Math. Proc. Camb. Phil. Soc. (2007), **142**, 133 © 2007 *Cambridge Philosophical Society*
doi:10.1017/S0305004106009777 *Printed in the United Kingdom*

Zeros of differences of meromorphic functions

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(*Received* 27 *June* 2005; *revised* 23 *January* 2006)

Abstract

Let f be a function transcendental and meromorphic in the plane, and define $g(z)$ by $g(z) = \Delta f(z) = f(z+1) - f(z)$. A number of results are proved concerning the existence of zeros of $g(z)$ or $g(z)/f(z)$, in terms of the growth and the poles of f. The results may be viewed as discrete analogues of existing theorems on the zeros of f' and f'/f.

1. *Introduction*

Let f be a function transcendental and meromorphic in the plane. The forward differences $\Delta^n f$ are defined in the standard way [**23**, p. 52] by

$$\Delta f(z) = f(z+1) - f(z), \quad \Delta^{n+1} f(z) = \Delta^n f(z+1) - \Delta^n f(z), \quad n = 0, 1, 2, \ldots. \quad (1{\cdot}1)$$

This paper is concerned with the question of whether the forward differences defined in (1·1) must have zeros, the principal motivations for this being twofold.

First, considerable recent attention has been given to meromorphic solutions $y = f(z)$ in the plane of difference equations

$$a_n(z)y(z+n) + \cdots + a_1(z)y(z+1) + a_0(z)y(z) = A(z),$$

as well as of functional equations of related type. A number of papers (including [**1**, **5**, **10**, **11**, **16**, **18**]) focus on the growth and zeros of solutions of such equations, investigating analogies and contrasts with the theory of linear differential equations in the complex plane. The second motivation is as a discrete analogue of the following theorem, in which the notation is that of [**13**].

† Supported by the G.I.F., the German–Israeli Foundation for Scientific Research and Development, Grant G -809-234.6/2003.

‡ Supported by a grant from the Alexander von Humboldt Stiftung.

https://doi.org/10.1017/S0305004106009777 Published online by Cambridge University Press

THEOREM 1·1 ([**7**, **17**]). *Let f be transcendental and meromorphic in the plane with*

$$\liminf_{r\to\infty} \frac{T(r, f)}{r} = 0. \tag{1·2}$$

Then f' has infinitely many zeros.

Theorem 1·1 is sharp, as shown by e^z, $\tan z$ and examples of arbitrary order greater than 1 constructed in [**6**]. The result was originally proved in [**7**] (see also [**4**]) with lim sup in (1·2), the improvement to lim inf being due to Hinchliffe [**17**]. For f as in the hypotheses of Theorem 1·1 it follows from Hurwitz' theorem that if z_1 is a zero of f' then $f(z+c)-f(z)$ has a zero near z_1, for all sufficiently small $c \in \mathbb{C}\setminus\{0\}$. This makes it natural to ask whether $f(z+c)-f(z)$, for such functions f, must always have infinitely many zeros. Here there is no loss of generality in assuming that $c = 1$, since otherwise $f(z)$ may be replaced by $F(z) = f(cz)$. Examples such as

$$f(z) = ze^{2\pi iz} + h(z), \quad \Delta f(z) = e^{2\pi iz},$$

where h is an entire function of period 1, show that attention must be restricted to functions of subexponential growth.

Consider first the case where f is a transcendental entire function of order less than 1. Then so is the first difference Δf [**23**] (see also Lemma 2·1) and by repetition of this argument each difference $\Delta^n f$, for $n \geqslant 1$, is transcendental entire of order less than 1 and so obviously has infinitely many zeros. Thus for entire f it is natural to consider zeros not of $\Delta^n f$ but rather of the divided difference $(\Delta^n f)/f$. This is analogous to the counterpart of Theorem 1·1 for the logarithmic derivative f'/f proved in [**6**, **17**]: if f is transcendental entire satisfying (1·2) or transcendental meromorphic with $\liminf_{r\to\infty} r^{-1/2}T(r, f) = 0$ then f'/f has infinitely many zeros.

The following result may be proved using a version of the Wiman–Valiron theory for differences developed in [**18**]: a proof based instead on the standard Wiman–Valiron theory [**14**, **22**] will be given in Section 5.

THEOREM 1·2. *Let $n \in \mathbb{N}$ and let f be a transcendental entire function of order of growth $\rho < 1/2$, and set*

$$G(z) = \frac{\Delta^n f(z)}{f(z)}. \tag{1·3}$$

If G is transcendental then G has infinitely many zeros. In particular if f has order less than $\min\{1/n, 1/2\}$ *then G is transcendental and has infinitely many zeros.*

The proof of Theorem 1·2 relies upon the classical $\cos \pi\rho$ minimum modulus theorem [**15**, theorem 6·13, p. 331], and breaks down if the order is at least $1/2$. However, for the first divided difference it is possible to extend Theorem 1·2 slightly beyond $\rho = 1/2$.

THEOREM 1·3. *There exists $\delta_0 \in (0, 1/2)$ with the following property. Let f be a transcendental entire function with order*

$$\rho(f) \leqslant \rho < \frac{1}{2} + \delta_0 < 1. \tag{1·4}$$

Then

$$G(z) = \frac{\Delta f(z)}{f(z)} = \frac{f(z+1)-f(z)}{f(z)} \tag{1·5}$$

has infinitely many zeros.

