lfloor nT \rfloor\}$$

that take values in

$$\Lambda \doteq \{e_{j+1} - e_j, 0 \leqslant j \leqslant I\} \cup \{0\}$$

and with distributions

$$P\{y^{i,n}(x) = v\} = \begin{cases} \rho_k\left(\frac{i}{n}, x\right) & \text{if } v = e_{k+1} - e_k, \quad 0 \leqslant k \leqslant I, \\ \rho_{I+1}\left(\frac{i}{n}, x\right) & \text{if } v = 0. \end{cases}$$

Finally, we define $X^n(l/n)$ recursively by

$$X^n((i+1)/n) = X^n(i/n) + \frac{1}{n} y^{i,n}(X^n(i/n)),$$

and the initial condition $X^n(0) = (1, 0, \ldots, 0)$. Observe that the increments $\{y^{i,n}(X^n(i/n))\}$ are conditionally distributed according to $\rho\left(\frac{i}{n}, X^n(i/n)\right)$, and thus the process X^n is obviously Markovian and will have the same distribution as the occupancy process described previously.

Often one is interested in the large deviations of the empirical occupancy measure at the terminal time T, namely $X^n(T)$. We study this by analysing the large-deviation properties of the whole process X^n and then using the Contraction Principle. The Laplace formulation will be used to perform the process-level analysis. Let F be any bounded and continuous function on $\mathscr{U}$. The processes X^n are said to satisfy the Laplace principle with rate function $\mathscr{I}$ if the following two conditions hold.

(1) For each $K < \infty$, the set $\{\varphi \in \mathscr{U} : \mathscr{I}(\varphi) \leqslant K\}$ is compact in $\mathscr{U}$.
(2)

$$\lim_{n\to\infty} -\frac{1}{n} \log E \exp[-nF(X^n)] = \inf_{\varphi\in\mathscr{U}} [\mathscr{I}(\varphi) + F(\varphi)].$$

Since X^n takes values in a Polish space, the notions of Laplace principle and large-deviation principle are equivalent [3, Theorem 1.2.1].

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

Define the $I+2$ by $I+2$ matrix

$$M = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Let $\varphi \in \mathscr{U}$ be given with $\varphi_0(0) = 1$. Suppose there is a Borel-measurable function $\theta : [0,T] \longmapsto \mathscr{S}_I$ such that, for any $t \in [0,T]$,

$$\varphi(t) = \varphi(0) + \int_0^t M\theta(s)\, ds. \tag{2.3}$$

We interpret $\theta_i(s)$ as the rate at which balls are thrown into urns that contain i balls at time s. Moreover, $\theta(s)$ is unique in the sense that, if another $\tilde{\theta} : [0,T] \longmapsto \mathscr{S}_I$ satisfies (2.3), then $\tilde{\theta} = \theta$ a.e. on $[0,T]$. We call φ a *valid* occupancy state process if there exists $\theta : [0,T] \longmapsto \mathscr{S}_I$ satisfying (2.3). In this case θ is called the occupancy rate process associated with φ. It is easy to observe that if φ is valid then $\varphi(s)$ satisfies (2.2) for all $s \in [0,T]$. This shows that $\rho(s,\varphi(s)) \in \mathscr{S}_I$.

The relative entropy function will be used throughout the paper and we define it now. For two probability measures α and β on a Polish space $\mathscr{A}$, the relative entropy of α with respect to β is defined by

$$R(\alpha\|\beta) \doteq \int_{\mathscr{A}} \left(\log \frac{d\alpha}{d\beta}\right) d\alpha$$

whenever α is absolutely continuous with respect to β (and with the convention that $0 \log 0 = 0$). In all other cases we set $R(\alpha\|\beta) = \infty$. When two probability vectors ρ and $v \in \mathscr{S}_I$ appear in the relative entropy function, we interpret them as probability measures on the simplex $\{0,1,\ldots,I,I+1\}$, and thus

$$R(\rho\|v) \doteq \sum_{i=0}^{I+1} \rho_i \log \frac{\rho_i}{v_i}.$$

As observed before, when $\varphi(s)$ is *valid*, $\rho(s,\varphi(s)) \in \mathscr{S}_I$, which makes $R(\theta(s)\|\rho(s,\varphi(s)))$ well defined. For such φ define

$$\mathscr{I}(\varphi) = \int_0^T R\big(\theta(s)\|\rho(s,\varphi(s))\big)\, ds. \tag{2.4}$$

If φ is not valid then define $\mathscr{I}(\varphi) = \infty$. In the next three sections we will prove that the urn models constructed in this section satisfy the Laplace principle with rate function $\mathscr{I}$. In particular, in Section 3 we will prove

$$\liminf_{n\to\infty} -\frac{1}{n} \log E \exp[-nF(X^n)] \geqslant \inf_{\varphi\in\mathscr{U}} [\mathscr{I}(\varphi) + F(\varphi)],$$

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

and in Section 5 we will prove

$$\limsup_{n\to\infty} -\frac{1}{n}\log E\exp[-nF(X^n)] \leqslant \inf_{\varphi\in\mathscr{U}}[\mathscr{I}(\varphi)+F(\varphi)].$$

These bounds are equivalent to the large-deviation upper and lower bounds [3]. In Section 4, we will prove several properties of the rate function $\mathscr{I}$, and in particular show that $\mathscr{I}$ has compact level sets.

It will turn out that certain representation formulas for exponential integrals simplify proving the Laplace principle. Consider a controlled process $\bar{X}^n(t)$ constructed as follows. The process dynamics are of the same general structure as those of X^n, save that $y^{i,n}(X^n(i/n))$ is replaced by a sequence of controlled random vectors $\bar{y}^{i,n}$. Let $(\mathscr{V},\mathscr{A})$ be a measurable space and $\mathscr{Y}$ a Polish space and let $\tau(dy|x)$ be a family of probability measures on $\mathscr{Y}$ parametrized by $x\in\mathscr{V}$. We call $\tau(dy|x)$ a *stochastic kernel* on $\mathscr{Y}$ given $\mathscr{V}$ if for every Borel subset E of $\mathscr{Y}$ the function mapping $x\in\mathscr{V}\mapsto\tau(E|x)\in[0,1]$ is measurable. The conditional distributions of the controlled random vectors will be specified by a sequence $\{v^{i,n}: i=0,1,\dots,\lfloor nT\rfloor\}$, where each quantity $v^{i,n}=v^{i,n}(x_0,x_1,x_2,\dots,x_i)$ is interpreted as a stochastic kernel on Λ given $(\mathscr{S}_I)^{i+1}$. We call such a sequence $\{v^{i,n}: i=0,1,\dots,\lfloor nT\rfloor\}$ an *admissible* control sequence. Each control $v^{i,n}$ will give rise to a corresponding relative entropy term in the representation formula.

The controlled process is determined by

$$\bar{X}^n((i+1)/n)=\bar{X}^n(i/n)+\frac{1}{n}\bar{y}^{i,n},\quad \text{for } i=0,1,\dots,\lfloor nT\rfloor,$$
$$\bar{X}^n(0)=(1,0,\dots,0),$$

where $\bar{y}^{i,n}$ has the conditional distribution $v^{i,n}(\bar{X}^n(0),\dots,\bar{X}^n(i/n))$. Random vectors $\bar{X}^n(i/n)$ and $\bar{y}^{i,n}$ are defined recursively in the following order:

$$\bar{X}^n(0),\ \bar{y}^{0,n},\ \bar{X}^n(1/n),\ \bar{y}^{1,n},\ \bar{X}^n(2/n),\dots,\bar{X}^n(\lfloor nT\rfloor/n),\ \bar{y}^{\lfloor nT\rfloor,n}.$$

For all $n\in\mathbb{N}$ the controlled random vectors $\bar{X}^n(i/n)$ and $\bar{y}^{i,n}$ are defined on a common probability space $(\bar{\Omega},\bar{\mathscr{F}},\bar{P})$, and expectation on this space is denoted by $\bar{E}$. Define

$$\rho^{i,n}\doteq\rho\big(i/n,\bar{X}^n(i/n)\big),$$

where $\rho(t,x)$ was defined previously in (2.1) . Then by [3, Proposition 1.4.2] and the chain rule for relative entropy [3, Theorem C.3.1],

$$-\frac{1}{n}\log E\exp[-nF(X^n)]=\inf_{\{v^{i,n}\}}\bar{E}\left[F(\bar{X}^n)+\frac{1}{n}\sum_{i=0}^{\lfloor nT\rfloor}R\big(v^{i,n}\|\rho^{i,n}\big)\right],\tag{2.5}$$

where the infimum is over all the admissible control sequences $\{v^{i,n}\}$.

A convention for the case $a\in-\mathbb{N}$. When $a\in-\mathbb{N}$ it is only possible to throw balls into the categories $0,1,\dots,-a-1$, and the only possible categories are $0,1,\dots,-a$. Thus, if there are n urns there can at most be $-an$ balls thrown, and therefore $T\leqslant -a$. When $T=-a$ all the urns have exactly $-a$ balls, which is not an interesting case to study. As a consequence, throughout this paper we assume $T<-a$. Also, because of the restriction on the possible categories we can (without loss) assume that $I=-a-1$. Thus, throughout the paper, for $a<0$ we assume

$$T<-a,\quad I=-a-1.\tag{2.6}$$

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

3. The large-deviation upper bound

In this section, we prove

$$\liminf_{n\to\infty} -\frac{1}{n}\log E\exp[-nF(X^n)] \geqslant \inf_{\varphi\in\mathscr{U}}[\mathscr{I}(\varphi)+F(\varphi)],$$

which corresponds to the large-deviation upper bound. By (2.5) it is enough to show that

$$\liminf_{n\to\infty}\inf_{\{v^{i,n}\}}\bar{E}\left[F(\bar{X}^n)+\frac{1}{n}\sum_{i=0}^{\lfloor nT\rfloor}R\big(v^{i,n}\|\rho^{i,n}\big)\right]\geqslant\inf_{\varphi\in\mathscr{U}}[\mathscr{I}(\varphi)+F(\varphi)].$$

For $0\leqslant l\leqslant\lfloor nT\rfloor$ and $t\in[l/n,l/n+1/n)$, define

$$\hat{X}^n(t)=\bar{X}^n(l/n).$$

Thus $\hat{X}^n$ is the piecewise constant interpolation of the occupancy process. Note that for all ω,

$$\sup_{t\in[0,T]}|\hat{X}^n(t)-\bar{X}^n(t)|\leqslant\frac{1}{n}.$$

Therefore, if $\bar{X}^n$ converges weakly to $\bar{X}$, then $\hat{X}^n$ also converges weakly to $\bar{X}$.

We do the same thing for the controlled measures, and for $t\in[l/n,l/n+1/n)$ set

$$\hat{v}^n(t)=v^{l,n}\quad\text{and}\quad\hat{\rho}^n(t)=\rho\left(\frac{l}{n},\hat{X}^n(t)\right).$$

Note that because relative entropy is non-negative and $(\lfloor nT\rfloor+1)/n\geqslant T$,

$$\frac{1}{n}\sum_{i=0}^{\lfloor nT\rfloor}R\big(v^{i,n}\|\rho^{i,n}\big)\geqslant\int_0^T R\big(\hat{v}^n(t)\|\hat{\rho}^n(t)\big)\,dt. \tag{3.1}$$

For an $\mathscr{S}_I$-valued process $\{y(t)\}$ we define $\langle y\rangle$ as its indefinite integral, *i.e.*,

$$\langle y\rangle(t)\doteq\int_0^t y(s)\,ds,\quad\text{for }0\leqslant t\leqslant T.$$

Then $\langle y\rangle$ can be viewed as a vector of sub-cumulative distribution functions taking values in the space

$$\mathscr{Q}\doteq\{\langle y\rangle, y:[0,T]\mapsto\mathscr{S}_I\ \text{is measurable}\}.$$

We consider $\mathscr{Q}$ as a subset of $C([0,T],\mathbb{R}^{I+2})$ with the inherited topology. Since each component of $\langle y\rangle$ is Lipschitz-continuous (with Lipschitz constant 1), the Arzela–Ascoli theorem implies that $\mathscr{Q}$ is precompact. Choose a convergent sequence $\langle y_n\rangle$, with limit z. Since each component of z inherits the Lipschitz-continuity and monotonicity of the $\langle y_n\rangle$, a vector of non-negative derivatives y exist, and for any $0\leqslant s\leqslant t\leqslant T$ these derivatives satisfy

$$\int_s^t\sum_{i=1}^{I+1}y_i(u)\,du=\sum_{i=1}^{I+1}[z_i(t)-z_i(s)]=t-s.$$

This implies that $y\in\mathscr{S}_I$ for almost every $t\in[0,T]$, and hence $z\in\mathscr{Q}$. We conclude that $\mathscr{Q}$ is compact.

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

Let $m \otimes y$ be the vector of sub-probability measures generated by $\langle y \rangle$, *i.e.*, for each $0 \leqslant i \leqslant I+1$ and $0 \leqslant a \leqslant T$,

$$(m \otimes y)_i((-\infty, a]) \doteq \frac{\langle y \rangle_i(a)}{T}.$$

Each component of $m \otimes y$ can be viewed as taking values in the space of sub-probability measures with the topology of weak convergence, and then $m \otimes y$ as taking values in the product space with corresponding product topology. However, we can also consider $m \otimes y$ as a probability measure on $[0, T] \times \{0, 1, \ldots, I, I+1\}$, with the topology of weak convergence on this space. These two topologies are clearly equivalent, and since uniform convergence of sub-cumulative distribution functions implies the weak convergence of the corresponding sub-probability measures, the mapping $\langle y \rangle \mapsto m \otimes y$ is continuous. The Continuous Mapping Theorem then implies the following result.

Lemma 3.1. *Let Y_n and Y be $\mathcal{S}_I$-valued random processes. If $\langle Y_n \rangle$ converges weakly to $\langle Y \rangle$ then $m \otimes Y_n$ converges weakly to $m \otimes Y$.*

We will also need conditions under which $\langle Y_n \rangle$ will converge weakly to $\langle Y \rangle$. Define $\mathcal{V} \doteq \mathcal{D}([0, T] : \mathcal{S}_I)$ to be the space of functions that map $[0, T]$ into $\mathcal{S}_I$, are right-continuous, and have left-hand limits. Note that $\mathcal{U} \subset \mathcal{V}$. We equip $\mathcal{V}$ with the standard *Skorohod metric* $s(\cdot, \cdot)$ so that $(\mathcal{V}, s)$ is a Polish space (see [1]). If $y_n \in \mathcal{V}$, $s(y_n, y) \to 0$, and if $y \in \mathcal{U}$, then in fact $y_n(t) \to y(t)$ uniformly in $t \in [0, T]$, and hence $\langle y_n \rangle \to \langle y \rangle$. Another application of the Continuous Mapping Theorem gives the following.

