SOME EIGHTH ORDER MOCK THETA FUNCTIONS

BASIL GORDON AND RICHARD J. McINTOSH

Dedicated to Tom M. Apostol

ABSTRACT

A method is developed for obtaining Ramanujan's mock theta functions from ordinary theta functions by performing certain operations on their q-series expansions. The method is then used to construct several new mock theta functions, including the first ones of eighth order. Summation and transformation formulae for basic hypergeometric series are used to prove that the new functions actually have the mock theta property. The modular transformation formulae for these functions are obtained.

1. *Introduction*

In Ramanujan's last letter to Hardy [**8**, pp. 354–355; **9**, pp. 127–131; **12**, pp. 56–61], he observes that the asymptotic expansions of certain q-series with exponential singularities at roots of unity 'close' in a striking manner. For example, let

$$H(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(1-q)(1-q^2)\dots(1-q^n)} = \prod_{m=1}^{\infty} \frac{1}{(1-q^{5m-3})(1-q^{5m-2})} \tag{1.1}$$

(where the last equality is the second Rogers–Ramanujan identity). If $q = e^{-t}$ and $t \to 0^+$ (so that q approaches 1 radially from inside the unit circle), then

$$H(q) = \sqrt{\frac{2}{5+\sqrt{5}}} \exp\left(\frac{\pi^2}{15t} + \frac{11t}{60}\right) + o(1). \tag{1.2}$$

In the same letter Ramanujan notes that it is only for certain special q-series $f(q)$ that the exponential *closes*, that is, its argument terminates with some power t^N. If $f(q)$ is not a theta function and if for each root of unity ρ there is an approximation of the form

$$f(q) = \sum_{\mu=1}^{M} t^{k_\mu} \exp\left(\sum_{\nu=-1}^{N} c_{\mu\nu} t^{\nu}\right) + O(1)$$

as $t \to 0^+$ with $q = \rho e^{-t}$, then he calls $f(q)$ a mock theta function. It appears from his letter, however, that he was actually concerned with functions having the (possibly) more restrictive property that for every root of unity ρ, there are modular forms $h_j^{(\rho)}(q)$ and real numbers α_j, $1 \leqslant j \leqslant J(\rho)$, such that

$$f(q) - \sum_{j=1}^{J(\rho)} q^{\alpha_j} h_j^{(\rho)}(q)$$

is bounded as q radially approaches ρ. We will call this the strong mock theta property.

Received 24 September 1997; revised 12 January 1999.

2000 *Mathematics Subject Classification* 11B65, 11F11, 33D10, 33D15, 41A60.

J. London Math. Soc. (2) 62 (2000) 321–335 © London Mathematical Society 2000.

https://doi.org/10.1112/S0024610700008735 Published online by Cambridge University Press

Ramanujan listed 17 such functions, to which he assigned orders 3, 5 and 7. (The order is analogous to the level of a modular form.) Watson [**12**] found three more mock theta functions of order 3, and two more of order 5 appear in the lost notebook [**9**, p. 9]. In studying the asymptotics of q-series, we found some mock theta functions of eighth order.

We use the standard notation

$$(a;q^k)_0 = 1,$$
$$(a;q^k)_n = (1-a)(1-aq^k)(1-aq^{2k})\dots(1-aq^{(n-1)k}),$$
$$(a;q^k)_\infty = \prod_{m=0}^{\infty}(1-aq^{mk}).$$

When $k = 1$ we usually write $(a)_n$ and $(a)_\infty$ instead of $(a;q)_n$ and $(a;q)_\infty$, respectively. For non-negative integers n we have

$$(a;q^k)_n = \frac{(a;q^k)_\infty}{(aq^{nk};q^k)_\infty},$$

and for other real n we take this as the definition of $(a;q^k)_n$. For convenience we write

$$(a_1, a_2, \dots, a_r; q^k)_\infty$$

for

$$(a_1;q^k)_\infty (a_2;q^k)_\infty \dots (a_r;q^k)_\infty.$$

Our new mock theta functions are

$$S_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n},$$

$$S_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n}{(-q^2;q^2)_n},$$

$$T_0(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}},$$

$$T_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}},$$

$$U_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^4;q^4)_n},$$

$$U_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(-q^2;q^4)_{n+1}},$$

$$V_0(q) = -1 + 2\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n} = -1 + 2\sum_{n=0}^{\infty} \frac{q^{2n^2}(-q^2;q^4)_n}{(q;q^2)_{2n+1}},$$

$$V_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n+1}(-q^4;q^4)_n}{(q;q^2)_{2n+2}}.$$

https://doi.org/10.1112/S0024610700008735 Published online by Cambridge University Press

The generalized Lambert series are

$$U_0(q) = 2\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(2n+1)}}{1+q^{4n}}, \tag{1.3}$$

$$U_1(q) = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(n+1)(2n+1)}}{1+q^{4n+2}}, \tag{1.4}$$

$$V_0(q) = -1+2\frac{(-q^2;q^4)_\infty}{(q^4;q^4)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n(2n+1)}}{1-q^{4n+1}}, \tag{1.5}$$

$$V_1(q) = \frac{(-q^4;q^4)_\infty}{(q^4;q^4)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(2n+1)^2}}{1-q^{4n+1}}. \tag{1.6}$$

The identities are

$$U_0(q) = S_0(q^2)+qS_1(q^2), \tag{1.7}$$

$$U_1(q) = T_0(q^2)+qT_1(q^2), \tag{1.8}$$

$$U_0(q)+2U_1(q) = (-q;q^2)_\infty^3(q^2;q^2)_\infty(q^2;q^4)_\infty, \tag{1.9}$$

$$V_0(q)+V_0(-q) = 2(-q^2;q^4)_\infty^4(q^8;q^8)_\infty, \tag{1.10}$$

$$V_1(q)-V_1(-q) = 2q(-q^2;q^2)_\infty^3(-q^4;q^4)_\infty^2(q^8;q^8)_\infty. \tag{1.11}$$

The equality of the two series in the definitions of $V_0(q)$ and $V_1(q)$ follows from the identity

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n(n-1)}z^n}{(z;q^2)_n} = \sum_{n=0}^{\infty}\frac{(-zq;q^4)_n q^{n(2n-1)}z^n}{(z;q^2)_{2n+1}} \tag{1.12}$$

with $z=q$ and $z=q^3$, respectively. The following proof of (1.12) is due to George Andrews [**1**].

