Ergod. Th. & Dynam. Sys. (2001), **21**, 915–955
Printed in the United Kingdom © *2001 Cambridge University Press*

Combinatorial configurations of fibered polynomials

OLIVIER SESTER

Equipe d'Analyse et de Mathématiques Appliquées, Université de Marne-la-Vallée, 5 boulevard Descartes, Champs sur Marne, F-77454 Marne-la-Vallée Cedex 2, France (e-mail: sester@math.univ-mlv.fr)

(*Received* 15 *May* 1999 *and accepted in revised form* 14 *December* 1999)

Abstract. We study from a combinatorial point of view quadratic fibered polynomials. The main object of this paper is the construction of a combinatorial model of a subset of the parameter space. This set (the configuration space) is proved to be a Hausdorff compact and connected space. Then, we prove that an abstract configuration which is non-recurrent can be uniquely realized by a quadratic fibered polynomial.

1. *Introduction*

A central problem in the study of dynamical systems is to know to what extent combinatorics forces dynamics. More precisely, the theory of iteration of rational functions succeeded in many cases in understanding the relationship between combinatorial properties and dynamical behaviors. This work is an attempt to generalize in higher dimension the successful combinatorial theory of the dynamics of quadratic polynomials.

Let X be a Hausdorff compact space, f a continuous self-mapping of X, and c a continuous complex-valued function on X, considered as a parameter. We deal with fibered polynomials:

$$\begin{aligned} P : X \times \mathbb{C} &\longrightarrow X \times \mathbb{C} \\ (x, z) &\longmapsto (f(x), P_x(z) = z^2 + c(x)). \end{aligned}$$

In some sense these are dynamical systems in dimension 1.5, since the first coordinate of P does not depend on z. Fibered polynomials have already been the object of many studies. Viana [**Via97**] and Alvez [**Alv98**] dealt with the real case where X is the circle and $f(\theta) = d\theta$, d a large integer. Jonsson [**Jon97**] and Heinneman [**Hei96**] considered the situation where f and c are both polynomials. In [**Ses97**] the focus was on hyperbolicity and we give a detailed characterization of this property.

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

There are several motivations for studying this kind of map. On the one hand, when $c : X \to \mathbb{C}$ is close to a constant c_0 with respect to the supremum norm, the corresponding fibered polynomial can be viewed as a 'random' perturbation of the polynomial $q_{c_0} : z \mapsto z^2 + c_0$. Thus a natural question is to know whether the dynamical properties of q_{c_0} are stable under perturbations in the class of fibered polynomials. Let us mention that many results on stochastic stability have already been obtained in the setting of random dynamical systems, see for instance [**Kif88**] or [**BM96**]. Nevertheless, the point of view is different, the maps P_x are supposed to be measurable 'independently identically distributed' and the results are of the 'almost everywhere' kind.

On the other hand, the second coordinate of the nth iterate of P is the composition of n quadratic polynomials $P_{f^{n-1}(x)} \circ P_{f^{n-2}(x)} \circ \cdots \circ P_x$. Thus the dynamical study of P reduces to the study of this family. In fact, given a sequence of bounded complex numbers $(c_i)_{i\in\mathbb{N}}$, the compositions of $z \mapsto z^2 + c_i$ can be considered as a fibered polynomial over the Stone–Cech compactification of $\mathbb{N}$.

Our goal is first to make sense of the word 'combinatoric' for a fibered polynomial. This will lead us to the construction of a topological space for the admissible combinatorics. Then we start answering the following related questions of realizability and rigidity:

Is any abstract combinatorics realizable by a fibered polynomial?

Is such a fibered polynomial unique?

When X is a single point, we are back to the study of a quadratic polynomial, and we refer to this case as the *constant case*. Let us start by recalling some well-known facts on the theory of iteration of quadratic polynomials. Given c a complex number, the *filled-in Julia set* of $q_c : z \mapsto z^2 + c$ is the set of points z such that the orbit of z under iteration of q_c is bounded. The boundary of the filled-in Julia set is called the *Julia set*. Then, the Mandelbrot set $\mathcal{M}$ is by definition the set of complex numbers c such that the Julia set of q_c is connected. Fix p/q a rational number. According to [**Gol92**], there exists a unique *degree-two rotation subset* of the circle of rotation number p/q. This is a finite subset of $\mathbb{R}/\mathbb{Z}$, labeled $R_0(p/q) = \{\theta_1, \ldots, \theta_q\}$ such that $2\theta_i = \theta_{i+p}$, taking indices modulo q. A complex number c belongs to the *(p/q)-limb* of the Mandelbrot set if the q external rays of $z \mapsto z^2 + c$ with argument in $R_0(p/q)$ land at the same repelling point, denoted traditionally by α. In fact, the limbs of $\mathcal{M}$ are exactly the connected components of $\mathcal{M}$ minus the closure of the main cardioid.

Whenever the parameter belongs to the (p/q)-limb, we can define a combinatorial Julia set by looking at how the the external rays which land at an inverse image of α are glued together. When c is not *renormalizable*, Yoccoz [**Yoc**] proved that the Julia set of q_c is locally connected. Consequently, this Julia set is homeomorphic to its combinatorial model. This construction in the dynamical plane enables us to define a *combinatorial Mandelbrot set* by identifying the parameters whose combinatorial Julia sets coincide. This combinatorial model can be viewed as $\mathcal{M}$ where the copies of $\mathcal{M}$ have been collapsed to points.

We shall construct in the setting of fibered polynomials a similar combinatorial model for the (p/q)-limb of the connected locus. To state our main results we first need to introduce some notation. Let $P_x^n = P_{f^{n-1}(x)} \circ P_{f^{n-2}(x)} \circ \cdots \circ P_x$. Then we can define for

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press



FIGURE 1. The fibered filled-in julia set.

all $x \in X$ the filled-in Julia set as

$$K_x = \left\{ z \in \mathbb{C} : \sup_{n \in \mathbb{N}} |P_x^n(z)| < +\infty \right\}.$$

Figure 1 shows the Julia set K_0 in the particular case $X = \mathbb{R}/\mathbb{Z}$, $f(x) = x + \theta_0$, θ_0 the golden mean, and $c(x) = -1 + 0.3\sin(2\pi x)$.

In this paper we focus on the connected case and we introduce the connected locus $\mathcal{M}(X, f)$ as the set of parameters such that K_x is connected for all x. Using the Riemann mapping theorem, one can find a unique conformal isomorphism $\varphi_x : \mathbb{C} \setminus K_x \to \mathbb{C} \setminus \overline{\mathbb{D}}$ such that φ_x is tangent to the identity at infinity. For $\theta \in \mathbb{R}/\mathbb{Z}$, the external ray of argument θ is defined as $\mathcal{R}_x(\theta) = \varphi_x^{-1}\{r \exp(2i\pi\theta),\ r > 1\}$. We say that $\mathcal{R}_x(\theta)$ lands at $\alpha(x)$ if $\varphi_x^{-1}(r \exp(2i\pi\theta))$ tends to $\alpha(x)$ as r tends to 1. In our fibered setting the (p/q)-limb of $\mathcal{M}(X, f)$, denoted by $\mathcal{M}_{p/q}(X, f)$, is also the set of parameters such that the external rays of argument in $R_0(p/q)$ land at the same point $\alpha(x)$. We shall restrict ourselves to subsets of $\mathcal{M}_{p/q}(X, f)$ called the *satellites*. The satellites will be define rigorously later. Similarly to the constant case, they are indexed by dyadic rationals j and they are denoted by $\mathcal{M}_{p/q}^j(X, f)$.

Our combinatorial model of $\mathcal{M}_{p/q}^j(X, f)$ will be obtained as an inverse limit space. To explain our approach we shall present an example kindly suggested to us by the referee and showing that many spaces can be obtained by this process (see [**Mun84**] for definitions and basic properties of simplicial complexes).

Example 1.1. Let Y be a finite simplicial complex, and let Σ_n be the nth barycentric subdivision. Let $i_n : Y \to \Sigma_n$ be the map which associates to every point the open simplex containing it. We endow Σ_n with the finest topology such that i_n is continuous. The topological space $\varprojlim \Sigma_n$ is then the set of nested sequence $\sigma_1 \supset \sigma_2 \cdots$ of open simplices. The intersection $\bigcap_n \sigma_n$ of such a sequence is either a point or it is empty, in which case the point of the inverse limit space accumulates on the intersection of the corresponding closed simplices. There is an obvious map $Y \to \varprojlim \Sigma_n$, which is a homeomorphism onto the set of closed points. For instance, if one considers the simplicial complex made up of $0, \frac{1}{2},]0; \frac{1}{2}[$ and $]\frac{1}{2}; 1[$ whose underlying space is the circle $\mathbb{R}/\mathbb{Z}$, the nth barycentric subdivision is the set of dyadic rationals of the form $j/2^n$ together with the connected

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

components of the complement. Passing to the limit and after restriction to the set of closed points we reconstruct $\mathbb{R}/\mathbb{Z}$.

In §2 we shall introduce Γ_n as the set of admissible equivalence relations on $Q^{-n}(R_0(p/q))$, where $Q : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is $Q(z) = 2z \bmod 1$. To be admissible means that they have to be *unlinked* and satisfy some symmetry properties (see Definition 3.1). These equivalence relations (on a finite subset) correspond exactly to the different ways the external rays may be glued together. In the constant case these relations need to be invariant under Q, but here they do not and there are many more possibilities.

Then, we consider the inverse limit space $\varprojlim \Gamma_n$ and we define $X_\infty^{p/q,j}$ (corresponding to the jth satellite of the (p/q)-limb) as the subset of closed points of the inverse limit space. The space $X_\infty^{p/q,j}$ is endowed with a natural continuous self-mapping $\mathcal{F}$ deduced from angle doubling. The elements of $X_\infty^{p/q,j}$ turn out to be closed equivalence relations on $\mathbb{R}/\mathbb{Z}$. They are analogous to the rational laminations of Thurston, i.e. closed equivalence relations of the circle whose classes are unlinked and totally disconnected (see [**BL99, Thu85, McM94**]). In §5, we prove the following key result.

THEOREM 1.2. *The space $X_\infty^{p/q,j}$ is a Hausdorff compact and connected space. It has the property that for each basis (X, f) and for all $c \in \mathcal{M}_{p/q}^j(X, f)$ there exists a continuous canonical map $h : X \to X_\infty^{p/q,j}$ which makes the following diagram commutative:*

$$\begin{array}{ccc} X & \xrightarrow{h} & X_\infty^{p/q,j} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{F}} \\ X & \xrightarrow{h} & X_\infty^{p/q,j} \end{array}$$

The point $h(x)$ is called the configuration of x.

The main difficulty in this result is to obtain compactness and connectedness. Note that, even in the constant case, the construction of a Hausdorff compact combinatorial space which is associated to the limbs of the Mandelbrot set is not obvious. Here, the configurations corresponding to the non-fibered dynamics are just the fixed points of $\mathcal{F}$. The compact $X_\infty^{p/q,j}$ seems to be enormous since it contains all the combinatorics of external rays independently of the basis (X, f). We conjecture it has infinite dimension.

The second part of this paper deals with both questions of realizability and rigidity attached to Theorem 1.2. They can be stated as follows. Given X an invariant compact subset of $X_\infty^{p/q,j}$ does there exist a continuous function $c \in \mathcal{M}_{p/q}^j(X, \mathcal{F}_{|X})$ such that the map h in Theorem 1.2 is the canonical injection? If yes, is c unique? Observe that, in this fibered framework, the question of realizability is not obvious. Although we can always realize the finite configurations (the elements of Γ_n), it is not always possible to pass to the limit since the parameter space is not locally compact. In the case of quadratic polynomials, realizability of abstract configurations (the fixed point of $\mathcal{F}$) results from the compactness of the Mandelbrot set. Moreover, since parameters in the same copy of $\mathcal{M}$ have the same combinatorial configuration, the question of uniqueness has a negative answer if the parameter is renormalizable. In fact, since $\mathcal{M}$ is locally connected

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

at non-renormalizable points (see [**Yoc**] or [**Hub93**]), there is a unique complex number realizing non-renormalizable configurations.

Therefore, we shall need to make some combinatorial assumptions that guarantee that the parameter (in the constant case) is not renormalizable to obtain rigidity. Mainly for technical reasons, we restrict our attention to the $(1/2)$-limb (also called the real limb since it contains some real parameters). We consider X_3, the invariant compact subset of $X_\infty^{1/2,1/2}$, made up of 'primitive' non-recurrent configurations. Roughly speaking, these are the most simple type of critically non-recurrent configurations. They correspond to fibered polynomials such that the post-critical orbits avoid a given neighborhood of the critical section defined through external rays of arguments $\frac{5}{24}$, $\frac{7}{24}$, $\frac{17}{24}$ and $\frac{19}{24}$. Observe that in the non-fibered case this assumption guarantees that the associated parameters are not renormalizable. We shall prove the following.

THEOREM 1.3. *The set X_3 is realizable and rigid.*

Let us mention that we are also able to give a positive answer to the questions of realizability and rigidity for the 'weakly-recurrent configurations'. This is the object of the preprint [**Ses00**]. For the general case the first step would be to have a suitable definition of renormalizable (or non-renormalizable) adapted to the fibered framework.

This paper is organized as follows. In §2 we recall without proofs some basic facts on fibered polynomials. The object of §3 is to define rigorously Γ_n, the topological space of admissible (finite) combinatorics. The main result of §4 states that although Γ_n is not Hausdorff we can separate distinct points in a suitable sense. §5 is devoted to the proof of Theorem 1.2. First, we prove that the closed points of the inverse limit of Γ_n is a Hausdorff compact and connected space (Theorem 5.3). The existence of the equivariant map h is the object of Proposition 5.5. In §6 we prove that Julia sets corresponding to combinatorial non-recurrent fibered polynomials are holomorphically removable. This enables us to prove the rigidity of non-recurrent configurations. Finally, we establish the realizability part of Theorem 1.3 in the last section of this paper.