Lemma 3.2. *Suppose a sequence of $\mathcal{V}$-valued processes Y_n converges weakly to the $\mathcal{U}$-valued process Y. Then $\langle Y_n \rangle$ converges weakly to $\langle Y \rangle$, and hence $m \otimes Y_n$ converges weakly to $m \otimes Y$.*

We will also need the following formula, which can be verified directly from the given definitions:

$$\int_0^T R\big(\hat{\nu}^n(t) \| \rho^n(t)\big)\, dt = TR\big(m \otimes \hat{\nu}^n \| m \otimes \rho^n\big). \tag{3.2}$$

Next we will prove the key weak convergence theorem used in the process-level analysis.

Theorem 3.3. *Define a sequence of controlled processes and controls $(\bar{X}^n(t), \hat{\nu}^n(t))$ as above. Then*

$$\{(\bar{X}^n, \langle \hat{\nu}^n \rangle), n \in \mathbb{N}\}$$

is tight. For any sequence from $\{(\bar{X}^n, \langle \hat{\nu}^n \rangle), n \in \mathbb{N}\}$, consider a further subsequence that converges in distribution to (X, η). Then the limit processes have the following properties.

(1) *There exists an S_I-valued process θ such that $\eta = \langle \theta \rangle$ with probability* 1.

(2) *The process X is a valid occupancy process and the process θ is the occupancy rate process associated with X, i.e., with probability* 1,

$$X(t) = X(0) + \int_0^t M\theta(s)\, ds, \quad \textit{for all } \ t \in [0, T].$$

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

Proof. Both $\bar{X}^n$ and $\langle \hat{v}^n \rangle$ are uniformly (in n and ω) Lipschitz-continuous. Hence, by the Arzela–Ascoli theorem, $\{(\bar{X}^n, \langle \hat{v}^n \rangle), n \in \mathbb{N}\}$ is tight. Let (X, η) denote the weak limit of a convergent subsequence. Since the second component takes values in $\mathcal{Q}$ and this space is compact, there exists a measurable $\mathcal{S}_I$-valued process $\theta(t)$ such that $\eta = \langle \theta \rangle$.

Notice that for each $0 \leqslant i \leqslant I+1$ and $0 \leqslant l \leqslant \lfloor nT \rfloor$, and with the notational conventions $e_{I+2} = e_{I+1}$ and $\hat{v}_{-1}(t) = 0$,

$$\begin{aligned} \bar{X}_i^n\left(\frac{l+1}{n}\right) - \bar{X}_i^n\left(\frac{l}{n}\right) &= \frac{1}{n} 1_{\{\bar{y}^{l,n} = e_i - e_{i-1}\}} - \frac{1}{n} 1_{\{\bar{y}^{l,n} = e_{i+1} - e_i\}} \\ &= \frac{1}{n} \hat{v}_{i-1}^n\left(\frac{l}{n}\right) - \frac{1}{n} \hat{v}_i^n\left(\frac{l}{n}\right) + \frac{1}{n} \hat{Y}_i^n\left(\frac{l}{n}\right), \end{aligned} \tag{3.3}$$

with $\hat{Y}_i^n\left(\frac{l}{n}\right)$ implicitly defined by

$$\hat{Y}_i^n\left(\frac{l}{n}\right) = \left[1_{\{\bar{y}^{l,n} = e_i - e_{i-1}\}} - \hat{v}_{i-1}^n\left(\frac{l}{n}\right)\right] - \left[1_{\{\bar{y}^{l,n} = e_{i+1} - e_i\}} - \hat{v}_i^n\left(\frac{l}{n}\right)\right].$$

In the same way that we defined $\hat{v}^n$ and $\hat{X}^n$ on the whole $[0, T]$ by piecewise constant interpolation, we can also define $\hat{Y}^n(t)$ on $[0, T]$. Let $\mathcal{F}_l^n$ be the natural filtration, *i.e.*, the σ-algebra generated by $\{\bar{X}^n(0), \bar{X}^n(1/n), \ldots, \bar{X}^n(l/n)\}$. Then

$$E\left[1_{\{\bar{y}^{l,n} = e_i - e_{i-1}\}} | \mathcal{F}_l^n\right] = \hat{v}_{i-1}^n\left(\frac{l}{n}\right),$$

which shows that $\{\hat{Y}_i^n\left(\frac{l}{n}\right)\}$, $0 \leqslant l \leqslant \lfloor nT \rfloor$ is a martingale difference with respect to $\mathcal{F}_l^n$.

We have observed that $\{\sum_{l=0}^k \hat{Y}_i^n\left(\frac{l}{n}\right) : 0 \leqslant k \leqslant \lfloor nT \rfloor\}$ is a martingale with respect to $\mathcal{F}_k^n$. It is also easy to see that $E\left[\hat{Y}_i^n\left(\frac{l}{n}\right)\right]^2 = O(1)$. Summing (3.3) shows that, for any $0 \leqslant l \leqslant \lfloor nT \rfloor$,

$$\bar{X}_i^n(l/n) - \bar{X}_i^n(0) = \langle \hat{v}_{i-1}^n \rangle(l/n) - \langle \hat{v}_i^n \rangle(l/n) + \langle \hat{Y}_i^n \rangle(l/n).$$

Owing to the fact that the jumps in the discrete time processes are uniformly bounded, if $t \in \left[\frac{l}{n}, \frac{l+1}{n}\right)$ for some $0 \leqslant l \leqslant \lfloor nT \rfloor$ then $\langle \hat{v}^n \rangle(t) = \langle \hat{v}^n \rangle(l/n) + O(1/n)$ and $\langle \hat{Y}^n \rangle(t) = \langle \hat{Y}^n \rangle(l/n) + O(1/n)$, where the $O(1/n)$-term is uniform in ω. Since the Lipschitz-continuity of $\bar{X}^n$ implies $|\bar{X}^n(t) - \bar{X}^n(s)| \leqslant |t - s|$, for any $s, t \in [0, T]$,

$$\bar{X}_i^n(t) - \bar{X}_i^n(0) = \langle \hat{v}_{i-1}^n \rangle(t) - \langle \hat{v}_i^n \rangle(t) + \langle \hat{Y}_i^n \rangle(t) + g_i^n(t),$$

where $g_i^n(t)$ converges to 0 uniformly in t and in ω. Recalling that $\{\hat{Y}_i^n\left(\frac{l}{n}\right)\}$, $0 \leqslant l \leqslant \lfloor nT \rfloor$ is a martingale difference and $E\left(\langle \hat{Y}_i^n \rangle^2\left(\frac{l}{n}\right)\right) = O(1/n)$, by a standard martingale inequality

$$\langle \hat{Y}_i^n \rangle(t) \to 0 \text{ uniformly for } t \in [0, T]$$

in probability. Since $\langle \hat{Y}_i^n \rangle(t) + g_i^n(t)$ converges to 0 in probability and $(\bar{X}_i^n(t) - \bar{X}_i^n(0), \langle \hat{v}_{i-1}^n \rangle(t) - \langle \hat{v}_i^n \rangle(t))$ converges weakly to $(X_i(t) - X_i(0), \eta_{i-1}(t) - \eta_i(t))$,

$$X_i(t) - X_i(0) = \eta_{i-1}(t) - \eta_i(t) \text{ with probability 1.}$$

Recall that we have proved the existence of a process θ so that $\eta(t) = \int_0^t \theta(s)\, ds$. Thus the last display can be rewritten

$$X_i(t) - X_i(0) = \langle \theta \rangle_{i-1}(t) - \langle \theta \rangle_i(t) \text{ with probability 1,}$$

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

which is indeed

$$X(t) = X(0) + \int_0^t M\theta(s)\,ds \text{ with probability 1.} \qquad \square$$

Theorem 3.4. *Define $\mathscr{I}$ by (2.4) for any of the occupancy models discussed in Section 2. If $F : \mathscr{U} \mapsto \mathbb{R}$ is bounded and continuous, then*

$$\liminf_{n\to\infty} -\frac{1}{n}\log E\exp[-nF(X^n)] \geqslant \inf_{\varphi\in\mathscr{U}}[\mathscr{I}(\varphi) + F(\varphi)].$$

Proof. Owing to the representation formula (2.5) it is enough to show that

$$\liminf_{n\to\infty} \inf_{\{v^{i,n}\}} \bar{E}\left[F(\bar{X}^n) + \frac{1}{n}\sum_{i=0}^{\lfloor nT\rfloor} R\big(v^{i,n}\|\rho^{i,n}\big)\right] \geqslant \inf_{\varphi\in\mathscr{U}}[\mathscr{I}(\varphi) + F(\varphi)].$$

Consider any admissible sequence $\{v^{i,n}\}$. Then (3.1) and (3.2) imply

$$\begin{aligned}\bar{E}\left[F(\bar{X}^n) + \frac{1}{n}\sum_{i=0}^{\lfloor nT\rfloor} R\big(v^{i,n}\|\rho^{i,n}\big)\right] &\geqslant \bar{E}\left[F(\bar{X}^n) + \int_0^T R\big(\hat{v}^n(t)\|\hat{\rho}^n(t)\big)\,dt\right]\\ &= \bar{E}\big[F(\bar{X}^n) + TR\big(m\otimes\hat{v}^n\|m\otimes\hat{\rho}^n\big)\big].\end{aligned}$$

By Theorem 3.3 we know that for any subsequence of $\mathbb{N}$ there exists a subsubsequence such that $\{(\hat{X}^n, \langle\hat{v}^n\rangle)\}$ converges in distribution to a limit $(X, \langle\theta\rangle)$.

Let

$$\mathscr{W} \doteq \left\{ x \in \mathscr{V} : \sum_{i=1}^{I+1} ix(t) \leqslant t, \quad \text{for } 0 \leqslant t \leqslant T \right\}.$$

Owing to our construction of the controlled process $\hat{X}^n$, we know that for each ω, $\hat{X}^n(\omega) \in \mathscr{W}$. For $a \in (0,\infty)\cup\{-1,-2,\dots\}$ and $x \in \mathscr{W}$, define $g(x)$ by

$$(g(x))_i(t) \doteq \frac{a+i}{a+t}x_i(t), \quad \text{for } 0 \leqslant i \leqslant I,$$

and

$$(g(x))_{I+1}(t) \doteq 1 - \sum_{i=0}^{I}\frac{a+i}{a+t}x_i(t).$$

Then g maps $\mathscr{W}$ to $\mathscr{V}$. The case $a = \infty$ is defined as the obvious limit. When $a \in (0,\infty]$, g is clearly bounded and continuous. When $a < 0$ the boundedness of g follows from (2.6), and the bound

$$\|g_i\| \leqslant \frac{-i-a}{-t-a} \leqslant \frac{-a}{-a-T} = \frac{a}{a+T}.$$

The argument to show continuity is similar, and omitted.

With these definitions we have $g(\hat{X}^n)(t) = \hat{\rho}^n(t)$ and $g(X)(t) = \rho(t, X(t))$. Since $\hat{X}^n, X \in \mathscr{W}$, we have $\hat{\rho}^n$ and $\rho(\cdot, X(\cdot)) \in \mathscr{V}$. By the Continuous Mapping Theorem and the definition of $\hat{\rho}^n$, weak convergence of $\hat{X}^n$ to X implies weak convergence of $\hat{\rho}^n$ to $\rho(t, X(t))$ in $\mathscr{V}$. Applying

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

Lemma 3.2, we have that $m \otimes \hat{\rho}^n$ converges weakly to $m \otimes \rho$. Similarly, by Lemma 3.1 the weak convergence of $\langle \hat{v}^n \rangle$ to $\langle \theta \rangle$ implies the weak convergence of $m \otimes \hat{v}^n$ to $m \otimes \theta$.

Now, applying Fatou's lemma (for weak convergence) and using the lower semi-continuity of relative entropy,

$$\begin{aligned}\liminf_{n\to\infty} \bar{E}\left[F(\bar{X}^n) + \frac{1}{n}\sum_{i=0}^{\lfloor nT\rfloor} R\big(v^{i,n}\|\rho^{i,n}\big)\right] &\geqslant \liminf_{n\to\infty} \bar{E}\big[F(\bar{X}^n) + TR\big(m\otimes\hat{v}^n\|m\otimes\hat{\rho}^n\big)\big] \\ &\geqslant \bar{E}\big[F(X) + TR\big(m\otimes\theta\|m\otimes\rho\big)\big] \\ &= \bar{E}\left[F(X) + \int_0^T R\big(\theta(t)\|\rho(t,X(t))\big)\,dt\right]. \qquad (3.4)\end{aligned}$$

As proved in Theorem 3.3, $X(t) = X(0) + \int_0^t M\theta(s)\,ds$, therefore, by the definition (2.4) of the rate function $\mathscr{I}(\varphi)$,

$$\int_0^T R\big(\theta(t)\|\rho(t,X(t))\big)\,dt = \mathscr{I}(X).$$

Thus (3.4) yields

$$\liminf_{n\to\infty} \inf_{\{v_i^n\}} \bar{E}\left[F(\bar{X}^n) + \frac{1}{n}\sum_{i=0}^{\lfloor nT\rfloor} R\big(v^{i,n}\|\rho^{i,n}\big)\right] \geqslant \inf_{\varphi\in\mathscr{U}}[\mathscr{I}(\varphi) + F(\varphi)].$$

Hence we complete the proof of the large-deviation upper bound. □

4. Properties of the rate function

In this section, we prove important properties of the rate function, some of which will be used later on to prove the large-deviation lower bound.