Let $L(z)$ and $R(z)$ represent the left-hand and right-hand sides of (1.12), respectively. Clearly both are analytic in z for $|z|<1$ provided that $|q|<1$ is fixed. Also $L(0)=R(0)=1$. The q-difference equation for $R(z)$ is given by

$$\begin{aligned}
R(z) &= \frac{1}{1-z}+\sum_{n=1}^{\infty}\frac{(-zq;q^4)_n q^{n(2n-1)}z^n}{(z;q^2)_{2n+1}}\\
&= \frac{1}{1-z}+\sum_{n=0}^{\infty}\frac{(-zq;q^4)_{n+1} q^{(n+1)(2n+1)}z^{n+1}}{(z;q^2)_{2n+3}}\\
&= \frac{1}{1-z}+\frac{zq(1+zq)}{(1-z)(1-zq^2)}\sum_{n=0}^{\infty}\frac{(-zq^5;q^4)_n q^{n(2n-1)}(zq^4)^n}{(zq^4;q^2)_{2n+1}}\\
&= \frac{1}{1-z}+\frac{zq(1+zq)}{(1-z)(1-zq^2)}R(zq^4).
\end{aligned} \tag{1.13}$$

Define the auxiliary function

$$F(z) = \sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n(n-1)}z^n}{(z;q^2)_{n+1}}$$

and observe that

$$F(z) = \sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n(n-1)}z^n(1-zq^{2n}+zq^{2n})}{(z;q^2)_{n+1}} = L(z)+\frac{z}{1-z}L(zq^2). \tag{1.14}$$

https://doi.org/10.1112/S0024610700008735 Published online by Cambridge University Press

We also have

$$\begin{aligned}F(z) &= \frac{1}{1-z}+\sum_{n=0}^{\infty}\frac{(-q;q^2)_{n+1}\,q^{n(n+1)}z^{n+1}}{(z;q^2)_{n+2}}\\ &= \frac{1}{1-z}+\frac{z}{(1-z)(1-zq^2)}\sum_{n=0}^{\infty}\frac{(-q;q^2)_n(1+q^{2n+1})\,q^{n(n-1)}q^{2n}z^n}{(zq^4;q^2)_n}\\ &= \frac{1}{1-z}+\frac{z}{1-z}\sum_{n=0}^{\infty}\frac{(-q;q^2)_n\,q^{n(n-1)}(zq^2)^n}{(zq^2;q^2)_{n+1}}\\ &\quad+\frac{zq}{(1-z)(1-zq^2)}\sum_{n=0}^{\infty}\frac{(-q;q^2)_n\,q^{n(n-1)}(zq^4)^n}{(zq^4;q^2)_n}\\ &= \frac{1}{1-z}+\frac{z}{1-z}F(zq^2)+\frac{zq}{(1-z)(1-zq^2)}L(zq^4).\end{aligned}\tag{1.15}$$

Using (1.14) to eliminate $F(z)$ and $F(zq^2)$ from (1.15) we obtain

$$L(z)=\frac{1}{1-z}+\frac{zq(1+zq)}{(1-z)(1-zq^2)}L(zq^4).\tag{1.16}$$

Comparing (1.16) and (1.13) we see that $L(z)$ and $R(z)$ satisfy the same first order non-homogeneous q-difference equation. Since they are both analytic in z for $|z|<1$ and $L(0)=R(0)=1$, it follows that $L(z)=R(z)$ for $|z|<1$, $|q|<1$. This completes the proof of (1.12).

To prove equations (1.3)–(1.6) we begin with the Watson–Whipple transformation [**2**, equation (III.17), p. 242]:

$$\begin{aligned}&{}_8\phi_7\left[\begin{matrix}a, qa^{1/2}, -qa^{1/2}, & b, & c, & d, & e, & f\\ a^{1/2}, & -a^{1/2}, & \frac{aq}{b}, & \frac{aq}{c}, & \frac{aq}{d}, & \frac{aq}{e}, & \frac{aq}{f}\end{matrix};q;\frac{a^2q^2}{bcdef}\right]\\ &\qquad=\frac{\left(aq,\frac{aq}{de},\frac{aq}{df},\frac{aq}{ef}\right)_\infty}{\left(\frac{aq}{d},\frac{aq}{e},\frac{aq}{f},\frac{aq}{def}\right)_\infty}\,{}_4\phi_3\left[\begin{matrix}\frac{aq}{bc}, d, e, f\\ \frac{aq}{b}, \frac{aq}{c}, \frac{def}{a}\end{matrix};q;q\right].\end{aligned}\tag{1.17}$$

Now

$$\frac{(qa^{1/2})_n(-qa^{1/2})_n}{(a^{1/2})_n(-a^{1/2})_n}=\frac{(1-aq^2)(1-aq^4)\dots(1-aq^{2n})}{(1-a)(1-aq^2)\dots(1-aq^{2n-2})}=\frac{1-aq^{2n}}{1-a}.$$

Let e and f tend to infinity (or equivalently, put $e=1/e'$, $f=1/f'$, simplify and then let $e'=f'=0$). Then $(aq/e)_n$ and $(aq/f)_n$ tend to 1. Also,

$$\begin{aligned}(e)_n &= (1-e)(1-eq)\dots(1-eq^{n-1})\\ &= (-e)^n\left(-\frac{1}{e}+1\right)\left(-\frac{1}{e}+q\right)\dots\left(-\frac{1}{e}+q^{n-1}\right)\\ &\sim (-e)^n\,q^{(1/2)\,n(n-1)}\end{aligned}$$

https://doi.org/10.1112/S0024610700008735 Published online by Cambridge University Press

as $e \to \infty$. Similarly, $(f)_n \sim (-f)^n q^{(1/2)n(n-1)}$ as $f \to \infty$. Hence, in the limit, (1.17) becomes

$$\sum_{n=0}^{\infty} \left(\frac{1-aq^{2n}}{1-a}\right) \frac{(a)_n (b)_n (c)_n (d)_n}{(q)_n \left(\frac{aq}{b}\right)_n \left(\frac{aq}{c}\right)_n \left(\frac{aq}{d}\right)_n} \left(\frac{a^2}{bcd}\right)^n q^{n(n+1)}$$

$$= \frac{(aq)_\infty}{\left(\frac{aq}{d}\right)_\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{aq}{bc}\right)_n (d)_n}{(q)_n \left(\frac{aq}{b}\right)_n \left(\frac{aq}{c}\right)_n} \left(-\frac{a}{d}\right)^n q^{(1/2)n(n+1)}. \quad (1.18)$$

Since $(a)_n = (1-a)(aq)_{n-1}$ for $n \geqslant 1$, we obtain

$$1 + \sum_{n=1}^{\infty} (1-aq^{2n}) \frac{(aq)_{n-1} (b)_n (c)_n (d)_n}{(q)_n \left(\frac{aq}{b}\right)_n \left(\frac{aq}{c}\right)_n \left(\frac{aq}{d}\right)_n} \left(\frac{a^2}{bcd}\right)^n q^{n(n+1)}$$

$$= \frac{(aq)_\infty}{\left(\frac{aq}{d}\right)_\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{aq}{bc}\right)_n (d)_n}{(q)_n \left(\frac{aq}{b}\right)_n \left(\frac{aq}{c}\right)_n} \left(-\frac{a}{d}\right)^n q^{(1/2)n(n+1)}. \quad (1.19)$$