2. *Preliminaries*

2.1. *Fibered Julia sets.* Let X be a compact Hausdorff space and $f : X \to X$ a continuous map. A (quadratic) fibered polynomial over (X, f) is a skew-product map of the form:

$$\begin{aligned} P : X \times \mathbb{C} &\longrightarrow X \times \mathbb{C} \\ (x, z) &\longmapsto (f(x), P_x(z) = z^2 + c(x)), \end{aligned}$$

where c is a continuous complex-valued function on X. In this section we recall some elementary results concerning Julia sets of fibered polynomials and some properties of the associated Green function and Böttcher coordinate. We refer to [**Ses97**] for proofs.

Let us denote by P^n the n-times composition of P and by P_x^n the second coordinate of P^n, i.e. $P_x^n = P_{f^{n-1}(x)} \circ \cdots \circ P_x$. The (global) filled-in Julia set is defined as the set of points with bounded orbits under iteration of P:

$$K = \left\{(x, z) \in X \times \mathbb{C} : \sup_{n \in \mathbb{N}} |P_x^n(z)| < +\infty\right\}.$$

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

The set K is compact, non-empty and backward invariant. We denote by K_x the fiber of K over x, i.e. $z \in K_x$ iff $(x, z) \in K$. It is a full compact subset of $\mathbb{C}$ which is included in the disk of center 0 and of radius $\sup_{x\in X}(1 + |c(x)|)$. If $\log_+(|z|) = \max(\log(|z|), 0)$ for all $z \in \mathbb{C}$, we set

$$G(x, z) = G_x(z) = \lim_{n\to+\infty} \frac{1}{2^n} \log_+ |P_x^n(z)|.$$

PROPOSITION 2.1. *The map $G : X \times \mathbb{C} \to \mathbb{R}^+$ is continuous and satisfies: G_x is harmonic in $\mathbb{C} \setminus K_x$, $K_x = G_x^{-1}(\{0\})$, $G_{f(x)}(P_x(z)) = 2G_x(z)$ and G_x is tangent to $z \mapsto \log|z|$ at infinity.*

The Julia set over x denoted by J_x is just the boundary of K_x. From this proposition we deduce that $x \mapsto J_x$ is lower-semi-continuous and $x \mapsto K_x$ is upper-semi-continuous (with respect to the Hausdorff metric on compact subsets of the plane). Another consequence is a generalization of the well-known Fatou criterion for the connectedness of Julia sets of complex polynomials.

COROLLARY 2.2. *$\forall x \in X$, K_x is connected iff $\forall x \in X$, the critical point $0 \in K_x$.*

By definition, the connected locus $\mathcal{M}(X, f)$ is the set of parameters $c : X \to \mathbb{C}$ such that the K_x are connected for all $x \in X$. The space $\mathcal{M}(X, f)$ is equipped with the supremum norm. Note the difference from the constant case: here $\mathcal{M}(X, f)$ is a subset of continuous functions and is not locally compact. The second essential tool for the study of fibered polynomials is the so-called Böttcher coordinate. Assume that the parameter belongs to $\mathcal{M}(X, f)$, then from the Riemann mapping theorem we obtain the following.

PROPOSITION 2.3. *There exists a unique holomorphic isomorphism $\varphi_x : \mathbb{C} \setminus K_x \to \mathbb{C} \setminus \overline{\mathbb{D}}$ such that:*

- *the following diagram is commutative*

$$\begin{array}{ccc} \mathbb{C} \setminus K_x & \xrightarrow{P_x} & \mathbb{C} \setminus K_{f(x)} \\ \varphi_x \downarrow & & \downarrow \varphi_{f(x)} \\ \mathbb{C} \setminus \overline{\mathbb{D}} & \xrightarrow{q} & \mathbb{C} \setminus \overline{\mathbb{D}} \end{array}$$

  where $q(z) = z^2$;
- *$G_x(z) = \log(|\varphi_x(z)|)$, and the map φ_x is tangent to the identity at infinity;*
- *$(x, z) \mapsto \varphi_x(z)$ is continuous on its domain of definition.*

As in the constant case, the external rays of K_x are the gradient lines of the Green function G_x. In fact, we are mainly interested in the connected case and so we define them using the Böttcher coordinate. The external ray of argument $\theta \in \mathbb{R}/\mathbb{Z}$ is

$$\mathcal{R}_x(\theta) = \varphi_x^{-1}(\{\rho \exp(2i\pi\theta),\ \rho \in]1; +\infty]\}).$$

For future use, we also denote for all $1 \le t < t'$:

$$\mathcal{R}_x(\theta;]t, t']) = \varphi_x^{-1}(\{\rho \exp(2i\pi\theta),\ \rho \in]t, t']\}).$$

Moreover, if there exists $\alpha(x) \in \mathbb{C}$ such that $\varphi_x^{-1}(\rho \exp(2i\pi\theta)) \to \alpha(x)$ as $\rho \to 1$, we say that $\mathcal{R}_x(\theta)$ lands at $\alpha(x)$. Note that this limit may not exist.

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

2.2. *Generalized (p/q)-limb.* Let p/q be an element of $\mathbb{Q}\cap\mathbb{R}/\mathbb{Z}$ with p and q relatively prime. We denote by $R_0(p/q)$ the (unique) quadratic rotation set of rotation number p/q. This is the subset of the circle $R_0(p/q) = \{\theta_1, \ldots, \theta_q\}$ such that $\theta_{j+p} = 2\theta_j$, taking subscripts modulo q. A complete and detailed study of rotation subsets can be found in [**Gol92**].

Similarly to the constant case, the (p/q)-limb labelled $\mathcal{M}_{p/q}(X, f)$ is the subset of the connected locus such that the q external rays of argument θ_j land at the same point. This property must hold for every $x \in X$, and the landing point of the external rays, denoted by $\alpha(x)$, will satisfy that $x \mapsto \alpha(x)$, is an invariant repelling section. Let us be more specific on this notion of invariant section.

Definition 2.4. A continuous function $\alpha : X \longrightarrow \mathbb{C}$ is called an invariant repelling section for P if there exist constants $A > 0$ and $\lambda > 1$ such that $\forall x \in X$

$$P_x(\alpha(x)) = \alpha(f(x)) \quad \text{and} \quad |(P_x^n)'(\alpha(x))| \geq A\lambda^n, \quad \forall n \in \mathbb{N}.$$

We now give a necessary and sufficient condition which guarantees the existence of such a repelling section.

PROPOSITION 2.5. *Let U and V be open subsets of $X \times \mathbb{C}$ such that:*
(1) $U = \bigcup_{x\in X}(x, U_x)$, $V = \bigcup_{x\in X}(x, V_x)$ *where U_x and V_x are topological disks of $\mathbb{C}$;*
(2) $V \Subset U$ *and there exists $m_0 > 0$ such that* $\operatorname{diam}(U_x) \leq m_0$, $\forall x \in X$; *and*
(3) $P_x : V_x \longrightarrow U_{f(x)}$ *is an holomorphic isomorphism.*
Then there exists a unique invariant repelling section in U. Conversely, if α is an invariant repelling section, we may find two open neighborhoods U and V of α satisfying the properties above.

Proof. The open sets U and V being given, let $V_x^0 = U_x$, $V_x^1 = V_x$ and V_x^n be the connected component of $(P_x^n)^{-1}(U_{f^n(x)})$ included in V_x^{n-1}. Since V is compactly included in U, the distances $d(V_x, \partial U_x)$ are uniformly bounded from below, and there exists μ_0 such that $\operatorname{mod}(U_x \setminus \overline{V_x}) \geq \mu_0$. Thus,

$$\operatorname{mod}(U_x \setminus \overline{V_x^n}) \geq \sum_{i=0}^{n} \operatorname{mod}(V_x^i \setminus \overline{V_x^{i+1}}) \geq n\mu_0,$$

and

$$\lim_{n\to+\infty} \operatorname{mod}(U_x \setminus \overline{V_x^n}) = +\infty.$$

Consequently, $\bigcap_{n\in\mathbb{N}} V_x^n = \{\alpha(x)\}$, with $x \mapsto \alpha(x)$ a continuous function satisfying $P_x(\alpha(x)) = \alpha(f(x))$. Furthermore, by virtue of the extremal Grötzsch theorem, there exist positive constants A_1 and A_2 which do not depend on x, such that

$$\operatorname{diam}(V_x^n) \leq A_1 e^{-nA_2}.$$

Then, we deduce from Koebe's theorem that there exist positive constants B_1 and B_2 such that

$$|(P_x^n)'(\alpha(x))| \geq B_1 e^{B_2 n}.$$

Conversely, assume that there exists an invariant repelling section α. In the standard way, one can find a continuous cocycle $\phi : X \times \mathbb{C} \to X \times \mathbb{C}$ of the form $\phi(x, z) =$

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

$(x, e^{\varphi(x)}z)$ which allows us to conjugate P to another fibered polynomial $Q : X \times \mathbb{C} \to X \times \mathbb{C}$ satisfying

- $Q(x, z) = (f(x), Q_x(z) = \tilde{c}_2(x)z^2 + \tilde{c}_0(x))$;
- $|(Q_x)'(e^{\varphi(x)}\alpha(x))| \geq \theta > 1$.

Thus we denote $\widetilde{\alpha}(x) = e^{\varphi(x)}\alpha(x)$ and we fix $\theta > \theta' > 1$ and $r > 0$ small enough to ensure that

$$D(\widetilde{\alpha}(f(x)), \theta' r) \subset Q_x(D(\widetilde{\alpha}(x), r)).$$

Denoting $\widetilde{U}_x = D(\widetilde{\alpha}(x), \theta' r)$ and $\widetilde{V}_x$ the connected component of $Q_x^{-1}(\widetilde{U}_{f(x)})$ included in $\widetilde{U}_x$, one has $\widetilde{U} = \bigcup_{x\in X}(x, \widetilde{U}_x)$ and $\widetilde{V} = \bigcup_{x\in X}(x, \widetilde{V}_x)$, two open subsets of $X \times \mathbb{C}$ satisfying the required properties of Proposition 2.5 for the map Q. Finally, we obtain the desired open sets by considering $U = \phi^{-1}(\widetilde{U})$ and $V = \phi^{-1}(\widetilde{V})$. □

We are now aiming to prove that the repelling section depends continuously on the parameter. To be more precise, denote by $\mathcal{S}(X, f)$ the set of pairs (c, α) such that α is a repelling section of the fibered polynomial $P : (x, z) \mapsto (f(x), z^2 + c(x))$. The space $\mathcal{S}(X, f)$ is endowed with the product topology, and $\Pi : \mathcal{S}(X, f) \to \mathcal{C}(X, \mathbb{C})$ denotes the first projection.

PROPOSITION 2.6. *Π is a local homeomorphism.*

Proof. By definition of the product topology, Π is continuous and from uniqueness in Proposition 2.5, we deduce that Π is locally injective.

Let us prove the openness of Π. Fix $(c, \alpha) \in \mathcal{S}(X, f)$. Consider $U = \bigcup_{x\in X}(x, U_x)$ and $V = \bigcup_{x\in X}(x, V_x)$ the open subsets of $X \times \mathbb{C}$ given by Proposition 2.5. For each c' in an open neighborhood of c, we set $U'_x = U_x$ and V'_x the connected component of $Q_x^{-1}(U'_{f(x)})$ included in U'_x where $Q_x(z) = z^2 + c'(x)$. By continuity, the open sets $U' = \bigcup_{x\in X}(x, U'_x)$ and $V' = \bigcup_{x\in X}(x, V'_x)$ satisfy the criterion of Proposition 2.5. Hence, $Q : (x, z) \mapsto (f(x), Q_x(z))$ admits a unique invariant repelling section α' which belongs to V'. Finally, we note that α' is arbitrary close to α provided that c' is close enough to c. This completes the proof of Proposition 2.6. □

Assume that P admits an invariant repelling section α, and denote by U and V the open sets given by Proposition 2.5. The next proposition provides a necessary and sufficient condition which guarantees that the external rays with arguments in $R_0(p/q)$ land at $\alpha(x)$.

PROPOSITION 2.7. *$\forall x \in X, \forall \theta \in R_0(p/q)$ the external rays $\mathcal{R}_x(\theta)$ land at $\alpha(x)$ iff there exists $t_0 > 0$ such that*

$$\mathcal{R}_x(\theta;]1, t_0]) \subset U_x, \quad \forall x \in X, \ \forall \theta \in R_0(p/q). \tag{1}$$

Proof. Recall that $(V_x^n)_{n\in\mathbb{N}}$ stands for the system of neighborhoods of $\alpha(x)$ introduced in the proof of Proposition 2.5. Under the hypothesis (1), one has $\forall n \in \mathbb{N}$

$$\mathcal{R}_x(\theta;]1, t_0^{2-n}]) \subset V_x^n.$$

Since the diameter of V_x^n converges to 0 as $n \to +\infty$, $\mathcal{R}_x(\theta)$ lands at $\alpha(x)$.

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

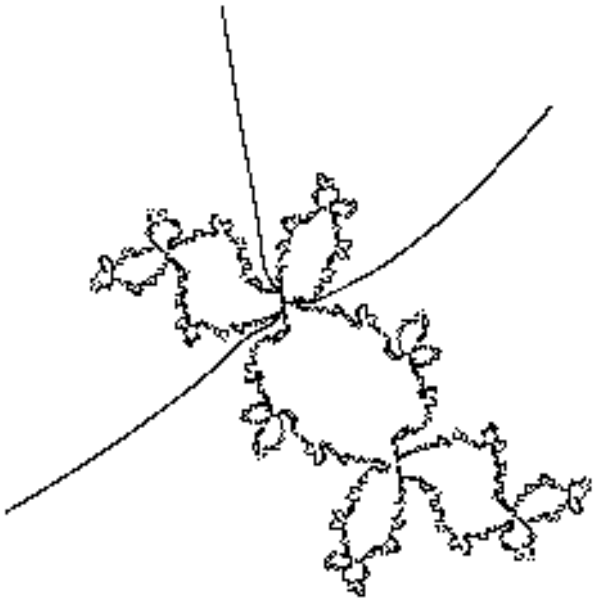


FIGURE 2. A small perturbation of the Douady's rabbit.