Theorem 4.1. *Let $\mathscr{I}$ be defined as in* (2.4)*. Then, for any $K \in [0,\infty)$, the level set $\{\varphi \in \mathscr{U} : \mathscr{I}(\varphi) \leqslant K\}$ is compact.*

Proof. As is always the case in the weak convergence approach, the proof of compactness of level sets is essentially a deterministic analogue of the proof of the large-deviation upper bound, and is hence omitted. See [3, Proposition 6.2.4] for the proof in an analogous situation. □

Theorem 4.2 (zero-cost trajectory). *For $t \in [0,T]$ let $f(t) = \big(1 + \frac{t}{a}\big)^{-a}$ ($f(t) = e^{-t}$ in the case $a = \infty$),*

$$\phi_i(t) = \frac{(-t)^i}{i!} f^{(i)}(t), \quad \text{for } 0 \leqslant i \leqslant I,$$

and let $\phi_{I+1}(t) = 1 - \sum_{i=0}^{I} \phi_i(t)$. Then $\mathscr{I}(\phi) = 0$.

Proof. We first assume $a \neq \infty$. It is easy to see that, for any $0 \leqslant i < \infty$,

$$\frac{(-t)^i}{i!} f^{(i)}(t) \geqslant 0 \quad \text{and} \quad \sum_{i=0}^{\infty} \frac{(-t)^i}{i!} f^{(i)}(t) = 1. \qquad (4.1)$$

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

Thus ϕ as defined in the statement of the theorem is indeed a probability vector. It is also clearly a continuously differentiable function. We will show that

$$\dot{\phi}(t) = M\rho(t, \phi(t)). \tag{4.2}$$

If so, then the occupancy rate process θ associated to ϕ is indeed $\rho(t, \phi(t))$, and thus, by the definition of rate function,

$$\mathcal{I}(\phi) = \int_0^T R\big(\theta(t) \| \rho(t, \phi(t))\big)\, dt = 0.$$

To show (4.2), we calculate $\phi_i(t) = \frac{(-t)^i}{i!} f^{(i)}(t)$ for $0 \leqslant i \leqslant I$ explicitly:

$$\phi_i(t) = \frac{t^i}{i!} \frac{\prod_{j=0}^{i-1}(a+j)}{a^i} \left(1 + \frac{t}{a}\right)^{-a-i}.$$

Hence the derivative satisfies

$$\begin{aligned}
\dot{\phi}_i(t) &= \frac{a+i-1}{a+t}\phi_{i-1}(t) - \frac{a+i}{a+t}\phi_i(t) \\
&= \rho_{i-1}(t, \phi(t)) - \rho_i(t, \phi(t)) \\
&= (M\rho(t, \phi(t)))_i,
\end{aligned}$$

where the second equality is due to the definition of $\rho(t, \phi(t))$. The case of $\phi_{I+1}(t)$ is also a straightforward calculation and hence omitted.

Next we consider the case when $a = \infty$. In this case $f(t) = e^{-t}$, and the validity of (4.2) can be directly verified. □

Lemma 4.3. *For every choice of the parameter a, there exist $\delta > 0$ and $0 < K < \infty$ such that*

$$\phi_i(t) \geqslant \delta t^K \tag{4.3}$$

for any $0 \leqslant i \leqslant I + 1$.

Proof. Note that when $a > 0$, $0 \leqslant i \leqslant I$ and $0 \leqslant t \leqslant T$,

$$\begin{aligned}
\phi_i(t) &= \frac{t^i}{i!} \frac{\prod_{j=0}^{i-1}(a+j)}{a^i} \left(1 + \frac{t}{a}\right)^{-a-i} \\
&\geqslant \frac{t^i}{I!} \left(1 + \frac{T}{a}\right)^{-a-I},
\end{aligned} \tag{4.4}$$

and because of (4.1) we have

$$\begin{aligned}
\phi_{I+1}(t) &= 1 - \sum_{i=0}^{I} \phi_i(t) \qquad (4.5) \\
&\geqslant \frac{(-t)^{I+1}}{(I+1)!} f^{(I+1)}(t) \\
&\geqslant \frac{t^{I+1}}{(I+1)!} \left(1 + \frac{T}{a}\right)^{-a-I-1}. \qquad (4.6)
\end{aligned}$$

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

For the case $a < 0$, by (2.6) $T < -a$ and $a = -I - 1$. If $0 \leqslant i \leqslant I$, then, since $a + j \leqslant -1$ for each $0 \leqslant j \leqslant I$,

$$\phi_i(t) = \frac{t^i}{i!} \frac{\prod_{j=0}^{i-1}(a+j)}{a^i} \left(1 + \frac{t}{a}\right)^{-a-i} \geqslant \frac{t^i}{I!} \frac{1}{(-a)^i} \left(1 + \frac{t}{a}\right)^{-a-i}.$$

Moreover, since $a < 0$ and $-a - I = 1$, for each fixed a and $i \leqslant I$, $\left(1 + \frac{t}{a}\right)^{-a-i}$ is monotone decreasing in t. Therefore

$$\phi_i(t) \geqslant \frac{t^i}{I!} \left(-\frac{1}{a}\right)^i \left(1 + \frac{T}{a}\right)^{-a-i}.$$

For $\phi_{I+1}(t)$ we have

$$\begin{aligned}
\phi_{I+1}(t) &= 1 - \sum_{i=0}^{I} \phi_i(t) \\
&\geqslant \frac{(-t)^{I+1}}{(I+1)!} f^{(I+1)}(t) \\
&= \frac{t^{I+1}}{(I+1)!} \frac{\prod_{j=0}^{I}(a+j)}{a^{I+1}} \left(1 + \frac{t}{a}\right)^{-a-I} \\
&\geqslant \frac{t^{I+1}}{(I+1)!} \left(-\frac{1}{a}\right)^{I+1} \left(1 + \frac{T}{a}\right)^{-a-I}.
\end{aligned}$$

The last inequality follows exactly the same reasoning as for $0 \leqslant i \leqslant I$.

Finally, for the case $a = \infty$ we just take limits on (4.4) and (4.5), and use that

$$\lim_{a \to \infty} \left(1 + \frac{T}{a}\right)^{-a-I} = e^{-T}.$$

It follows that for every choice of the parameter a there exist $\delta > 0$ and $0 < K < \infty$ such that the zero-cost trajectory $\phi(t)$ satisfies (4.3) for any $0 \leqslant i \leqslant I + 1$. □

Lemma 4.4. *For a given value of a let the parameters δ and K be as in* (4.3). *Let $\varphi \in \mathscr{U}$ satisfy $\mathscr{I}(\varphi) < \infty$. Then, for any $\varepsilon > 0$ there exists $\varphi^\varepsilon \in \mathscr{U}$ such that:*

(1) $\mathscr{I}(\varphi^\varepsilon) \leqslant \mathscr{I}(\varphi)$,
(2) $d(\varphi, \varphi^\varepsilon) \leqslant \varepsilon$,
(3) $\varphi_i^\varepsilon(t) \geqslant \varepsilon \delta t^K$ *for all* $t \in [0, T]$ *and* $i = 0, 1, \ldots, I, I + 1$.

Proof. For any $\varepsilon > 0$ and $\varphi \in \mathscr{U}$, let

$$\varphi^\varepsilon = (1 - \varepsilon)\varphi + \varepsilon \bar{\varphi},$$

where $\bar{\varphi}$ is the zero-cost trajectory. Then $\varphi^\varepsilon \in \mathscr{U}$. From the definition of $\rho(t, x)$ in (2.1) it follows that $\rho(t, x)$ is linear in x. Also, recalling the definition of $\mathscr{I}(\varphi)$ in (2.4) and the joint convexity of

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

relative entropy, we find that $\mathscr{I}(\varphi)$ is convex in φ. Therefore

$$\begin{aligned}\mathscr{I}(\varphi^\varepsilon) &\leqslant (1-\varepsilon)\mathscr{I}(\varphi) + \varepsilon\mathscr{I}(\bar{\varphi}) \\ &= (1-\varepsilon)\mathscr{I}(\varphi) \\ &\leqslant \mathscr{I}(\varphi).\end{aligned}$$

Since $d(\varphi, \bar{\varphi}) \leqslant 1$,

$$d(\varphi, \varphi^\varepsilon) \leqslant \varepsilon d(\varphi, \bar{\varphi}) \leqslant \varepsilon,$$

and also $\varphi^\varepsilon \geqslant \varepsilon\bar{\varphi} \geqslant \varepsilon\delta t^K$. □

The final theorem of this section is essential in proving the large-deviation lower bound.

Definition (good path). We call an occupancy process $\varphi \in \mathscr{U}$ a *good path* if there exist constants $0 < \delta', K' < \infty$ so that $\varphi_i(t) \geqslant \delta' t^{K'}$ for $t \in [0, T]$ and $0 \leqslant i \leqslant I+1$.

Definition (good control). We call an occupancy rate process θ a *good control* if the process θ is piecewise constant on $[0, T]$, with a finite number of intervals of constancy. In other words, there exist a finite number of intervals $[r_i, s_i], 1 \leqslant i \leqslant m$ so that $[0, T] = \cup_{i=1}^m [r_i, s_i]$, and $\theta(t)$ is a constant vector on each (r_i, s_i). In addition, we assume there exists $0 < \sigma < T$ so that θ is 'pure' on $[0, \sigma)$, in the sense that, for any interval of constancy $(r, s) \subset [0, \sigma)$, there exists $i, 0 \leqslant i \leqslant I+1$ such that $\theta_i(t) = 1$ for $t \in (r, s)$.

Theorem 4.5. *For a good path $\varphi \in \mathscr{U}$ assume $\mathscr{I}(\varphi) < \infty$. Let δ', K' be the associated constants in the definition of a good path. For any $\varepsilon > 0$ there exists a good control θ^* and associated $\sigma > 0$ so that, if φ^* is the occupancy path associated to θ^*, then*

(1) $\mathscr{I}(\varphi^*) \leqslant \mathscr{I}(\varphi) + \varepsilon$,
(2) $d(\varphi^*, \varphi) \leqslant \varepsilon$,
(3) *if $t < \sigma$ and $\theta_i^*(t) = 1$ then $\varphi_i^*(t) > \delta'\sigma^{K'}$.*

Proof. For a $\sigma > 0$ that will be specified later on, we construct a pure control $\theta^*(t), t \in [0, \sigma)$ as follows. For $0 \leqslant i \leqslant I$, let $\theta_i^*(t) = 1$ if

$$\sum_{j=0}^{i} j\varphi_j(\sigma) + \sum_{k=i+1}^{I+1} i\varphi_k(\sigma) \leqslant t < \sum_{j=0}^{i} j\varphi_j(\sigma) + \sum_{k=i+1}^{I+1} (i+1)\varphi_k(\sigma),$$

and let $\theta_{I+1}^*(t) = 1$ if

$$\sum_{j=0}^{I} j\varphi_j(\sigma) + (I+1)\varphi_{I+1}(\sigma) \leqslant t < \sigma.$$

Observe that the component φ_i^* will increase only during the interval when $\theta_{i-1}^*(t) = 1$, and that it decreases to its final value while $\theta_i^*(t) = 1$. Observe also that $\varphi^*(\sigma) = \varphi(\sigma)$. Hence, for $t < \sigma$, if $\theta_i^*(t) = 1$ then $\varphi_i^*(t) \geqslant \varphi_i^*(\sigma) \geqslant \delta'\sigma^{K'}$.

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

Now assume that $0 < a < \infty$. For such i and t,

$$\begin{aligned}\rho_i(t, \varphi^*(t)) &= \frac{a+i}{a+t}\varphi_i^*(t) \\ &\geqslant \frac{a}{a+T}\delta'\sigma^{K'} \\ &= \delta''\sigma^{K'}, \end{aligned} \tag{4.7}$$

where $\delta'' \doteq \frac{a}{a+T}\delta'$. Note that the above bound is true for all $0 \leqslant i \leqslant I+1$.

Recall that when $a < 0$ we assume without loss that $I = -a-1$, and that no balls are placed in urns that currently contain more than I balls. It follows that

$$\sum_{j=0}^{I+1} j\varphi_j(\sigma) = \sigma,$$

and that $\theta_{I+1}(t)$ is always zero. Hence the same will be true of $\theta_{I+1}^*(t)$, *i.e.*, $\theta_{I+1}^*(t) = 0$ for all $t \in [0, \sigma]$. For $0 \leqslant i \leqslant I$, we have

$$\begin{aligned}\rho_i(t, \varphi^*(t)) &\geqslant \frac{a+i}{a+t}\delta'\sigma^{K'} \\ &\geqslant \frac{a+I}{a+t}\delta'\sigma^{K'} \\ &\geqslant -\frac{1}{a}\delta'\sigma^{K'}. \end{aligned}$$

Thus for such i there exists a constant $\delta'' > 0$ such that $\rho_i(t, \varphi^*(t)) \geqslant \delta''\sigma^{K'}$ when $\theta_i^*(t) = 1$.

Finally, when $a = \infty$ we can choose $\delta'' = \delta'$ and (4.7) will hold.

This completes the construction of θ^* and φ^* on $[0, \sigma)$. The lower bounds on the ρ_i and the fact that θ^* is pure on $[0, \sigma]$ imply

$$\int_0^\sigma R\big(\theta^*(t) \| \rho(t, \varphi^*(t))\big)\, dt \leqslant -\sigma \log(\delta''\sigma^{K'}).$$

Now let us choose σ small enough that $\int_0^\sigma R(\theta^*(t)\|\rho(t, \varphi^*(t)))\, dt \leqslant \varepsilon/2$ and $\sup_{t\in[0,\sigma)} |\varphi^*(t) - \varphi(t)| < \varepsilon$. Also, recall that under the construction $\varphi^*(\sigma) = \varphi(\sigma)$.

The construction of controls on $[\sigma, T]$ is easier. Let $\theta(t)$ be the rate process associated with $\varphi(t)$ by (2.3). For $M \in \mathbb{N}$ we partition $[\sigma, T]$ into M sub-intervals of length $c_M = (T-\sigma)/M$. For each s that $\sigma + lc_M \leqslant s \leqslant \sigma + (l+1)c_M$ where $0 \leqslant l \leqslant (M-1)$, let

$$\theta^{(M)}(s) = \frac{\int_{\sigma+lc_M}^{\sigma+(l+1)c_M} \theta(t)\, dt}{c_M}.$$

Let $\varphi^{(M)}(t)$ be the occupancy path associated with $\theta^{(M)}(t)$. Then it is easy to check that $\varphi^{(M)}(t)$ coincides with $\varphi(t)$ on the 'partition points' in $[\sigma, T]$, *i.e.*, those points of the form $\{\sigma + lc_M : 0 \leqslant l \leqslant (M-1)\}$. Thus, for M large enough (*e.g.*, $M > (T-\sigma)/\varepsilon$), $\sup_{t\in[\sigma,T]} |\varphi^{(M)}(t) - \varphi(t)| < \varepsilon$.

