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Elastic wave fields generated by scalar wave functions

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Abstract. In this paper we obtain a representation of elastic wave fields (i.e. solutions of the equations of motion of classical elastokinetics) in terms of three functions satisfying scalar wave equations. Although the form of the representation implies no restriction upon the choice of coordinate system or upon the shape of the elastic body, it is found that the result can only be applied advantageously to initial-boundary-value problems having spherical polar coordinates as a natural frame of reference. The representation is shown to be complete in the sense that every (sufficiently smooth) elastic wave field in a homogeneous, isotropic body bounded by two concentric spheres can be expressed in the given form.

Under conditions of axial symmetry the representation generates wave fields which can be decomposed into poloidal and toroidal constituents, the former arising from two scalar wave functions and comprising both dilatational and rotational waves, and the latter being associated with a single scalar wave function and a state of pure shear of the elastic solid. Finally, the representation is used to obtain a formal solution describing the elastic pulse generated in an infinite body by the application of time-dependent tractions to the surface of a spherical cavity.

1. *Introduction.* One of the most widely used procedures for solving initial-boundary-value problems in classical elastokinetics and boundary-value problems in the theory of steady (i.e. time-harmonic) elastic vibrations involves the introduction of so-called potential functions which generate displacement fields satisfying the equations of motion and themselves satisfy partial differential equations for which solutions are known. An excellent account of the theory underlying potential function methods has been given by Sternberg (1); see also (2). In the present paper we consider solutions of the equations of classical elastokinetics which are generated by functions satisfying scalar wave equations, and we refer to such functions as *scalar wave functions* rather than potentials.

The equations governing small amplitude motions of a homogeneous, isotropic elastic body can be stated in the vectorial form

$$v_P^2 \operatorname{grad} \operatorname{div} \mathbf{u} - v_S^2 \operatorname{curl}^2 \mathbf{u} = \partial^2 \mathbf{u}/\partial t^2. \tag{1·1}$$

Here t denotes time, $\mathbf{u}$ the displacement vector and v_P, v_S the speeds of propagation of

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dilatational and rotational body waves, respectively; for simplicity it is supposed that no body forces act. We make use of the notation

$$\Box_P^2 = \nabla^2 - \frac{1}{v_P^2}\frac{\partial^2}{\partial t^2}, \quad \Box_S^2 = \nabla^2 - \frac{1}{v_S^2}\frac{\partial^2}{\partial t^2} \tag{1·2}$$

for the wave operators with characteristic propagation speeds v_P, v_S.

Let Φ be a scalar wave function satisfying

$$\Box_P^2 \Phi = 0, \tag{1·3}$$

and let $\mathbf{A}$ be a vector function satisfying the equations

$$\Box_S^2 \mathbf{A} = \mathbf{0}, \quad \operatorname{div} \mathbf{A} = 0. \tag{1·4}$$

Then, provided that Φ and $\mathbf{A}$ possess sufficient numbers of continuous derivatives, it may easily be verified that the vector

$$\mathbf{u} = \operatorname{grad} \Phi + \mathbf{A} \tag{1·5}$$

satisfies equation (1·1). This result was first obtained by Poisson (see (1), pages 34–35) who subsequently proved it to be complete in the sense that every sufficiently smooth solution of equation (1·1) admits the representation (1·5) where Φ and $\mathbf{A}$ can be chosen to satisfy equations (1·3) and (1·4).

Now, except when referred to rectangular Cartesian coordinates, the vector wave equation is much more difficult to solve than its scalar counterpart. A way of surmounting this difficulty is to derive expressions for vector wave functions in terms of scalar wave functions and cases in which this can be accomplished in a convenient form have been listed by Morse and Feshbach ((3), pages 1759–1767). The analysis given by these authors is based upon the vector Helmholtz equation, but the results apply equally to the more general vector wave equation $(1·4)_1$ and, with appropriate modifications, can be summarized as follows.

Any solution of the vector wave equation can be expressed as the sum of irrotational and solenoidal parts: in view of equation $(1·4)_2$, we are concerned here only with the latter. Let (ξ_1, ξ_2, ξ_3) denote a system of orthogonal curvilinear coordinates, normally defined in such a manner that the boundaries of the region of space on which the solution of equation $(1·4)_1$ is required belong to one family of coordinate surfaces, say $\xi_1 = \text{constant}$; let $\mathbf{a}_1$ be the unit vector acting in the direction of ξ_1 increasing. Then, subject to certain conditions on the coordinate system, there is a function $w(\xi_1)$ such that

$$\mathbf{A}_1 = -\operatorname{curl}(w\Omega\mathbf{a}_1) \quad \text{and} \quad \mathbf{A}_2 = -\operatorname{curl}^2(w\Psi\mathbf{a}_1) \tag{1·6}$$

are solutions of (1·4) whenever Ψ and Ω are scalar wave functions satisfying

$$\Box_S^2 \Psi = 0, \quad \Box_S^2 \Omega = 0. \tag{1·7}$$

Furthermore, the vectors $\mathbf{A}_1$ and $\operatorname{curl} \mathbf{A}_2$ are directed tangentially to the coordinate surfaces $\xi_1 = \text{constant}$, a property which indicates that the expressions (1·6) are advantageous for satisfying boundary conditions.