Conversely, assume that $\mathcal{R}_x(\theta)$ lands at $\alpha(x)$. From compactness of X and openness of U, we deduce that there exists $t_0 > 1$ such that

$$\mathcal{R}_x(\theta;]t_0^{1/2}, t_0]) \subset U_x, \quad \forall x \in X,\ \forall \theta \in R_0(p/q).$$

Hence, $\forall n \in \mathbb{N}$, $\mathcal{R}_x(\theta;]t_0^{2-n}, t_0^{2-n+1}]) \subset U_x$ and (1) follows. □

We know that $\mathcal{M}_{p/q}(X, f)$ is not empty, since it contains the constant functions with values in the (p/q)-limb of the Mandelbrot set. Combining Propositions 2.5–2.7 we deduce that $\mathcal{M}_{p/q}(X, f)$ is a relatively open subset of $\mathcal{M}(X, f)$.

Example 2.8. We consider fibered polynomials over $(\mathbb{R}/\mathbb{Z}, f)$ where f is the rotation $x \mapsto x + \theta_0 \pmod 1$ with θ_0 the golden mean. In the case where $c(x) = c_0 = -0.122\,561\ldots + 0.744\,861i\ldots$ the Julia set over any $x \in X$ is the well-known Douady's rabbit. This Julia set admits a repelling fixed point which is the landing point of external rays of arguments $\frac{1}{7}$, $\frac{2}{7}$ and $\frac{4}{7}$. Now, if we take a parameter c closed to c_0, say $c(x) = c_0 + 0.15(1+i)\sin(2\pi x)$, the external rays of arguments $\frac{1}{7}$, $\frac{2}{7}$ and $\frac{4}{7}$ keep on landing at a repelling section. Figure 2 shows the Julia set over $x = 0$ together with the external rays $\frac{1}{7}$, $\frac{2}{7}$ and $\frac{4}{7}$.

3. *Configurations of finite order*

In this section we define in an abstract way a set of admissible equivalence relations that turn out to be precisely those obtained by looking at how external rays with angles in R_n are glued together. Next, we endow this set of equivalence relations with an order relation and a compatible topological structure.

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

3.1. *Definition and characterization of* Γ_n. Fix a rational number p/q, and denote $R_0(p/q)$ the associated rational subset of the circle. Our construction depends on the rational number p/q but we shall omit this number from the notations, without ambiguity. Let us introduce some notation:

- $Q(z) = 2z$ modulo 1;
- $R_0 = R_0(p/q)$ and $R_n = Q^{-n}(R_0)$;
- R_n' is the set of connected components of $\mathbb{R}/\mathbb{Z} \setminus R_n$ and $E_n = R_n \sqcup R_n'$.

We have a natural projection $p_n : \mathbb{R}/\mathbb{Z} \to E_n$ defined by the following: if $\theta \in R_n$ we set $p_n(\theta) = \theta$, otherwise $p_n(\theta)$ is the unique element of R_n' containing θ.

The space E_n is endowed with the finest topology such that the map p_n is continuous. Thus the elements of R_n are closed in E_n whereas those of R_n' are open. Moreover, the map p_n is open but not closed since the image of any element $\theta \in \mathbb{R}/\mathbb{Z} \setminus R_n$ is open in E_n. Consequently, the space E_n is connected and compact but not Hausdorff.

Next, we introduce the maps $\widehat{Q}_{n,n-1}$ and $\hat{\imath}_{n,n-1} : E_n \to E_{n-1}$ making the following diagrams commutative:

$$\begin{array}{ccc} \mathbb{R}/\mathbb{Z} & \xrightarrow{Q} & \mathbb{R}/\mathbb{Z} \\ {\scriptstyle p_n}\downarrow & & \downarrow{\scriptstyle p_{n-1}} \\ E_n & \xrightarrow{\widehat{Q}_{n,n-1}} & E_{n-1} \end{array} \qquad \begin{array}{ccc} \mathbb{R}/\mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{R}/\mathbb{Z} \\ {\scriptstyle p_n}\downarrow & & \downarrow{\scriptstyle p_{n-1}} \\ E_n & \xrightarrow{\hat{\imath}_{n,n-1}} & E_{n-1} \end{array}$$

Obviously, $\widehat{Q}_{n,n-1}$ and $\hat{\imath}_{n,n-1}$ are continuous and open. Moreover, $\widehat{Q}_{n,n-1}$ carries R_n and R_n' surjectively onto R_{n-1} and R_{n-1}' respectively, hence $\widehat{Q}_{n,n-1}$ is a closed map. Concerning $\hat{\imath}_{n,n-1}$, we have that $\hat{\imath}_{n,n-1}(r)$ belongs to R_{n-1}', whenever $r \in R_n \setminus R_{n-1}$, hence $\hat{\imath}_{n,n-1}$ is not a closed map.

If $x \in R_n$ we denote by $\mathrm{ord}(x)$ the smallest integer m such that $Q^m(x) \in R_n$. We come now to the definition of Γ_n.

Definition 3.1. We say that γ_n, an equivalence relation on R_n, is admissible if it satisfies:

(1) each class of γ_n contains at least q distinct elements;
(2) the classes of γ_n are uncrossed;
(3) whenever $x \sim_{\gamma_n} y$ then $\mathrm{ord}(x) = \mathrm{ord}(y) \leq n$;
(4) still assuming $x \sim_{\gamma_n} y$, then for all $1 \leq k < \mathrm{ord}(x)$ there exists l an odd integer such that $l < 2^k$ and

$$x + \frac{1}{2^k} \sim_{\gamma_n} y + \frac{l}{2^k}.$$

The space Γ_n is by definition the set of admissible equivalence relations. If $\gamma_n \in \Gamma_n$, we denote $\pi_{\gamma_n} : R_n \to R_n/\gamma_n$ the canonical projection, thus the notation $x \sim_{\gamma_n} y$ just means that $\pi_{\gamma_n}(x) = \pi_{\gamma_n}(y)$.

To be uncrossed (or unlinked) means that if c_1 and c_2 are different classes then they are contained in disjoint subintervals of $\mathbb{R}/\mathbb{Z}$. Note that Γ_0, Γ_1 and Γ_2 are reduced to a single point and that $\forall n \in \mathbb{N}$, Γ_n is finite.

Now, we can naturally define two maps $Q_{n,n-1}$ and $i_{n,n-1} : \Gamma_n \to \Gamma_{n-1}$ corresponding to the doubling and the forgetting of some angle, respectively.

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

Definition 3.2. Let $\gamma_n \in \Gamma_n$, we define:

- $\gamma_{n-1} = i_{n,n-1}(\gamma_n)$ by: $x, y \in R_{n-1} \subset R_n$ then $x \sim_{\gamma_{n-1}} y$ iff $x \sim_{\gamma_n} y$;
- $\gamma^*_{n-1} = Q_{n,n-1}(\gamma_n)$ by: $x, y \in R_{n-1} \subset R_n$ then $x \sim_{\gamma^*_{n-1}} y$ if there exist $x',\ y' \in R_n$ such that $Q(x') = x$, $Q(y') = y$ and $x' \sim_{\gamma_n} y'$.

PROPOSITION 3.3. *If $\gamma_n \in \Gamma_n$ then both $i_{n,n-1}(\gamma_n)$ and $Q_{n,n-1}(\gamma_n)$ belong to Γ_{n-1}.*

Proof. It is clear from the definition that $i_{n,n-1}(\gamma_n)$ is an admissible equivalence relation. Concerning $\gamma_{n-1} = Q_{n,n-1}(\gamma_n)$, we first prove that it is an equivalence relation. Reflexivity and symmetry are obvious. For the transitivity, assume that $x \sim_{\gamma_{n-1}} y$ and $y \sim_{\gamma_{n-1}} z$. This yields $x' \sim_{\gamma_n} y'$ and $y'' \sim_{\gamma_n} z'$ with $Q(x') = x$, $Q(y') = Q(y'') = y$ and $Q(z') = z$. Thus, either $y' = y''$ then $x' \sim_{\gamma_n} z'$ and one has $x \sim_{\gamma_{n-1}} z$; or $y' = y'' + \frac{1}{2}$, then by property (4) in Definition 3.1 with $k = 1$, one has $x' + \frac{1}{2} \sim_{\gamma_n} y' + \frac{1}{2}$. Hence $x' + \frac{1}{2} \sim_{\gamma_n} z'$, and since $Q(x' + \frac{1}{2}) = x$ we obtain $x \sim_{\gamma_{n-1}} z$.

Let us check that γ_{n-1} is admissible. It easily satisfies (1), (2) and (3) in Definition 3.1. Then, assume that $x \sim_{\gamma_{n-1}} y$. Let $x' \sim_{\gamma_n} y'$ with $Q(x') = x$ and $Q(y') = y$. Fix $1 \le k < \mathrm{ord}(x) = \mathrm{ord}(x') - 1$, since γ_n satisfies (4), there exists an odd integer $l < 2^{k+1}$ such that

$$x' + \frac{1}{2^{k+1}} \sim_{\gamma_n} y' + \frac{l}{2^{k+1}}.$$

If $l < 2^k$, one has immediately $x + 1/2^k \sim_{\gamma_n} y + l/2^k$. Otherwise, $l = 2^k + l'$ with $l' < 2^k$ an odd integer. Thus

$$x' + \frac{1}{2^{k+1}} \sim_{\gamma_n} y' + \frac{1}{2} + \frac{l'}{2^{k+1}}$$

and $x + 1/2^k \sim_{\gamma_n} y + l'/2^k$. In any case, $Q(\gamma_n)$ satisfies property (4). □

It is convenient to represent an element of Γ_n by a schematic diagram $D(\gamma_n)$ joining the points in the same class by disjoint arcs as shown in Figure 3. In [**Mil98**, p. 11] an explanation is given of how to obtain such diagrams. This enables us to extend γ_n to E_n by the following. Two elements r and r' of R' are in the same class if they belong to the same connected component of

$$\overline{\mathbb{D}} \setminus \overline{D(\gamma_n)}.$$

Of course, this definition is independent of the choice of the diagram $D(\gamma_n)$. We still denote by γ_n the extended equivalence relation.

3.2. *Critical and post-critical classes.* Often in complex dynamics the behavior of the critical points is crucial. Here, for each admissible equivalence relation we can also define a class which plays the role of the critical point.

Definition 3.4. Fix $n \ge 1$ and let γ_n be an element of Γ_n. We denote $\gamma_{n-1} = Q_{n,n-1}(\gamma_n)$. There exists a unique element of E_n/γ_n which is invariant under $s_{1/2} : \theta \mapsto \theta + \frac{1}{2} \bmod 1$. This element is called the critical class of γ_n and labeled $v(\gamma_n) \subset \mathbb{R}/\mathbb{Z}$. The post-critical class of γ_n denoted by $c(\gamma_n) \subset \mathbb{R}/\mathbb{Z}$ is the image of $v(\gamma_n)$ under $\widehat{Q}_{n,n-1}$.

Note that $c(\gamma_n) \in E_{n-1}/\gamma_{n-1}$ and that $\hat{Q}_{n,n-1} : v(\gamma_n) \to c(\gamma_n)$ is a double covering. The uniqueness in this definition just follows from the fact that γ_n is unlinked. Moreover,

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

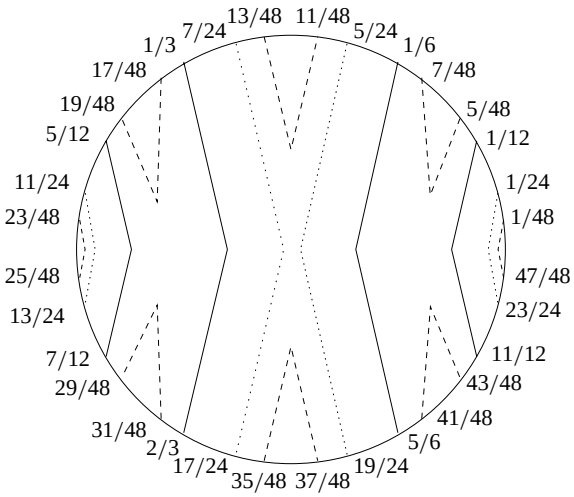


FIGURE 3. The element of Γ_4 associated to the $(1/2)$-limb.

either there exists an invariant class under $s_{1/2}$ included in R_n, then this is the critical one; or the number of classes of γ_n included in R_n is even. This implies that the number of classes included in R'_n is odd and thus there exists an element in R'_n invariant under $s_{1/2}$.

We shall now relate the admissible equivalence relations to the combinatorics of external rays. Fix X to be a Hausdorff compact set, $f : X \to X$ continuous and $c \in \mathcal{M}_{p/q}(X, f)$. Let $\alpha : X \to \mathbb{C}$ be the associated repelling section. For all $n \in \mathbb{N}$, we define $h_n(x)$ to be an equivalence relation on R_n by declaring that r and r' are in the same equivalence class of $h_n(x)$ iff the associated external rays $\mathcal{R}_x(r)$ and $\mathcal{R}_x(r')$ land at the same point (an inverse image of $\alpha(f^n(x))$). Hence we have defined a map $h_n : X \to \Gamma_n$ that actually depends on (X, f) and $c \in \mathcal{M}_{p/q}(X, f)$.