Now take $a = 1$, $b = i$, $c = -i$ and $d = -q^{1/2}$. Then

$$(b)_n (c)_n = (-1; q^2)_n \quad \text{and} \quad \left(\frac{aq}{b}\right)_n \left(\frac{aq}{c}\right)_n = (-iq)_n (iq)_n = (-q^2; q^2)_n.$$

Hence (1.19) becomes

$$1 + \sum_{n=1}^{\infty} \frac{(-1)^n (1-q^{2n}) (q)_{n-1} (-1; q^2)_n q^{n^2+(1/2)n}}{(q)_n (-q^2; q^2)_n} = \frac{(q)_\infty}{(-q^{1/2})_\infty} \sum_{n=0}^{\infty} \frac{(-q^{1/2})_n q^{(1/2)n^2}}{(-q^2; q^2)_n}.$$

Observe that

$$\frac{(1-q^{2n}) (q)_{n-1} (-1; q^2)_n}{(q)_n (-q^2; q^2)_n} = \left(\frac{1-q^{2n}}{1-q^n}\right)\left(\frac{2}{1+q^{2n}}\right) = \frac{2(1+q^n)}{1+q^{2n}}.$$

Therefore

$$1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n (1+q^n) q^{n^2+(1/2)n}}{1+q^{2n}} = \frac{(q)_\infty}{(-q^{1/2})_\infty} \sum_{n=0}^{\infty} \frac{(-q^{1/2})_n q^{(1/2)n^2}}{(-q^2; q^2)_n}.$$

Replacing q by q^2 we get

$$1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n (1+q^{2n}) q^{2n^2+n}}{1+q^{4n}} = \frac{(q^2; q^2)_\infty}{(-q; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2}}{(-q^4; q^4)_n}. \quad (1.20)$$

https://doi.org/10.1112/S0024610700008735 Published online by Cambridge University Press

The left-hand side of (1.20) simplifies to

$$\begin{aligned}1+2\sum_{n=1}^{\infty}\frac{(-1)^n(1+q^{2n})q^{2n^2+n}}{1+q^{4n}} &= 2\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2+n}}{1+q^{4n}}+2\sum_{n=1}^{\infty}\frac{(-1)^n q^{2n^2+3n}}{1+q^{4n}}\\ &= 2\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2+n}}{1+q^{4n}}+2\sum_{n=1}^{\infty}\frac{(-1)^n q^{2n^2+3n}q^{-4n}}{(1+q^{4n})q^{-4n}}\\ &= 2\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2+n}}{1+q^{4n}}+2\sum_{n=1}^{\infty}\frac{(-1)^n q^{2n^2-n}}{1+q^{-4n}}\\ &= 2\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n^2+n}}{1+q^{4n}}.\end{aligned}$$

Hence

$$2\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n^2+n}}{1+q^{4n}}=\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2}}{(-q^4;q^4)_n},$$

so

$$\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(-q^4;q^4)_n}=2\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(2n+1)}}{1+q^{4n}},$$

which completes the proof of (1.3).

In a similar manner (1.4) follows from (1.18) with the substitutions $a=q$, $b=iq^{1/2}$, $c=-iq^{1/2}$, $d=-q^{1/2}$, and then $q\to q^2$. Substituting $a=1$, $b=q^{1/4}$, $c=q^{-1/4}$ and $d=-q^{1/2}$ in (1.19) and then replacing q by q^4 yields

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n(2n+1)}}{1-q^{4n+1}}=\frac{(q^4;q^4)_\infty}{(-q^2;q^4)_\infty}\sum_{n=0}^{\infty}\frac{q^{2n^2}(-q^2;q^4)_n}{(q;q^2)_{2n+1}},$$

which implies (1.5). Substituting $a=q$, $b=q^{1/4}$, $c=q^{3/4}$ and $d=-q$ in (1.18) and then replacing q by q^4 yields

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{4n(n+1)}}{1-q^{4n+1}}=\frac{(q^4;q^4)_\infty}{(-q^4;q^4)_\infty}\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}(-q^4;q^4)_n}{(q;q^2)_{2n+2}},$$

which implies (1.6).

Equation (1.7) also follows from (1.18), this time with $a=q^{1/2}$, $b=iq^{1/2}$, $c=-iq^{1/2}$ and $d=q$. We obtain

$$\sum_{n=0}^{\infty}\left(\frac{1-q^{2n+1/2}}{1-q^{1/2}}\right)\frac{(-q;q^2)_n}{(-q^2;q^2)_n}q^{n^2}=\frac{1}{1-q^{1/2}}\sum_{n=0}^{\infty}\frac{(q^{1/2};q)_n}{(-q^2;q^2)_n}(-1)^n q^{1/2n^2}. \tag{1.21}$$

The left-hand side of (1.21) is equal to $(S_0(q)-q^{1/2}S_1(q))/(1-q^{1/2})$ and the right-hand side is equal to $U_0(-q^{1/2})/(1-q^{1/2})$. Thus

$$S_0(q^2)-qS_1(q^2)=U_0(-q).$$

Replacing q by $-q$ completes the proof of (1.7).

To establish (1.8) we set $a=q^{3/2}$, $b=iq$, $c=-iq$ and $d=q$ in (1.18). This gives

$$\sum_{n=0}^{\infty}\left(\frac{1-q^{2n+(3/2)}}{1-q^{3/2}}\right)\frac{(-q^2;q^2)_n}{(-q^3;q^2)_n}q^{n(n+1)}=\frac{1}{1-q^{3/2}}\sum_{n=0}^{\infty}\frac{(q^{1/2};q)_n}{(-q^3;q^2)_n}(-1)^n q^{(1/2)n(n+2)}.$$

https://doi.org/10.1112/S0024610700008735 Published online by Cambridge University Press