One of the coordinate systems which meets the requirements referred to above is the spherical polar system. Let (r, θ, ϕ) denote such a system with origin O and let $\mathbf{x}$

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be the position vector of a typical point P relative to O. In this instance $w = r$ and it follows from (1·6) that the vectors

$$\mathbf{A}_1 = -\operatorname{curl}(\mathbf{x}\Omega) \quad \text{and} \quad \mathbf{A}_2 = -\operatorname{curl}^2(\mathbf{x}\Psi), \tag{1·8}$$

where Ψ and Ω again satisfy (1·7), are solutions of the vector wave equation $(1·4)_1$ and also fulfil the condition $(1·4)_2$. Combining these results with the Poisson representation (1·5), we conclude that if Φ, Ψ and Ω are sufficiently smooth scalar wave functions satisfying equations (1·3) and (1·7), then the vector field

$$\mathbf{u} = \operatorname{grad}\Phi - \operatorname{curl}^2(\mathbf{x}\Psi) - \operatorname{curl}(\mathbf{x}\Omega) \tag{1·9}$$

is a solution of the equation of motion (1·1).

It is with the representation (1·9) that we are mainly concerned in this paper. As stated it has an invariant form which implies no restriction upon the choice of coordinate system or upon the boundary of the elastic body. It is evident, however, from the analysis outlined above, that this representation of elastic wave fields is likely to be useful for solving boundary-value problems only in situations where the motion can conveniently be referred to spherical polar coordinates.

When the displacement field $\mathbf{u}$ is assumed to have harmonic time dependence, equation (1·9) may be regarded as a concise statement of a method, usually attributed to Lamb (see (4), pages 278–287), of constructing solutions describing vibrations of elastic bodies with spherical boundaries. Recently Einspruch, Witterholt and Truell (5) have derived this special case of (1·9) from the results of Morse and Feshbach and have applied it to the diffraction of plane rotational elastic waves by a spherical obstacle. In its general form, however, the representation (1·9) finds scant mention in the literature. It is implicit in work by Allen and Robinson (6) on the scattering of plane elastic pulses by a spherical cavity, and Hook (7) gives a somewhat restricted version of (1·9) before extending results of the form (1·6), referred to an orthogonal system of cylindrical coordinates, to certain types of inhomogeneous elastic solids (see also (8)).

In none of the work referred to above is the question of the completeness of the representation (1·9) considered, but in connexion with an investigation of self-sustaining flows of spherical bodies of conducting fluid, Backus ((9), pages 379–384) has proved that the sum of the vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ given by (1·8) provides a complete representation of solenoidal vector fields in a region bounded by two concentric spheres. In section 2 we make use of ideas from Backus's work in establishing a completeness theorem for the representation (1·9).

The remainder of the paper is concerned with questions of the effectiveness of this representation in solving particular problems. In section 3 we consider briefly the particular case in which the displacement field $\mathbf{u}$ is axially symmetric; the results of this section have been applied by the present authors to a study of the possible modes of vibration of a rigid sphere embedded in an infinite elastic solid (10, 11). Finally, we point out in section 4 that the solution of a problem discussed by Eringen (12) can be obtained in a particularly simple manner with the aid of (1·9).

2. *A completeness theorem.* First we dispose of some preliminaries. Let r_1, r_2 and t_1, t_2 be pairs of real numbers such that $0 < r_1 < r_2 < \infty$ and $0 \leqslant t_1 < t_2 < \infty$, and let

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I denote the time interval $t_1 \leqslant t \leqslant t_2$. The results obtained in this section refer, in the first instance, to a homogeneous, isotropic elastic body in the form of a spherical shell with centre O, inner radius r_1 and outer radius r_2. Referred to the system of spherical polar coordinates introduced in section 1 the points of the body thus correspond, in aggregate, to the set

$$S = \{(r, \theta, \phi): r_1 \leqslant r \leqslant r_2,\ 0 \leqslant \theta \leqslant \pi,\ 0 \leqslant \phi \leqslant 2\pi\}.$$

We shall also have occasion to refer to the set

$$R = \{(\theta, \phi): 0 \leqslant \theta \leqslant \pi,\ 0 \leqslant \phi \leqslant 2\pi\},$$

essentially the unit sphere with centre O, and the region of space-time $\mathscr{S} = S \times I$. When, in the sequel, it is stated that derivatives exist and differential equations are satisfied on the closed set S or $\mathscr{S}$, it is presumed that these properties hold on some open set including S or $\mathscr{S}$ (cf. (13), section 2·1). Finally, a function is said to be *Hölder continuous* on a certain domain when it satisfies a Hölder condition with a positive exponent $\alpha < 1$ (see (14), page 136).