PROPOSITION 3.5. *If γ_n is any formal admissible equivalence relation then there exists a fibered polynomial which realizes γ_n, i.e. there exist a basis (X, f), a parameter $c \in \mathcal{M}_{p/q}(X, f)$ and $x \in X$ such that $h_n(x) = \gamma_n$.*

Proof. We proceed by induction, the realizability of Γ_1 and Γ_2 is immediate since constant parameters have the required combinatorics. Next, we assume that Γ_{n-1} is realizable. Fix $\gamma_n \in \Gamma_n$, $\gamma_{n-1} = Q_{n,n-1}(\gamma_n)$ and denote by $c(\gamma_n)$ the post-critical class of γ_n. By assumption, there exist (X, f), $c \in \mathcal{M}_{p/q}(X, f)$ and $x_0 \in X$ such that $h_{n-1}(x_0) = \gamma_{n-1}$. Let $\hat{X} := X \cup \{*\}$ be the compact Hausdorff space obtained from X by adding a point denoted $*$. The set $\{*\}$ is both an open and closed subset of $\hat{X}$. Then we define a continuous mapping $\hat{f} : \hat{X} \to \hat{X}$ by $\hat{f}_{|X} = f$ and $\hat{f}(*) = x_0$. Either, $c(\gamma_n) \subset R_n$ then we set $\hat{c}_{|X} = c$ and $\hat{c}(*)$ the landing point of external rays with argument in $c(\gamma_n)$; otherwise, $c(\gamma_n)$ induces a unique connected component, B, of

$$\overline{\mathbb{C}} \setminus \bigcup_{r \in R_n} \overline{\mathcal{R}_{x_0}(r)}.$$

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

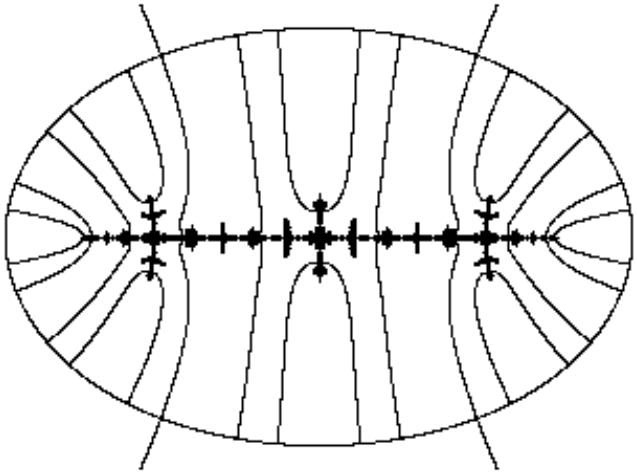


FIGURE 4. Realization of an element of Γ_4.

We define $\hat{c}(*)$ as any element of $B \cap K_{x_0}$. The fibered polynomial $P : \hat{X} \times \mathbb{C} \to \hat{X} \times \mathbb{C}$ with $P(x, z) = (\hat{f}(x), z^2 + \hat{c}(x))$ realizes γ_n. □

Thus the elements of Γ_n are exactly the equivalence relations obtained by the combinatorics of external rays. As already mentioned in the introduction, the elements of Γ_n do not need to be invariant under angle doubling. For example, the element γ_4 of Figure 3 is not realizable by a constant parameter since $i_{4,3}(\gamma_4) \neq Q_{4,3}(\gamma_4)$. Nevertheless, if one consider $\mathbb{R}/\mathbb{Z}$ equipped with f the rotation $x \mapsto x + \theta_0 \pmod 1$ with θ_0 the golden mean, the fibered polynomial $P(x, z) = (f(x), z^2 - 1.6 + 0.3\sin(2\pi x))$ realizes γ_4. Figure 4 shows the filled-in Julia sets over 0 and their external rays with angles in R_4.

3.3. *Order relation and topological structure.* The space Γ_n is equipped with a (partial) order relation in the following way.

Definition 3.6. We say that γ_n is smaller than γ_n' iff

$$\forall r, r' \in R_n, \quad \pi_{\gamma_n}(r) = \pi_{\gamma_n}(r') \text{ implies } \pi_{\gamma_n'}(r) = \pi_{\gamma_n'}(r').$$

Clearly, this defines an order relation that we denote by $\preceq$.

We now endow Γ_n with two topologies that in fact turn out to be equivalent, but which will provide different perspectives of the same space. We consider $\mathcal{T}_1$ the topological structure of Γ_n generated by the family of open sets

$$U_{r,r'} = \{\gamma_n \in \Gamma_n : \pi_{\gamma_n}(r) \neq \pi_{\gamma_n}(r')\}, \quad \text{where } r, r' \in R_n.$$

Similarly $\mathcal{T}_2$ is the topological structure generated by the family of closed sets

$$F_{\gamma_n} = \{\gamma_n' \in \Gamma_n : \gamma_n \preceq \gamma_n'\}, \quad \text{where } \gamma_n \in \Gamma_n.$$

PROPOSITION 3.7. *The topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ are equivalent.*

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

Proof. The main point is that, since Γ_n is finite, a union of closed sets is closed and an intersection of open sets is open. Let $U_{r,r'}$ be an open set for $\mathcal{T}_1$. Suppose that $\gamma \notin U_{r,r'}$ and consider $\gamma' \in \Gamma_n$ such that $\gamma \preceq \gamma'$. Then $\pi_\gamma(r) = \pi_\gamma(r') = \pi_{\gamma'}(r) = \pi_{\gamma'}(r')$, and one has

$$\Gamma_n \setminus U_{r,r'} = \bigcup_{\gamma \notin U_{r,r'}} F_\gamma.$$

This proves that any open set for $\mathcal{T}_1$ is also an open set for $\mathcal{T}_2$. Conversely, consider F_γ a closed set for $\mathcal{T}_2$. We denote by $\Lambda(\gamma)$ the set of pairs $(r_j, r_l) \in R_n \times R_n$ such that $r_j \neq r_l$ and $\pi_\gamma(r_j) = \pi_\gamma(r_l)$. Then we obtain

$$F_\gamma = \bigcap_{(r_j,r_l)\in\Lambda(\gamma)} \Gamma_n \setminus U_{r_j,r_l}.$$

This completes the proof of Proposition 3.7. □

Note that h_n is continuous for this topology. Indeed, if $\mathcal{R}_x(r)$ and $\mathcal{R}_x(r')$ do not land at the same point, neither do $\mathcal{R}_{x'}(r)$ and $\mathcal{R}_{x'}(r')$ for every x' sufficiently closed to x. In fact, the topology on Γ_n is the finest, turning each h_n to a continuous map. Note that the topology is not trivial since Γ_n is not a Hausdorff space.

Now, we present some elementary properties of the critical and post-critical classes in relation to the topology.

LEMMA 3.8. *Let γ_n and $\beta_n \in \Gamma_n$, then the following properties hold:*
(1) $\hat{\imath}_{n,n-1}(v(\gamma_n)) = v(i_{n,n-1}(\gamma_n))$;
(2) *if $\gamma_n \in \overline{\{\beta_n\}}$ then $v(\gamma_n) \cap \overline{v(\beta_n)} \neq \emptyset$.*

Proof. The first assertion clearly follows from the definition. For the second point, we distinguish the following two cases.

(i) $v(\gamma_n) \subset R_n'$, then $c(\gamma_n) \subset c(\beta_n)$ hence $v(\gamma_n) \subset v(\beta_n)$.

(ii) $v(\gamma_n) \subset R_n$. Suppose by contradiction that $v(\gamma_n) \cap \overline{v(\beta_n)} = \emptyset$. Let Λ be the set of pairs (r_i, r_j) of $R_n \times R_n$ such that r_i and r_j belong to $\overline{v(\beta_n)}$ and $\pi_{\beta_n}(r_i) = \pi_{\beta_n}(r_j)$. Denoting

$$F = \bigcap_{(r_i,r_j)\in\Lambda} \Gamma_n \setminus U_{r_i,r_j}$$

we obtain a closed subset of Γ_n such that $\beta_n \in F$ whereas $\gamma_n \notin F$, giving a contradiction. □

We turn now to the regularity of $Q_{n,n-1}$ and $i_{n,n-1}$.

PROPOSITION 3.9. *The maps $Q_{n,n-1}$ and $i_{n,n-1} : \Gamma_n \to \Gamma_{n-1}$ are both continuous and open, and they make the following diagram commutative:*

$$\begin{array}{ccc} \Gamma_n & \xrightarrow{i_{n,n-1}} & \Gamma_{n-1} \\ {\scriptstyle Q_{n,n-1}}\downarrow & & \downarrow{\scriptstyle Q_{n-1,n-2}} \\ \Gamma_{n-1} & \xrightarrow{i_{n-1,n-2}} & \Gamma_{n-2} \end{array}$$

Moreover, $Q_{n,n-1}$ is a closed map.

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

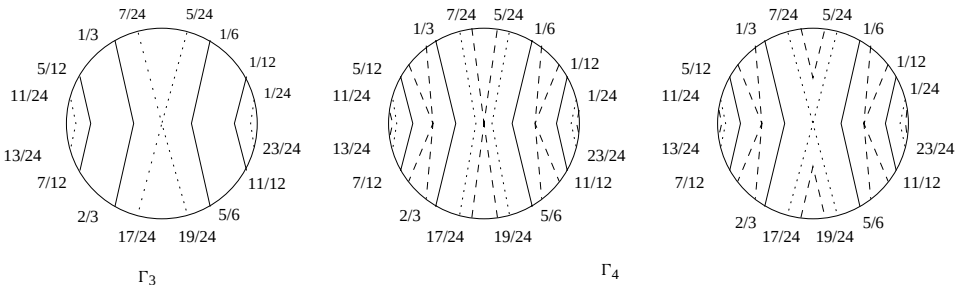


FIGURE 5. Closed points of Γ_3 and Γ_4 (in the (1/2)-limb).

Proof. If $\gamma_n \preceq \gamma_n'$ then

$$i_{n,n-1}(\gamma_n) \preceq i_{n,n-1}(\gamma_n') \quad \text{and} \quad Q_{n,n-1}(\gamma_n) \preceq Q_{n,n-1}(\gamma_n').$$

This ensures that the inverse image under $Q_{n,n-1}$ or $i_{n,n-1}$ of a closed subset of Γ_n is closed in Γ_{n-1}.

Concerning the openness of $Q_{n,n-1}$, we show that $Q_{n,n-1}(U_{r,r'})$ is an open subset of Γ_n. Let us denote $\rho = \widehat{Q}_{n,n-1}(r)$, $\rho' = \widehat{Q}_{n,n-1}(r')$ and consider $\delta \in Q_{n,n-1}(U_{r,r'})$. First, if $\pi_\delta(\rho) \neq \pi_\delta(\rho')$ it is straightforward that $\delta \in U_{\rho,\rho'} \subset Q_{n,n-1}(U_{r,r'})$. Thus assume that $\pi_\delta(\rho) = \pi_\delta(\rho')$. We claim that under this condition Γ_{n-1} is equal to $Q_{n,n-1}(U_{r,r'})$. Take $\delta'_{n-1} \in \Gamma_{n-1}$, then we shall construct by hand an element δ'_n belonging to $Q^{-1}_{n,n-1}(\delta'_{n-1}) \cap U_{r,r'}$. If $\pi_{\delta'_{n-1}}(\rho) \neq \pi_{\delta'_{n-1}}(\rho')$, any $\delta'_n \in Q^{-1}_{n,n-1}(\delta'_{n-1})$ will satisfy $\pi_{\delta'_n}(r) \neq \pi_{\delta'_n}(r)$. Otherwise, we select a critical value c^* in the same connected component of $E_n \setminus \{\rho, \rho'\}$ as $c(\delta)$. Hence we construct $\delta'_n \in \Gamma_n$ such that $c(\delta'_n) = c^*$ and $Q_{n,n-1}(\delta'_n) = \delta'_{n-1}$. It is immediate to see that $\delta'_n \in U_{r,r'}$. In both cases we obtain that $Q_{n,n-1}(U_{r,r'})$ is open since it contains an open neighborhood of any δ.

Arguing as above, we show that $i_{n,n-1}(U_{r,r'})$ is an open subset of Γ_{n-1} and that $Q_{n,n-1}(\Gamma_n \setminus U_{r,r'})$ is a closed subset of Γ_n for any $r, r' \in R_n$. □

The reason why $i_{n,n-1}$ fails to be a closed map is that $\hat{\imath}_{n,n-1}(R_n) \neq R_{n-1}$. Figure 5 shows a closed point (the middle one) of Γ_4 whose image under $i_{4,3}$ is an open subset of Γ_3.

For all $n, l \in \mathbb{N}$ we denote by $i_{n+l,n} : \Gamma_{n+l} \longrightarrow \Gamma_n$ and $\hat{\imath}_{n+l,n} : E_{n+l} \longrightarrow E_n$ the compositions

$$i_{n+l,n} = i_{n+l,n+l-1} \circ \cdots \circ i_{n+1,n},$$
$$\hat{\imath}_{n+l,n} = \hat{\imath}_{n+l,n+l-1} \circ \cdots \circ \hat{\imath}_{n+1,n}.$$

4. *Three properties of* Γ_n

The main result of this section (Theorem 4.2) states that, although Γ_n is not Hausdorff, we can separate distinct elements in a suitable sense.

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

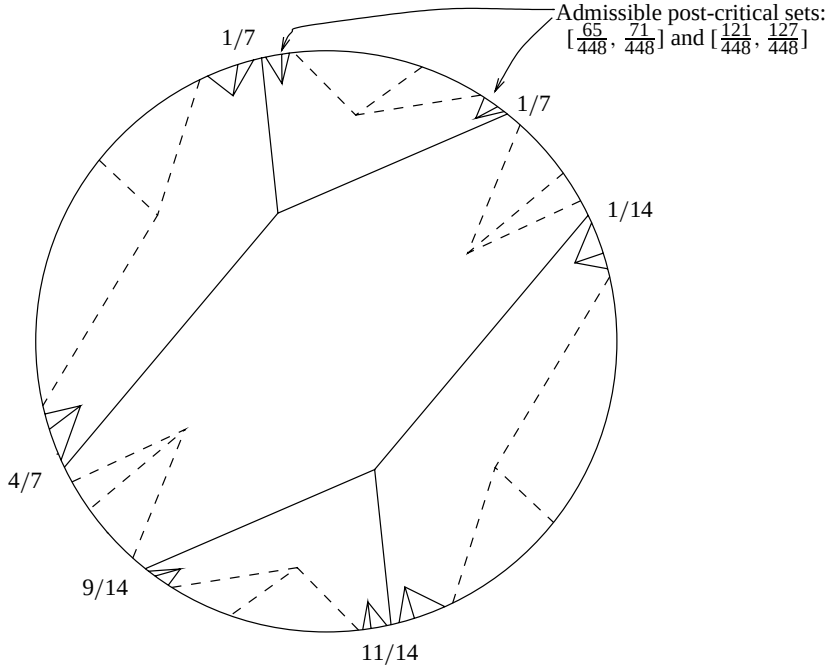


FIGURE 6. The two satellites of the (1/3)-limb with rank two.

4.1. *Satellite assumption.* Recall that p/q is a given rational number and that Γ_n describes the combinatorics of the (p/q)-limb. From now on we shall make the additional assumption that the configurations of Γ_n are in the same satellite.