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It will be seen from the proof of Theorem 1·3 in Section 6 that the constant δ_0 is extremely small. It seems reasonable to conjecture that the conclusion of Theorem 1·3 in fact holds for $\rho(f) < 1$, but the present proof, which is based on an estimate of Miles and Rossi [**20**] for the size of the set where f'/f is large, will not give this.

In considering the existence of zeros of $g(z) = f(z+c) - f(z)$ when f is meromorphic, complications appear to arise from the poles of f, which may or may not be poles of g. The following theorem will be deduced in Section 7 from Theorem 1·1, using an approximation of $g(z)$ in terms of $f'(z)$ which will be developed in Section 3.

THEOREM 1·4. *Let f be a function transcendental and meromorphic of lower order $\lambda(f) < 1$ in the plane. Let $c \in \mathbb{C} \setminus \{0\}$ be such that at most finitely many poles z_j, z_k of f satisfy $z_j - z_k = c$. Then $g(z) = f(z+c) - f(z)$ has infinitely many zeros.*

It is clear that for a given f all but countably many $c \in \mathbb{C}$ satisfy the hypotheses of Theorem 1·4, but the following construction shows that Theorem 1·4 fails without the hypothesis on c, even for lower order 0.

THEOREM 1·5. *Let $\phi(r)$ be a positive non-decreasing function on $[1, \infty)$ which satisfies $\lim_{r\to\infty} \phi(r) = \infty$. Then there exists a function f transcendental and meromorphic in the plane with*

$$\limsup_{r\to\infty} \frac{T(r, f)}{r} < \infty \tag{1·6}$$

and

$$\liminf_{r\to\infty} \frac{T(r, f)}{\phi(r)\log r} < \infty, \tag{1·7}$$

such that

$$g(z) = \Delta f(z) = f(z+1) - f(z) \tag{1·8}$$

has only one zero. Moreover, the function g satisfies

$$\limsup_{r\to\infty} \frac{T(r, g)}{\phi(r)\log r} < \infty. \tag{1·9}$$

On the other hand, for transcendental meromorphic functions of sufficiently small growth, it is possible to show that either the first difference or the first divided difference has infinitely many zeros.

THEOREM 1·6. *Let f be a function transcendental and meromorphic in the plane with*

$$T(r, f) = O(\log r)^2 \quad \text{as } r \to \infty \tag{1·10}$$

and set

$$g(z) = \Delta f(z) = f(z+1) - f(z), \quad G(z) = \frac{\Delta f(z)}{f(z)} = \frac{f(z+1) - f(z)}{f(z)}. \tag{1·11}$$

Then at least one of g and G has infinitely many zeros.

The proofs of Theorems 1·3, 1·4, 1·5 and 1·6 will be given in Sections 6, 7, 8 and 9 respectively.

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2. *Preliminaries*

LEMMA 2·1. *Let f be a function transcendental and meromorphic in the plane which satisfies* (1·2)*, and with the notation* (1·1) *let $g = \Delta f$ and $G = g/f$. Then g and G are both transcendental.*

The assertion concerning g may be found in [**23**, p. 101], but a proof will be given for completeness.

Proof. Suppose first that G is a rational function. Then (1·5) gives

$$f(z+1) = R_0(z)f(z), \quad f(z-1) = R_1(z)f(z), \tag{2·1}$$

where R_0 and R_1 are rational functions, neither identically zero. Take $r_0 > 0$, so large that R_0 and R_1 have no zeros or poles in $|z| > r_0$. Suppose that z_0 is a zero of f with $|z_0| > r_0$. Then (2·1) shows that either $z_0+1, z_0+2, \ldots$, or $z_0-1, z_0-2, \ldots$, are zeros of f, depending on the sign of $\operatorname{Re} z_0$, and both contradict (1·2). The same argument shows that f cannot have a pole z_0 with $|z_0| > r_0$. But (1·2) shows that f must have infinitely many zeros or infinitely many poles, and this is a contradiction.

The proof that g is transcendental is similar. Assume that g is a rational function. Then there exist rational functions R_2 and R_3 such that

$$f(z+1) = f(z) + R_2(z), \quad f(z-1) = f(z) + R_3(z). \tag{2·2}$$

If f has infinitely many poles then a contradiction arises exactly as in the proof that G is transcendental. Assume henceforth that f has finitely many poles. Then there exists a rational function R_4 such that $h = f - R_4$ is transcendental entire, and by considering h in place of f it may be assumed that R_2 is a polynomial in (2·2). But then by [**23**, p. 21] there exists a polynomial P such that $P(z+1) - P(z) = R_2(z)$, and so by considering $f - P$ in place of f it may now be assumed that $R_2 \equiv 0$ in (2·2). Hence f has period 1, which contradicts (1·2).

A key role in the proof of Theorem 1·3 will be played by the following result of Miles and Rossi [**20**].

LEMMA 2·2 ([**20**]). *Let f be a transcendental entire function of order $\rho(f) \leqslant \rho < \infty$. Let $0 < \gamma < 1$, and for $r > 0$ let*

$$U_r = \left\{\theta \in [0, 2\pi] : \left|\frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})}\right| \geqslant \gamma n(r, 1/f)\right\}. \tag{2·3}$$

Let $M > 3$. Then there exists a set $E_M \subseteq [1, \infty)$ satisfying

$$\underline{\operatorname{logdens}}\, E_M = \liminf_{r\to\infty}\left(\frac{1}{\log r}\int_{[1,r]\cap E_M}\frac{dt}{t}\right) \geqslant 1 - \frac{3}{M}, \tag{2·4}$$

such that

$$m(U_r) > \left(\frac{1-\gamma}{7M(\rho+1)}\right)^2 \quad \text{for} \quad r \in E_M, \tag{2·5}$$

in which $m(U_r)$ denotes the Lebesgue measure of U_r .