Because $\varphi(t)$ is good, when $t > \sigma$, we have $\varphi(t) \geqslant \delta' t^{K'} \geqslant \delta'\sigma^{K'} > 0$. Therefore $\varphi(t)$ is uniformly bounded away from the boundary after time σ. As $M \to \infty$, $\theta^{(M)}(t)$ converges to $\theta(t)$ and $\varphi^{(M)}(t)$ converges to $\varphi(t)$ a.e., and thus, by the Lebesgue Dominated Convergence Theorem,

$$\lim_{M\to\infty} \int_\sigma^T R\big(\theta^M(t) \| \rho(t|\varphi^M(t))\big)\, dt = \int_\sigma^T R\big(\theta(t) \| \rho(t|\varphi(t))\big)\, dt.$$

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

Now choose $M < \infty$ sufficiently large that

$$\int_\sigma^T R\big(\theta^M(t)\|\rho(t|\varphi^M(t))\big)\,dt < \int_\sigma^T R\big(\theta(t)\|\rho(t|\varphi(t))\big)\,dt + \varepsilon/2.$$

Let θ^* be defined as it was previously on $[0,\sigma]$, and set it equal to θ^M on $[\sigma, T]$. We have

$$\begin{aligned}\mathscr{I}(\varphi^*) &= \int_\sigma^T R\big(\theta^M(t)\|\rho(t|\varphi^M(t))\big)\,dt + \int_0^\sigma R\big(\theta^*(t)\|\rho(t|\varphi^*(t))\big)\,dt \\ &< \int_\sigma^T R\big(\theta(t)\|\rho(t|\varphi(t))\big)\,dt + \varepsilon/2 + \varepsilon/2 \\ &\leqslant \mathscr{I}(\varphi) + \varepsilon.\end{aligned}$$

Thus we complete the proof. □

5. The large-deviation lower bound

Theorem 5.1. *Define $\mathscr{I}$ by* (2.4) *for any of the occupancy models discussed in Section 2. If $F : \mathscr{U} \mapsto \mathbb{R}$ is bounded and continuous, then*

$$\limsup_{n\to\infty} -\frac{1}{n}\log E\exp[-nF(X^n)] \leqslant \inf_{\varphi\in\mathscr{U}}[\mathscr{I}(\varphi) + F(\varphi)].$$

Proof. According to (2.5), the theorem follows if

$$\limsup_{n\to\infty}\inf_{\{v^{i,n}\}} \bar{E}\left[F(\bar{X}^n) + \frac{1}{n}\sum_{i=0}^{\lfloor nT\rfloor} R\big(v^{i,n}\|\rho^{i,n}\big)\right] \leqslant \inf_{\varphi\in\mathscr{U}}[\mathscr{I}(\varphi) + F(\varphi)].$$

For any $\varphi \in \mathscr{U}$ such that $\mathscr{I}(\varphi) < \infty$, Lemma 4.4 and Theorem 4.5 imply that for any $\varepsilon > 0$ there exists (φ^*, θ^*) with the properties described in Theorem 4.5. Since F is continuous in $\mathscr{U}$, we only need to show that there exists a sequence of admissible controls $\{v^n\}$ so that

$$\limsup_{n\to\infty} \bar{E}\left[F(\bar{X}^n) + \frac{1}{n}\sum_{i=0}^{\lfloor nT\rfloor} R\big(v^{i,n}\|\rho^{i,n}\big)\right] \leqslant \mathscr{I}(\varphi^*) + F(\varphi^*).$$

The latter inequality will follow if we can find a sequence of admissible $\{v^n\}$ such that

$$\limsup_{n\to\infty} \bar{E}\left[\sum_{i=0}^{\lfloor nT\rfloor} R\big(v^{i,n}\|\rho^{i,n}\big)\right] \leqslant \mathscr{I}(\varphi^*), \tag{5.1}$$

and such that if $\bar{X}^n$ is the occupancy process constructed under $\{v^n\}$ then, for any small $b > 0$,

$$\limsup_{n\to\infty} \bar{P}\{d(\bar{X}^n, \varphi^*) > b\} = 0. \tag{5.2}$$

In other words, $\bar{X}^n$ converges to φ^* in probability.

To prove the desired inequalities (5.1) and (5.2) we need to construct the proper $\{v^n\}$. Recall that $\{v^n\}$ can depend in any measurable way on the 'past', and so we could, in principle, use such information in constructing the controls. However, it turns out that we can construct the controls without reference to the controlled process (so-called 'open loop' controls). Let θ^* be the good

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

control as described in Theorem 4.5. We know that θ^* is piecewise constant and pure up to time $\sigma > 0$. We also know that, before time σ, if $\theta_i^*(t) = 1$ then both $\rho_i(t, \varphi^*(t))$ and $\varphi_i^*(t)$ are greater than a fixed value $\zeta > 0$. We can also assume, for the same value of ζ, that both $\rho_i(t, \varphi^*(t))$ and $\varphi_i^*(t)$ are greater than ζ for all $i \in [0, 1, \ldots, I, I+1]$ and all $t \in [\sigma, T]$.

Although the limit trajectory stays away from the boundary after time σ, there is no guarantee that the random process $\bar{X}^n$ is uniformly bounded away. In order to handle this possibility, we use a stopping time argument similar to one used in [4].

Let (l_n/n) be the minimum of the first time such that, for some i, $\bar{X}_i^n(l_n/n) \leqslant \zeta/2$ and $\theta_i^*(l_n/n) > 0$, and the fixed deterministic time T. This is the first time the random process is close to the boundary, and hence there is the possibility of a large contribution to the total cost (note that when $\theta_i^*(l_n/n) = 0$ there is no contribution to the cost regardless of the value of $\bar{X}_i^n(l_n/n)$). The control $\{v^n\}$ is then defined by

$$v^{i,n} = \begin{cases} \theta^*(i/n) & \text{if } i \leqslant l_n, \\ \rho(i/n, \bar{X}^n(i/n)) & \text{if } i > l_n. \end{cases}$$

Prior to the stopping time, we use exactly what θ^* suggests, and after the stopping time we follow the law of large number trajectory (and therefore incur no additional cost).

Now we apply Theorem 3.3. Thus, given any subsequence we have convergence along a further subsequence as indicated in the theorem, with limit $(\bar{X}, [\bar{\theta}])$. Using a standard argument by contradiction, it will be enough to prove (5.1) and (5.2) for this convergent processes. Let $\tau^n = (l_n/n) \leqslant T$. Note that, because the applied controls are pure, the process $\bar{X}^n(t)$ is deterministic prior to σ, and also that, prior to this time, the time derivatives of $\bar{X}^n(t)$ and $\varphi^*(t)$ are piecewise constant. In fact, the two derivatives are identical, except possibly on a bounded number of intervals, each of length less than $1/n$ (the points where they may disagree are all located with within distance $1/n$ of the endpoints of the intervals of constancy of $\dot{\varphi}^*(t)$). Thus, for large n, we cannot have $\tau^n < \sigma$. Since the range of τ^n is a bounded set in $\mathbb{R}$, we can also assume that τ^n converges in distribution to a limit τ, and without loss we assume the convergence is along the same subsequence. Since $\tau^n \geqslant \sigma$ for large n, we have $\tau \geqslant \sigma$ with probability 1. It is easy to check that the limit control processes, with probability 1, satisfy

$$\bar{\theta}(t) = \begin{cases} \theta^*(t) & \text{if } t \leqslant \tau, \\ \rho(t, \bar{X}(t)) & \text{if } t > \tau. \end{cases}$$

Owing to the definition of τ^n, if $\tau < T$ then $\bar{X}_i(\tau) \leqslant \zeta/2$ for some $i \in [0, 1, \ldots, I, I+1]$ (although $\varphi_i^*(t) \geqslant \zeta$ when $t \in [\sigma, T]$).

We use that $\bar{\theta}(t) = \theta^*(t)$ when $t \leqslant \tau$ and that $\theta^*(t)$ is deterministic. As shown in Theorem 3.3, $(\bar{X}, \theta^*)$ satisfies (2.3) for $t \in [0, \tau]$. Thus, for $t \in [0, \tau]$,

$$\bar{X}(t) = \varphi^*(t) \text{ with probability 1.}$$

This forms a contradiction since

$$\bar{X}_i(\tau) \leqslant \zeta/2 < \zeta \leqslant \varphi_i^*(\tau).$$

Therefore $\tau = T$, and thus, for all $t \in [0, T]$,

$$\bar{X}(t) = \varphi^*(t) \text{ with probability 1.}$$

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

This also indicates that the weak limit of the random processes is indeed limit (φ^*, θ^*), which implies (5.2). To prove (5.1), we use the weak convergence and the Dominated Convergence Theorem:

$$\limsup_{n\to\infty} \bar{E}\left[\frac{1}{n}\sum_{i=0}^{\lfloor nT\rfloor} R\big(v^{i,n}\|\rho^{i,n}\big)\right] = \int_0^T R\big(\theta^*(t)\|\rho(t,\varphi^*(t))\big)\,dt$$
$$= \mathscr{I}(\varphi^*).$$

This completes the proof. □

6. Explicit solution to the calculus of variations problem

In the previous sections we identified the process-level large-deviation rate function (2.4) for a class of occupancy problems. The large-deviation principle for the process at a given fixed time can then be expressed in terms of the solution to a calculus of variations problem. In most cases this calculus of variations problem will not have a closed-form solution. However, for the class of occupancy models studied here it can be identified with the solution to a related finite-dimensional minimization problem. This latter problem can be solved by the standard Lagrange multiplier method, which is easily implemented numerically. In this section we give the precise statement of this equivalence. The explicit formula generalizes that obtained in [4] for the special case of MB statistics. The techniques we use are quite different, based as they are on dynamic programming and control theory rather than methods from the calculus of variations.

6.1. Theorem statement

Suppose the current occupancy state is $x \in S_I$ and that t is the number of balls per urn among all categories. If $y_i, i = 0, 1, \ldots, I, I+1, \ldots$ are the fraction in category i, then $x_i = y_i$ for $i \leqslant I$ and

$$t = \sum_{k=0}^{\infty} k y_k.$$

Note that $t \geqslant \sum_{k=0}^{I+1} k x_k$.

Although the proofs in the previous sections considered the particular case where all urns start empty, one can without difficulty extend as in [4] to the case of a general (feasible) initial time and initial condition (x, t). Feasibility in this context depends on the underlying parameter a.

Definition (feasible domain). Define $\mathscr{D}_a$, the *feasible domain* for the occupancy model with parameter a, as follows.

- When $a > 0$,

$$\mathscr{D}_a \doteq \left\{(x,t) \in \mathscr{S}_I \times [0,T) : x_{I+1} > 0 \text{ and } t \geqslant \sum_{i=0}^{I+1} i x_i\right\}$$
$$\bigcup \left\{(x,t) \in \mathscr{S}_I \times [0,T) : x_{I+1} = 0 \text{ and } t = \sum_{i=0}^{I} i x_i\right\}.$$

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

- When $a < 0$ and $I = -a - 1$,

$$\mathscr{D}_a \doteq \left\{ (x,t) \in \mathscr{S}_I \times [0,T) : t = \sum_{i=0}^{I+1} i x_i \right\}.$$

As before, when $a < 0$ we restrict to $T < -a$. In the first case the second set in the union reflects the fact that when $x_{I+1} = 0$ the number of balls thrown is exactly $\sum_{i=0}^{I} i x_i$, and similarly for the second case.

Consider a valid occupancy process $\varphi \in C([t,T], \mathscr{S}_I)$ with $\varphi(t) = x$. Making the dependence on (x,t) explicit, the rate function $\mathscr{I}(x,t;\varphi)$ for such paths can be written

$$\mathscr{I}(x,t;\varphi) \doteq \int_t^T R\big(\theta(s) \| \rho(s,\varphi(s))\big)\, ds,$$

where

$$\dot{\varphi}(s) = M\theta(s) \tag{6.1}$$

and

$$\rho_k(s,y) \doteq \frac{a+k}{a+s} y_k, \quad k = 0,1,\ldots,I,$$
$$\rho_{I+1}(s,y) \doteq 1 - \sum_{k=0}^{I} \rho_k(s,y).$$

The relevant calculus of variations problem for a point in the feasible domain is

$$\mathscr{J}(x,t;\omega) \doteq \inf_{\substack{\varphi \in C([t,T],\mathscr{S}_I) \\ \varphi(t)=x, \varphi(T)=\omega}} \mathscr{I}(x,t;\varphi). \tag{6.2}$$

The formula for the finite-dimensional minimization problem requires some notation. For all $a \in \mathbb{R}$, $a \neq 0$ and $i \in \mathbb{N}$, let

$$\binom{a}{i} \doteq \frac{\prod_{j=0}^{i-1}(a-j)}{i!}.$$

Note that if $a \in \mathbb{N}$ and $i > a$ then $\binom{a}{i} = 0$, and that if $a \notin \mathbb{N} \cup \{0\}$ and $i \in \mathbb{N}$, then $\binom{a}{i} \neq 0$. We will use the fact that if $a \in \mathbb{R}$ and $|z| < 1$, then the binomial expansion

$$(1+z)^{-a} = \sum_{i=0}^{\infty} \binom{-a}{i} z^i$$

is valid, and if $-a \in \mathbb{N}$ then the sum contains only a finite number of non-zero terms and is valid for all $z \in \mathbb{R}$.