Therefore

$$\begin{aligned}\frac{q^{-1/2}+q^{1/2}}{1-q^{3/2}}[q^{1/2}T_1(q)-T_0(q)] &= \frac{1+q}{1-q^{3/2}}[T_1(q)-q^{-1/2}T_0(q)]\\ &= \frac{1+q}{1-q^{3/2}}\sum_{n=0}^{\infty}(1-q^{2n+(3/2)})\frac{(-q^2;q^2)_n}{(-q;q^2)_{n+1}}q^{n(n+1)}\\ &= \sum_{n=0}^{\infty}\left(\frac{1-q^{2n+(3/2)}}{1-q^{3/2}}\right)\frac{(-q^2;q^2)_n}{(-q^3;q^2)_n}q^{n(n+1)}\\ &= \frac{1}{1-q^{3/2}}\sum_{n=0}^{\infty}\frac{(q^{1/2};q)_n}{(-q^3;q^2)_n}(-1)^n q^{(1/2)n(n+2)}\\ &= \frac{1+q}{1-q^{3/2}}\sum_{n=0}^{\infty}\frac{(q^{1/2};q)_n}{(-q;q^2)_{n+1}}(-1)^n q^{(1/2)n(n+2)}\\ &= \frac{q^{-1/2}+q^{1/2}}{1-q^{3/2}}\sum_{n=0}^{\infty}\frac{(q^{1/2};q)_n}{(-q;q^2)_{n+1}}(-1)^n q^{(1/2)(n+1)^2}\\ &= \frac{q^{-1/2}+q^{1/2}}{1-q^{1/2}}[-U_1(-q^{1/2})],\end{aligned}$$

which implies that

$$T_0(q^2)-qT_1(q^2) = U_1(-q).$$

Replacing q by $-q$ completes the proof of (1.8).

To prove (1.9) we first observe that by (1.3) and (1.4),

$$U_0(-q)+2U_1(-q) = 2\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(1/2)n(n+1)}}{1+q^{2n}}. \tag{1.22}$$

By Bailey's summation of a well-poised bilateral basic hypergeometric series ${}_3\psi_3$ [**2**, equation (II.31), p. 239] we have

$$\begin{aligned}{}_3\psi_3\left[\begin{matrix}b, c, d,\\ \frac{q}{b}, \frac{q}{c}, \frac{q}{d}\end{matrix}; q; \frac{q}{bcd}\right] &= \sum_{n=-\infty}^{\infty}\frac{(b)_n(c)_n(d)_n}{\left(\frac{q}{b}\right)_n\left(\frac{q}{c}\right)_n\left(\frac{q}{d}\right)_n}\left(\frac{q}{bcd}\right)^n\\ &= \frac{\left(q, \frac{q}{bc}, \frac{q}{bd}, \frac{q}{cd}\right)_\infty}{\left(\frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{bcd}\right)_\infty}.\end{aligned} \tag{1.23}$$

Let $b = i$ and $c = -i$. Then

$$\frac{(b)_n(c)_n}{\left(\frac{q}{b}\right)_n\left(\frac{q}{c}\right)_n} = \frac{(-1;q^2)_n}{(-q^2;q^2)_n} = \frac{2}{1+q^{2n}}.$$

Hence, as $d\to\infty$, (1.23) becomes

$$2\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(1/2)n(n+1)}}{1+q^{2n}} = \frac{(q)_\infty^2}{(-q^2;q^2)_\infty}.$$

https://doi.org/10.1112/S0024610700008735 Published online by Cambridge University Press

It now follows from (1.22) that

$$U_0(-q)+2U_1(-q)=\frac{(q)_\infty^2(q;q^2)_\infty}{(q^4;q^4)_\infty}. \tag{1.24}$$

Replacing q by $-q$ completes the proof of (1.9).

In §4 we obtain the modular transformation formulae (4.5)–(4.8) and (4.13)–(4.16) from the generalized Lambert series (1.3)–(1.6). Adding (4.7) and (4.15) we get

$$V_0(q)+V_0(-q)=\sqrt{\frac{\pi}{t}}q_1^{-1/16}(U_0(q_1^{1/2})+2U_1(q_1^{1/2})), \tag{1.25}$$

where $q=e^{-t}$, $q_1=e^{-\pi^2/t}$. Substituting (1.9) with q replaced by $q_1^{1/2}$ into (1.25) yields

$$\begin{aligned}V_0(q)+V_0(-q)&=\sqrt{\frac{\pi}{t}}q_1^{-1/16}(-q_1^{1/2};q_1)_\infty^3(q_1;q_1)_\infty(q_1;q_1^2)_\infty\\&=\sqrt{\frac{\pi}{t}}q_1^{-1/16}\frac{(q_1;q_1)_\infty^8}{(q_1^{1/2};q_1^{1/2})_\infty^3(q_1^2;q_1^2)_\infty^4}.\end{aligned} \tag{1.26}$$

Applying the transformation formula for the Dedekind η-function to the right-hand side of (1.26) completes the proof of (1.10). Similarly, (1.11) can be obtained by (4.8), (4.16) and (1.24).

2. *Obtaining mock theta functions from theta functions*

As a motivation for the discussion, we first prove closure of the exponential factor in the asymptotic expansion as $q=e^{-t}\to 1^-$ of the function

$$F_0(q)=\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q;q^2)_n} \tag{2.1}$$

listed in Ramanujan's letter. This is tantamount to showing that $\log F_0(q)$ has an asymptotic expansion of the form

$$\log F_0(q)=\frac{A}{t}+B\log t+\sum_{k=0}^{N}C_k t^k+O(t^p) \tag{2.2}$$

for all $p>N$. (Here N is a fixed positive integer; it will actually turn out that $N=1$.) To do this we rewrite (2.1) in the form

$$(q;q^2)_\infty F_0(q)=\sum_{n=0}^{\infty}q^{2n^2}(q^{2n+1};q^2)_\infty=\sum_{n=0}^{\infty}a_n, \tag{2.3}$$

say. The term a_n on the right-hand side of (2.3) is defined for all real n. We make a *right half-shift*, and sum a_n over the positive half-integers $\frac{1}{2}, \frac{3}{2}, \frac{5}{2},\ldots$ instead of the non-negative integers. More precisely we define $b_n=a_{n+1/2}$ and form the sum

$$\sum_{n=0}^{\infty}b_n=\sum_{n=0}^{\infty}q^{2n^2+2n+1/2}(q^{2n+2};q^2)_\infty=q^{1/2}(q^2;q^2)_\infty\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q^2;q^2)_n}=q^{1/2}(q^2;q^2)_\infty H(q^2), \tag{2.4}$$

where $H(q)$ is the series (1.1). Hence $(q;q^2)_\infty F_0(q)$ can be obtained from $q^{1/2}(q^2;q^2)_\infty H(q^2)$ by applying a *left half-shift*, that is, by substituting $b_{n-1/2}$ for b_n in

https://doi.org/10.1112/S0024610700008735 Published online by Cambridge University Press

(2.4). Since the sequences (a_n) and (b_n) are unimodal for each fixed $q \in (0,1)$, it follows from [**6**, pp. 126–127] that

$$\sum_{n=0}^{\infty} a_n = \left(\sum_{n=0}^{\infty} b_n\right)[1+O(e^{-\delta/t})],$$

where δ is a positive constant. Hence, by (2.3) and (2.4),

$$\log(q;q^2)_\infty + \log F_0(q) = -\tfrac{1}{2}t + \log(q^2;q^2)_\infty + \log H(q^2) + O(e^{-\delta/t}) \tag{2.5}$$

as $t \to 0^+$.