Our main result is the following

THEOREM. *Let* $\mathbf{u}$ *be a vector field which is three times continuously differentiable on* $\mathscr{S}$ *and satisfies there equation* (1·1). *Then there are scalar wave functions* Φ, Ψ *and* Ω *satisfying equations* (1·3) *and* (1·7) *such that* $\mathbf{u}$ *admits the representation* (1·9) *on* $\mathscr{S}$.

In order to prove this theorem two lemmas are needed. The proofs given here are similar to, but simpler than, those of Backus ((9), pages 381–384).

LEMMA 1. *Let* $\mathbf{v}$ *be a vector field which is twice continuously differentiable on* S. *If, in addition,* $\mathbf{v}$ *satisfies on* S *the conditions*

$$\operatorname{div} \mathbf{v} = 0, \mathbf{x}.\quad \mathbf{v} = 0, \quad \mathbf{x}.\operatorname{curl} \mathbf{v} = 0, \tag{2·1}$$

then it vanishes there.

Proof. Set $\mathbf{w} = \mathbf{v} \sin\theta$. Then equations (2·1) imply that the components of $\mathbf{w}$ referred to the spherical polar coordinate system (r, θ, ϕ) satisfy the conditions

$$w_r = 0, \quad \Lambda^2 w_\theta = 0, \quad \Lambda^2 w_\phi = 0,$$

where the differential operator Λ^2 is defined by

$$\sin^2\theta \Lambda^2 = \sin\theta \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{\partial^2}{\partial\phi^2}. \tag{2·2}$$

(Apart from the factor r^{-2}, Λ^2 is the transverse part of the Laplacian operator.) We now assert that the partial differential equation $\Lambda^2 f = 0$ has no bounded solutions on R other than constants. Then since $\mathbf{v}$, and hence $\mathbf{w}$, is bounded on S it follows that $\mathbf{w}$ is a fixed vector. But if $\mathbf{w} \neq \mathbf{0}$, $\mathbf{v}$ is unbounded at $\theta = 0, \pi$, contrary to hypothesis. Hence the result.

It remains to justify the assertion. Let ρ be a 'dummy' radial variable forming with θ and ϕ a system of spherical polar coordinates relative to which the Laplacian operator is

$$\nabla^2 = \frac{1}{\rho^2}\left\{\frac{\partial}{\partial\rho}\left(\rho^2 \frac{\partial}{\partial\rho}\right) + \Lambda^2\right\}.$$

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Since f is independent of ρ it is harmonic on the entire 3-space spanned by the coordinate system (ρ, θ, ϕ). But f is also bounded on this space. Hence f is a constant ((15), page 40).

LEMMA 2. *Let* $\mathbf{A}$ *be a vector field possessing partial derivatives of orders up to two which are Hölder continuous on* S. *Then if, in addition,* $\operatorname{div}\mathbf{A} = 0$ *on* S, *there are scalars* Ψ *and* Ω *such that*

$$\mathbf{A} = -\operatorname{curl}^2(\mathbf{x}\Psi) - \operatorname{curl}(\mathbf{x}\Omega) \tag{2·3}$$

on S.

Proof. Let Ψ and Ω be solutions of the partial differential equations

$$\Lambda^2\Psi = \mathbf{x}.\mathbf{A}, \quad \Lambda^2\Omega = \mathbf{x}.\operatorname{curl}\mathbf{A}.$$

Since Λ^2 is an elliptic operator, such solutions exist and possess partial derivatives of orders up to four and three respectively which are Hölder continuous on S ((14), page 136). Define

$$\mathbf{v} = \mathbf{A} + \operatorname{curl}^2(\mathbf{x}\Psi) + \operatorname{curl}(\mathbf{x}\Omega).$$

Then, in view of the identities

$$-\mathbf{x}.\operatorname{curl}^2(\mathbf{x}f) = \Lambda^2 f, \quad \operatorname{curl}(\mathbf{x}\Omega) = -\mathbf{x}\wedge\operatorname{grad}\Omega,$$

$$\operatorname{curl}^3(\mathbf{x}\Psi) = \mathbf{x}\wedge\operatorname{grad}(\nabla^2\Psi), \tag{2·4}$$

$\mathbf{v}$ satisfies the three conditions (2·1) as well as the hypotheses of Lemma 1. Hence $\mathbf{v}$ vanishes on S and the proof is complete.

We now proceed to the proof of the theorem. This falls into two parts, (a) and (b), the first of which is a completeness proof for the Poisson representation (1·5). The version given here is a condensed form of Poisson's own proof as reproduced by Todhunter and Pearson ((16), pages 273–276). To avoid repetition we remark that throughout the proof changes in the order of partial differentiation are justified by the smoothness of the functions involved.