Definition 4.1. We say that Γ_n satisfies the satellite hypothesis if there exists an integer ν_0 (called the rank) such that if $r_0 = q\,\nu_0$, then:

- $\Gamma_{r_0} = \{\gamma_{r_0}\}$ with $R_0(p/q) \subset \overline{c(\gamma_{r_0})}$;
- $\Gamma_{r_0+1} = \{\gamma^1_{r_0+1}, \dots \gamma^q_{r_0+1}\}$ with $\bigcup_{i=1}^{q} c(\gamma^i_{r_0+1})$ a closed connected subset of E_{r_0+1} disjoint from $R_0(p/q)$.

To say that γ_{r_0+1} is in a given satellite is equivalent to saying that the post-critical class of γ_{r_0+1} is contained in the closure of a given connected component of $c(\gamma_{r_0}) \setminus c^*$ where $c^* \subset c(\gamma_{r_0})$ is the unique class of γ_{r_0} such that $R_0(p/q)$ is contained in the closure of c^* (see in Figure 6 the two satellites of rank two in the (1/3)-limb). Since, for a given ν_0 there are exactly 2^{ν_0-1} such classes, the satellites of rank ν_0 are indexed by a dyadic rational number j of the form $j'/2^{\nu_0}$.

Up to the level $r_0 + 1$, the configurations of Γ_n are exactly those appearing in the non-fibered case. A description of the satellites of the Mandelbrot set can be found in [**Hub93**] or [**Mil98**]. We fix the rational number p/q and the dyadic number j, and we assume that Γ_n satisfies the corresponding satellite assumption. We still denote $\Gamma_n = \Gamma_n(p/q, j)$.

THEOREM 4.2. *For all $n \geq 1$ the following properties hold.*

(1) *The space Γ_n is connected.*

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

(2) *Suppose that* $\gamma_n,\ \gamma'_n \in \Gamma_n$ *with* $\overline{\{\gamma_n\}} = \overline{\{\gamma'_n\}}$. *Then* $\gamma_n = \gamma'_n$.
(3) *Suppose that* $\overline{\{\gamma_n\}} \cap \overline{\{\gamma'_n\}} = \emptyset$ *and choose* γ_{n+r_0} *and* γ'_{n+r_0} *such that*

$$\gamma_{n+r_0} \in (i_{n+r_0,n})^{-1}(\gamma_n) \quad \textit{and} \quad \gamma'_{n+r_0} \in (i_{n+r_0,n})^{-1}(\gamma'_n).$$

Then the smallest open subsets of Γ_{n+r_0} *containing respectively* γ_{n+r_0} *and* γ'_{n+r_0} *are disjoint.*

The three assertions of Theorem 4.2 constitute the main ingredients in the construction of the configuration space to obtain the expected properties of compactness and connectedness. Since Γ_{n+r_0} is finite the smallest open set containing γ_{n+r_0} is defined without ambiguity as the intersection of all open subsets of Γ_{n+r_0} containing γ_{n+r_0}. In view of proving Theorem 4.2, we need to introduce the space

$$\Delta_n = \{(\gamma, x) : \gamma \in \Gamma_n \text{ and } x \in E_n/\gamma\}.$$

The introduction of Δ_n is motivated by the following remark: a configuration of order $n+1$, γ_{n+1}, is completely determined by γ_n, its image under $Q_{n+1,n}$ and the knowledge of a post-critical class which belongs to E_n/γ_n. We shall make this rigorous by showing that Γ_{n+1} is homeomorphic to a closed subset of Δ_n. Then, we shall deduce properties (1), (2) and (3) of Γ_{n+1} from the corresponding properties of Δ_n.

The first step is to endow Δ_n with a topology which is compatible with its fibered structure. To that purpose, we first analyze how the open and closed subsets of E_n/γ_n depend on the configuration γ_n. We begin with some notation.

- For all $\gamma \in \Gamma_n$, we denote by $U(\gamma)$ the minimal (with respect to the inclusion), open subset of Γ_n containing γ.
- Moreover, if $\gamma \in \Gamma_n$ and $x \in E_n/\gamma$, we denote by $w_\gamma(x)$ the minimal open subset of E_n/γ containing x.
- Given a subset $\mathcal{H}_\gamma$ of E_n/γ, we denote by $\widetilde{\mathcal{H}}_\gamma$ the subset of E_n defined by $\widetilde{\mathcal{H}}_\gamma = \pi_\gamma^{-1}(\mathcal{H}_\gamma)$. Sometimes when no confusion is possible, we identify $\mathcal{H}_\gamma$ with $\widetilde{\mathcal{H}}_\gamma$.

The next proposition states that it is possible to follow an open subset $\mathcal{O}_\gamma$ of E_n/γ as γ' ranges in $U(\gamma)$. Similarly, it is possible to follow the closed subsets of E_n/γ as γ' varies in the closure $\overline{\{\gamma\}}$ (which is generally not reduced to γ).

PROPOSITION 4.3. *Let* $\gamma \in \Gamma_n$ *and* $\mathcal{O}_\gamma$ *be an open subset of* E_n/γ. *If* $\gamma' \in U(\gamma)$ *then* $\widetilde{\mathcal{O}}_{\gamma'} = \pi_{\gamma'}^{-1}(\pi_{\gamma'}(\widetilde{\mathcal{O}}_\gamma))$ *is an open subset of* E_n. *Moreover,* $\widetilde{\mathcal{O}}_{\gamma'} \setminus \widetilde{\mathcal{O}}_\gamma$ *is included in* R'_n.

PROPOSITION 4.4. *Let* $\gamma \in \Gamma_n$ *and* F_γ *be a closed subset of* E_n/γ. *If* $\gamma' \in \overline{\{\gamma\}}$, *then* $\widetilde{F}_{\gamma'} = \pi_{\gamma'}^{-1}(\pi_{\gamma'}(\widetilde{F}_\gamma))$ *is a closed subset of* E_n. *Moreover,* $\widetilde{F}_{\gamma'} \setminus \widetilde{F}_\gamma$ *is included in* R_n.

We call $\mathcal{O}_{\gamma'} = \pi_{\gamma'}(\widetilde{\mathcal{O}}_{\gamma'})$ (respectively $F_{\gamma'} = \pi_{\gamma'}(\widetilde{F}_{\gamma'})$) the trace of $\mathcal{O}_\gamma$ (respectively F_γ). It is by definition the saturated set of $\mathcal{O}_\gamma$ (respectively F_γ) under the relation γ'.

Proof of Proposition 4.3. First, we note that if $\gamma \in U_{r,r'}$ with $r, r' \in R_n$ then by minimality of $U(\gamma)$ we get $U(\gamma) \subset U_{r,r'}$. Next, we consider $\gamma' \in U(\gamma)$ and denote $\mathcal{O} = \widetilde{\mathcal{O}}_{\gamma'} \setminus \widetilde{\mathcal{O}}_\gamma$. We claim that $\mathcal{O} \cap R_n = \emptyset$. Suppose this is not true and consider $r_0 \in \mathcal{O} \cap R_n$. Then, there exists $r_1 \in \widetilde{\mathcal{O}}_\gamma$ such that

$$\pi_{\gamma'}(r_0) = \pi_{\gamma'}(r_1) \quad \text{but} \quad \pi_\gamma(r_0) \neq \pi_\gamma(r_1).$$

Hence, $\gamma \in U_{r_0,r_1}$ whereas $\gamma' \notin U_{r_0,r_1}$. This contradicts the note above. □

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

The corresponding statement for closed sets is proved in a completely symmetrical way. We leave it to the reader. The following lemma extends Lemma 3.8 and gives a useful characterization of the closure $\overline{\{\gamma_n\}}$ whenever $\gamma_n \in \Gamma_n$.

LEMMA 4.5. *Let $\gamma_n,\ \gamma_n' \in \Gamma_n$, and denote $\gamma_{n-1} = Q_{n,n-1}(\gamma_n)$ and $\gamma_{n-1}' = Q_{n,n-1}(\gamma_n')$. The following conditions are equivalent:*

(i) $\gamma_n' \in \overline{\{\gamma_n\}}$;

(ii) $\gamma_{n-1}' \in \overline{\{\gamma_{n-1}\}}$ *and* $c(\gamma_n') \in F_{\gamma_{n-1}'}(c(\gamma_n))$, *where* $F_{\gamma_{n-1}'}(c(\gamma_n))$ *is the trace of* $\overline{\{c(\gamma_n)\}}$ *in* E_{n-1}.

Proof. That (i) implies (ii) is a consequence of the continuity of $Q_{n,n-1}$ and of Proposition 3.8.

To prove that (ii) implies (i), it suffices to show that if $r, r' \in R_n$ with $\pi_{\gamma_n}(r) = \pi_{\gamma_n}(r')$ then $\pi_{\gamma_n'}(r) = \pi_{\gamma_n'}(r')$, whenever γ_n' satisfies (ii). Denote $\rho = \widehat{Q}_{n,n-1}(r)$ and $\rho' = \widehat{Q}_{n,n-1}(r')$. Let us consider two cases. First, assume that $\rho = \rho'$. This means that the critical class is included in R_n and that $r, r' \in v(\gamma_n)$. Hence $c(\gamma_n) \subset R_{n-1}$ and $\rho = \rho' \in c(\gamma_n')$, thus $\pi_{\gamma_n'}(r) = \pi_{\gamma_n'}(r')$. Suppose on the contrary that $\rho \neq \rho'$, and consider C_1 and C_2, the two connected components of $E_{n-1} \setminus \{\rho, \rho'\}$. If, for instance, $c(\gamma_n) \in C_1$ the assumption $c(\gamma_n') \in F_{\gamma_{n-1}'}(c(\gamma_n))$ guarantees that $c(\gamma_n') \in \overline{C_1}$, thus $\pi_{\gamma_n'}(r) = \pi_{\gamma_n'}(r')$. □

Now, we are aiming to endow Δ_n with a topological structure. Again, we define it in two different ways (the first by a collection of open sets, the second by a collection of closed sets), and we check immediately that the two notions coincide.

Fix $(\gamma_0, x_0) \in \Delta_n$ and denote $w_\gamma(x_0)$ the trace of by $w_{\gamma_0}(x_0)$ for all $\gamma \in U(\gamma_0)$. According to Proposition 4.3, $w_\gamma(x_0)$ is an open subset of E_n/γ. We define $\mathcal{E}_1$, a topology on Δ_n by, its basis of open neighborhoods,

$$W(\gamma_0, x_0) = \{(\gamma, x) \in \Delta_n : \gamma \in U(\gamma_0) \text{ and } x \in w_\gamma(x_0)\}.$$

Similarly, we define the topology $\mathcal{E}_2$ by its basis of closed sets

$$F(\gamma_0, x_0) = \{(\gamma, x) \in \Delta_n : \gamma \in \overline{\{\gamma_0\}} \text{ and } x \in F_\gamma(x_0)\},$$

where $F_\gamma(x_0)$ is the trace of $\overline{\{x_0\}}$, i.e. $F_\gamma(x_0) = \pi_\gamma(\pi_{\gamma_0}^{-1}(\overline{\{x_0\}}))$.

PROPOSITION 4.6. *The topologies $\mathcal{E}_1$ and $\mathcal{E}_2$ are equivalent.*

Proof. To prove that $\mathcal{E}_1$ is finer than $\mathcal{E}_2$ it suffices to show that the complement of $F(\gamma_0, x_0)$ is open in $\mathcal{E}_1$, i.e. $W(\gamma_1, x_1) \cap F(\gamma_0, x_0) = \emptyset$ whenever $(\gamma_1, x_1) \notin F(\gamma_0, x_0)$. In the case where $\overline{\{\gamma_0\}} \cap U(\gamma_1) = \emptyset$ it follows from the definition that $W(\gamma_1, x_1) \cap F(\gamma_0, x_0) = \emptyset$. So, assume that $\overline{\{\gamma_0\}} \cap U(\gamma_1) \neq \emptyset$ or equivalently that $\gamma_1 \in \overline{\{\gamma_0\}}$. Since $(\gamma_1, x_1) \notin F(\gamma_0, x_0)$, x_1 does not belong to $F_{\gamma_1}(x_0)$. Thus $w_{\gamma_1}(x_1)$, the minimal open subset of E_n/γ_1 containing x_1, satisfies

$$w_{\gamma_1}(x_1) \cap F_{\gamma_1}(x_0) = \emptyset. \tag{2}$$

According to Propositions 4.3 and 4.4, we may write for all $\gamma \in U(\gamma_1) \cap \overline{\{\gamma_0\}}$:

- $\widetilde{F}_\gamma(x_0) = \widetilde{F}_{\gamma_0}(x_0) \cup \{r_1, \ldots, r_i\}$ with $r_k \in R_n$;
- $\widetilde{w}_\gamma(x_1) = \widetilde{w}_{\gamma_1}(x_1) \cup \{r_1', \ldots, r_j'\}$ with $r_k' \in R_n'$.

We want to check that $w_\gamma(x_1) \cap F_\gamma(x_0)$ is empty whenever $\gamma \in U(\gamma_1) \cap \overline{\{\gamma_0\}}$.

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

CLAIM. *For all* $\gamma \in U(\gamma_1) \cap \overline{\{\gamma_0\}}$ *we have* $\widetilde{F}_\gamma(x_0) \subset \widetilde{F}_{\gamma_1}(x_0)$.

Proof. Indeed, if $\gamma_1 \in U_{r,r'}$ then $U(\gamma_1)$ is included in $U_{r,r'}$. Consequently, if $r_k \notin \widetilde{F}_{\gamma_1}(x_0)$ then $r_k \notin \widetilde{F}_\gamma(x_0)$ for all $\gamma \in U(\gamma_1)$. □

Next, assume by contradiction that there exists $\gamma \in U(\gamma_1) \cap \overline{\{\gamma_0\}}$ satisfying $\widetilde{w}_\gamma(x_1) \cap \widetilde{F}_\gamma(x_0) \neq \emptyset$. From the claim and the assumption that $\widetilde{w}_{\gamma_1}(x_1) \cap \widetilde{F}_{\gamma_1}(x_0) = \emptyset$, we deduce that there exist ρ and $\rho^* \in R'_n$ such that

$$\rho \in \widetilde{w}_\gamma(x_1) \cap \widetilde{F}_\gamma(x_0), \quad \rho^* \in \widetilde{w}_{\gamma_1}(x_1) \quad \text{and} \quad \pi_\gamma(\rho) = \pi_\gamma(\rho^*).$$

Since $\widetilde{F}_\gamma(x_0)$ is a saturated set under γ, ρ^* actually belongs to $\widetilde{F}_\gamma(x_0)$. Hence, $\rho^* \in \widetilde{F}_{\gamma_1}(x_0) \cap \widetilde{w}_{\gamma_1}(x_1)$, which contradicts (2).