For $i \in \mathbb{N} \cup \{0\}$ and $a > 0$, $s \geqslant 0$ or $a \in -\mathbb{N}$, $0 \leqslant s \leqslant -a$, define

$$Q_i^a(s) \doteq \left(-\frac{s}{a}\right)^i \binom{-a}{i} \left(1 + \frac{s}{a}\right)^{-a-i}.$$

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

If $a > 0, s \geqslant 0$ and $|s\theta/(a+s)| < 1$, then the binomial expansion gives

$$\begin{aligned}\sum_{i=0}^{\infty} Q_i^a(s)\theta^i &= \sum_{i=0}^{\infty}\left(-\frac{s}{a}\right)^i\binom{-a}{i}\left(1+\frac{s}{a}\right)^{-a-i}\theta^i \\ &= \left(1+\frac{s}{a}\right)^{-a}\sum_{i=0}^{\infty}\left(-\frac{s}{s+a}\theta\right)^i\binom{-a}{i} \\ &= \left(1+\frac{s}{a}\right)^{-a}\left(1-\frac{s\theta}{s+a}\right)^{-a} \\ &= \left(1+\frac{s}{a}-\frac{s\theta}{a}\right)^{-a} \\ &= \left(1+\frac{s}{a}(1-\theta)\right)^{-a}.\end{aligned}$$

We thus have the following expressions, where the second one may be justified by a very similar calculation (when $|s\theta/(a+s)| < 1$):

$$\sum_{i=0}^{\infty} Q_i^a(s)\theta^i = \left(1+\frac{s}{a}(1-\theta)\right)^{-a}, \qquad \sum_{i=0}^{\infty} iQ_i^a(s)\theta^i = s\theta\left(1+\frac{s}{a}(1-\theta)\right)^{-a-1}. \tag{6.3}$$

Note also that when $-a \in \mathbb{N}$ and $0 \leqslant s \leqslant -a$, the number of non-zero summands is finite and the formulas again hold. If $-a \in \mathbb{N}$ and $0 \leqslant s \leqslant -a$ or $a > 0$ and $s \geqslant 0$, then $Q_i^a(s) \geqslant 0$ for $i \in \mathbb{N} \cup \{0\}$. Letting $\theta \uparrow 1$ in the first expression shows that under these conditions $Q_i^a(s)$ defines a probability measure on $\mathbb{N} \cup \{0\}$. When $a = 0$ we use the limiting values

$$Q_0^0(s) = 1, \qquad Q_i^0(s) = 0$$

for all $i \in \mathbb{N}$ and $s \geqslant 0$.

Denote $\pi^k = \{\pi_0^k, \pi_1^k, \dots\} \in \mathbb{R}_\infty$ for all $0 \leqslant k \leqslant I+1$, where π_i^k represents the probability of throwing i additional balls into the kth category. Denote $\pi = (\pi^0, \pi^1, \dots, \pi^I, \pi^{I+1})$, so that $\pi \in \mathbb{R}_\infty^{I+2}$. For any given $x \in \mathscr{S}_I$, we say $\pi = (\pi^0, \pi^1, \dots, \pi^I, \pi^{I+1}) \in \mathscr{F}(x, t; \omega, T)$ if

$$x_k \sum_{j=0}^{\infty} \pi_j^k = x_k, \quad 0 \leqslant k \leqslant I+1, \qquad \sum_{k=0}^{I+1} x_k \sum_{j=0}^{\infty} j\pi_j^k = T - t, \tag{6.4}$$

and

$$\omega_i = \sum_{k=0}^{i} x_k \pi_{i-k}^k, \quad 0 \leqslant i \leqslant I, \qquad \omega_{I+1} = 1 - \sum_{k=0}^{I} \omega_k.$$

We will use $\omega \doteq x \times \pi$ as shorthand for the last display. A terminal point ω is *feasible* (for the given initial time and condition) if $\mathscr{F}(x, t; \omega, T)$ is not empty.

Now we are ready to state the theorem. For $s > 0$ let $P(s)$ denote the Poisson distribution with parameter s, and if $s = 0$ let $P(s)$ denote the probability measure on $\{0, 1, \dots\}$ with mass one on $\{0\}$.

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

Theorem 6.1 (explicit formula for the rate function). *Consider an initial condition* $(x,t) \in \mathscr{D}_a$*, and a feasible terminal condition* ω*. If* $a \in (0,\infty)$*, then for* $x_{I+1} > 0$ *let*

$$\tau(x,t) \doteq \frac{(t - \sum_{k=0}^{I} k x_k)}{x_{I+1}}$$

(*so that* $\tau(x,t)$ *is the average number of balls per urn distributed in the* $I+$ *categories for the initial condition* (x,t)*) and if* $x_{I+1} = 0$ *let* $\tau(x,t) = 0$*. Then the quantity* $\mathscr{J}(x,t;\omega)$ *defined in* (6.2) *has the representation*

$$\mathscr{J}(x,t;\omega) = \min_{\pi \in \mathscr{F}(x,t;\omega,T)} \left\{ \sum_{k=0}^{I} x_k R\left(\pi^k \middle\| Q^{a+k}\left(\frac{a+k}{a+t}(T-t) \right) \right) \right.$$
$$\left. + x_{I+1} R\left(\pi^{I+1} \middle\| Q^{a+\tau(x,t)}\left(\frac{a+\tau(x,t)}{a+t}(T-t) \right) \right) \right\}.$$

If $a \in -\mathbb{N}$ *with* $I = -a-1$ *then* $\tau(x,t) = I+1$*, and*

$$\mathscr{J}(x,t;\omega) = \min_{\pi \in \mathscr{F}(x,t;\omega,T)} \left\{ \sum_{k=0}^{I+1} x_k R\left(\pi^k \middle\| Q^{a+k}\left(\frac{a+k}{a+t}(T-t) \right) \right) \right\}.$$

In the final case of $a = \infty$*, we have*

$$\mathscr{J}(x,t;\omega) = \min_{\pi \in \mathscr{F}(x,t;\omega,T)} \left\{ \sum_{k=0}^{I+1} x_k R\big(\pi^k \| P(T-t)\big) \right\}.$$

Although these minimization problems as stated appear to be infinite-dimensional, they can in fact be reduced to finite-dimensional problems. This is because if π^k is the minimizer, then π^k_j takes a prescribed form for $j > I$. In fact, all π^k_j can be represented in terms of no more than $I+3$ Lagrange multipliers as in (6.10) below.

6.2. The Hamilton–Jacobi–Bellman equation

Since the Maxwell–Boltzmann case ($a = \infty$) is rigorously analysed in [4], we will assume $a < \infty$ from now on. However, the case $a = \infty$ can easily be proved using the arguments developed here, and can also be obtained as a limit.

The calculus of variations problem (6.2) has a natural control interpretation, where $\theta(s)$, $t \leqslant s \leqslant T$ is the control, $\dot{\varphi}(s) = M\theta(s)$ are the dynamics, $R(\theta(s)\|\rho(s,\varphi(s)))$ is the running cost and $g(x) = 1_{\{x \neq \omega\}} \cdot \infty$ is the terminal cost. It is expected that if we define

$$V(x,t) \doteq \inf_{\varphi \in C([t,T],\mathscr{S}_I),\varphi(t)=x} \left\{ \int_t^T R\big(\theta(s)\|\rho(s,\varphi(s))\big)\, ds + g(\varphi(T)) \right\}, \tag{6.5}$$

then $V(x,t)$ is a weak-sense solution to the *Hamilton–Jacobi–Bellman* (HJB) equation

$$W_t + H(W_x, x, t) = 0,$$

and terminal condition

$$W(x,T) = 1_{\{x \neq \omega\}} \cdot \infty.$$

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

Here the Hamiltonian $H(p,x,t)$ is defined by

$$H(p,x,t) \doteq \inf_{\theta\in\mathscr{S}_I}\left[\langle p, M\theta\rangle + R(\theta\|\rho(t,x))\right]$$

and W_t and W_x denote the partial derivative with respect to t and gradient in x, respectively. Note that by the representation formula [3, Proposition 1.4.2], the infimum in the definition of $H(p,x,t)$ can be evaluated, yielding

$$\begin{cases} W_t = \log\left(\sum_{k=0}^{I} x_k\left(\frac{a+k}{a+t}\right)\exp(W_{x_k} - W_{x_{k+1}}) + \left(\frac{a+\tau(x,t)}{a+t}\right)x_{I+1}\right), \\ W(x,T) = g(x). \end{cases} \tag{6.6}$$

The problems of interest in applications often have a terminal condition of the form

$$g(x) = 1_{\mathscr{A}^c}(x)\cdot\infty,$$

where $\mathscr{A}$ is some convex set. For example, $\mathscr{A}$ can be a single point of ω as was discussed previously. Such terminal conditions usually yield only a weak-sense solution, and not a classical-sense C^1 solution to the HJB equation. However, it turns out for the occupancy problem that there are terminal conditions that yield classical-sense solutions, and as we will see, these solutions can be used to carry out a fairly complete analysis of the problem for general terminal conditions, including the very singular terminal conditions that arise in defining the large-deviation rate $\mathscr{J}(x,t;\omega)$.

In the next result we state a representation for the calculus of variations problem with affine terminal cost $\langle l,\omega\rangle + b$. To simplify, we first observe that W is a solution of just the PDE alone (*i.e.*, without the terminal condition) if and only if $W + c$ is a solution for any real number c. Since ω is a probability vector, it suffices to prove the representation when $l_{I+1} = 0$ and $b = 0$.

Proposition 6.2. *Consider* $(x,t)\in\mathscr{D}_a$ *and* $g(\omega) = \langle l,\omega\rangle$, *where* $l\in\mathbb{R}^{I+2}$ *and* $l_{I+1} = 0$. *Define*

$$V(x,t) \doteq \inf_{\varphi\in C([t,T],\mathscr{S}_I),\varphi(t)=x}\left\{\int_t^T R\big(\theta(s)\|\rho(s,\varphi(s))\big)\,ds + g(\varphi(T))\right\},$$

and

$$\begin{aligned} U(x,t) \doteq \min_{\pi\in\mathscr{F}(x,t;T)}\Bigg\{ &\sum_{k=0}^{I} x_k R\left(\pi^k\middle\|Q^{a+k}\left(\frac{a+k}{a+t}(T-t)\right)\right) \\ &+ x_{I+1}R\left(\pi^{I+1}\middle\|Q^{a+\tau(x,t)}\left(\frac{a+\tau(x,t)}{a+t}(T-t)\right)\right) + g(x\times\pi)\Bigg\}, \end{aligned} \tag{6.7}$$

where $\pi\in\mathscr{F}(x,t;T)$ *means that* π *satisfies the constraints in* (6.4). *Then* $V(x,t) = U(x,t)$.

The fact that Theorem 6.1 follows from this proposition can be argued as follows. For a general (proper) convex terminal cost $g(\omega)$, the Legendre transform gives a representation of the form

$$g(\omega) = \sup_{\beta\in\mathbb{R}^{I+2}}[\langle\beta,\omega\rangle - h(\beta)].$$

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

Let $V^g(x,t)$ denote the solution to the calculus of variations problem (6.5) with terminal cost $g(\cdot)$. Then one can show

$$V^g(x,t) = \sup_{\beta \in \mathbb{R}^{I+2}} V^{\{\langle \beta,\omega\rangle - h(\beta)\}}(x,t),$$

and an analogous formula for U^g. Given Proposition 6.2, $V^g = U^g$ then follows.

6.3. Analysis of the finite-dimensional minimization problem

We now focus on proving Proposition 6.2. We will do so by proving that $U(x,t)$ is a classical-sense solution to the HJB equation (6.6). A modification of the standard verification argument [7] can then be used to show that $V(x,t) = U(x,t)$. The classical verification argument consists of two parts. One first considers any valid occupancy process and control (φ,θ) for the initial condition (x,t). If U is a smooth solution to the PDE (6.6) in neighbourhood of $\{(\varphi(s),s) : t \leqslant s \leqslant T\}$ and if $U(\varphi(T),T) = \langle \varphi(T), l\rangle$, then the chain rule implies that the cost along this trajectory is at least $U(x,t)$. The reverse inequality is proved by defining an optimal feedback control through the HJB equation, using this control to construct a trajectory, and then verifying (once again via the chain rule) that the cost for this control is $U(x,t)$. The characterization of $V(x,t)$ as an infimum over all valid occupancy processes and controls that start at (x,t) then gives $V(x,t) = U(x,t)$. However, we have to clarify here what is meant by a 'classical-sense' solution to (6.6). The difficulty is that $U(x,t)$ is only well defined on the set $\mathscr{D}_a$, whose interior is empty.

Given any point $(x,t) \in \mathscr{D}_a$, we will prove that one can we extend $U(x,t)$ smoothly to a neighbourhood of (x,t) in $\mathbb{R}^{I+2} \times \mathbb{R}$. To be more precise, for any such (x,t) we will show there exists a neighbourhood $\mathscr{U} \subset \mathbb{R}^{I+2} \times \mathbb{R}$ of (x,t) and a function $\bar{U} \in C^\infty(\mathscr{U},\mathbb{R})$, such that $\bar{U}(y,s) = U(y,s)$ for $(y,s) \in \mathscr{U} \cap \mathscr{D}_a$, and that $\bar{U}$ satisfies (6.6) in $\mathscr{U} \cap \mathscr{D}_a$. One can then use $\bar{U}$ in place of U in the verification argument, since any feasible trajectory will never leave $\mathscr{D}_a$.

To analyse $U(x,t)$ we formulate an appropriate Lagrangian. Let

$$\begin{aligned} f(x,t;\pi) \doteq \sum_{k=0}^{I} x_k R\left(\pi^k \middle\| Q^{a+k}\left(\frac{a+k}{a+t}(T-t)\right)\right) \\ + x_{I+1} R\left(\pi^{I+1} \middle\| Q^{a+\tau(x,t)}\left(\frac{a+\tau(x,t)}{a+t}(T-t)\right)\right) + \langle l, x \times \pi\rangle \end{aligned} \tag{6.8}$$

and for a set of Lagrange multipliers $\Lambda \doteq (\lambda,\mu) = (\lambda_0,\lambda_1,\ldots,\lambda_I,\lambda_{I+1},\mu)$, let

$$L(x,t;\Lambda;\pi) \doteq f(x,t;\pi) + \sum_{k=0}^{I+1} \lambda_k x_k \left(1 - \sum_{j=0}^{\infty} \pi_j^k\right) + \mu\left(T - t - \sum_{k=0}^{I+1} x_k \sum_{j=0}^{\infty} j\pi_j^k\right). \tag{6.9}$$

It follows from the definition of $U(x,t)$ that $U(x,t) = \inf_\pi \sup_\Lambda L(x,t;\Lambda;\pi)$.