(a) Define

$$\Phi = v_P^2\int_{t'}^{t}\int_{t'}^{\eta}(\operatorname{div}\mathbf{u})_{t=\xi}\,d\xi\,d\eta + \alpha + (t-t')\beta, \quad \mathbf{A} = \mathbf{u} - \operatorname{grad}\Phi, \tag{2·5}$$

where $t' \epsilon I$ and α, β are functions of r, θ and ϕ satisfying the equations

$$\nabla^2\alpha = (\operatorname{div}\mathbf{u})_{t=t'}, \quad \nabla^2\beta = (\partial/\partial t\operatorname{div}\mathbf{u})_{t=t'}.$$

The theory of elliptic partial differential equations shows that such functions exist.

Now, taking the divergence of each term in equation (1·1), we have

$$v_P^2\nabla^2(\operatorname{div}\mathbf{u}) = \partial^2/\partial t^2(\operatorname{div}\mathbf{u}).$$

Hence
$$\nabla^2\Phi = \int_{t'}^{t}\int_{t'}^{\eta}\left(\frac{\partial^2}{\partial t^2}\operatorname{div}\mathbf{u}\right)_{t=\xi}d\xi\,d\eta + (\operatorname{div}\mathbf{u})_{t=t'} + (t-t')\left(\frac{\partial}{\partial t}\operatorname{div}\mathbf{u}\right)_{t=t'}$$

$$= \operatorname{div}\mathbf{u}. \tag{2·6}$$

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and this result, in conjunction with equations (2·5) and (1·1), gives

$$\Box_P^2 \Phi = \operatorname{div} \mathbf{u} - \frac{1}{v_P^2} \frac{\partial^2 \Phi}{\partial t^2} = 0,$$

$$\operatorname{div} \mathbf{A} = \operatorname{div} \mathbf{u} - \nabla^2 \Phi = 0,$$

$$\Box_S^2 \mathbf{A} = -\operatorname{curl}^2 \mathbf{A} - \frac{1}{v_S^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\operatorname{curl}^2 \mathbf{u} - \frac{1}{v_S^2} \frac{\partial^2 \mathbf{u}}{\partial t^2} + \frac{v_P^2}{v_S^2} \operatorname{grad} \operatorname{div} \mathbf{u}$$
$$= \mathbf{0}.$$

We have thus proved that the Poisson representation (1·5) is complete.

(*b*) The smoothness hypothesis on $\mathbf{u}$ implies that the spatial derivatives of $\mathbf{u}$ of orders up to two are Hölder continuous on S. It then follows from equations (2·6) and $(2\cdot5)_1$ in turn that the scalar and vector wave functions Φ and $\mathbf{A}$ have spatial partial derivatives of orders up to three and two respectively which are Hölder continuous on S. $\mathbf{A}$ is therefore a solenoidal vector field which meets the hypotheses of Lemma 2 for all $t \in I$. Accordingly there are scalars Ψ' and Ω' such that

$$\mathbf{A} = -\operatorname{curl}^2(\mathbf{x}\Psi') - \operatorname{curl}(\mathbf{x}\Omega')$$

on $\mathscr{S}$. On substituting this expression into the vector wave equation $(1\cdot4)_1$ we obtain

$$\operatorname{curl}^2(\mathbf{x}\Box_S^2 \Psi') + \operatorname{curl}(\mathbf{x}\Box_S^2 \Omega') = \mathbf{0},$$

and recalling the identities (2·4) there follow

$$\Lambda^2(\Box_S^2 \Psi') = 0, \quad \Lambda^2(\Box_S^2 \Omega') = 0.$$

Thus $\Box_S^2 \Psi' = p, \quad \Box_S^2 \Omega' = q, \quad$ where $\quad \Lambda^2 p = 0, \quad \Lambda^2 q = 0.$

Define

$$P(r,\theta,\phi,t) = \frac{v_S}{2r} \int_{t'}^{t} \int_{r-v_S(t-\eta)}^{r+v_S(t-\eta)} \varpi p(\varpi,\theta,\phi,\eta)\, d\varpi\, d\eta,$$

$$Q(r,\theta,\phi,t) = \frac{v_S}{2r} \int_{t'}^{t} \int_{r-v_S(t-\eta)}^{r+v_S(t-\eta)} \varpi q(\varpi,\theta,\phi,\eta)\, d\varpi\, d\eta,$$

where $t' \in I$ and, in case the range of integration with respect to ϖ is not included in the interval $r_1 \leqslant r \leqslant r_2$, we suppose p and q to be smoothly continued outside $\mathscr{S}$. Then

$$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial P}{\partial r}\right) - \frac{1}{v_S^2} \frac{\partial^2 P}{\partial t^2} = -p, \quad \Lambda^2 P = 0,$$

so that $\Box_S^2 P = -p$, and similarly $\Box_S^2 Q = -q$.