The proof of Theorem 1·3 will also require the following variant of a standard estimate for harmonic measure [**21**, p. 116–7].

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LEMMA 2·3. *Let H be a transcendental entire function of order $\rho < \infty$. For large $r > 0$ define $r\theta(r)$ to be the length of the longest arc of the circle $|z| = r$ on which $|H(z)| > 1$, with $\theta(r) = 2\pi$ if the minimum modulus*

$$m_0(r, H) = \min\{|H(z)| : |z| = r\} \tag{2·6}$$

satisfies $m_0(r, H) > 1$. Then at least one of the following is true:

(i) *there exists a set $F \subseteq [1, \infty)$ having positive upper logarithmic density such that $m_0(r, H) > 1$ for $r \in F$;*

(ii) *for each $\tau \in (0, 1)$ the set*

$$F_\tau = \{r : \theta(r) > 2\pi(1 - \tau)\} \tag{2·7}$$

satisfies

$$\underline{\text{logdens}}\, F_\tau \geqslant \frac{1 - 2\rho(1 - \tau)}{\tau}. \tag{2·8}$$

Note that when $\rho = 1/2$ the right hand side of (2·8) is 1, and that when H has lower order less than $1/2$ it follows from Barry's lower order version of the classical cos $\pi\rho$ theorem [**3**] (see also [**15**, p. 331]) that conclusion (i) always holds.

Proof. Assume that conclusion (i) does not hold. Define $\theta^*(r)$ to be the same as $\theta(r)$, except that $\theta^*(r) = \infty$ if $m_0(r, H) > 1$. Then $\theta(r) = \theta^*(r)$ on a set of logarithmic density 1. Since

$$\frac{1}{\theta(r)} \leqslant \frac{1}{\theta^*(r)} + \frac{1}{2\pi}$$

for all large r, the standard Carleman–Tsuji estimate for harmonic measure [**21**, pp. 116–7] gives a large positive R such that

$$\int_R^r \frac{\pi\, dt}{t\theta(t)} \leqslant \int_R^r \frac{\pi\, dt}{t\theta^*(t)} + o(\log r) \leqslant (\rho + o(1)) \log r \tag{2·9}$$

as $r \to \infty$. Hence if $\tau \in (0, 1)$ then (2·9) leads to, as $r \to \infty$,

$$\begin{aligned}
(2\rho + o(1)) \log r &\geqslant \int_{[R,r]\cap F_\tau} \frac{2\pi\, dt}{t\theta(t)} + \int_{[R,r]\setminus F_\tau} \frac{2\pi\, dt}{t\theta(t)} \\
&\geqslant \int_{[R,r]\cap F_\tau} \frac{dt}{t} + \frac{1}{1-\tau} \int_{[R,r]\setminus F_\tau} \frac{dt}{t} \\
&= \int_{[R,r]\cap F_\tau} \frac{dt}{t} + \frac{1}{1-\tau} \left(\log r - O(1) - \int_{[R,r]\cap F_\tau} \frac{dt}{t} \right) \\
&= -\frac{\tau}{1-\tau} \int_{[R,r]\cap F_\tau} \frac{dt}{t} + \frac{1}{1-\tau} \log r - O(1),
\end{aligned}$$

from which (2·8) follows.

3. *An estimate of Cartan type*

Denote by $B(a, r)$ the open disc of centre a and radius r, and by $S(a, r)$ the corresponding boundary circle. Following Hayman [**12**], define an ε-set to be a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. If E is an ε-set then [**12**, pp. 75–76] the set of $r \geqslant 1$ for which the circle $S(0, r)$ meets E has finite logarithmic measure and hence zero logarithmic density, and for almost all real θ the intersection of E with the ray $\arg z = \theta$ is bounded.

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The term ε-set originated in a theorem of Hayman [**12**] on the asymptotic behaviour of subharmonic functions of small growth, from which the following result was deduced by Anderson and Clunie [**2**].

THEOREM 3·1 ([**2**]). *Let h be a function transcendental and meromorphic in the plane, with*

$$T(r,h) = O(\log r)^2 \quad as \quad r \longrightarrow \infty,$$

and assume that the Nevanlinna deficiency $\delta(\infty, h)$ of the poles of h is positive. Then there exists an ε-set E such that

$$\log|h(z)| \geqslant (\delta(\infty,h) - o(1))T(|z|,h) \quad as\ z \longrightarrow \infty\ in\ \mathbb{C}\setminus E.$$

In the particular setting of Theorem 3·1, the ε-set E may be chosen so that only finitely many discs of E meet any given bounded set [**12**, p. 77]. The same property is true of of all ε-sets occurring in this paper.