Note that by the joint convexity of relative entropy, $L(x,t;\Lambda;\pi)$ is convex in π. Thus (6.7) is a standard convex programming problem with linear constraints, except that the minimization is over a variable π which is infinite-dimensional. Hence the standard Lagrange multiplier method does not apply directly. If we temporarily ignore this issue, then to guess the form of the

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

minimizer one would of course set $D_\pi L(x,t;\Lambda;\pi)=0$ to get $\pi=\pi(x,t;\Lambda)$, where

$$\begin{aligned}\pi_j^k(x,t;\Lambda) &= Q_j^{a+k}\left(\frac{a+k}{a+t}(T-t)\right)\exp(\lambda_k-1+j\mu-l_{k+j}),\\ &\qquad k=0,1,\dots,I \quad \text{and} \quad j\geqslant 0,\\ \pi_j^{I+1}(x,t;\Lambda) &= Q_j^{a+\tau(x,t)}\left(\frac{a+\tau(x,t)}{a+t}(T-t)\right)\exp(\lambda_{I+1}-1+j\mu).\end{aligned} \tag{6.10}$$

Here, for notational simplicity, we extend l in Proposition 6.2 to $\mathbb{R}_\infty$ and set $l_i=0$ when $i>I$. Note in particular that $\{\pi^k\}$ will depend on x only when $k=I+1$. Observe also that setting $D_\Lambda L(x,t;\Lambda;\pi)=0$ gives the constraints (6.4). For any $(x,t)\in\mathbb{R}^{I+2}\times\mathbb{R}$ and $\Lambda\in\mathbb{R}^{I+3}$, let $\pi(x,t;\Lambda)$ be determined by (6.10) and define $G:\mathbb{R}^{I+2}\times\mathbb{R}\times\mathbb{R}^{I+3}\mapsto\mathbb{R}^{I+3}$ by

$$\begin{aligned}G_k(x,t;\Lambda) &= \left(1-\sum_{j=0}^{\infty}\pi_j^k(x,t;\Lambda)\right), \quad k=0,1,\dots,I+1,\\ G_{I+2}(x,t;\Lambda) &= \left(T-t-\sum_{k=0}^{I+1}x_k\sum_{j=0}^{\infty}j\pi_j^k(x,t;\Lambda)\right).\end{aligned}$$

In the next theorem we show that the $\pi(x,t;\lambda)$ defined in (6.10) indeed give the minimizer of (6.7).

Theorem 6.3. *For any $(x,t)\in\mathscr{D}_a$ define $U(x,t)$ by* (6.7)*. Then there exists $\Lambda\in\mathbb{R}^{I+3}$ so that $G(x,t;\Lambda)=0$, and $\pi(x,t;\Lambda)$ is a minimizer of* (6.7)*. Thus $U(x,t)=L\big(x,t;\Lambda;\pi(x,t;\Lambda)\big)$. In addition, the Λ that satisfies $G(x,t;\Lambda)=0$ is unique. Hence if $G(x,t;\Lambda)=0$ for some $\Lambda\in\mathbb{R}^{I+3}$, then $\pi(x,t;\Lambda)$ is a minimizer of* (6.7)*.*

The proof is divided into two lemmas.

Lemma 6.4. *For any $(x,t)\in\mathscr{D}_a$ define $U(x,t)$ by* (6.7)*. Then there exists $\Lambda\in\mathbb{R}^{I+3}$ so that $G(x,t;\Lambda)=0$, and $\pi(x,t;\Lambda)$ is a minimizer of* (6.7)*.*

Proof. For a point $(x,t)\in\mathscr{D}_a$, quantities of the following sort will appear frequently in the proof:

$$\begin{aligned}\bar\pi_j^k &\doteq Q_j^{a+k}\left(\frac{a+k}{a+t}(T-t)\right),\\ \bar\pi_j^{I+1} &\doteq Q_j^{a+\tau(x,t)}\left(\frac{a+\tau(x,t)}{a+t}(T-t)\right),\\ &\qquad \text{for } k=0,1,\dots,I,\ \ j=0,1,\dots.\end{aligned} \tag{6.11}$$

In particular, it will often be the case that (6.3) must be invoked, with a there replaced by $a+k$ [or $a+\tau(x,t)$] and s there replaced by the corresponding argument in the expression above. We note that the conditions required for (6.3) will always hold as long as $t\in[0,T]$. This is straightforward to check in the case of $a>0$. For the case $-a\in\mathbb{N}$, it uses that $-a=I+1$, $T\leqslant -a$, and that always $\tau(x,t)=I+1$. Thus, for example, for any $k\in\{0,\dots,-a\}$ and

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

$t \in [0, T]$, $(a+k)(T-t)/(a+t) \geqslant 0$ and $(T-t)/(-a-t) \leqslant 1$ shows that $(a+k)(T-t)/(a+t) \leqslant -a-k$, as required for (6.3).

As a first application of this observation, we note that by (6.3) the quantities in (6.11) are probabilities that satisfy (6.4). Hence $\mathscr{F}(x,t;T)$ is non-empty. Next note that with this notation, we can rewrite (6.8) as

$$f(x,t;\pi) = R\big(x \otimes \pi \| x \otimes \bar{\pi}\big) + \langle l, x \times \pi \rangle. \tag{6.12}$$

Since the relative entropy has compact level sets in the first argument [3, Lemma 1.4.3(c)], the existence of a minimizer of (6.7) follows. In addition, because of the strict convexity in that argument we know that the minimizer is unique up to those $\{\pi^k_\cdot\}$ with $x_k > 0$.

For a general initial condition (x,t) let $\mathscr{K} \doteq \{k : x_k > 0\}$. Then the choice of $\{\pi^k_\cdot : k \notin \mathscr{K}\}$ will not affect either the constraint (6.4) or the objective function (6.12). Hence we can consider the equivalent minimization problem over $\mathscr{M}_{(x,t)} = \{\pi^k_j : k \in \mathscr{K}, j = 0, 1, \dots\}$. As discussed in the previous paragraph, a minimizer in $\mathscr{M}_{(x,t)}$ exists and is unique. Let this minimizer be denoted π^*.

We next show that, for all $k = 0, 1, \dots, I+1$, $j = 0, 1, \dots$,

$$\pi^{*k}_j > 0 \quad \text{if } \bar{\pi}^k_j > 0. \tag{6.13}$$

First observe that if $\bar{\pi}^k_j = 0$ for some k and j, then π^{*k}_j must be 0, since otherwise the relative entropy will be ∞. Let $\pi^\epsilon \doteq (1-\epsilon)\pi^* + \epsilon\bar{\pi}$, where $\bar{\pi}$ is defined in (6.11) and let $f(\epsilon) = f(x,t;\pi^\epsilon)$. By computing the derivative of $f(\epsilon)$ explicitly, it is readily observed that if (6.13) does not hold then $f'(\epsilon) \to -\infty$ as $\epsilon \to 0$. Thus (6.13) must be true, since otherwise π^* is not the minimizer.

We want to argue that the minimizers must take the form of (6.10). However, there is a difficulty since $M_{(x,t)}$ can be infinite-dimensional. To deal with this we use a truncation argument adapted from one in [4]. For any $N \in \mathbb{N}$ let

$$T^{(N)} \doteq \sum_{k \in \mathscr{K}} x_k \sum_{j=0}^{N} j\pi^{*k}_j, \qquad \alpha^{(N)}_k \doteq \sum_{j=0}^{N} \pi^{*k}_j, \qquad k \in \mathscr{K}$$

and also let

$$f^{(N)}(x,t;\pi) \doteq f(x,t;\hat{\pi}),$$

where

$$\begin{aligned} \hat{\pi}^k_j &= \pi^k_j, \quad \text{for } k \in \mathscr{K},\ j \leqslant N \\ \hat{\pi}^k_j &= \pi^{*k}_j, \quad \text{for } k \in \mathscr{K},\ j > N. \end{aligned}$$

Since π^* is the minimizer of (6.7), automatically

$$U(x,t) = \min_{\pi} f^{(N)}(x,t;\pi), \tag{6.14}$$

where the minimum is subject to the constraints

$$\sum_{k \in \mathscr{K}} x_k \sum_{j=0}^{N} j\pi^k_j = T^{(N)}, \qquad \sum_{j=0}^{N} \pi^k_j = \alpha^{(N)}_k, \qquad k \in \mathscr{K}.$$

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

We can now apply the standard Lagrange multiplier method to (6.14). The first step is to formulate the Lagrangian for this finite-dimensional problem:

$$L^N(x,t;\Lambda;\pi) \doteq f^N(x,t;\pi) + \sum_{k\in\mathscr{K}} \lambda_k^{(N)} x_k \left(\alpha_k^{(N)} - \sum_{j=0}^{N} \pi_j^k\right) + \mu^{(N)} \left(T^{(N)} - \sum_{k=0}^{I+1} x_k \sum_{j=0}^{N} j\pi_j^k\right).$$

We have that $\{\pi_j^{*k} : k \in \mathscr{K}, j \leqslant N\}$ satisfies the constraints in (6.14), and by (6.13) we know that $\pi_j^{*k} > 0$ if $\bar{\pi}_j^k > 0$. Hence, by [10, Corollary 28.2.2] and [10, Theorem 28.3] applied to (6.14), there must exist a set of Lagrange multipliers so that the minimizer of (6.14) π_j^{*k} has the form

$$\pi_j^{*k} = \bar{\pi}_j^k \exp\bigl(\lambda_k^{(N)} - 1 + j\mu^{(N)} - l_{k+j}\bigr) \tag{6.15}$$

for $k \in \mathscr{K}$ and $0 \leqslant j \leqslant N$. If $k + j > I$, then since $l_{k+j} = 0$

$$\frac{\pi_{j+1}^{*k}}{\pi_j^{*k}} = C \cdot e^{\mu^{(N)}},$$

where C does not depend on N. Thus $\mu^{(N)}$ is independent of N, and hence $\lambda^{(N)}$ is also independent of N. Since the choice of N is arbitrary, we then know that for all $k \in \mathscr{K}$ and $j = 0, 1, \ldots$, π_j^{*k} indeed has the form in (6.10) for a suitable choice of λ_k and μ. For $k \notin \mathscr{K}$, we can simply define π_j^{*k} as in (6.15) and then solve for λ_k from the normalization constraint $\sum_{j=0}^{\infty} \pi_j^{*k} = 1$. When defined in this way, $\Lambda^* = (\lambda_0, \ldots, \lambda_{I+1}, \mu)$ automatically satisfies $G(x,t;\Lambda^*) = 0$.

For $k \in \mathscr{K}$ the corresponding λ_k are a Kuhn–Tucker vector as in [10, Corollary 28.2.2], and hence each $\lambda_k < \infty$. However, for $k \notin \mathscr{K}$ the finiteness of λ_k is not automatic.

To show the finiteness, we first insert the explicit form of $\pi_j^{I+1}(x,t;\Lambda)$ from (6.10) into $G_{I+1}(x,t,\Lambda) = 0$ to obtain

$$\sum_{j=0}^{\infty} Q_j^{a+\tau(x,t)}\left(\frac{a+\tau(x,t)}{a+t}(T-t)\right) \exp(\lambda_{I+1} - 1 + j\mu) = 1.$$

Using (6.3) to evaluate the sum gives

$$\lambda_{I+1} = (a+\tau(x,t)) \log\left(\frac{a+T-e^{\mu}(T-t)}{a+t}\right) + 1.$$

For notational simplicity define

$$\eta(t,\mu) \doteq \log\left(\frac{a+T-e^{\mu}(T-t)}{a+t}\right), \quad \lambda(x,t;\mu) \doteq (a+\tau(x,t))\eta(t,\mu) + 1. \tag{6.16}$$

Then $\lambda_{I+1} = \lambda(x,t;\mu)$. Choose $C < \infty$ such that $|l_k| \leqslant C$ for $0 \leqslant k \leqslant I$. Then

$$\sum_{j=0}^{\infty} \bar{\pi}_j^k \exp(\lambda_k - 1 + j\mu - C) \leqslant \sum_{j=0}^{\infty} \pi_j^k(x,t;\Lambda) \leqslant \sum_{j=0}^{\infty} \bar{\pi}_j^k \exp(\lambda_k - 1 + j\mu + C).$$

A calculation of the same sort that gave (6.16) gives

$$(a+k)\eta(t,\mu) + 1 - C \leqslant \lambda_k \leqslant (a+k)\eta(t,\mu) + 1 + C, \quad k = 0, \ldots, I.$$

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

Hence $\lambda_k < \infty$ as long as $\lambda_i < \infty$ for some $i = 0, 1, \ldots, I, I+1$, which is true by [10, Corollary 28.2.2]. This completes the proof that, for any $(x,t) \in \mathscr{D}_a$ there exists $\Lambda \in \mathbb{R}^{I+3}$ so that $G(x,t;\Lambda) = 0$ and $\pi(x,t;\Lambda)$ is a minimizer of (6.7). □

The next lemma will focus on the claim that, for $(x,t) \in \mathscr{D}_a$, there is only one Λ that satisfies $G(x,t,\Lambda) = 0$, which together with the previous lemma completes the proof of Theorem 6.3.