It is known [**12**, pp. 57–58] that

$$(q)_\infty = \sqrt{\frac{2\pi}{t}}\exp\left(-\frac{\pi^2}{6t}+\frac{t}{24}\right)+o(1).$$

Hence

$$(q^2;q^2)_\infty = \sqrt{\frac{\pi}{t}}\exp\left(-\frac{\pi^2}{12t}+\frac{t}{12}\right)+o(1) \tag{2.6}$$

and

$$(q;q^2)_\infty = \frac{(q)_\infty}{(q^2;q^2)_\infty} = \sqrt{2}\exp\left(-\frac{\pi^2}{12t}-\frac{t}{24}\right)+o(1). \tag{2.7}$$

From (1.2), (2.6) and (2.7) we find, in view of (2.5), that

$$\log F_0(q) = \frac{\pi^2}{30t} - \tfrac{1}{2}\log t + \tfrac{1}{2}\log\frac{\pi}{5+\sqrt{5}} - \frac{t}{120} + O(t^p)$$

for every positive integer p, which has the form (2.2). Numerical calculations suggest that in fact

$$F_0(q) = \sqrt{\frac{\pi}{(5+\sqrt{5})\,t}}\exp\left(\frac{\pi^2}{30t}-\frac{t}{120}\right)+t-\frac{9}{2}t^2+\frac{205}{6}t^3-\frac{2907}{8}t^4+O(t^5).$$

The series

$$F_1(q) = \sum_{n=1}^{\infty}\frac{q^{2n(n-1)}}{(q;q^2)_n}$$

in Ramanujan's list can be similarly obtained from the function

$$G(q) = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n} = \prod_{m=1}^{\infty}\frac{1}{(1-q^{5m-4})(1-q^{5m-1})}$$

of the first Rogers–Ramanujan identity. More precisely, the two series

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} q^{2n(n-1)}(q^{2n+1};q^2)_\infty = (q;q^2)_\infty[1+F_1(q)]$$

and

$$\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} q^{2n^2-1/2}(q^{2n+2};q^2)_\infty = q^{-1/2}(q^2;q^2)_\infty\, G(q^2)$$

are related by the half-shift $a_n = b_{n-1/2}$.

https://doi.org/10.1112/S0024610700008735 Published online by Cambridge University Press

There is a close analogy between the Rogers–Ramanujan identities and the so-called Göllnitz–Gordon identities [**3**; **4**; **11**, equations (36), (34)]

$$A(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q;q^8)_\infty (q^4;q^8)_\infty (q^7;q^8)_\infty} \tag{2.8}$$

and

$$B(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q^3;q^8)_\infty (q^4;q^8)_\infty (q^5;q^8)_\infty}. \tag{2.9}$$

From (2.8) and (2.9) we have

$$\frac{(q;q)_\infty}{(-q;q^2)_\infty} A(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(q^{2n+2};q^2)_\infty}{(-q^{2n+1};q^2)_\infty} \tag{2.10}$$

and

$$\frac{(q;q)_\infty}{(-q;q^2)_\infty} B(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(q^{2n+2};q^2)_\infty}{(-q^{2n+1};q^2)_\infty}. \tag{2.11}$$

By analogy with the above discussion of how the mock theta functions $F_0(q)$ and $F_1(q)$ can be obtained from the 'genuine' theta functions $H(q)$ and $G(q)$, we replace n by $n-\frac{1}{2}$ in the series (2.11) and (2.10). The resulting functions are

$$\frac{q^{-3/4}(q;q^2)_\infty}{(-1;q^2)_\infty}(1+2T_0(-q))$$

and

$$\frac{q^{1/4}(q;q^2)_\infty}{(-1;q^2)_\infty}(1+2T_1(-q)),$$

where $T_0(q)$ and $T_1(q)$ are eighth order mock theta functions listed in the introduction. (In §3 we will show that they indeed have the strong mock theta property.) Thus the analogies

$$G(q) \leftrightarrow A(q) \quad \text{and} \quad H(q) \leftrightarrow B(q)$$

between fifth and eighth order theta functions lead to analogies

$$F_1(q) \leftrightarrow T_1(-q) \quad \text{and} \quad F_0(q) \leftrightarrow T_0(-q)$$

between fifth and eighth order mock theta functions. These analogies are the reason why we have called the new functions eighth order.

The mock theta functions $V_0(q)$ and $V_1(q)$ listed in the introduction were obtained by half-shifting the series in identities [**11**, (8), (12)].

If the half-shift $n \to n+\frac{1}{2}$ is applied twice to a q-series $f(q)$, the composite shift $n \to n+1$ leads back to $f(q)$ minus its first term (or to $f(q)$ itself in the case of a bilateral q-series). Thus nothing essentially new is obtained by iterated half-shifts. However, the involution $n \to -n$, when applied to the terms of a q-series, can lead to a new mock theta function. For example, if we apply this involution to $F_0(q)$ we get

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_{-n}} = \sum_{n=0}^{\infty} \frac{q^{2n^2}(q^{1-2n};q^2)_\infty}{(q;q^2)_\infty} = \sum_{n=0}^{\infty} (-1)^n q^{n^2}(q;q^2)_n = \phi_0(-q),$$

where $\phi_0(q)$ appears in Ramanujan's list. Applying this involution to

$$1+2T_0(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-1;q^2)_n}{(-q;q^2)_n} \quad \text{and} \quad 1+2T_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)}(-1;q^2)_n}{(-q;q^2)_n},$$

we obtain the eighth order mock theta functions $S_0(q)$ and $S_1(q)$ listed in the introduction.

https://doi.org/10.1112/S0024610700008735 Published online by Cambridge University Press

3. *The strong mock theta property for the functions of order* 8

One way of showing that a function $f(q)$ has the strong mock theta property is to produce another function $g(q)$ such that

(i) $f(q)+g(q)=\sum_{j=1}^{J} q^{\alpha_j}h_j(q)$, where the $h_j(q)$, $1\leqslant j\leqslant J$, are modular forms (with $q=e^{\pi i\tau}$);

(ii) $g(q)$ is bounded under radial approach to each exponential singularity of $f(q)$.

In practice we show that the exponential singularities of $f(q)$ and $g(q)$ are disjoint, and each of these is bounded as q radially approaches an exponential singularity of the other. This implies that both $f(q)$ and $g(q)$ have the strong mock theta property.