LEMMA 3·1. *Let $a_1, a_2, \ldots$ be complex numbers satisfying the conditions $|a_k| \leqslant |a_{k+1}|$ and $\lim_{k\to\infty}|a_k| = \infty$. For $r > 0$ let $n(r)$ be the number of a_k, taking account of repetition, with $|a_k| \leqslant r$. Let $\alpha > 1$. Then there exist a positive constant d_α, depending only on α, and an ε-set $E = E_\alpha$ such that for large z with $z \notin E$ and $|z| = r$,*

$$\sum_{|a_k|<\alpha r} \frac{1}{|z-a_k|} < d_\alpha \frac{n(\alpha^2 r)}{r}(\log r)^\alpha \log n(\alpha^2 r). \tag{3·1}$$

Moreover, if

$$\sum_{a_k\neq 0} \frac{1}{|a_k|} < \infty, \tag{3·2}$$

then for any positive constant h it is possible to choose E so that $|z - a_k| \geqslant 2h$ for all large $z \notin E$ and for all k.

Proof. The first part is proved, though not explicitly stated, by Gundersen [**9**, lemma 2]. In particular [**9**, (5·8)] shows that (3·1) holds outside an ε-set. Suppose now that (3·2) holds. Then if k_0 is large the set $E' = \bigcup_{k\geqslant k_0} B(a_k, 2h)$ is an ε-set, and it is only necessary to replace E by $E \cup E'$.

The next lemma is standard and can be found, for example, in [**19**, p. 65].

LEMMA 3·2 ([**19**]). *Let g be non-constant and meromorphic in the plane and let $\beta > 1$. Then for $|z| = r$ sufficiently large,*

$$\left|\frac{g'(z)}{g(z)}\right| \leqslant d_\beta \frac{T(\beta r, g)}{r} + \sum_{|a_k|<\beta r} \frac{2}{|z-a_k|}, \tag{3·3}$$

in which d_β is a positive constant depending only on β, and the a_k are the zeros and poles of g, repeated according to multiplicity.

LEMMA 3·3. *Let g be a function transcendental and meromorphic in the plane of order less than 1. Let $h > 0$. Then there exists an ε-set E such that*

$$\frac{g'(z+c)}{g(z+c)} \longrightarrow 0 \quad and \quad \frac{g(z+c)}{g(z)} \longrightarrow 1 \quad as\ z \longrightarrow \infty\ in\ \mathbb{C}\setminus E, \tag{3·4}$$

uniformly in c for $|c| \leqslant h$. Further, E may be chosen so that for large z not in E the function g has no zeros or poles in $|\zeta - z| \leqslant h$.

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Proof. Since g has order less than 1 the sequence (a_k) of zeros and poles of g, with repetition according to multiplicity, evidently satisfies (3·2). Apply Lemmas 3·1 and 3·2 with $\alpha = 4$ and $\beta = 2$. In particular, Lemma 3·1 gives an ε-set E such that for large $z \notin E$ the estimate (3·1) holds, as well as $|z - a_k| \geqslant 2h$ for all k. Let z be large, not in E, set $r = |z|$, and let $|\zeta - z| \leqslant h$. Then

$$|\zeta| \leqslant 2r \quad \text{and} \quad |\zeta - a_k| \geqslant |z - a_k| - h \geqslant \frac{|z - a_k|}{2} \quad \text{for all } k. \tag{3·5}$$

In particular, ζ is not a pole or zero of g. Now (3·1), (3·3) and (3·5) give an absolute constant $d > 0$ such that

$$\begin{aligned}\left|\frac{g'(\zeta)}{g(\zeta)}\right| &\leqslant d\frac{T(4r, g)}{r} + \sum_{|a_k|<4r} \frac{2}{|\zeta - a_k|} \\ &\leqslant d\frac{T(4r, g)}{r} + \sum_{|a_k|<4r} \frac{4}{|z - a_k|} \\ &\leqslant d\frac{T(4r, g)}{r} + 4d_4\frac{n(16r)}{r}(\log r)^4 \log n(16r) = o(1),\end{aligned}$$

where $n(r) = n(r, g) + n(r, 1/g)$. The first assertion of (3·4) now follows immediately on setting $\zeta = z + c$, while the second assertion follows on writing

$$\log \frac{g(z + c)}{g(z)} = \int_z^{z+c} \frac{g'(\zeta)}{g(\zeta)} d\zeta = o(1).$$

The example $g(z) = \sin z$ and the remark following the definition of an ε-set show that Lemma 3·3 fails for functions of order 1, since for any $r > 0$ there exists $c \in (0, \pi)$ such that $g(r + c)/g(r)$ is either 0 or ∞.

LEMMA 3·4. *Let f be a function transcendental and meromorphic in the plane of order less than* 1. *Let $h > 0$. Then there exists an ε-set E such that*

$$|f(z + c) - f(z) - cf'(z)| \leqslant |c|^2 \frac{|f''(z)|}{2}(1 + o(1)) \quad \text{as } z \to \infty \text{ in } \mathbb{C} \setminus E, \tag{3·6}$$

uniformly in c for $|c| \leqslant h$.

Proof. Apply Lemma 3·3, with $g = f''$. This gives an ε-set E satisfying the second assertion of (3·4) with $g = f''$, and for large z not in E there are no zeros or poles of f'' in $|\zeta - z| \leqslant h$.

Let z be large, not in E. Then (3·4) gives, for $|u| \leqslant h$,

$$|f'(z + u) - f'(z)| \leqslant \int_z^{z+u} |f''(\zeta)|\,|d\zeta| \leqslant |uf''(z)|(1 + o(1))$$

and so, for $|c| \leqslant h$,

$$|f(z + c) - f(z) - cf'(z)| = \left|\int_0^c (f'(z + u) - f'(z))du\right| \leqslant (1 + o(1))|f''(z)| \int_0^{|c|} t\,dt,$$

from which (3·6) follows at once.