Now set

$$\Psi = \Psi' + P, \quad \Omega = \Omega' + Q.$$

Then Ψ and Ω satisfy equations (1·7), and

$$\mathbf{A} = -\operatorname{curl}^2(\mathbf{x}\Psi) - \operatorname{curl}(\mathbf{x}\Omega) + \mathbf{A}', \tag{2·7}$$

where

$$\mathbf{A}' = \operatorname{curl}^2(\mathbf{x}P) + \operatorname{curl}(\mathbf{x}Q). \tag{2·8}$$

Also $$\Lambda^2\Psi = \Lambda^2\Psi' + \Lambda^2 P = \mathbf{x}.\mathbf{A}, \quad \Lambda^2\Omega = \Lambda^2\Omega' + \Lambda^2 Q = \mathbf{x}.\operatorname{curl}\mathbf{A},$$

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and it follows, as in the proof of Lemma 2, that, for all $t \in I$, Ψ and Ω possess spatial derivatives of orders up to four and three respectively which are Hölder continuous on S. Thus, from equation (2·7), the spatial derivatives of $\mathbf{A}'$ of orders up to two have this property and, from equation (2·8),

$$\operatorname{div} \mathbf{A}' = 0, \quad \mathbf{x} \cdot \mathbf{A}' = -\Lambda^2 P = 0, \quad \mathbf{x} \cdot \operatorname{curl} \mathbf{A}' = -\Lambda^2 Q = 0.$$

$\mathbf{A}'$ therefore satisfies the hypotheses of Lemma 1 for all $t \in I$ and so vanishes on $\mathscr{S}$.

Combining equations (1·5) and (2·7) with $\mathbf{A}' = \mathbf{0}$ we now arrive at the representation (1·9), thus establishing the theorem.

Before turning to particular properties of the wave function representation (1·9), we examine briefly the possibility of relaxing the conditions under which the completeness theorem has been proved. These conditions are of two types: restrictions on the class of displacement fields; and restrictions on the region $\mathscr{S}$ of space-time.

It seems likely that, so long as the theorem is proved by the methods of classical analysis, the smoothness condition imposed on $\mathbf{u}$ is minimal. If, however, the problem of the completeness of (1·9) is set within the theory of generalized functions, it may be possible to prove a theorem covering a much wider class of elastic wave fields. We have shown elsewhere (10) that the representation (1·9) has interesting applications in which even the first partial derivatives of the displacement field fail to be continuous everywhere. However, the fact that the completeness theorem is inapplicable in such situations does not prevent (1·9) being used to *construct* a displacement field which is then shown to solve the problem.

With regard to the region of space-time on which the theorem holds, we first note that the restrictions $t_1 \geqslant 0$, $t_2 < \infty$ are not essential so long as $\mathbf{u}$ is bounded on $\mathscr{S}$. In particular, the result holds for all t when $\mathbf{u}$ is time-harmonic. The removal of the restriction $r_1 > 0$, thus extending the theorem to a closed ball of radius r_2, requires an investigation of the behaviour of the rotational wave functions Ψ and Ω as $r \to 0$. Relevant results have been obtained by Backus ((9), pages 382–383). The extension of the theorem to an infinite region bounded internally by a sphere ($r_2 \to \infty$) again requires a boundedness condition on $\mathbf{u}$ and in addition the formulation of regularity conditions in the limit $r \to \infty$.

3. *Axially symmetric wave fields.* In this section we consider some properties of elastic wave fields which are symmetrical about a fixed axis L. Assuming that the displacement field admits the representation (1·9), its components, referred to a system of spherical polar coordinates (r, θ, ϕ) having L as base line, are given by

$$\left.\begin{aligned} u_r &= \frac{\partial \Phi}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Psi}{\partial \theta}\right), \\ u_\theta &= \frac{1}{r}\frac{\partial \Phi}{\partial \theta} - \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \Psi}{\partial \theta}\right), \\ u_\phi &= \frac{\partial \Omega}{\partial \theta}. \end{aligned}\right\} \tag{3·1}$$

We notice immediately that, under conditions of axial symmetry, the pair Φ, Ψ and the single wave function Ω generate displacement fields which are orthogonal.

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We apply to these fields a terminology used in electromagnetic theory ((9), page 384), the displacement field associated with Φ and Ψ being referred to as a *poloidal* wave field and the field arising from Ω as a *toroidal* wave field. At a typical point P of the elastic solid, the poloidal field acts in the meridian plane through P and L and the toroidal field at right angles to this plane. Since toroidal waves are generated by the single wave function Ω they are entirely rotational in character and may easily be shown to be associated with a state of pure shear of the elastic solid. Poloidal waves, on the other hand, have both a dilatational and a rotational constituent, associated respectively with the wave functions Φ and Ψ.

Toroidal and poloidal waves produced in an infinite elastic solid by the motion of a rigid spherical inclusion have been studied by the present authors (10, 11), and the generation of such waves by the application of time-dependent tractions to the surface of a spherical cavity is discussed in section 4.

The terminology introduced in this section need not of course be confined to elastic wave fields having axial symmetry. In general, however, the poloidal and toroidal components are not orthogonal and so cannot be decoupled in the manner exhibited above.