We first show that $S_0(q)+2T_0(q)=R_0(q)$ and $S_1(q)+2T_1(q)=R_1(q)$, where $R_0(q)$ and $R_1(q)$ are powers of q times modular forms. To this end we note that

$$S_0(q)+2T_0(q)=\sum_{n=-\infty}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n} \tag{3.1}$$

and

$$S_1(q)+2T_1(q)=\sum_{n=-\infty}^{\infty}\frac{q^{n(n+2)}(-q;q^2)_n}{(-q^2;q^2)_n}. \tag{3.2}$$

By Bailey's summation of a very well-poised bilateral basic hypergeometric series ${}_6\psi_6$ [**2**, p. 239, equation (II.33)] we have

$${}_6\psi_6\left[\begin{matrix} qa^{1/2}, -qa^{1/2}, & b, & c, & d, & e \\ a^{1/2}, & -a^{1/2}, & \dfrac{aq}{b}, & \dfrac{aq}{c}, & \dfrac{aq}{d}, & \dfrac{aq}{e}\end{matrix};q;\frac{qa^2}{bcde}\right]$$
$$=\frac{\left(aq,\dfrac{aq}{bc},\dfrac{aq}{bd},\dfrac{aq}{be},\dfrac{aq}{cd},\dfrac{aq}{ce},\dfrac{aq}{de},q,\dfrac{q}{a}\right)_\infty}{\left(\dfrac{aq}{b},\dfrac{aq}{c},\dfrac{aq}{d},\dfrac{aq}{e},\dfrac{q}{b},\dfrac{q}{c},\dfrac{q}{d},\dfrac{q}{e},\dfrac{qa^2}{bcde}\right)_\infty}. \tag{3.3}$$

Now

$$\frac{(qa^{1/2})_n(-qa^{1/2})_n}{(a^{1/2})_n(-a^{1/2})_n}=\frac{(1-aq^2)(1-aq^4)\dots(1-aq^{2n})}{(1-a)(1-aq^2)\dots(1-aq^{2n-2})}=\frac{1-aq^{2n}}{1-a}.$$

Let d and e tend to infinity. Then the left-hand side of (3.3) becomes

$$\sum_{n=-\infty}^{\infty}\left(\frac{1-aq^{2n}}{1-a}\right)\frac{(b)_n(c)_n}{\left(\dfrac{aq}{b}\right)_n\left(\dfrac{aq}{c}\right)_n}\left(\frac{a^2q}{bc}\right)^n q^{n(n-1)}.$$

Now take $a=q^{1/2}$, $b=iq^{1/2}$, $c=-iq^{1/2}$. Then

$$(b)_n(c)_n=(-q;q^2)_n,\quad \left(\frac{aq}{b}\right)_n\left(\frac{aq}{c}\right)_n=(-iq)_n(iq)_n=(-q^2;q^2)_n\quad\text{and}\quad\frac{a^2q}{bc}=q.$$

Hence (3.3) becomes

$$\frac{1}{1-q^{1/2}}[R_0(q)-q^{1/2}R_1(q)]=\sum_{n=-\infty}^{\infty}\left(\frac{1-q^{2n+1/2}}{1-q^{1/2}}\right)\frac{(-q;q^2)_n}{(-q^2;q^2)_n}q^{n^2}$$
$$=\left(\frac{1}{1-q^{1/2}}\right)\frac{(q^{1/2})_\infty^3(q)_\infty}{(-q^2;q^2)_\infty(-q;q^2)_\infty},$$

https://doi.org/10.1112/S0024610700008735 Published online by Cambridge University Press

where $R_0(q)$ and $R_1(q)$ are the bilateral series in (3.1) and (3.2) respectively. Replacing q by q^2 we obtain

$$R_0(q^2)-qR_1(q^2)=\frac{(q;q^2)_\infty^3(q^2;q^2)_\infty}{(-q^4;q^4)_\infty(-q^2;q^4)_\infty}=\frac{(q)_\infty^3}{(q^2;q^2)_\infty(q^4;q^4)_\infty}=\frac{q^{1/8}\eta\left(\frac{\tau}{2}\right)^3}{\eta(\tau)\,\eta(2\tau)}.$$

Since the even and odd parts of the Fourier series of a modular form are modular forms, it follows that $R_0(q)$ and $R_1(q)$ are powers of q times modular forms. This completes the proof that the pairs $f(q)=S_0(q)$, $g(q)=2T_0(q)$ and $f(q)=S_1(q)$, $g(q)=2T_1(q)$ satisfy property (i).

To show that $f(q)=S_0(q)$ and $g(q)=2T_0(q)$ satisfy property (ii), we first note that the exponential singularities of $S_0(q)$ are at the roots of unity $\rho=e^{\pi iM/N}$ with M odd and N even. If ρ is such a root, the odd powers of ρ are bounded away from -1, say $|1+\rho^k|\geqslant\delta>0$ for all odd integers k. As shown in [**10**, equation (V), p. 4] we have

$$|1+r_1e^{i\theta}|\geqslant\sqrt{\frac{r_1}{r_2}}|1+r_2e^{i\theta}|$$

for all real r_1, r_2, θ with $0\leqslant r_1\leqslant r_2\leqslant 1$. In this inequality we take $r_1=e^{-(2j-1)t}$, $r_2=1$ and $\theta=\pi(2j-1)M/N$, getting

$$|1+e^{-(2j-1)t}\rho^{2j-1}|\geqslant e^{-(2j-1)t/2}|1+\rho^{2j-1}|\geqslant\delta e^{-(2j-1)t/2}.$$

Thus, if $q=\rho e^{-t}$, we have

$$|(-q;q^2)_n|\geqslant\delta^n\exp\left(-\sum_{j=1}^{n}\frac{(2j-1)t}{2}\right)=\delta^ne^{-n^2t/2},$$

and hence (since $|(-1;q^2)_n|\leqslant 2^n$)

$$\left|\frac{q^{n(n+1)}(-1;q^2)_n}{(-q;q^2)_n}\right|\leqslant\left(\frac{2}{\delta}\right)^n e^{-n(n+2)t/2}.$$

Therefore

$$|2T_0(q)|=\left|\sum_{n=1}^{\infty}\frac{q^{n(n+1)}(-1;q^2)_n}{(-q;q^2)_n}\right|\leqslant\sum_{n=0}^{\infty}\left(\frac{2}{\delta}\right)^n e^{-n(n+2)t/2}=O(1)$$

as $t\to 0^+$. In the same way, we find that $S_0(q)$ is bounded as q radially approaches the exponential singularities $q=e^{\pi iM/N}$ (M, N odd) of $T_0(q)$. This shows that both $S_0(q)$ and $T_0(q)$ have the strong mock theta property. The same argument shows that $S_1(q)$ and $T_1(q)$ also have the strong mock theta property. The strong mock theta property for the functions $U_0(q)$, $U_1(q)$, $V_0(q)$ and $V_1(q)$ follows from (1.9), (1.10) and (1.11).