LEMMA 3·5. *Let f and h be as in Lemma* 3·4. *Then there exists an ε-set E' such that*

$$f(z + c) - f(z) = cf'(z)(1 + o(1)) \quad \text{as } z \to \infty \text{ in } \mathbb{C} \setminus E', \tag{3·7}$$

uniformly in c for $|c| \leqslant h$.

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Proof. Lemma 3·5 follows immediately from Lemma 3·4, it being only necessary to adjoin to the ε-set E of Lemma 3·4 an ε-set E'' outside which $f''(z)/f'(z) \to 0$, which is possible by Lemma 3·3.

For functions of lower order less than 1 the condition (3·2) may fail, but the following weaker assertion will suffice for subsequent application.

LEMMA 3·6. *Let f be a function transcendental and meromorphic in the plane of lower order $\lambda(f) < \lambda < 1$. Then there exist arbitrarily large R with the following properties. First,*

$$T(32R, f') < R^{\lambda}. \tag{3·8}$$

Second, there exists a set $J_R \subseteq [R/2, R]$ of linear measure $(1 - o(1))R/2$ such that, for $r \in J_R$,

$$f(z+1) - f(z) \sim f'(z) \quad \text{on} \quad |z| = r. \tag{3·9}$$

Proof. Let (a_k) be the sequence of all zeros and poles of f', with repetition according to multiplicity, and let $n(r)$ be the counting function of the a_k as in Lemma 3·1. Let E be the ε-set arising from Lemma 3·1, with the choice $\alpha = 4$.

It is clear from the hypotheses that there exist arbitrarily large R satisfying (3·8). For such R let E_R be the union of discs given by

$$E_R = \bigcup_{|a_k| \leqslant 4R} B(a_k, 2). \tag{3·10}$$

Then by (3·8) the discs of E_R have sum of radii $o(R)$. Combining this observation with the remark following the definition of an ε-set shows that there exists a subset J_R of $[R/2, R]$, of measure $(1 - o(1))R/2$, such that for $r \in J_R$ the circle $S(0, r)$ does not meet $E \cup E_R$.

Let $|z| = r \in J_R$ and let $|\zeta - z| \leqslant 1$. Then $|\zeta| \leqslant 2r$ and $|\zeta - a_k| \geqslant |z - a_k|/2$ for $|a_k| \leqslant 4R$, by (3·10). Thus Lemma 3·2 with $\beta = 2$, (3·1) and (3·8) give, for some positive constants c_j,

$$\begin{aligned}\left|\frac{f''(\zeta)}{f'(\zeta)}\right| &\leqslant c_1 \frac{T(4r, f')}{r} + \sum_{|a_k|<4r} \frac{2}{|\zeta - a_k|} \\ &\leqslant c_1 \frac{T(4r, f')}{r} + \sum_{|a_k|<4r} \frac{4}{|z - a_k|} \\ &\leqslant c_1 \frac{T(4r, f')}{r} + c_2 \frac{n(16r)}{r} (\log r)^4 \log n(16r) \\ &= o(1). \end{aligned} \tag{3·11}$$

For $|\zeta - z| \leqslant 1$ and $|z| = r \in J_R$, integration of (3·11) now leads to

$$f'(\zeta) \sim f'(z), \quad f(z+1) - f(z) = \int_z^{z+1} f'(\zeta) d\zeta = \int_z^{z+1} f'(z)(1 + o(1))\, d\zeta \sim f'(z)$$

which gives (3·9).

Remark. The recent papers [**5**] and [**10**] include independently obtained estimates for the proximity function $m(r, g(z+c)/g(z))$, when g is a meromorphic function of finite order. Applications of these estimates appear in [**5**, **10**, **11**]. The paper [**5**], of which the authors became aware after writing this paper, also contains pointwise estimates for the modulus

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$|g(z+c)/g(z)|$ outside an ε-set, obtained via the Poisson-Jensen formula and valid for meromorphic g of arbitrary growth. However for the applications of the present paper it is necessary to show as in (3·4) that the function $g(z+c)/g(z)$ itself, rather than just its modulus, tends to 1 outside an ε-set.

4. *Higher differences*

The aim of this section is to prove an asymptotic formula for the higher differences $\Delta^n f$, for $n \geqslant 2$, when f is a transcendental meromorphic function in the plane of order less than 1. It will be convenient to write

$$g_n(z) = \Delta^n f(z), \quad n \in \mathbb{N}. \tag{4·1}$$

LEMMA 4·1. *With the notation* (1·1) *and* (4·1),

$$g_n'(z) = (\Delta^n f')(z), \quad n \in \mathbb{N}. \tag{4·2}$$

Proof. The relation (4·2) for $n = 1$ follows immediately on writing

$$(\Delta f')(z) = f'(z+1) - f'(z) = g_1'(z).$$

Assume now that $m \in \mathbb{N}$ and that (4·2) is true for $1 \leqslant n \leqslant m$. Then (1·1) gives

$$\begin{aligned}(\Delta^{m+1} f')(z) &= (\Delta^m f')(z+1) - (\Delta^m f')(z)\\ &= g_m'(z+1) - g_m'(z)\\ &= (\Delta g_m')(z) = (\Delta g_m)'(z) = (\Delta^{m+1} f)'(z).\end{aligned}$$