Lemma 6.5. *For $(x,t) \in \mathscr{D}_a$, there is only one $\Lambda \in \mathbb{R}^{I+3}$ such that $G(x,t,\Lambda) = 0$.*

Proof. Recalling the definition of $\bar{\pi}$ in (6.11), notice that (6.10) is simply

$$\pi_j^k(x,t;\Lambda) = \bar{\pi}_j^k \exp(\lambda_k - 1 + j\mu - l_{k+j}) \quad \text{for } k = 0,1,\ldots,I+1,\ j = 0,1,\ldots .$$

As noted previously, for any $(x,t) \in \mathscr{D}_a$ we can assume that each $\{\bar{\pi}^k\}$ is a valid probability vector. Thus $\pi_j^k(x,t;\Lambda) \geqslant 0$ and for each k at least one of $\{\pi_j^k(x,t;\Lambda), j = 0,1,\ldots\}$ is strictly positive. If

$$\alpha_k \doteq \sum_{j=0}^{\infty} \pi_j^k(x,t;\Lambda), \qquad T_k \doteq \sum_{j=0}^{\infty} j\pi_j^k(x,t;\Lambda),$$

then $\alpha_k > 0$ for each $k = 0,1,\ldots,I,I+1$ and any $\Lambda \in \mathbb{R}^{I+3}$. Using the particular dependency of $\pi_j^k(x,t;\Lambda)$ on Λ, one can compute

$$\begin{cases} \frac{\partial \pi_j^k(x,t;\Lambda)}{\partial \lambda_k} = \pi_j^k(x,t;\Lambda), \\ \frac{\partial \pi_j^k(x,t;\Lambda)}{\partial \mu} = j\pi_j^k(x,t;\Lambda), \\ \frac{\partial \pi_j^k(x,t;\Lambda)}{\partial \lambda_l} = 0, \quad l \neq k. \end{cases}$$

It is straightforward to construct a dominating function of the form $\bar{\pi}_j^k \cdot C \cdot D^j$ for suitable constants C and D, and hence, by the Lebesgue Dominated Convergence Theorem, one can compute $D_\Lambda G(x,t;\Lambda)$ explicitly as

$$\begin{pmatrix} -\alpha_0 & \cdots & 0 & -T_0 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & -\alpha_{I+1} & -T_{I+1} \\ -x_0 T_0 & \cdots & -x_{I+1} T_{I+1} & -\sum_{k=0}^{I+1} x_k \sum_{j=0}^{\infty} j^2 \pi_j^k(x,t;\Lambda) \end{pmatrix}.$$

Observe that $\{-\alpha_k : k = 0,1,\ldots,I+1\}$ and

$$\sum_{k=0}^{I+1} \frac{x_k}{\alpha_k} T_k^2 - \sum_{k=0}^{I+1} x_k \sum_{j=0}^{\infty} j^2 \pi_j^k(x,t;\Lambda)$$

are the eigenvalues of $D_\Lambda G(x,t;\Lambda)$. We have already observed that, for all $k = 0,1,\ldots,I+1$, $\alpha_k > 0$. Also, for every $k = 0,1,\ldots,I+1$, the Cauchy–Schwarz inequality implies

$$\left(\sum_{j=0}^{\infty} j^2 \pi_j^k(x,t;\Lambda)\right)\left(\sum_{j=0}^{\infty} \pi_j^k(x,t;\Lambda)\right) \geqslant \left(\sum_{j=0}^{\infty} j\pi_j^k(x,t;\Lambda)\right)^2.$$

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

It is easy to verify that the necessary condition for equality does not hold. Hence the inequality is strict, and therefore

$$\left(\sum_{j=0}^{\infty} j^2 \pi_j^k(x,t;\Lambda)\right) > \frac{T_k^2}{\alpha_k}.$$

Thus $D_\Lambda G(x,t;\Lambda)$ is negative definite for all $\Lambda \in \mathbb{R}^{I+3}$.

Now we can prove the uniqueness of Λ. Suppose there are two different $\Lambda_1, \Lambda_2 \in \mathbb{R}^{I+3}$ such that $G(x,t;\Lambda_i) = 0$, $i = 1,2$. Define $\Lambda(\epsilon) \doteq \epsilon\Lambda_1 + (1-\epsilon)\Lambda_2$ and

$$h(\epsilon) \doteq \langle G(x,t;\Lambda(\epsilon)), \Lambda_1 - \Lambda_2 \rangle.$$

Then $h'(\epsilon) = (\Lambda_1 - \Lambda_2)^T \cdot D_\Lambda G \cdot (\Lambda_1 - \Lambda_2)$. Since $D_\Lambda G(x,t;\Lambda)$ is always negative definite, $h'(\epsilon) < 0$ for all $0 < \epsilon < 1$. However, $h(0) = h(1) = 0$. This contradiction shows that $G(x,t;\Lambda) = 0$ has a unique solution of Λ. □

The next theorem considers differentiability properties of $U(x,t)$. As mentioned previously, we first extend the definition of $U(x,t)$ to a neighbourhood of (x,t) in $\mathbb{R}^{I+2} \times \mathbb{R}$, label this extension $\bar{U}(x,t)$, and then show $\bar{U}(x,t)$ is differentiable in the normal Euclidean sense. For our needs (a verification argument) the function $\bar{U}(x,t)$ can be used in lieu of $U(x,t)$.

Theorem 6.6. *Fix $(x,t) \in \mathscr{D}_a$ and define $U(x,t)$ by* (6.7)*. Then there is an open neighbourhood $\mathscr{U}$ of (x,t) and an extension $\bar{U}$ of U from $\mathscr{D}_a \cap \mathscr{U}$ to $\mathscr{U}$ which is differentiable on $\mathscr{U}$.*

Proof. By Theorem 6.3, for any $(x,t) \in \mathscr{D}_a$ there exists Λ so that $G(x,t;\Lambda) = 0$ and $U(x,t) = L(x,t;\Lambda,\pi(x,t;\Lambda))$. There is a problem with a direct application of the Implicit Function Theorem, owing to the fact that the non-smooth function $\tau(x,t)$ appears in the last constraint G_{I+1}. To avoid this difficulty we consider an equivalent but less obvious formulation of the constraint.

Since the Lagrange multiplier λ_{I+1} is determined by the multiplier μ via (6.16), we can work with the reduced set of multipliers $\tilde{\Lambda} \doteq \{\lambda_0, \ldots, \lambda_I, \mu\}$ and the definition

$$\Lambda(x,t;\tilde{\Lambda}) \doteq \{\lambda_0, \lambda_1, \ldots, \lambda_I, \lambda(x,t;\mu), \mu\}. \tag{6.17}$$

Define

$$H(x,t;\tilde{\Lambda}) \doteq L\big(x,t;\Lambda(x,t;\tilde{\Lambda});\pi(x,t;\Lambda(x,t;\tilde{\Lambda}))\big). \tag{6.18}$$

Then $U(x,t) = H(x,t;\tilde{\Lambda})$. Observe that by (6.3) and (6.16)

$$\begin{aligned} T_{I+1} &= \sum_{j=1}^{\infty} j\pi_j^{I+1}(x,t;\Lambda(x,t;\tilde{\Lambda})) \\ &= \sum_{j=1}^{\infty} jQ_j^{a+\tau(x,t)}\left(\frac{a+\tau(x,t)}{a+t}(T-t)\right)\exp(\lambda_{I+1} - 1 + j\mu) \\ &= \frac{T-t}{a+t}(a+\tau(x,t))\exp(\lambda_{I+1}-1+\mu)\left(\frac{a+T-e^{\mu}(T-t)}{a+t}\right)^{-(a+\tau(x,t)+1)} \end{aligned}$$

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

$$= \frac{T-t}{a+t}(a+\tau(x,t))\exp\big(\lambda_{I+1} - 1 - \eta(t,\mu)(a+\tau(x,t)+1)+\mu\big)$$
$$= \frac{T-t}{a+t}(a+\tau(x,t))\exp(\mu-\eta(t,\mu)). \tag{6.19}$$

Then $G(x,t;\Lambda)=0$ is equivalent to $\tilde{G}(x,t;\tilde{\Lambda})=0$, where $\tilde{G}_i(x,t;\tilde{\Lambda}) = G_i(x,t;\Lambda)$ for $i=0,\dots,I$ and

$$\tilde{G}_{I+1}(x,t;\tilde{\Lambda}) \doteq T-t-\sum_{k=0}^{I} x_k \sum_{j=0}^{\infty} j\pi_j^k(x,t;\tilde{\Lambda}) - \frac{(T-t)(ax_{I+1}+t-\sum_{k=0}^{I} kx_k)e^{\mu}}{a+T-e^{\mu}(T-t)},$$

where one uses (6.16), (6.19) and the definition of $\tau(x,t)$ to show that the last item above is just $x_{I+1}T_{I+1}$. Strictly, the π_j^k-term in the last formula should be written $\pi_j^k(x,t;\Lambda(x,t;\tilde{\Lambda}))$, but since $\pi_j^k(\cdot)$ does not depend on λ_{I+1} when $k \leqslant I$, we will abuse notation.

Owing to the form (6.17) and Theorem 6.3, since for $(x,t)\in\mathscr{D}_a$ the Λ such that $G(x,t;\Lambda)=0$ exists and is unique, we have the $\tilde{\Lambda}$ such that $\tilde{G}(x,t;\tilde{\Lambda})=0$ exists and is also unique.

$D_{\tilde{\Lambda}}\tilde{G}(x,t;\tilde{\Lambda})$ equals

$$\begin{pmatrix} -\alpha_0 & \cdots & 0 & -T_0 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & -\alpha_I & -T_I \\ -x_0T_0 & \cdots & -x_IT_I & D_\mu\tilde{G}_{I+1}(x,t;\tilde{\Lambda}) \end{pmatrix}.$$

The first two equalities below are due to definitions, and only the third must be verified:

$$\begin{aligned} D_\mu\tilde{G}_{I+1}(x,t;\tilde{\Lambda}) &= -D_\mu\frac{(T-t)(ax_{I+1}+t-\sum_{k=0}^{I}kx_k)e^{\mu}}{a+T-e^{\mu}(T-t)} \\ &= -x_{I+1}D_\mu\frac{(T-t)(a+\tau(x,t))e^{\mu}}{a+T-e^{\mu}(T-t)} \\ &= x_{I+1}\sum_{j=0}^{\infty} j^2\pi_j^{I+1}\big(x,t;\Lambda(x,t;\tilde{\Lambda})\big) - T_{I+1}^2x_{I+1}. \end{aligned}$$

The expression (6.19) and the definition of $\eta(t,\mu)$ in (6.16) give

$$T_{I+1} = \sum_{j=0}^{\infty} j\pi_j^{I+1}\big(x,t;\Lambda(x,t;\tilde{\Lambda})\big) = \frac{e^{\mu}(a+\tau(x,t))(T-t)}{a+T-e^{\mu}(T-t)}.$$

Another use of the binomial expansion gives

$$\begin{aligned} &\sum_{j=0}^{\infty} j^2\pi_j^{I+1}(x,t;\Lambda) - \sum_{j=0}^{\infty} j\pi_j^{I+1}(x,t;\Lambda) \\ &\quad= \sum_{j=2}^{\infty} j(j-1)\left(\frac{t-T}{a+t}\right)\binom{-(a+\tau(x,t))}{j}\left(\frac{a+T}{a+t}\right)^{-(a+\tau(x,t)+j)} e^{\lambda_{I+1}+j\mu-1} \\ &\quad= \left(\frac{t-T}{a+t}\right)^2(a+\tau(x,t))(a+\tau(x,t)+1)e^{\lambda_{I+1}+2\mu-1}\left[\frac{e^{\mu}(t-T)+a+T}{a+t}\right]^{-(a+\tau(x,t)+2)} \end{aligned}$$

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

$$= \left(\frac{t-T}{a+t}\right)^2 (a+\tau(x,t))(a+\tau(x,t)+1)e^{2\mu-2\eta}$$
$$= \left(\frac{t-T}{a+T-e^{\mu}(T-t)}\right)^2 (a+\tau(x,t))(a+\tau(x,t)+1)e^{2\mu}.$$

One can then check by direct substitution that the given expression for $D_\mu \tilde{G}_{I+1}(x,t;\tilde{\Lambda})$ is valid.

It follows that the eigenvalues of $D_{\tilde{\Lambda}}\tilde{G}(x,t;\tilde{\Lambda})$ are $-\alpha_0, -\alpha_1, \ldots, -\alpha_I$ and

$$\sum_{k=0}^{I+1} \frac{x_k}{\alpha_k} T_k^2 - \sum_{k=0}^{I} x_k \sum_{j=0}^{\infty} j^2 \pi_j^k(x,t;\tilde{\Lambda}) - x_{I+1} \sum_{j=0}^{\infty} j^2 \pi_j^{I+1}\big(x,t;\Lambda(x,t;\tilde{\Lambda})\big),$$

which by the same argument for $D_\Lambda G(x,t;\Lambda)$ are all negative, and hence $D_{\tilde{\Lambda}}\tilde{G}_{I+1}(x,t;\tilde{\Lambda})$ is invertible.

We claim that $D_{\tilde{\Lambda}}G$ is smooth in (x,t). One can check that the only potentially difficult component is

$$D_\mu \tilde{G}_{I+1}(x,t;\tilde{\Lambda}) = -x_{I+1} D_\mu \frac{(T-t)(a+\tau(x,t))e^{\mu}}{a+T-e^{\mu}(T-t)}$$
$$= -x_{I+1}\left[\frac{(T-t)(a+\tau(x,t))e^{\mu}}{a+T-e^{\mu}(T-t)} + \frac{(T-t)^2(a+\tau(x,t))e^{2\mu}}{[a+T-e^{\mu}(T-t)]^2}\right].$$

However, since $\eta(t,\mu) < \infty$ the denominators are non-zero, and while $\tau(x,t)$ is not smooth the product $x_{I+1}\tau(x,t)$ is. Therefore $\tilde{G}(\cdot;\cdot)$ is smooth in a neighbourhood of $(x,t;\tilde{\Lambda})$. By the Implicit Function Theorem, for any $(x,t) \in \mathscr{D}_a$ there exists a neighbourhood $\mathscr{U} \subset \mathbb{R}^{I+1} \times \mathbb{R}$ of (x,t), a neighbourhood $\mathscr{V} \subset \mathbb{R}^{I+2}$ of $\tilde{\Lambda}$, and a C^∞-function $g : \mathscr{U} \mapsto \mathscr{V}$, so that $\tilde{\Lambda} = g(x,t)$ and for every $(y,s) \in \mathscr{U}$, $\tilde{G}(y,s;g(y,s)) = 0$. Define

$$\bar{U}(y,s) = H(y,s;g(y,s)).$$

Since $g(y,s)$ is smooth in $\mathscr{U}$, $\bar{U} \in C^\infty(\mathscr{U},\mathbb{R})$, and by Theorem 6.3, $\bar{U}(y,s) = U(y,s)$ for $(y,s) \in \mathscr{U} \cap \mathscr{D}_a$. □

Our final theorem shows that $\bar{U}$ satisfies the HJB equation (6.6) in the classical sense on $\mathscr{U}$. When combined with a standard verification argument as in [7], this will imply $V(x,t) = \bar{U}(x,t) = U(x,t)$ on $\mathscr{D}_a$.

Theorem 6.7. *$\bar{U}$ satisfies* (6.6) *on $\mathscr{D}_a$.*

Proof. The basic idea is simply to represent the derivatives in terms of the Lagrange multipliers and directly show satisfaction of the PDE. Consider any point $(x,t) \in \mathscr{D}_a$ and let $\tilde{\Lambda}^* = g(x,t)$. By Theorem 6.6 there exists $\mathscr{U} \subset \mathbb{R}^{I+1} \times \mathbb{R}$ a neighbourhood of (x,t), $\mathscr{V} \subset \mathbb{R}^{I+2}$ a neighbourhood of $\tilde{\Lambda}^*$, and a C^∞-function $g : \mathscr{U} \mapsto \mathscr{V}$ such that and $\bar{U}(y,s) \doteq H(y,s;g(y,s))$ satisfies $\bar{U}(y,s) = U(y,s)$ for any $(y,s) \in \mathscr{U} \cap \mathscr{D}_a$.