4. *Transformation formulae*

Since any substitution of the modular group can be decomposed into a finite number of substitutions of the forms

$$\tau'=\tau+1\quad\text{and}\quad\tau'=-1/\tau,$$

it is sufficient to construct the transformations that express the 16 functions $S_0(\pm q),\ldots$ in terms of similar functions of q_1 (or powers of q_1), where $q=e^{-\alpha}$ and $q_1=e^{-\beta}$ with $\alpha\beta=\pi^2$.

https://doi.org/10.1112/S0024610700008735 Published online by Cambridge University Press

The modular transformation formulae for our mock theta functions are

$$q^{-1/16}S_0(q) = \sqrt{\frac{\pi}{4\alpha}}\,V_0(q_1) + \sqrt{\frac{2\pi}{\alpha}}\,q_1^{-1/4}\,V_1(q_1) + \sqrt{\frac{4\alpha}{\pi}}\,K_0(\alpha), \tag{4.1}$$

$$q^{7/16}S_1(q) = \sqrt{\frac{\pi}{4\alpha}}\,V_0(q_1) - \sqrt{\frac{2\pi}{\alpha}}\,q_1^{-1/4}\,V_1(q_1) - \sqrt{\frac{4\alpha}{\pi}}\,K_1(\alpha), \tag{4.2}$$

$$q^{-1/16}T_0(q) = \sqrt{\frac{\pi}{16\alpha}}\,V_0(-q_1) - \sqrt{\frac{\pi}{2\alpha}}\,q_1^{-1/4}\,V_1(-q_1) - \sqrt{\frac{\alpha}{\pi}}\,K_0(\alpha), \tag{4.3}$$

$$q^{7/16}T_1(q) = \sqrt{\frac{\pi}{16\alpha}}\,V_0(-q_1) + \sqrt{\frac{\pi}{2\alpha}}\,q_1^{-1/4}\,V_1(-q_1) + \sqrt{\frac{\alpha}{\pi}}\,K_1(\alpha), \tag{4.4}$$

$$q^{-1/8}U_0(q) = \sqrt{\frac{\pi}{2\alpha}}\,V_0(q_1^{1/2}) + \sqrt{\frac{\alpha}{2\pi}}\,J\left(\frac{\alpha}{2}\right), \tag{4.5}$$

$$q^{-1/8}U_1(q) = \sqrt{\frac{\pi}{8\alpha}}\,V_0(-q_1^{1/2}) - \sqrt{\frac{\alpha}{8\pi}}\,J\left(\frac{\alpha}{2}\right), \tag{4.6}$$

$$V_0(q) = \sqrt{\frac{\pi}{\alpha}}\,q_1^{-1/16}\,U_0(q_1^{1/2}) - \sqrt{\frac{16\alpha}{\pi}}\,J(4\alpha), \tag{4.7}$$

$$q^{-1/4}V_1(q) = \sqrt{\frac{\pi}{8\alpha}}\,q_1^{-1/16}\,U_0(-q_1^{1/2}) - \sqrt{\frac{\alpha}{\pi}}\,K(2\alpha), \tag{4.8}$$

$$q^{-1/16}S_0(-q) = \sqrt{\frac{\pi(2-\sqrt{2})}{\alpha}}\,q_1^{-1/16}\,T_0(-q_1) + \sqrt{\frac{\pi(2+\sqrt{2})}{\alpha}}\,q_1^{7/16}\,T_1(-q_1) + \sqrt{\frac{4\alpha}{\pi}}\,J_0(\alpha), \tag{4.9}$$

$$q^{7/16}S_1(-q) = \sqrt{\frac{\pi(2+\sqrt{2})}{\alpha}}\,q_1^{-1/16}\,T_0(-q_1) - \sqrt{\frac{\pi(2-\sqrt{2})}{\alpha}}\,q_1^{7/16}\,T_1(-q_1) + \sqrt{\frac{4\alpha}{\pi}}\,J_1(\alpha), \tag{4.10}$$

$$q^{-1/16}T_0(-q) = \sqrt{\frac{\pi(2-\sqrt{2})}{16\alpha}}\,q_1^{-1/16}\,S_0(-q_1) + \sqrt{\frac{\pi(2+\sqrt{2})}{16\alpha}}\,q_1^{7/16}\,S_1(-q_1) - \sqrt{\frac{\alpha}{\pi}}\,J_0(\alpha), \tag{4.11}$$

$$q^{7/16}T_1(-q) = \sqrt{\frac{\pi(2+\sqrt{2})}{16\alpha}}\,q_1^{-1/16}\,S_0(-q_1) - \sqrt{\frac{\pi(2-\sqrt{2})}{16\alpha}}\,q_1^{7/16}\,S_1(-q_1) - \sqrt{\frac{\alpha}{\pi}}\,J_1(\alpha), \tag{4.12}$$

$$q^{-1/8}U_0(-q) = \sqrt{\frac{4\pi}{\alpha}}\,q_1^{-1/8}\,V_1(q_1^{1/2}) + \sqrt{\frac{2\alpha}{\pi}}\,K(\alpha), \tag{4.13}$$

$$q^{-1/8}U_1(-q) = -\sqrt{\frac{\pi}{\alpha}}\,q_1^{-1/8}\,V_1(-q_1^{1/2}) - \sqrt{\frac{\alpha}{2\pi}}\,K(\alpha), \tag{4.14}$$

$$V_0(-q) = \sqrt{\frac{4\pi}{\alpha}}\,q_1^{-1/16}\,U_1(q_1^{1/2}) + \sqrt{\frac{16\alpha}{\pi}}\,J(4\alpha), \tag{4.15}$$

$$q^{-1/4}V_1(-q) = -\sqrt{\frac{\pi}{2\alpha}}\,q_1^{-1/16}U_1(-q_1^{1/2}) - \sqrt{\frac{\alpha}{\pi}}\,K(2\alpha), \tag{4.16}$$

https://doi.org/10.1112/S0024610700008735 Published online by Cambridge University Press

where the Mordell integrals J, K, J_0, K_0, J_1, K_1 are defined by

$$J(\alpha)=\int_0^\infty \frac{e^{-\alpha x^2}}{\cosh \alpha x}\,dx, \qquad K(\alpha)=\int_0^\infty e^{-(1/2)\alpha x^2}\frac{\cosh \frac{1}{2}\alpha x}{\cosh \alpha x}\,dx,$$

$$J_0(\alpha)=\int_0^\infty e^{-\alpha x^2}\frac{\cosh \frac{3}{2}\alpha x}{\cosh 2\alpha x}\,dx, \quad K_0(\alpha)=\int_0^\infty e^{-\alpha x^2}\frac{\sinh \frac{3}{2}\alpha x}{\sinh 2\alpha x}\,dx,$$

$$J_1(\alpha)=\int_0^\infty e^{-\alpha x^2}\frac{\cosh \frac{1}{2}\alpha x}{\cosh 2\alpha x}\,dx, \quad K_1(\alpha)=\int_0^\infty e^{-\alpha x^2}\frac{\sinh \frac{1}{2}\alpha x}{\sinh 2\alpha x}\,dx.$$