LEMMA 4·2. *Let $n \in \mathbb{N}$. Let f be transcendental and meromorphic of order less than* 1 *in the plane. Then there exists an ε-set E_n such that*

$$\Delta^n f(z) \sim f^{(n)}(z) \quad \text{as } z \longrightarrow \infty \text{ in } \mathbb{C} \setminus E_n. \tag{4·3}$$

Proof. For $n = 1$ the conclusion (4·3) follows at once from (1·1) and Lemma 3·5. Assume now that $n \in \mathbb{N}$ and that the lemma has been proved for n. Then g_n is a transcendental meromorphic function of order less than 1, by Lemma 2·1, and so there exists an ε-set F_0 such that

$$\Delta^{n+1} f(z) = (\Delta g_n)(z) \sim g_n'(z) \quad \text{as } z \longrightarrow \infty \text{ in } \mathbb{C} \setminus F_0.$$

Since f' also has order less than 1 the induction hypothesis gives an ε-set F_n such that

$$(\Delta^n f')(z) \sim f^{(n+1)}(z) \quad \text{as } z \longrightarrow \infty \text{ in } \mathbb{C} \setminus F_n.$$

But $E_{n+1} = F_0 \cup F_n$ is an ε-set and so the result for $n+1$ follows using (4·2).

5. *Proof of Theorem* 1·2

Let $n \in \mathbb{N}$ and let f be a transcendental entire function of order less than $1/2$. Let G be defined by (1·3). Then Lemma 4·2 gives an ε-set E_n such that (4·3) holds. Since f is transcendental entire the Wiman–Valiron theory [**14**, **22**] may be applied to f. Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

be the Maclaurin series of f. For $r > 0$ the maximum term $\mu(r)$ and central index $N(r)$ are defined by

$$\mu(r) = \max\{|a_k| r^k : k = 0, 1, 2, \ldots\}, \quad N(r) = \max\{k : |a_k| r^k = \mu(r)\}.$$

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The Wiman–Valiron theory [**14**, **22**] then gives a subset F_0 of $[1, \infty)$ of finite logarithmic measure such that, for large r not in F_0,

$$\frac{f^{(n)}(z)}{f(z)} \sim \frac{N(r)^n}{z^n} \quad \text{for all } z \text{ satisfying} \quad |z| = r, \quad |f(z)| = M(r, f). \tag{5·1}$$

By the remark following the definition of an ε-set it may be assumed that for large r not in F_0 the circle $S(0, r)$ does not meet the ε-set E_n of (4·3). Combining (4·3) and (5·1) then gives, for large r not in F_0,

$$z^n G(z) \sim N(r)^n \quad \text{for all } z \text{ satisfying} \quad |z| = r, \quad |f(z)| = M(r, f). \tag{5·2}$$

Suppose first that f has order of growth $\rho < 1/n$. Standard results from the Wiman–Valiron theory [**14**, **22**] imply that

$$\lim_{r\to\infty} N(r) = \infty \quad \text{and} \quad \limsup_{r\to\infty} \frac{\log N(r)}{\log r} = \rho, \tag{5·3}$$

so that

$$N(r)^n \to \infty \quad \text{and} \quad z^{-1}N(r)^n \to 0 \quad \text{as} \quad |z| = r \to \infty.$$

Hence it follows using (5·2) that there exists a sequence (z_j) tending to infinity such that $z_j^n G(z_j) \to \infty$ but $z_j^{n-1} G(z_j) \to 0$. Thus G cannot be a rational function.

Assume henceforth that G is transcendental, but has finitely many zeros. Then $1/G$ is transcendental of order less than $1/2$ with finitely many poles. The classical cos $\pi\rho$ theorem [**15**, p. 331] now gives a positive constant c_1 and a subset F_1 of $[1, \infty)$, of positive lower logarithmic density, such that for large $r \in F_1$,

$$\log |G(z)| < -c_1 T(r, G) < -n \log r \quad \text{on } |z| = r, \tag{5·4}$$

using the fact that G is transcendental. Since F_0 has finite logarithmic measure it may be assumed without loss of generality that $F_1 \cap F_0$ is empty, so that (5·2) holds for large $r \in F_1$. But (5·3) shows that (5·2) and (5·4) are incompatible, and this contradiction completes the proof of Theorem 1·2.

6. *Proof of Theorem* 1·3

Let f be a transcendental entire function of order $\rho < 1$, let G be defined by (1·5), and assume that G has finitely many zeros. By Lemma 2·1, the function G is transcendental.

Since $\rho(f) < 1$, Lemmas 3·3 and 3·5 give a set $G_0 \subseteq [1, \infty)$ of logarithmic density 1 such that

$$G(z) \sim \frac{f'(z)}{f(z)} = o(1) \quad \text{as } z \longrightarrow \infty \text{ with } |z| \in G_0. \tag{6·1}$$

Since G has finitely many zeros by assumption there exists a rational function R_0 with $R_0(\infty)$ finite such that

$$H(z) = \frac{1}{2z}\left(\frac{1}{G(z)} - R_0(z)\right) \tag{6·2}$$

is entire and transcendental, of order at most ρ, and there exists $r_1 > 0$ such that

$$|G(z)| < \frac{1}{|z|} \quad \text{for } |z| \geqslant r_1, \ |H(z)| > 1. \tag{6·3}$$

Apply Lemma 2·3 to H, and suppose first that conclusion (i) of that lemma holds, so that

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there exists a set $J_\delta \subseteq [1, \infty)$, of positive upper logarithmic density δ, on which the minimum modulus $m_0(r, H)$ exceeds 1, where $m_0(r, H)$ is defined by (2·6). There is no loss of generality in assuming that $J_\delta \subseteq G_0$, where G_0 is as in (6·1). Let γ be small and positive, and apply Lemma 2·2 to f. Then (6·1) and (6·3) show that the set U_r as defined in (2·3) is empty for large $r \in J_\delta$, so that with E_M as defined in Lemma 2·2 the intersection $E_M \cap J_\delta$ is bounded, for any choice of $M > 3$, by (2·5). Since M may be chosen so large that $3/M < \delta$, this contradicts (2·4).