We pause to introduce a convention which will be used in the remainder of the proof. Whenever a differential operator of the form D_{x_k} precedes a composed function, the derivative is computed via the chain rule for precisely those instances where a composed dependence on x_k is made

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

explicit in the notation. For example,

$$\begin{aligned} D_{x_k}\bar{U}(x,t) &= D_{x_k}H(x,t;g(x,t)) \\ &= D_{x_k}H(x,t;\tilde{\Lambda}^*) + D_{\tilde{\Lambda}}H(x,t;\tilde{\Lambda}^*)D_{x_k}g(x,t). \end{aligned}$$

Here, in the first line, $D_{x_k}H(x,t;\tilde{\Lambda}(x,t))$ means $H(x,t;\tilde{\Lambda}(x,t))$ is considered as the composed function of (x,t) (which by definition is $\bar{U}(x,t)$), and we take derivatives with respect to x_k and then evaluate it at (x,t). In the second line, $D_{x_k}H(x,t;\tilde{\Lambda}^*)$ means $H(x,t;\tilde{\Lambda}^*)$ is now a function of the independent variables $(x,t,\tilde{\Lambda}^*)$ and we take derivatives with respect to x_k and then evaluate it at $(x,t;\tilde{\Lambda}^*)$. In all calculations, vectors are interpreted as row vectors.

With the notation established, we can proceed. Note that by definition (6.18),

$$D_{\tilde{\Lambda}}H(x,t;\tilde{\Lambda}) = D_{\Lambda}L(x,t;\Lambda;\pi)D_{\tilde{\Lambda}}\Lambda(x,t;\tilde{\Lambda}) + D_{\pi}L(x,t;\Lambda;\pi)D_{\tilde{\Lambda}}\pi\big(x,t;\Lambda(x,t;\tilde{\Lambda})\big).$$

Since $\Lambda^* = \Lambda(x,t;\tilde{\Lambda}^*)$ and π^* are chosen so that

$$D_{\pi}L(x,t;\Lambda^*;\pi^*) = D_{\Lambda}L(x,t;\Lambda^*;\pi^*) = 0,$$

we have $D_{\tilde{\Lambda}}H(x,t;\tilde{\Lambda}^*) = 0$. This gives

$$D_{x_k}\bar{U}(x,t) = D_{x_k}H(x,t;\tilde{\Lambda}^*), \quad k = 0,1,\ldots,I+1. \tag{6.20}$$

By the same argument, we have $D_t\bar{U}(x,t) = D_tH(x,t;\tilde{\Lambda}^*)$.

Next we insert the explicit form of $\pi(x,t;\Lambda)$ from (6.10) into (6.9) to get

$$L(x,t;\Lambda,\pi(x,t;\Lambda)) = \sum_{k=0}^{I+1} x_k\lambda_k + \mu(T-t) - \sum_{k=0}^{I} x_k \sum_{j=0}^{\infty} \pi_j^k(x,t;\Lambda) - x_{I+1}\sum_{j=0}^{\infty}\pi_j^{I+1}(x,t;\Lambda).$$

In the definition of $H(x,t;\tilde{\Lambda})$, λ_{I+1} is replaced by $\lambda(x,t;\mu)$, so that automatically

$$\sum_{j=0}^{\infty}\pi_j^{I+1}(x,t;\Lambda) = 1.$$

Using $x_{I+1}\lambda(x,t;\mu) - x_{I+1} = x_{I+1}(a+\tau(x,t))\eta(t,\mu)$ from (6.16) and the definition of $\tau(x,t)$,

$$\begin{aligned} H(x,t;\tilde{\Lambda}) &= \sum_{k=0}^{I} x_k\lambda_k + \mu(T-t) - \sum_{k=0}^{I} x_k\sum_{j=0}^{\infty}\pi_j^k(x,t;\tilde{\Lambda}) \\ &\quad + x_{I+1}(a+\tau(x,t))\eta(t,\mu) \\ &= \sum_{k=0}^{I} x_k\lambda_k + \mu(T-t) - \sum_{k=0}^{I} x_k\sum_{j=0}^{\infty}\pi_j^k(x,t;\tilde{\Lambda}) \\ &\quad + a\left(1-\sum_{k=0}^{I}x_k\right)\eta(t,\mu) + \left(t-\sum_{k=0}^{I}kx_k\right)\eta(t,\mu). \end{aligned}$$

Recall from the explicit expression (6.10) that for $k = 0,1,\ldots,I$, $\pi_j^k(x,t;\Lambda)$ does not depend on x or on λ_{I+1}. Hence the x dependence can be omitted in $\pi_j^k(x,t;\tilde{\Lambda})$ in the last display, and we do so from now on. By (6.20),

$$\bar{U}_{x_k} = \lambda_k^* - 1 - (a+k)\eta^*, \quad k = 0,1,\ldots,I,$$

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

and $\bar{U}_{x_{I+1}} = 0$, where $\eta^* = \eta(t, \mu^*)$. Thus

$$\begin{cases} D_{x_k}\bar{U}(x,t) - D_{x_{k+1}}\bar{U}(x,t) = \lambda_k^* - \lambda_{k+1}^* + \eta^*, \quad k = 0, 1, \dots, I-1, \\ D_{x_I}\bar{U}(x,t) - D_{x_{I+1}}\bar{U}(x,t) = \lambda_I^* - 1 - (a+I)\eta^*. \end{cases}$$

Similarly

$$\begin{aligned} D_t\bar{U}(x,t) &= D_t H(x,t;\tilde{\Lambda}^*) \\ &= \eta^* - \mu^* - D_t\left\{\sum_{k=0}^{I} x_k \sum_{j=0}^{\infty} \pi_j^k(t;\tilde{\Lambda}^*)\right\} \\ &\quad + \left\{a\left(1 - \sum_{k=0}^{I} x_k\right) + \left(t - \sum_{k=0}^{I} k x_k\right)\right\} D_t\eta(t,\mu^*) \\ &= \eta^* - \mu^* - D_t\left\{\sum_{k=0}^{I} x_k \sum_{j=0}^{\infty} \pi_j^k(t;\tilde{\Lambda}^*)\right\} + x_{I+1}(a+\tau(x,t))D_t\eta(t,\mu^*). \end{aligned}$$

We now will verify that

$$-D_t\left\{\sum_{k=0}^{I} x_k \sum_{j=0}^{\infty} \pi_j^k(t;\tilde{\Lambda}^*)\right\} + x_{I+1}(a+\tau(x,t))D_t\eta(t;\mu^*) = 0, \tag{6.21}$$

which implies $D_t U(x,t) = \eta^* - \mu^*$. To do so, first recall that by (6.19)

$$\sum_{j=1}^{\infty} j\pi_j^{I+1}(x,t;\Lambda^*) = \frac{T-t}{a+t}(a+\tau(x,t))\exp(\mu^* - \eta^*).$$

Next, by the explicit formula for $\pi_j^k(t;\tilde{\Lambda})$ in (6.10), for all $k = 0, 1, \dots, I$, $j = 0, 1, \dots$,

$$D_t\pi_j^k(t;\tilde{\Lambda}) = j\pi_j^k(t;\tilde{\Lambda})\frac{a+T}{(t-T)(a+t)} + \frac{(a+k+j)}{a+t}\pi_j^k(t;\tilde{\Lambda}).$$

A suitable dominating function can be found, and thus, by the Lebesgue Dominated Convergence Theorem,

$$-D_t\left\{\sum_{k=0}^{I} x_k \sum_{j=0}^{\infty} \pi_j^k(t;\tilde{\Lambda})\right\} = \sum_{k=0}^{I} x_k \sum_{j=0}^{\infty}\left[j\pi_j^k(t;\tilde{\Lambda})\frac{a+T}{(t-T)(a+t)} + \frac{(a+k+j)}{a+t}\pi_j^k(t;\tilde{\Lambda})\right].$$

Notice that $\pi(x,t;\Lambda^*)$ satisfies the constraint (6.4), and therefore

$$-D_t\left\{\sum_{k=0}^{I} x_k \sum_{j=0}^{\infty} \pi_j^k(t;\tilde{\Lambda}^*)\right\} = \frac{x_{I+1}\sum_{j=0}^{\infty} j\pi_j^{I+1}(x,t;\Lambda^*)}{t-T} + \frac{x_{I+1}(a+\tau(x,t))}{a+t}.$$

Inserting (6.19) into the last equation produces

$$-D_t\left\{\sum_{k=0}^{I} x_k \sum_{j=0}^{\infty} \pi_j^k(t;\tilde{\Lambda}^*)\right\} = \frac{a+\tau(x,t)}{a+t}x_{I+1}(1 - \exp(\mu^* - \eta^*)). \tag{6.22}$$

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

The definition of $\eta(t,\mu)$ in (6.16) and $(a+t)e^{\eta} = a + T - e^{\mu}(T-t)$ gives

$$D_t\eta(t,\mu) = \frac{e^{\mu}}{a+T-(T-t)e^{\mu}} - \frac{1}{a+t} = \frac{\exp(\mu-\eta)-1}{a+t}.$$

Finally, combining this with (6.22) gives (6.21).

Having derived various expressions for the derivatives of $\bar{U}$ in terms of the Lagrange multipliers, to show that $\bar{U}(x,t)$ satisfies the PDE (6.6) it remains to show

$$e^{-\mu^*+\eta^*} = \sum_{k=0}^{I-1} x_k \frac{a+k}{a+t} e^{\lambda_k^*-\lambda_{k+1}^*+\eta^*} + x_I \frac{a+I}{a+t} e^{\lambda_I^*-1-(a+I)\eta^*} + x_{I+1}\frac{a+\tau(x,t)}{a+t}. \tag{6.23}$$

Recall from (6.10) that for $k = 0, 1, \ldots, I$ and $j \geqslant 0$

$$\pi_j^{*k} = Q_j^{a+k}\left(\frac{a+k}{a+t}(T-t)\right)\exp(\lambda_k^* - 1 + j\mu^* - l_{k+j}).$$

Using the definition of Q_j^{a+k}, for $k = 0, 1, \ldots, I-1$,

$$\pi_j^{*k+1} = \frac{(j+1)(a+t)}{(a+k)(T-t)}\pi_{j+1}^{*k}\exp(\lambda_{k+1}^* - \lambda_k^*)e^{-\mu^*}.$$

Now sum both sides from $j = 0$ to ∞ and use the fact that $\sum_{j=0}^{\infty}\pi_j^{*k} = 1$ to get

$$\exp(\lambda_k^* - \lambda_{k+1}^*) = e^{-\mu^*}\cdot\frac{(a+t)\sum_{j=1}^{\infty} j\pi_j^{*k}}{(T-t)(a+k)}, \quad \text{for } k = 0, 1, \ldots, I. \tag{6.24}$$

Inserting (6.24) into (6.23), satisfaction of the PDE is equivalent to

$$\begin{aligned}(T-t) = &\sum_{k=0}^{I-1} x_k \sum_{j=0}^{\infty} j\pi_j^{*k} + x_I\frac{a+I}{a+t}(T-t)\exp(\lambda_I^* - 1 - (a+I+1)\eta^* + \mu^*)\\ &+ x_{I+1}\frac{a+\tau(x,t)}{a+t}(T-t)\exp(-\eta^* + \mu^*).\end{aligned} \tag{6.25}$$

Since $\pi_j^{*I} = Q_j^{a+I}\left(\frac{a+I}{a+t}(T-t)\right)\exp(\lambda_I^* - 1 + j\mu^*)$ for $j \geqslant 1$, by the same argument as the one used in (6.19) gives

$$\begin{aligned}\sum_{j=1}^{\infty} j\pi_j^{*I} &= \sum_{j=1}^{\infty} jQ_j^{a+I}\left(\frac{a+I}{a+t}(T-t)\right)\exp(\lambda_I^* - 1 + j\mu^*)\\ &= \frac{T-t}{a+t}(a+I)\exp(\lambda_I^* - 1 - \eta(a+I+1) + \mu^*),\end{aligned} \tag{6.26}$$

where the last equality uses (6.16).

Inserting (6.19) and (6.26) into $\sum_{k=0}^{I+1} x_k \sum_{j=0}^{\infty} j\pi_j^k = T - t$, we then have

$$\begin{aligned}(T-t) = &\sum_{k=0}^{I-1} x_k \sum_{j=0}^{\infty} j\pi_j^{*k} + x_I\frac{a+I}{a+t}(T-t)\exp(\lambda_I^* - 1 - \eta^*(a+I+1) + \mu^*)\\ &+ x_{I+1}\frac{a+\tau(x,t)}{a+t}(T-t)\exp(-\eta^* + \mu^*).\end{aligned}$$

We have thus verified (6.25), which completes the proof that $\bar{U}$ satisfies (6.6). □

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press

References

[1] Billingsley, P. (1968) *Convergence of Probability Measures*, Wiley, New York.
[2] Charalambides, A. (1997) A unified derivation of occupancy and sequential occupancy distributions. In *Advances in Combinatorial Methods and Applications to Probability and Statistics* (N. Balakrishnan, ed.), pp. 259–273.
[3] Dupuis, P. and Ellis, R. (1997) *A Weak Convergence Approach to Large Deviations*, Wiley–Interscience, New York.
[4] Dupuis, P., Nuzman, C. and Whiting, P. (2004) Large deviations asymptotics for occupancy problems. *Ann. Probab.* **32** 2765–2818.
[5] Feller, W. (1968) *An Introduction to Probability Theory and its Applications*, Vol. 1, Wiley, New York.
[6] Feller, W. (1971) *An Introduction to Probability Theory and its Applications*, Vol. 2, Wiley, New York.
[7] Fleming, W. H. and Rishel, R. W. (1982) *Deterministic and Stochastic Optimal Control*, Springer.
[8] Holst, L. (1986) On the coupon collectors and other urn problems. *Internat. Statist. Review* **54** 15–27.
[9] Johnson, N. L. and Kotz, S. (1977) *Urn Models and their Applications*, Wiley.
[10] Rockafellar, R. T. (1970) *Convex Analysis*, Princeton University Press.
[11] Ross, S. (2002) *Introduction to Probability Models*, 8th edn, Academic Press.
[12] Zhang, J. X., Nuzman, C. and Whiting, P. (2005) Importance sampling in a general urn occupancy model. Preprint.

https://doi.org/10.1017/S0963548307008681 Published online by Cambridge University Press