4. *An application.* A homogeneous, isotropic elastic solid of infinite extent contains a single spherical cavity of radius a and is initially everywhere undisplaced and at rest. Beginning at time $t = 0$, specified time-dependent tractions are applied to the surface of the cavity so that an elastic pulse is propagated outwards into the elastic solid. A formal solution of this problem has been given by Eringen (12), and our aim in this section is to show that a parallel, but much simpler, treatment can be given with the aid of the wave function representation (1·9).

The problem merits further attention, however, for at least two reasons. First, both Eringen's work and the alternative discussion given here are wanting in rigour. The initial conditions have not been formulated satisfactorily (cf. (10)) and the final solution is expressed as an infinite series of infinite integrals the analytic properties of which have not been investigated. Secondly, the formal solution has been interpreted in physical terms only in two simple special cases: uniform normal loading resulting in a spherically symmetric dilatational pulse ((12), pages 265–269; (17), pages 98–111); and axially symmetric torsional shear loading resulting in a toroidal pulse having first-harmonic dependence upon the co-latitude (10, 18). The problems of interpretation are particularly severe when part of the cavity wall remains stress-free. The elastic pulse generated on the loaded part is then diffracted into shadow zones adjoining the stress-free part and a complicated pattern of wavefronts develops within the disturbance. An axially symmetric problem of this type has been studied by Tupholme (19) who has also given a detailed account of an analogous, but simpler, problem in acoustics (20).

For simplicity we assume here that the distribution of applied tractions on the surface of the cavity is symmetrical about an axis L. Our results can be generalized without difficulty to allow arbitrary loading as in Eringen's work (see (11), pages 105–111).

Relative to a system of spherical polar coordinates (r, θ, ϕ) based upon the centre O

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of the cavity and the axis of symmetry L, we wish to solve the vector equation of motion (1·1) subject to the initial conditions

$$\mathbf{u} = \mathbf{0}, \quad \partial\mathbf{u}/\partial t = \mathbf{0} \quad \text{at} \quad t = 0 \quad \text{for all} \quad r > a \tag{4·1}$$

and the boundary conditions

$$\sigma_{rr} = \sigma(\theta, t), \quad \sigma_{r\theta} = \tau_\theta(\theta, t), \quad \sigma_{r\phi} = \tau_\phi(\theta, t) \quad \text{for all} \quad t \geqslant 0 \quad \text{at} \quad r = a. \tag{4·2}$$

Here σ_{rr}, $\sigma_{r\theta}$, $\sigma_{r\phi}$ are components of the stress tensor $\boldsymbol{\sigma}$ referred to the coordinates r, θ, ϕ, and σ, τ_θ and τ_ϕ are prescribed functions.

We seek a solution of the form (1·9), the displacement components being given in terms of the wave functions by equations (3·1). Using the stress-displacement relations of classical elastokinetics, referred to spherical polar coordinates, the stress components σ_{rr}, $\sigma_{r\theta}$ and $\sigma_{r\phi}$ can be expressed in terms of Φ, Ψ and Ω, and with the aid of the axially symmetric wave equations satisfied by Φ and Ψ these results can be cast into the form

$$\left.\begin{aligned}
\sigma_{rr} &= \mu\left\{\left(\frac{1}{v_S^2} - \frac{2}{v_P^2}\right)\frac{\partial^2\Phi}{\partial t^2} + 2\frac{\partial^2\Phi}{\partial r^2} + \frac{2}{v_S^2}\frac{\partial^2}{\partial t^2}\left(r\frac{\partial\Psi}{\partial r} + \Psi\right) - 2\left(r\frac{\partial^3\Psi}{\partial r^3} + 3\frac{\partial^2\Psi}{\partial r^2}\right)\right\}, \\
\sigma_{r\theta} &= \mu\frac{\partial}{\partial\theta}\left\{\frac{2}{r}\left(\frac{\partial\Phi}{\partial r} - \frac{\Phi}{r}\right) + \frac{1}{v_S^2}\frac{\partial^2\Psi}{\partial t^2} - 2\left(\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r}\frac{\partial\Psi}{\partial r} - \frac{\Psi}{r^2}\right)\right\}, \\
\sigma_{r\phi} &= \mu\frac{\partial}{\partial\theta}\left(\frac{\partial\Omega}{\partial r} - \frac{\Omega}{r}\right),
\end{aligned}\right\} \tag{4·3}$$

where μ is the shear modulus of the elastic solid. Define

$$\gamma(\theta, t) = \frac{1}{\mu}\sigma(\theta, t) \quad \kappa(\theta, t) = \frac{1}{\mu}\int_0^\theta \tau_\theta(\chi, t)\,d\chi, \quad \pi(\theta, t) = \frac{1}{\mu}\int_0^\theta \tau_\phi(\chi, t)\,d\chi. \tag{4·4}$$