Assume henceforth that H satisfies conclusion (ii) of Lemma 2·3. Let $M > 3$ and again let γ be small and positive, and define τ by

$$2\pi\tau = \left(\frac{1-\gamma}{7M(\rho+1)}\right)^2. \tag{6·4}$$

Let F_τ and $\theta(r)$ be as in Lemma 2·3, and again apply Lemma 2·2 to f. This gives a subset E_M of $[1, \infty)$ satisfying (2·4) and (2·5), and there is no loss of generality in assuming that $E_M \subseteq G_0$, where G_0 is as in (6·1). But (2·3), (2·7), (6·1), (6·3), (6·4) and the definition of $\theta(r)$ show that the intersection $E_M \cap F_\tau$ is bounded, which by (2·4) and (2·8) forces

$$1 - 2\rho(1-\tau) \leqslant \frac{3\tau}{M}$$

and hence

$$2\rho - 1 \geqslant \frac{\tau}{1-\tau}\left(1 - \frac{3}{M}\right) \geqslant \tau\left(1 - \frac{3}{M}\right).$$

Since $\rho < 1$ and γ is small, it follows using (6·4) that ρ must satisfy

$$2\rho - 1 \geqslant \frac{1}{2\pi}\left(\frac{1}{14M}\right)^2\left(1 - \frac{3}{M}\right) = h(M).$$

In the last inequality the right-hand side $h(M)$ has a maximum relative to the interval $(3, \infty)$ at $M = 9/2$, with $h(9/2) = 1/23814\pi$.

7. *Proof of Theorem* 1·4

Let f and c be as in the hypotheses, and let $\lambda(f) < \lambda < 1$. There is no loss of generality in assuming that $c = 1$. By the hypotheses there exist arbitrarily large R satisfying the conclusions of Lemma 3·6. For such R let

$$E_R = \{r \in [R/2, R] : n(r, f) = n(r-1, f)\}. \tag{7·1}$$

Then E_R has linear measure

$$m(E_R) \geqslant (1 - o(1))R/2. \tag{7·2}$$

To see this, note that there are at most $o(R)$ points $s_k \in [R/4, R]$ at which $n(t, f)$ is discontinuous, by (3·8). But if $r \in [R/2, R]$ is such that $n(r) > n(r-1)$ then $r \in [s_k, s_k + 1]$ for some k. This proves (7·2).

Since R satisfies the conclusions of Lemma 3·6, it follows using (7·2) that there exists r in $E_R \cap J_R$ such that (3·9) holds, and such that $f(z)$, $f(z+1)$ and $f'(z)$ have no zeros or poles on $|z| = r$. But by the hypotheses there exists $r_0 > 0$, independent of R and r, such that if f has a pole of multiplicity m at z_0 and $r_0 \leqslant |z_0| \leqslant r - 1$ then $g(z)$ has poles at z_0

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and $z_0 - 1$ of multiplicity m. Thus (3·9) and Rouché's theorem give

$$\begin{aligned} n(r, 1/g) &= n(r, 1/f') - n(r, f') + n(r, g) \\ &\geqslant n(r, 1/f') - n(r, f') + 2n(r-1, f) - O(1) \\ &= n(r, 1/f') - n(r, f') + 2n(r, f) - O(1) \\ &\geqslant n(r, 1/f') - O(1), \end{aligned}$$

and the result now follows since f' has infinitely many zeros by Theorem 1·1.

8. *Proof of Theorem* 1·5

Let $n_1, n_2, \ldots$ be positive integers with

$$\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = \infty. \tag{8·1}$$

Let

$$H(z) = \prod_{k=1}^{\infty} \left(1 + \frac{z}{A_k}\right), \quad A_k = 4n_k^4. \tag{8·2}$$

Then (8·1) shows that H is an entire function with $T(r, H) = O(\log r)^2$ so that by Theorem 3·1 there exists an ε-set E such that

$$\log |H(z)| \geqslant (1 - o(1))T(|z|, H) \quad \text{as } z \to \infty \text{ with } z \notin E. \tag{8·3}$$

Let k be large and set $H_k(z) = H(z)/(z + A_k)$. Then (8·1) and (8·3) imply that there exists $d_k \in (2, 3)$ such that

$$\lim_{k\to\infty} \frac{\log m_k}{\log A_k} = +\infty, \quad m_k = \min\{|H_k(z)| : |z| = d_k^{\pm 1} A_k\}.$$

In particular, since $H_k(-A_k) = H'(-A_k)$ and H_k has no zeros in $d_k^{-1} A_k \leqslant |z| \leqslant d_k A_k$ it follows using (8·1) and (8·2) and the minimum principle that

$$\sum_{k=1}^{\infty} \left| \frac{A_k^N}{H'(-A_k)} \right| < \infty \tag{8·4}$$

for any choice of $N > 0$.