The strategy for proving that $\mathcal{E}_2$ is finer than $\mathcal{E}_1$ is similar. Details are left to the reader. □

Recall that ν_0 stands for the rank of any $\gamma_n \in \Gamma_n$ and $r_0 = q\nu_0$. By assumption, $\Gamma_{r_0+1} = \{\gamma^1_{r_0+1}, \ldots, \gamma^q_{r_0+1}\}$. Let us denote by $E^{cr}_{r_0} = c(\gamma^1_{r_0+1}) \cup \cdots \cup c(\gamma^q_{r_0+1})$ the closed connected subset of E_{r_0} corresponding to the post-critical locus of the configurations. The set $E^{cr}_{r_0}$ characterizes completely the satellite we are dealing with (there are exactly 2^{ν_0-1} choices for $E^{cr}_{r_0}$ corresponding to the 2^{ν_0-1} possible satellites). For all $n \geq r_0$ we denote by $E^{cr}_n = \hat{\imath}^{-1}_{n,r}(E^{cr}_{r_0})$ and Δ^{cr}_n the subset of Δ_n which consists of

$$\Delta^{cr}_n = \{(\gamma, x) : \gamma \in \Gamma_n \text{ and } x \in E^{cr}_n/\gamma\}.$$

We now have all the ingredients to identify Γ_n with Δ^{cr}_{n-1}.

THEOREM 4.7. *For all* $n \geq r_0 + 1$, *the map* $\Psi_n : \Gamma_n \longrightarrow \Delta^{cr}_{n-1}$ *defined by*

$$\Psi_n(\gamma_n) = (Q_{n,n-1}(\gamma_n), c(\gamma_n))$$

is a homeomorphism.

Proof. Obviously, Ψ_n is injective and surjective. Let us focus on the continuity of Ψ_n. Pick $(\gamma_0, x_0) \in \Delta^{cr}_{n-1}$ and consider $F(\gamma_0, x_0)$, the corresponding minimal closed subset of Δ^{cr}_{n-1}. We wish to prove that $\Psi_n^{-1}(F(\gamma_0, x_0))$ is also a closed subset of Γ_n. To that purpose, fix γ_n as an element of $\Psi_n^{-1}(F(\gamma_0, x_0))$ and consider $\widehat{\gamma}_n$ an element of the closure $\overline{\{\gamma_n\}}$. We denote $\gamma_{n-1} = Q_{n,n-1}(\gamma_n)$ and $\widehat{\gamma}_{n-1} = Q_{n,n-1}(\widehat{\gamma}_n)$. Since $Q_{n,n-1}$ is continuous, $\widehat{\gamma}_{n-1} \in \overline{\{\gamma_{n-1}\}}$. Thus we deduce from Lemma 3.8 that

$$F := c(\widehat{\gamma}_n) \cap \overline{c(\gamma_n)} \neq \emptyset. \tag{3}$$

Let us distinguish the two following cases.

Case 1. $c(\widehat{\gamma}_n) \subset R_{n-1}$. Take ρ as an element of $R_{n-1} \cap F$. Since $\overline{c(\gamma_n)} \subset \widetilde{F}_{\gamma_{n-1}}(x_0)$ and since by definition $\widetilde{F}_{\gamma_{n-1}}(x_0)$ is saturated under γ_{n-1}, there exists $\rho_0 \in \widetilde{F}_{\gamma_0}(x_0) \cap R_{n-1}$ such that $\pi_{\gamma_{n-1}}(\rho) = \pi_{\gamma_{n-1}}(\rho_0)$. Thus

$$\pi_{\hat{\gamma}_{n-1}}(\rho) = \pi_{\hat{\gamma}_{n-1}}(\rho_0), \quad \forall \hat{\gamma}_{n-1} \in \overline{\{\gamma_{n-1}\}}.$$

Hence, the post-critical class $c(\widehat{\gamma}_n)$ is included in $F_{\hat{\gamma}_{n-1}}(x_0)$, and it follows that $\Psi_n(\widehat{\gamma}_n) \in F(\gamma_0, x_0)$.

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

Case 2. $c(\widehat{\gamma}_n) \subset R'_{n-1}$. From (3) we deduce that $c(\gamma_n) \subset R'_{n-1}$ and $c(\gamma_n) \cap c(\widehat{\gamma}_n) \neq \emptyset$. Consequently, if $\rho \in c(\gamma_n) \cap c(\widehat{\gamma}_n)$ then $\rho \in \widetilde{F}_{\hat{\gamma}_{n-1}}(x_0)$. Hence $c(\widehat{\gamma}_n) \in F_{\hat{\gamma}_n}(x_0)$.

In any case $\Psi_n^{-1}(F(\gamma_0, x_0))$ is a closed subset of Γ_n and thus Ψ_n is continuous.

To prove that Ψ_n is closed, note first that the image under Ψ_n of the closure $\overline{\{\gamma_n\}}$ is by definition

$$\Psi_n(\overline{\{\gamma_n\}}) = \{(\gamma, x) : \gamma \in Q_{n,n-1}(\overline{\{\gamma_n\}}),\ x = c(\gamma)\}.$$

Using Lemma 4.5, we deduce the equality

$$\Psi_n(\overline{\{\gamma_n\}}) = F(Q_{n,n-1}(\gamma_n), c(\gamma_n)).$$

This completes the proof of Theorem 4.7. □

In what follows, it will often be sufficient to restrict ourselves to the closed points γ_n of Γ_n, i.e. $\overline{\{\gamma_n\}} = \{\gamma_n\}$. The lemma below explains why this restriction is relevant and provides a characterization of the closed points of Γ_n.

LEMMA 4.8. *For all $\gamma \in \Gamma_n$ there exists a closed point in $\overline{\{\gamma\}}$. Furthermore, β is a closed point of Γ_n iff $Q_{n,n-1}(\beta)$ is a closed point of Γ_{n-1} and the critical class $v(\beta)$ is included in R_n.*

Proof. Since the space Γ_n is finite, every non-empty and closed subset of Γ_n contains a minimal (with respect to the inclusion), non-empty closed subset labeled F. So, it remains to show that F is reduced to a single point. Assume that β and β' belong to F. Then, by minimality $\overline{\{\beta\}} = \overline{\{\beta'\}} = F$. Consequently, $\beta' \preceq \beta$ and $\beta \preceq \beta'$ hence $\beta = \beta'$.

The second assertion results immediately from Lemma 4.5. □

4.2. *Proof of Theorem 4.2.* The key result from which we shall deduce the third assertion of Theorem 4.2 is the following.

PROPOSITION 4.9. *Fix $n \geq 1$ and γ_n as a closed point of Γ_n. Let F_{γ_n} and F'_{γ_n} be two disjoint and closed subsets of E_n/γ_n. If $\gamma_{n+r_0} \in i^{-1}_{n+r_0,n}(\gamma_n)$, then $\mathcal{O}_{\gamma_{n+r_0}}$ and $\mathcal{O}'_{\gamma_{n+r_0}}$, the minimal open subsets of E_{n+r_0}/γ_{n+r_0} containing F_{γ_n} and F'_{γ_n} respectively, are disjoint. Moreover, the traces $\mathcal{O}_{\delta_{n+r_0}}$ and $\mathcal{O}'_{\delta_{n+r_0}}$ of $\mathcal{O}_{\gamma_{n+r_0}}$ and $\mathcal{O}'_{\gamma_{n+r_0}}$, respectively, remain disjoint when $\delta_{n+r_0} \in U(\gamma_{n+r_0})$.*

Proof. One may assume, without loss of generality, that F_{γ_n} and F'_{γ_n} are the closures of only one class of E_n/γ_n. Then we proceed by induction. If $n = 1$, Γ_1 is reduced to $\{\gamma_1\}$ and the proposition follows from the satellite assumption, since two classes of R_1/γ_1 are contained in two disjointed open subsets of E_{r_0+1}/γ_{r_0+1}. Now fix $n \geq 2$. We denote by $Q_{\gamma_n} : E_n/\gamma_n \to E_{n-1}/Q_{n,n-1}(\gamma_n)$ the map $\pi_{Q_{n,n-1}(\gamma_n)} \circ \widehat{Q}_{n,n-1} \circ \pi^{-1}_{\gamma_n}$. We consider two cases.

(1) $Q_{\gamma_n}(F_{\gamma_n}) \cap Q_{\gamma_n}(F'_{\gamma_n}) = \emptyset$. The inductive hypothesis at order $n-1$ and the openness of $Q_{n,n-1}$ provide the desired result.

(2) Suppose now that the previous intersection is not empty. Observe that one of the two closed sets, say for instance F_{γ_n}, does not intersect $\overline{v(\gamma_n)}$. Hence, $E_n \setminus v(\gamma_n)$ is made up of n_0 open connected components labeled $\mathcal{B}_1, \ldots, \mathcal{B}_{n_0}$. We assume that

$$F_{\gamma_n} \subset \mathcal{B}_1 \quad \text{and} \quad F'_{\gamma_n} \subset \mathcal{B}_2 \cup v(\gamma_n).$$

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

Similar to the previous case, the inductive hypothesis enables us to separate F_{γ_n} from $\overline{v(\gamma_n)}$. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be the disjoint open subsets of E_{n+r_0}/γ_{n+r_0} containing F_{γ_n} and $\overline{v(\gamma_n)}$, respectively. Finally, $\mathcal{O}_1$ and $\mathcal{O}_2 \cup \mathcal{B}_2$ are two open sets satisfying the required properties. □

We move now to the proof of Theorem 4.2.

Claim (1). We show by induction that Γ_n and Δ_{n-1} are both connected. The case $n = 2$ is obvious since $\Gamma_2 = \{\gamma_2\}$. Next, let $h : \Delta_n \to \{0, 1\}$ be a continuous map. For all $\gamma_0 \in \Gamma_n$, the set $\{(\gamma_0, x) : x \in E_n/\gamma_0\}$ is connected, hence there exists a map $h^* : \Gamma_n \to \{0, 1\}$ such that

$$h = h^* \circ q_n,$$

where $q_n : \Delta_n \to \Gamma_n$ denotes the first projection. Since q_n is an open map, h^* is continuous. By the inductive hypothesis, h^* is constant and the same holds for h. Therefore Δ_n is connected. Since the sets $\{(\gamma_0, x) : x \in E_n^{cr}/\gamma_0\}$ are also connected, the same argument as before combined with Theorem 4.7 yields that Γ_{n+1} is also connected.

Claim (2). This has already been proved in Lemma 4.8.

Claim (3). This part of the theorem requires a little more work. The claim for $n = r_0 + 1$ results from the satellite assumption. Then we proceed by induction. Let γ_n and $\hat{\gamma}_n$ be two elements of Γ_n such that

$$\overline{\{\gamma_n\}} \cap \overline{\{\hat{\gamma}_n\}} = \emptyset.$$

Fix $\gamma_{n+r_0} \in i_{n+r_0,n}^{-1}(\gamma_n)$ and $\hat{\gamma}_{n+r_0} \in i_{n+r_0,n}^{-1}(\hat{\gamma}_n)$. We may still limit ourselves to the case of closed points. Indeed, it is sufficient to prove that

$$\Omega_n := \bigcup_{\substack{\gamma_n^f \text{ closed point} \\ \gamma_n^f \subset \overline{\{\gamma_n\}}}} \; \bigcup_{\gamma_{n+r_0}^f \in i_{n+r_0,n}^{-1}(\gamma_n^f)} U(\gamma_{n+r_0}^f)$$

and

$$\widehat{\Omega}_n := \bigcup_{\substack{\gamma_n^f \text{ closed point} \\ \gamma_n^f \subset \overline{\{\gamma_n\}}}} \; \bigcup_{\hat{\gamma}_{n+r_0}^f \in i_{n+r_0,n}^{-1}(\hat{\gamma}_n^f)} U(\hat{\gamma}_{n+r_0}^f)$$

are disjoint, since either $U(\gamma_{n+r_0}) \subset \Omega_n$ and $U(\hat{\gamma}_{n+r_0}) \subset \widehat{\Omega}_n$ or (assume for instance $U(\gamma_{n+r_0}) \not\subset \Omega_n$) γ_{n+r_0} and $\hat{\gamma}_{n+r_0}$ are contained in two different connected components of

$$\Gamma_{n+r_0} \setminus \left(\bigcup_{\substack{\gamma_n^f \text{ closed point} \\ \gamma_n^f \subset \overline{\{\gamma_n\}}}} i_{n+r_0,n}^{-1}(\gamma_n^f) \right).$$

So from now on we assume that γ_n and $\hat{\gamma}_n$ are closed points of Γ_n. We transpose the difficulties in Δ_{n-1} by considering

$$(\gamma_{n-1}, x_{n-1}) := \Psi_n(\gamma_n) \quad \text{and} \quad (\hat{\gamma}_{n-1}, \hat{x}_{n-1}) := \Psi_n(\hat{\gamma}_n),$$
$$(\gamma_{n+r_0-1}, x_{n+r_0-1}) := \Psi_{n+r_0}(\gamma_{n+r_0}) \quad \text{and} \quad (\hat{\gamma}_{n+r_0-1}, \hat{x}_{n+r_0-1}) := \Psi_{n+r_0}(\hat{\gamma}_{n+r_0}).$$

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

Recall that $W(\gamma_{n+r_0-1}, x_{n+r_0-1})$ (respectively $W(\hat{\gamma}_{n+r_0-1}, \hat{x}_{n+r_0-1})$) denotes the smallest open subset of Δ_{n+r_0-1} containing $(\gamma_{n+r_0-1}, x_{n+r_0-1})$ (respectively $(\hat{\gamma}_{n+r_0-1}, \hat{x}_{n+r_0-1})$). We want to prove that $W(\gamma_{n+r_0-1}, x_{n+r_0-1}) \cap W(\hat{\gamma}_{n+r_0-1}, \hat{x}_{n+r_0-1}) = \emptyset$. To that purpose, let us consider the two following cases.