Then the boundary conditions (4·2) hold if

$$\left.\begin{aligned}
\left(\frac{1}{v_S^2} - \frac{2}{v_P^2}\right)\frac{\partial^2\Phi}{\partial t^2} + 2\frac{\partial^2\Phi}{\partial r^2} + \frac{2}{v_S^2}\frac{\partial^2}{\partial t^2}\left(r\frac{\partial\Psi}{\partial r} + \Psi\right) - 2\left(r\frac{\partial^3\Psi}{\partial r^3} + 3\frac{\partial^2\Psi}{\partial r^2}\right) &= \gamma, \\
\frac{2}{r}\left(\frac{\partial\Phi}{\partial r} - \frac{\Phi}{r}\right) + \frac{1}{v_S^2}\frac{\partial^2\Psi}{\partial t^2} - 2\left(\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r}\frac{\partial\Psi}{\partial r} - \frac{\Psi}{r^2}\right) &= \kappa, \\
\frac{\partial\Omega}{\partial r} - \frac{\Omega}{r} &= \pi,
\end{aligned}\right\} \quad \begin{matrix}\text{for all} \quad t \geqslant 0 \\ \text{at} \quad r = a,\end{matrix} \tag{4·5}$$

and the initial conditions (4·1) are satisfied if

$$\Phi = 0, \quad \frac{\partial\Phi}{\partial t} = 0, \quad \Psi = 0, \quad \frac{\partial\Psi}{\partial t} = 0, \quad \Omega = 0, \quad \frac{\partial\Omega}{\partial t} = 0 \quad \text{at} \quad t = 0 \quad \text{for all} \quad r > a. \tag{4·6}$$

It now remains to find solutions of the axially symmetric wave equations (1·3) and (1·7) which formally satisfy the subsidiary conditions (4·5) and (4·6), and this can be accomplished by standard procedures. First, an integral transform with respect to time is applied and the transforms of the wave functions expressed as infinite series of products of Bessel functions and Legendre polynomials with disposable coefficients.

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Next the three sets of coefficients are determined by expanding the transforms of the boundary data γ, κ, π in series of Legendre polynomials and using the boundary conditions (4·5). Finally, the transforms are inverted to yield the formal solution. In Eringen's paper a Fourier transform is taken with respect to t, but the Laplace transform is better suited to the solution of problems involving pulses and is preferred here.

Taking the (one-sided) Laplace transform with respect to t of each term in the axially symmetric form of equation (1·3) and bearing in mind the initial conditions (4·6) on Φ, we obtain

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\bar{\Phi}}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\bar{\Phi}}{\partial\theta}\right)-\frac{s^2}{v_P^2}\bar{\Phi}=0,$$

where bars are used to denote Laplace transforms and s is the transform variable. Solutions of this equation which remain bounded as $r\to\infty$ are, with the usual notation for modified Bessel and Legendre functions, $r^{-\frac{1}{2}}K_{n+\frac{1}{2}}(sr/v_P)\,P_n(\cos\theta)$ $(n=0,1,\ldots)$. Treating equations (1·7) in a similar manner and forming linear aggregates of the three families of solutions, we have

$$\left.\begin{aligned}\bar{\Phi}&=\sum_{n=0}^{\infty}a_n r^{-\frac{1}{2}}K_{n+\frac{1}{2}}(sr/v_P)\,P_n(\cos\theta),\\ \bar{\Psi}&=\sum_{n=0}^{\infty}b_n r^{-\frac{1}{2}}K_{n+\frac{1}{2}}(sr/v_S)\,P_n(\cos\theta),\\ \bar{\Omega}&=\sum_{n=0}^{\infty}c_n r^{-\frac{1}{2}}K_{n+\frac{1}{2}}(sr/v_S)\,P_n(\cos\theta).\end{aligned}\right\}\tag{4·7}$$

Next we represent the Laplace transform of γ in the series form

$$\bar{\gamma}=\sum_{n=0}^{\infty}\bar{\gamma}_n P_n(\cos\theta),\quad\text{where}\quad\bar{\gamma}_n=(n+\tfrac{1}{2})\int_0^{\pi}\bar{\gamma}(\theta)\,P_n(\cos\theta)\sin\theta\,d\theta,$$

and deal similarly with $\bar{\kappa}$ and $\bar{\pi}$. Substitution of the series (4·7) into the Laplace transforms of the boundary conditions (4·5) then gives three sets of equations for the coefficients a_n, b_n and c_n. With the notation

$$\nu_P=sa/v_P,\quad \nu_S=sa/v_S,\quad \mathsf{K}_{n+\frac{1}{2}}(x)=3K_{n+\frac{1}{2}}(x)-2xK'_{n+\frac{1}{2}}(x),$$

the solutions are given by

$$a^{-\frac{5}{2}}\Delta a_n=\bar{\gamma}_n\{(2n^2+2n-4+\nu_S^2)\,K_{n+\frac{1}{2}}(\nu_S)+\mathsf{K}_{n+\frac{1}{2}}(\nu_S)\}+\bar{\kappa}_n n(n+1)\,\mathsf{K}_{n+\frac{1}{2}}(\nu_S),$$