Let

$$h(z) = \frac{H(z^4)}{z}. \tag{8·5}$$

Then it follows from (8·2) that h has zeros at the points $\pm n_k \pm i n_k$. Also if β is a zero of h then so is $i\beta$ and

$$h'(\beta) = 4\beta^2 H'(\beta^4) \neq 0, \quad h'(i\beta) = -h'(\beta). \tag{8·6}$$

By (8·2), (8·4), (8·5) and (8·6),

$$\sum_{k=1}^{\infty} n_k |c_k| < \infty, \quad c_k = \frac{1}{h'(-n_k + i n_k)}. \tag{8·7}$$

Set

$$g(z) = \sum_{k=1}^{\infty} c_k \left[\frac{1}{z + n_k - i n_k} - \frac{1}{z - n_k - i n_k} - \frac{1}{z + n_k + i n_k} + \frac{1}{z - n_k + i n_k} \right]. \tag{8·8}$$

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The series in (8·8) converges absolutely and uniformly in each bounded region of the plane, by (8·7), and the function g is meromorphic in the plane. By (8·6) and (8·7), the function $G = g - 1/h$ is entire. But (8·1), (8·3) and (8·7) imply that there exist $c > 0$ and $n_k' \in (4n_k, 8n_k)$ with

$$M(n_k', G) \leqslant o(1) + M(n_k', g) \leqslant o(1) + c \sum_{j=1}^{\infty} \frac{|c_j|}{n_k} = o(1),$$

and so $G \equiv 0$ and g has one zero, by (8·5).

Finally, set

$$f(z) = \sum_{k=1}^{\infty} c_k \left[\sum_{j=-n_k}^{n_k-1} \frac{1}{z + j - in_k} - \sum_{j=-n_k}^{n_k-1} \frac{1}{z + j + in_k} \right], \tag{8·9}$$

the series again convergent by (8·7). Then f and g satisfy (1·8). A result of Keldysh [**8**, p. 327] (see also [**6**, **7**]) gives

$$m(r, f) + m(r, g) = o(1) \quad \text{as} \quad r \longrightarrow \infty. \tag{8·10}$$

But (8·1), (8·8) and (8·9) give

$$n(r, f) = O(n_k) = O(r) \quad \text{and} \quad n(r, g) = O(k) \quad \text{for} \quad n_k \leqslant r < n_{k+1}. \tag{8·11}$$

Hence (1·6) follows using (8·10), and since the sequence (n_k) may be chosen to grow arbitrarily fast in (8·1), applying (8·10) and (8·11) again gives (1·7) and (1·9).

9. *Proof of Theorem* 1·6

Assume that f, g and G are as in the hypotheses, but that G has finitely many zeros. By Lemma 2·1, the function G is transcendental, and $T(r, G) = O(\log r)^2$ as $r \to \infty$, using (1·10). By Lemma 3·5 and Theorem 3·1, there exists an ε-set E such that

$$G(z) \sim \frac{f'(z)}{f(z)} \quad \text{and} \quad \log |G(z)| \leqslant (-1 + o(1))T(r, G) \quad \text{as } z \longrightarrow \infty \text{ in } \mathbb{C} \setminus E. \tag{9·1}$$

Choose $t \in [0, 2\pi]$ such that the ray $\arg z = t$ has bounded intersection with E. Let r_0 be large and positive. Integrating f'/f using (9·1) along the ray $z = re^{it}, r \geqslant r_0$, and then around circles $S(0, r)$ which do not intersect E then shows that there exist a constant $b \in \mathbb{C} \setminus \{0\}$ and a set $E_0 \subseteq [1, \infty)$ of finite logarithmic measure such that

$$f(z) = b + o(1) \quad \text{for } |z| = r \in [1, \infty) \setminus E_0. \tag{9·2}$$

Set

$$F(z) = f(z) - b, \quad H(z) = \frac{\Delta F(z)}{F(z)} = \frac{\Delta f(z)}{f(z) - b}, \tag{9·3}$$

and assume that H has finitely many zeros. Then the same reasoning as above shows that H is transcendental and that there exists a non-zero constant d such that $F(z) \sim d$ for $|z| = r$ large and lying outside a set of finite logarithmic measure. This contradicts (9·2), and so H must have infinitely many zeros.

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Let z_0 be a zero of H with $|z_0|$ large. Then z_0 is not a pole of f, because otherwise the formula

$$G(z) = \frac{\Delta f(z)}{f(z)} = H(z)\frac{f(z)-b}{f(z)}$$

shows that z_0 is a zero of G, which contradicts the assumption that G has finitely many zeros. It now follows from (9·3) that z_0 is a zero of Δf, and Theorem 1·6 is proved.

Remarks. First, it seems highly unlikely that the hypothesis (1·10) in Theorem 1·6 is sharp. However the ε-set E' arising from Lemma 3·5 may be reasonably large, at least locally, so that for f of larger growth than (1·10) difficulties may arise in integrating f'/f on the set where G is small.

Second, if f is a transcendental meromorphic function in the plane satisfying (1·10) and

$$\delta(\infty, f) + \delta(0, f) > 0,$$

and if G is defined by (1·11), then G must have infinitely many zeros, since otherwise Theorem 3·1 may be applied to f or $1/f$, contradicting (9·2).

Acknowledgements. The authors are grateful to the referee for the observation above and for a number of helpful suggestions to improve the readability of the paper.

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