(1) $\gamma_{n-1} \neq \hat{\gamma}_{n-1}$. The inductive hypothesis at order $n-1$ implies $U(\gamma_{n+r_0-1}) \cap U(\hat{\gamma}_{n+r_0-1}) = \emptyset$, hence

$$W(\gamma_{n+r_0-1}, x_{n+r_0-1}) \cap W(\hat{\gamma}_{n+r_0-1}, \hat{x}_{n+r_0-1}) = \emptyset.$$

(2) $\gamma_{n-1} = \hat{\gamma}_{n-1}$. In view of Proposition 4.9, $\mathcal{O}_\gamma$ and $\widehat{\mathcal{O}}_\gamma$ the traces in E_{n+r_0-1}/γ of the smallest open subsets containing $\{\hat{x}_{n-1}\}$ and $\{x_{n-1}\}$ are disjoint for all $\gamma \in U(\gamma_{n+r_0-1})$. This guarantees that

$$W(\gamma_{n+r_0-1}, x_{n+r_0-1}) \cap W(\hat{\gamma}_{n+r_0-1}, \hat{x}_{n+r_0-1}) = \emptyset. \qquad \square$$

5. *Configuration space*

5.1. *Inverse limit space.* Recall that our construction depends on a rational number p/q and a given satellite (i.e. a dyadic number j) which are omitted from the notation. We consider the inverse limit space of Γ_n:

$$\Gamma_\infty = \varprojlim (\Gamma_n \xrightarrow{i_{n,n-1}} \Gamma_{n-1}).$$

This is a subset of the Cartesian product of Γ_n whose elements $(\gamma_n)_{n\in\mathbb{N}}$ satisfy $i_{n,n-1}(\gamma_n) = \gamma_{n-1}$ for all $n \geq 1$ (see [**Bou50**] or [**Dug66**] for details about inverse limit spaces). The space Γ_∞ is naturally endowed with a self-continuous map

$$\mathcal{F} : \Gamma_\infty \longrightarrow \Gamma_\infty, \quad \mathcal{F}((\gamma_n)_{n\in\mathbb{N}}) = (Q_{n+1,n}(\gamma_{n+1}))_{n\in\mathbb{N}}.$$

We denote by $i_n : \Gamma_\infty \longrightarrow \Gamma_n$ the projection defined by $i_n((\gamma_p)_{p\in\mathbb{N}}) = \gamma_n$. We shall see that Γ_∞ inherits the compactness of Γ_n, but Γ_∞ is not Hausdorff because Γ_n are not.

PROPOSITION 5.1. *The space Γ_∞ is compact.*

Proof. Let (U_α) be an open covering of Γ_∞. We denote

$$\widehat{\Gamma}_n = \{\gamma \in \Gamma_n : i_n^{-1}(\{\gamma\}) \text{ is not contained in any of the } U_\alpha\}.$$

If there exists n_0 in $\mathbb{N}$ such that $\widehat{\Gamma}_{n_0} = \emptyset$, then

$$\Gamma_\infty \subset \bigcup_{\gamma\in\Gamma_{n_0}} i_{n_0}^{-1}(\{\gamma\}) \subset \bigcup_{j=0}^{k} U_{\alpha_j}$$

and we are done.

Therefore, suppose by contradiction that for all $n \geq 0$, $\widehat{\Gamma}_n \neq \emptyset$. Then by induction we can construct a sequence $\delta_\infty = (\delta_n)_{n\in\mathbb{N}}$ such that $\delta_n \in \widehat{\Gamma}_n$ and $i_{n,n-1}(\delta_n) = \delta_{n-1}$ for all $n \in \mathbb{N}$. Thus $\delta_\infty \in \Gamma_\infty$, and by assumption there exists α_0 such that $\delta_\infty \in U_{\alpha_0}$. Moreover, since U_{α_0} is an open subset of Γ_∞, it may be written in the form $i_N^{-1}(V_{\alpha_0})$ where $N \in \mathbb{N}$ and V_{α_0} is an open subset of Γ_N. Thus $i_N^{-1}(\{\delta_N\}) \subset U_{\alpha_0}$, which means that $\delta_N \notin \widehat{\Gamma}_N$ and gives a contradiction. $\square$

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

The following proposition deals with the minimal closed subsets of Γ_∞ and is analogous to Lemma 4.8.

PROPOSITION 5.2. *Every non-empty closed subset of Γ_∞ contains a non-empty closed subset which is minimal with respect to the inclusion. Furthermore, every minimal closed subset of Γ_∞ is reduced to a single point.*

Proof. Let F be a non-empty closed subset of Γ_∞. We denote by $\mathcal{K}(F)$ the collection of non-empty closed subsets of F. In view of the quasi-compactness of Γ_∞, $\mathcal{K}(F)$ is inductive. Thus, Zorn's lemma yields the existence of a minimal closed subset of F.

On the other hand, assume that a minimal closed subset G of Γ_∞ contains two distinct elements γ and γ'. Then, there exists an integer $n_0 > 0$ such that $i_{n_0}(\gamma) = \gamma_{n_0}$ and $i_{n_0}(\gamma') = \gamma'_{n_0}$ are two distinct elements of Γ_{n_0}. According to Theorem 4.2, we may assume, for instance, that $\gamma_{n_0} \notin \overline{\{\gamma'_{n_0}\}}$. Thus $\widehat{G} = i_{n_0}^{-1}(\overline{\{\gamma'_{n_0}\}}) \cap G$ is a non-empty closed set, strictly contained in G. This contradicts the minimality of G. □

5.2. *Proof of Theorem 1.2.* The configuration space, $X_\infty^{p/q,j}$, is by definition the set of closed points of Γ_∞:

$$X_\infty^{p/q,j} = \{\gamma_\infty \in \Gamma_\infty : \overline{\{\gamma_\infty\}} = \{\gamma_\infty\}\}.$$

It is equipped with the relative topology of Γ_∞. Since $Q_{n,n-1}$ is closed and open, $X_\infty^{p/q,j}$ is invariant under $\mathcal{F}$. The main result concerning the topology of $X_\infty^{p/q,j}$ is the following.

THEOREM 5.3. *The space $X_\infty^{p/q,j}$ is a Hausdorff compact and connected space.*

In view of Proposition 5.2, $X_\infty^{p/q,j}$ has the fundamental property that any open subset of Γ_∞ containing $X_\infty^{p/q,j}$ is actually equal to Γ_∞. Therefore, the compactness of Γ_∞ implies that of $X_\infty^{p/q,j}$. To show the separation of $X_\infty^{p/q,j}$, we start with the following lemma.

LEMMA 5.4. *Let $F \subset \Gamma_\infty$, and denote*

$$\widetilde{F} = \bigcap_{n \geq 0} i_n^{-1}(\overline{i_n(F)}).$$

Then $\overline{F}$, the closure of F, is actually equal to $\widetilde{F}$.

Proof. The inclusion $\overline{F} \subset \widetilde{F}$ is obvious.

Conversely, if $y \in \widetilde{F}$ we denote by V an open neighborhood of y of the form $V = i_m^{-1}(V_m)$, where V_m is an open subset of Γ_m. Since $i_m(y) \in \overline{i_m(F)}$ we get that $V_m \cap i_m(F) \neq \emptyset$. Therefore, $V \cap F \neq \emptyset$ which implies $y \in \overline{F}$. □

Proof of Theorem 5.3. Suppose now that x and x' are two distinct elements of $X_\infty^{p/q,j}$. As a consequence of the previous lemma we obtain that $\overline{i_n(x)} \cap \overline{i_n(x')} = \emptyset$, provided that n is large enough. In view of Theorem 4.2, the smallest open subsets of Γ_{n+r_0} containing $\overline{i_{n+r_0}(x)}$ and $\overline{i_{n+r_0}(x')}$, respectively, are disjoint. This yields two disjoint open subsets of Γ_∞ containing x and x', respectively.

The connectedness of $X_\infty^{p/q,j}$ follows from that of Γ_n for all $n \geq 1$. □

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

Observe that, by the previous proof, we have that for all $\gamma \in \Gamma_\infty$ the closed point contained in $\overline{\{\gamma\}}$ is actually unique. This enables us to define a continuous projection

$$p_\infty : \Gamma_\infty \longrightarrow X_\infty^{p/q,j}$$

by associating to every $\gamma \in \Gamma_\infty$ the unique closed point in $\overline{\{\gamma\}}$.

Now we can state the universal property of $X_\infty^{p/q,j}$, i.e. independent of the underlying basis (X, f). Let us denote by $\mathcal{M}_{p/q}^j(X, f)$ the subset of $\mathcal{M}_{p/q}(X, f)$ such that $\forall x \in X$, $h_{r_0+1}(x) \in \Gamma_{r_0+1} = \Gamma_{r_0+1}(p/q, j)$.

PROPOSITION 5.5. *For all $c \in \mathcal{M}_{p/q}^j(X, f)$ and for all $x \in X$, the closure in Γ_∞ of the sequence $(h_n(x))_{n\in\mathbb{N}}$ intersects $X_\infty^{p/q,j}$ at a unique point $h(x)$ called the configuration of x. The map h is continuous and makes the following diagram commutative:*

$$\begin{array}{ccc} X & \xrightarrow{h} & X_\infty^{p/q,j} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{F}} \\ X & \xrightarrow{h} & X_\infty^{p/q,j} \end{array}$$

Proof. The map h is obtained by setting $h(x) = p_\infty((h_n(x))_{n\in\mathbb{N}})$, for all $x \in X$. Hence it is straightforward that it has the required properties. □

The combination of Theorem 5.3 and Proposition 5.5 completes the proof of Theorem 1.2 stated in the introduction.

Remark 5.6. If $c : X \to \mathbb{C}$ is a bounded function (not necessarily continuous) we can consider $\hat{X}$ as the Stone–Cech compactification of X endowed with the discrete topology. Then we introduce $\hat{f} : \hat{X} \to \hat{X}$, $\hat{c} : \hat{X} \to \mathbb{C}$, the continuations of f and c, respectively, to $\hat{X}$. This enables us to define $h_{\hat{c}} : \hat{X} \to X_\infty^{p/q,j}$ (provided that $\hat{c} \in \mathcal{M}_{p/q}^j(\hat{X}, \hat{f})$), and thus $h_c = h_{\hat{c}|\hat{X}}$ is well defined even if c is just a bounded function.

5.3. *Abstract Julia set.* In the previous section, we introduced Γ_n as a parameter space, and E_n/γ_n played the role of the dynamical space associated to the parameter $\gamma_n \in \Gamma_n$. We define in this section the dynamical space corresponding to an element $(\gamma_n)_{n\in\mathbb{N}}$ of $X_\infty^{p/q,j}$.

Recall that $p_n : \mathbb{T}^1 \to E_n$ and $\pi_{\gamma_n} : E_n \to E_n/\gamma_n$ are the canonical projections. We denote by $p_{\gamma_n} : \mathbb{T}^1 \to E_n/\gamma_n$ the map $p_{\gamma_n} = \pi_{\gamma_n} \circ p_n$. Thus γ naturally induces an equivalence relation $\sim_\gamma$ on $\mathbb{T}^1$ in the following way: if $x, y \in \mathbb{T}^1$

$$x \sim_\gamma y \iff \forall n \in \mathbb{N}, \quad p_{\gamma_n}(x) = p_{\gamma_n}(y).$$

PROPOSITION 5.7. *For all $\gamma \in X_\infty^{p/q,j}$, $\sim_\gamma$ is a closed equivalence relation.*

Proof. It is an immediate consequence of the definition of $X_\infty^{p/q,j}$. □

We now consider the abstract Julia set associated to an element $\gamma \in X_\infty^{p/q,j}$. It is by definition the quotient space $E_\gamma = \mathbb{T}^1/\sim_\gamma$. Due to the proposition above, E_γ is a

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

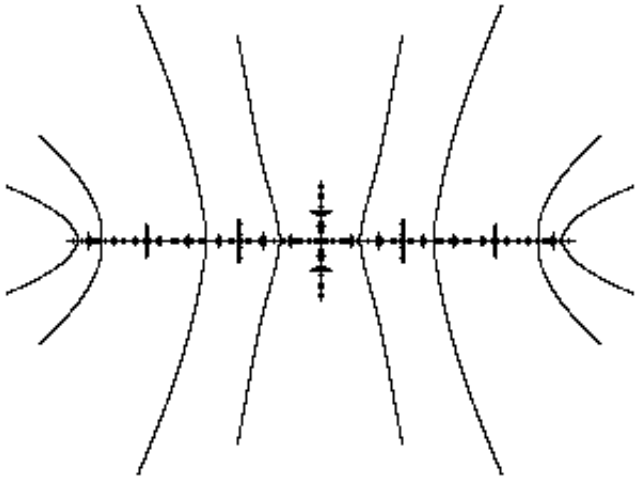


FIGURE 7. Parameter in $\mathcal{M}^*_{1/2}(X, f)$.

(Hausdorff) compact and connected space. Observe that E_γ can also be viewed as the inverse limit space of the system

$$\hat{\imath}_{n,n-1} : E_n/\gamma_n \to E_{n-1}/\gamma_{n-1}.$$

From this point of view, we may define the critical point of γ, $v_\infty(\gamma) = (v(\gamma_n))_{n\in\mathbb{N}}$, as an element of E_γ. The critical value of γ is just $c_\infty(\gamma) = (c(\gamma_n))_{n\in\mathbb{N}}$ and belongs to $E_{\mathcal{F}(\gamma)}$. Finally, the angle doubling Q induces a dynamical map on the abstract Julia set $\mathcal{Q}_\gamma : E_\gamma \longrightarrow E_{\mathcal{F}(\gamma)}$. If $e = (e_n)_{n\in\mathbb{N}}$ belongs to E_γ we set $\mathcal{Q}_\gamma(e) = (\widehat{Q}_{n,n-1}(e_n))_{n\geq 1}$.