Transformations formulae (4.5)–(4.8) and (4.13)–(4.16) can be obtained from the Lambert series (1.3)–(1.6) by contour integration (see [**5**; **12**, pp. 73–79]). From identities (1.7) and (1.8) we see that

$$S_0(q)=\tfrac{1}{2}(U_0(q^{1/2})+U_0(-q^{1/2})), \quad S_1(q)=\tfrac{1}{2}q^{-1/2}(U_0(q^{1/2})-U_0(-q^{1/2})),$$
$$T_0(q)=\tfrac{1}{2}(U_1(q^{1/2})+U_1(-q^{1/2})), \quad T_1(q)=\tfrac{1}{2}q^{-1/2}(U_1(q^{1/2})-U_1(-q^{1/2})).$$

Hence (4.1)–(4.4) can be easily deduced from (4.5)–(4.8) and (4.13)–(4.16). The construction of formulae (4.9)–(4.12) is somewhat more difficult. In addition to contour integration, we must transform the infinite products occurring in (1.3) and (1.4) (with q replaced by $\pm iq^{1/2}$) to infinite products involving powers of q_1. This is accomplished with the aid of the transformation theory of the Dedekind η-function.

The transformation formulae for the Mordell integrals can be obtained from the transformation formulae for the mock theta functions. In particular, we have

$$J(\alpha)=\sqrt{\frac{\pi^3}{\alpha^3}}J(\beta) \quad \text{and} \quad K(\alpha)=\sqrt{\frac{\pi^3}{\alpha^3}}K(\beta),$$

where $\alpha\beta=\pi^2$.

5. *Concluding remarks*

It is not known whether Ramanujan's mock theta functions differ from linear combinations

$$\sum_{j=1}^{J} q^{\alpha_j}h_j(q)$$

(where the $h_j(q)$ are modular forms) by functions which are bounded in the unit disk $|q|\leqslant 1$. The same applies to the eighth order functions discussed in this paper.

We note that the functions $R_0(q)$ and $R_1(q)$ satisfy

$$q^{-1/16}R_0(-q)=\sqrt{\frac{\pi(2-\sqrt{2})}{4t}}q_1^{-1/16}R_0(-q_1)+\sqrt{\frac{\pi(2+\sqrt{2})}{4t}}q_1^{7/16}R_1(-q_1)$$

and

$$q^{7/16}R_1(-q)=\sqrt{\frac{\pi(2+\sqrt{2})}{4t}}q_1^{-1/16}R_0(-q_1)-\sqrt{\frac{\pi(2-\sqrt{2})}{4t}}q_1^{7/16}R_1(-q_1),$$

where $q=e^{-t}$, $q_1=e^{-\pi^2/t}$. These transformation formulae are analogous to the formulae [**6**, p. 134]

$$q^{-1/60}G(q)=\sqrt{\frac{5+\sqrt{5}}{10}}q_1^{-1/60}G(q_1)+\sqrt{\frac{5-\sqrt{5}}{10}}q_1^{11/60}H(q_1)$$

https://doi.org/10.1112/S0024610700008735 Published online by Cambridge University Press

and

$$q^{11/60}H(q) = \sqrt{\frac{5-\sqrt{5}}{10}}q_1^{-1/60}G(q_1) - \sqrt{\frac{5+\sqrt{5}}{10}}q_1^{11/60}H(q_1),$$

where $q = e^{-t}$, $q_1 = e^{-4\pi^2/t}$. Here $G(q)$ and $H(q)$ are the functions in the Rogers–Ramanujan identities.

Acknowledgements. Support by the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged.

Note added in proof, *August* 2000. Subsequent to the writing of this paper, it was brought to the authors' attention by Dean Hickerson that $V_1(q)$ appears, in a different form, in the first equation on p. 8 of Ramanujan's lost notebook. The function $U_0(q)$ appears in the sixth equation on p. 29 of the lost notebook.

References

1. G. E. Andrews, private correspondence, 1997.
2. G. Gasper and M. Rahman, *Basic hypergeometric series* (Cambridge University Press, Cambridge, 1990).
3. H. Göllnitz, 'Partitionen mit Differenzenbedingungen', *J. Reine Angew. Math.* 225 (1967) 154–190.
4. B. Gordon, 'Some continued fractions of the Rogers–Ramanujan type', *Duke Math.* 32 (1965) 741–748.
5. B. Gordon and R. J. McIntosh, 'Modular transformations of Ramanujan's fifth and seventh order mock theta functions (in preparation).
6. R. J. McIntosh, 'Some asymptotic formulae for q-hypergeometric series', *J. London Math. Soc.* (2) 51 (1995) 120–136.
7. R. J. McIntosh, 'Asymptotic transformations of q-series', *Canad. J. Math.* 50 (1998) 412–425.
8. S. Ramanujan, *Collected papers* (Cambridge University Press, 1927) (reprint: Chelsea, New York, 1962).
9. S. Ramanujan, *The lost notebook and other unpublished papers* (Narosa, New Delhi, 1988).
10. A. Selberg, 'Über die Mock-Thetafunktionen siebenter Ordnung', *Arch. Math. Naturvidenskab* 41 (1938) 3–15.
11. L. J. Slater, 'Further identities of the Rogers–Ramanujan type', *Proc. London Math. Soc.* (2) 54 (1951) 147–167.
12. G. N. Watson, 'The final problem: an account of the mock theta functions', *J. London Math. Soc.* 11 (1936) 55–80.
13. G. N. Watson, 'The mock theta functions (2)', *Proc. London Math. Soc.* (2) 42 (1937) 274–304.

Department of Mathematics
University of California
Los Angeles
CA 90024
USA

bg@sonia.math.ucla.edu

Department of Mathematics and Statistics
University of Regina
Regina
Canada S4S 0A2

mcintosh@math.uregina.ca

https://doi.org/10.1112/S0024610700008735 Published online by Cambridge University Press