$$a^{-\frac{5}{2}}\Delta b_n=-\bar{\gamma}_n\,\mathsf{K}_{n+\frac{1}{2}}(\nu_P)-\bar{\kappa}_n\{(2n^2+2n-4+\nu_S^2)\,K_{n+\frac{1}{2}}(\nu_P)+2\mathsf{K}_{n+\frac{1}{2}}(\nu_P)\},$$

where

$$\Delta=\{(2n^2+2n-4+\nu_S^2)\,K_{n+\frac{1}{2}}(\nu_P)+2\,\mathsf{K}_{n+\frac{1}{2}}(\nu_P)\}\times\{(2n^2+2n-4+\nu_S^2)\,K_{n+\frac{1}{2}}(\nu_S)+\mathsf{K}_{n+\frac{1}{2}}(\nu_S)\}$$
$$-n(n+1)\,\mathsf{K}_{n+\frac{1}{2}}(\nu_P)\,\mathsf{K}_{n+\frac{1}{2}}(\nu_S)$$

and

$$c_n=-2a^{\frac{3}{2}}\bar{\pi}_n/\mathsf{K}_{n+\frac{1}{2}}(\nu_S).$$

The formal solution is completed by entering the above expressions for a_n, b_n and c_n into equations (4·7) and applying the inversion theorem for Laplace transforms.

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A glance at equations (3·1) shows that the leading terms of the series for the rotational wave functions Ψ and Ω do not contribute to the displacement field.

Since our purpose here is to demonstrate the effectiveness of the wave function representation (1·9) in a preliminary analysis of the problem described at the beginning of this section, we do not embark upon the much more difficult task of determining the structure of the elastic pulse which the solution portrays. It is clear from our results that the decoupling of the poloidal and toroidal constituents of the pulse extends to the boundary conditions, the poloidal part being excited by the components $\sigma = \mu\gamma$ and $\tau_\theta = \mu\,\partial\kappa/\partial\theta$ of the applied traction and the toroidal part by the component $\tau_\phi = \mu\,\partial\pi/\partial\theta$. The toroidal part has much the simpler structure and there is a close mathematical similarity between this constituent and the class of sound pulses studied by Tupholme (20).

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REFERENCES

(1) Sternberg, E. *Arch. Rational Mech. Anal.* **6** (1960), 34–50.
(2) Sternberg, E. and Gurtin, M. E. *Proc. 4th U.S. Nat. Congr. Appl. Mech.* (1962), vol. 2, 793–797. Amer. Soc. Mech. Engrs., New York.
(3) Morse, P. M. and Feshbach, H. *Methods of theoretical physics*, Part 2 (McGraw-Hill, New York, etc., 1953).
(4) Love, A. E. H. *A treatise on the mathematical theory of elasticity*, 4th edn. (Cambridge University Press, 1927).
(5) Einspruch, N. G., Witterholt, E. J. and Truell, R. *J. Appl. Phys.* **31** (1960), 806–818.
(6) Allen, D. E. and Robinson, A. R. *Technical Report.* Contract Nonr 1834 (03), Office of Naval Research. University of Illinois, Urbana, 1966.
(7) Hook, J. F. *J. Acoust. Soc. Amer.* **33** (1961), 302–313, 967.
(8) Hook, J. F. *J. Acoust. Soc. Amer.* **34** (1962), 354–355, 946–953.
(9) Backus, G. *Ann. Physics* **4** (1958), 372–447.
(10) Chadwick, P. and Trowbridge, E. A. *Proc. Cambridge Philos. Soc.* **63** (1967), 1189–1227.
(11) Trowbridge, E. A. Thesis, University of Sheffield, 1964.
(12) Eringen, A. C. *Quart. J. Mech. Appl. Math.* **10** (1957), 257–270.
(13) Chadwick, P. and Powdrill, B. *Int. J. Engng. Sci* **3** (1965), 561–595.
(14) Bers, L., John F. and Schechter, M. *Partial differential equations* (Interscience, New York, etc., 1964).
(15) Hellwig, G. *Partial differential equations* (Blaisdell, New York, etc., 1964).
(16) Todhunter, I. and Pearson, K. *A history of the theory of elasticity and of the strength of materials*, vol. 1 (Cambridge University Press, 1886).
(17) Hopkins, H. G. *Progress in solid mechanics* (eds. I. N. Sneddon and R. Hill), vol. 1, pp. 83–164 (North-Holland, Amsterdam, 1960).
(18) Jeffreys, H. *Mon. Not. Roy. Astron. Soc. Geophys. Suppl.* **2** (1931), 407–416.
(19) Tupholme, G. E. Thesis, University of East Anglia, 1967.
(20) Tupholme, G. E. *Proc. Cambridge Philos. Soc.* Forthcoming.

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