6. *Rigidity of non-recurrent configurations*

From now on, we pay attention to the parameters in the $(1/2)$-limb (also called the real limb) whose configurations have rank 1, i.e. are in the satellite $\mathcal{M}^{1/2}_{1/2}(X, f)$. We also assume that the critical values $c(x)$ are not the landing points of the rays of argument $\frac{5}{12}$ and $\frac{7}{12}$. Hence, $\forall x \in X$ the configuration of external rays with arguments in $Q^{-3}(\frac{1}{3}; \frac{2}{3})$ looks as shown in Figure 7. This subset of parameters is denoted by $\mathcal{M}^*_{1/2}(X, f)$.

In this section, we introduce a sequence of partitions of a neighborhood of the filled-in Julia sets, called the puzzles. They are a direct generalization of the non-fibered puzzles introduced by Branner, Hubbard and Yoccoz (cf. [**BH88, Yoc**]). They will be the main tools to deduce topological properties of the Julia sets from combinatorial assumptions.

The main statement of this section says that the Julia sets associated to non-recurrent fibered polynomials are holomorphically removable. As a consequence of this removability, we find that the function c realizing non-recurrent configurations is automatically unique and continuous.

6.1. *Puzzles.*

Fix X to be a Hausdorff compact set, $f : X \to X$ a continuous map and $c \in \mathcal{M}^*_{1/2}(X, f)$. Recall that $\varphi_x : \mathbb{C} \setminus K_x \to \mathbb{C} \setminus \overline{\mathbb{D}}$ denotes the Böttcher coordinate, and

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

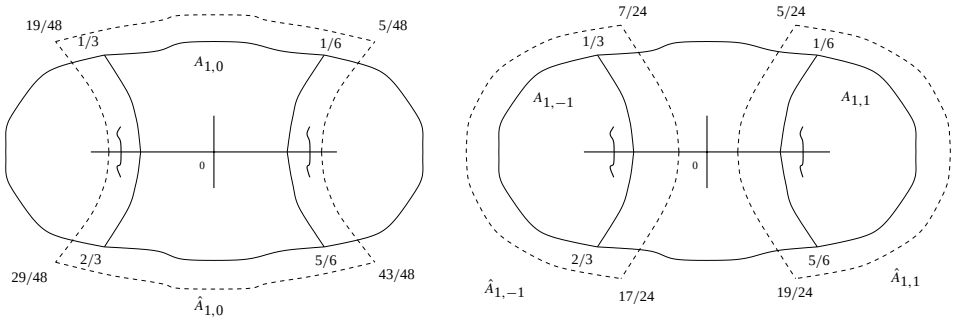


FIGURE 8. Puzzle and thickened puzzle (dotted line) at depth 1.

G_x denotes the Green function of K_x. Let $H_r(x)$ be the topological disk defined by

$$H_r(x) = \{z \in \mathbb{C} : G_x(z) \leq r\}.$$

By assumption, $\mathcal{R}_{1/3}(x)$ and $\mathcal{R}_{2/3}(x)$ land at $\alpha(x)$, hence $\mathcal{R}_{1/6}(x)$ and $\mathcal{R}_{5/6}(x)$ both land at $-\alpha(x)$. Consequently, $H_1(x)$ is cut by the closures of these four external rays into three pieces. These domains labeled $A_{1,-1}(x)$, $A_{1,0}(x)$ and $A_{1,1}(x)$ are chosen by convention closed and constitute the puzzle at depth 1, labeled $\mathcal{P}_1(x)$. We define by induction the puzzle at depth n as the set of components $P_x^{-1}(A)$, where A varies in $\mathcal{P}_{n-1}(f(x))$. The pieces of $\mathcal{P}_n(x)$ are closed with disjoint interiors and provide a covering of $H_{1/2^{n-1}}(x)$. For any $n \in \mathbb{N}$ and any $x \in X$ we denote by $j_n : \mathcal{P}_n(x) \to \mathcal{P}_{n-1}(x)$ the map such that $j_n(A_n)$ is the unique piece of $\mathcal{P}_{n-1}(x)$ that contains A_n. It obviously satisfies $j_{n-1} \circ P_x = P_x \circ j_n$.

We shall also make use of a sequence of open coverings of a neighborhood of K_x which are slight extensions of the previous puzzles. They are a generalization of the thickened puzzle pieces of Milnor (see [**Mil92**]). We denote by $\widehat{A}_{1,-1}(x)$, $\widehat{A}_{1,0}(x)$ and $\widehat{A}_{1,1}(x)$ the open subsets of $H_{3/2}(x)$ defined by:

- the boundary of $\widehat{A}_{1,-1}(x)$ is made up of the external rays of argument $\frac{7}{24}$, $\frac{17}{24}$ and the Green line of level $3/2$;
- the boundary of $\widehat{A}_{1,0}(x)$ is made up of the external rays of argument $\frac{5}{48}$, $\frac{19}{48}$, $\frac{29}{48}$, $\frac{43}{48}$ and the Green line of level $3/2$;
- the boundary of $\widehat{A}_{1,1}(x)$ is made up of the external rays of argument $\frac{5}{24}$, $\frac{19}{24}$ and the Green line of level $3/2$.

Figure 8 illustrates these definitions. We denote by $\widehat{\mathcal{P}}_1(x)$ the collection of these three open domains and by induction $\widehat{\mathcal{P}}_n(x)$ stands for the set of pre-images under P_x of the elements of $\widehat{\mathcal{P}}_{n-1}(f(x))$. Note that $\widehat{\mathcal{P}}_n(x)$ is an open covering of $H_{3/2^n}(x)$ such that each element is compactly contained in those of $\widehat{\mathcal{P}}_{n-1}(x)$.

This last property which is not true for the puzzle $\mathcal{P}_n(x)$ is one of the reasons why $\widehat{\mathcal{P}}_n(x)$ will turn out to be fruitful in the fibered setting. The next proposition enables us to associate in a natural way an element of $\widehat{\mathcal{P}}_n(x)$ to any piece of $\mathcal{P}_n(x)$ in which it is contained.

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

PROPOSITION 6.1. *For all $n \geq 1$ there exists a unique map $\pi_n : \mathcal{P}_n(x) \to \widehat{\mathcal{P}}_n(x)$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{P}_n(x) & \xrightarrow{P_x} & \mathcal{P}_{n-1}(f(x)) \\ \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_{n-1}} \\ \widehat{\mathcal{P}}_n(x) & \xrightarrow{P_x} & \widehat{\mathcal{P}}_{n-1}(f(x)) \end{array}$$

and $A_n \Subset \pi_n(A_n)$ whenever $A_n \in \mathcal{P}_n(x)$.

To avoid heavy notation we omit the x from the notation of π_n. Similarly we still denote by P_x the map $\mathcal{P}_n(x) \to \mathcal{P}_{n-1}(f(x))$ induced by $P_x : \mathbb{C} \to \mathbb{C}$.

Proof. First, note that π_1 is well defined since by construction

$$\pi_1(A_{1,i}(x)) = \widehat{A}_{1,i}(x), \quad \text{for } i = -1, 0, 1.$$

Next, by induction we assume that π_{n-1} is already well defined $\forall x \in X$. We set $\pi_n(A_n)$ to be the unique connected component of $P_x^{-1}(\pi_{n-1}(P_x(A_n)))$ which contains A_n. The commutativity of the diagram obviously results from the definition. □

On the other hand, for any $\widehat{A}_n \in \widehat{\mathcal{P}}_n(x)$ we set $\hat{\jmath}_n(\widehat{A}_n)$ to be the unique element of $\widehat{\mathcal{P}}_{n-1}(x)$ such that $\widehat{A}_n$ is compactly included in $\hat{\jmath}_n(\widehat{A}_n)$. It is straightforward that this defines a map $\hat{\jmath}_n : \widehat{\mathcal{P}}_n(x) \longrightarrow \widehat{\mathcal{P}}_{n-1}(x)$ which actually depends on x and which makes the following diagrams commutative:

$$\begin{array}{ccc} \mathcal{P}_n(x) & \xrightarrow{j_n} & \mathcal{P}_{n-1}(f(x)) \\ \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_{n-1}} \\ \widehat{\mathcal{P}}_n(x) & \xrightarrow{\hat{\jmath}_n} & \widehat{\mathcal{P}}_{n-1}(f(x)) \end{array} \qquad \begin{array}{ccc} \widehat{\mathcal{P}}_n(x) & \xrightarrow{P_x} & \widehat{\mathcal{P}}_{n-1}(f(x)) \\ \downarrow{\scriptstyle \hat{\jmath}_n} & & \downarrow{\scriptstyle \hat{\jmath}_{n-1}} \\ \widehat{\mathcal{P}}_{n-1}(x) & \xrightarrow{P_x} & \widehat{\mathcal{P}}_{n-2}(f(x)) \end{array}$$

The critical and post-critical pieces of $\mathcal{P}_n(x)$ and $\widehat{\mathcal{P}}_n(x)$ respectively will play a central role in what follows (as is often the case in complex dynamics). Assume that 0 is not a pre-image of $\alpha(f^n(x))$, then $J_n(x)$ stands for the critical piece, i.e. the unique element of $\mathcal{P}_n(x)$ such that $0 \in J_n(x)$. The post-critical piece $I_n(x)$ is the image under P_x of $J_{n+1}(x)$ and belongs to $\mathcal{P}_n(f(x))$. We set

$$\widehat{J}_n(x) = \pi_n(J_n(x)) \quad \text{and} \quad \widehat{I}_n(x) = \pi_n(I_n(x)).$$

Observe that 0 might be contained in several different pieces of $\widehat{\mathcal{P}}_n(x)$. On the other hand, if $f : X \to X$ is not one-to-one there might be several post-critical pieces in $\widehat{\mathcal{P}}_n(f(x))$. Finally, if 0 is a pre-image of $\alpha(f^n(x))$, the critical piece of $\mathcal{P}_n(x)$ does not make sense. Nevertheless, in this case we define $\widehat{J}_n(x)$ as the unique pre-image of $\widehat{A}_{1,0}(f^n(x))$ under P_x^n that contains 0. We set $\widehat{I}_n(x) = P_x(\widehat{J}_{n+1}(x))$.

One of the reasons for introducing the thickened puzzles $\widehat{\mathcal{P}}_n(x)$ is that the associated critical piece is always well-defined. Contrary to the constant case, we cannot avoid the occurrence where 0 is preperiodic in this fibered framework.

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

Now, we shall see how the puzzles enable us to relate the filled-in Julia set K_x to the associated configuration γ. Recall that if $\gamma \in X_\infty$ is viewed as a closed equivalence relation on $\mathbb{R}/\mathbb{Z}$, E_γ denotes the corresponding abstract Julia set which is the quotient $\mathbb{R}/\mathbb{Z}/\gamma$.

PROPOSITION 6.2. *For each $x \in X$, there exists a continuous map $\chi_x : K_x \longrightarrow E_\gamma$ such that the diagram*

$$\begin{array}{ccc} K_x & \xrightarrow{\chi_x} & E_\gamma \\ {\scriptstyle P_x}\downarrow & & \downarrow{\scriptstyle Q_\gamma} \\ K_{f(x)} & \xrightarrow{\chi_{f(x)}} & E_{\mathcal{F}(\gamma)} \end{array}$$

commutes.

Proof. For all $n \in \mathbb{N}$, the puzzle $\mathcal{P}_n(x)$ induces canonically a continuous map $\chi_n : K_x \longrightarrow E_n/\gamma_n$. Thus, for all $z \in K_x$ we set $\chi_\gamma(z) = (\chi_1(z), \dots, \chi_n(z), \dots)$. By construction $\chi_\gamma(z)$ is an element of the projective system $(E_n/\gamma_n, i_{\gamma_n})_{n\in\mathbb{N}}$. Hence $\chi_\gamma(z) \in E_\gamma$ and the continuity follows from that of χ_n. □

LEMMA 6.3. *Let $P : X \times \mathbb{C} \to X \times \mathbb{C}$, $P(x, z) = (f(x), z^2 + c(x))$ be a fibered polynomial with $c \in \mathcal{M}^*_{1/2}(X, f)$. Then, there exists a constant $M_0 > 0$ such that $\forall x \in X$*

$$\operatorname{mod}(\widehat{A}_1(x) \setminus \overline{\widehat{A}_2(x)}) \geq M_0$$

whenever $\widehat{A}_1(x) \in \widehat{\mathcal{P}}_1(x)$, $\widehat{A}_2(x) \in \widehat{\mathcal{P}}_2(x)$ and $\widehat{A}_2(x) \subset \widehat{A}_1(x)$.

Remark. Following ideas of Yoccoz, we proved in [**Ses97**] that the constant M_0 can be chosen independently of both $x \in X$ and $c \in \mathcal{M}^*_{1/2}(X, f)$.

Proof. The first observation is that $\forall x \in X$, there exists $M(x) > 0$ such that

$$\operatorname{mod}(\widehat{A}_1(x) \setminus \overline{\widehat{A}_2(x)}) \geq M(x).$$

Next combining Proposition 2.5 with the continuity of the Green function G_x, we deduce that $\partial\widehat{A}_1(x)$ and $\partial\widehat{A}_2(x)$ are Jordan curves depending continuously on x. Therefore, $x \mapsto \operatorname{mod}(\widehat{A}_1(x) \setminus \overline{\widehat{A}_2(x)})$ is continuous and thus $\inf_{x\in X} \operatorname{mod}(\widehat{A}_1(x) \setminus \overline{\widehat{A}_2(x)}) \geq M_0$. □

6.2. *Removability in the non-recurrent case.* Now, we deal with critically non-recurrent fibered polynomials and associated configurations. Let us define this notion of non-recurrence more precisely.

Definition 6.4. Fix $m_0 \in \mathbb{N}$. A fibered polynomial $P : (x, z) \mapsto (f(x), z^2 + c(x))$ is said to be m_0-non-recurrent if $c \in \mathcal{M}^*_{1/2}(X, f)$ and

$$P_x^k(0) \notin \widehat{J}_{m_0}(f^k(x)), \quad \forall x \in X, \ \forall k \in \mathbb{N}^*.$$

We call a (finite) configuration $\gamma_n \in \Gamma_n$ m_0-non-recurrent if each associated fibered polynomial is m_0-non-recurrent. Finally, a configuration $\gamma_\infty = (\gamma_n)_{n\in\mathbb{N}} \in X_\infty^{1/2,1/2}$ is said to be m_0-non-recurrent if each γ_n is.

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

Our main result states that whenever P is m_0-non-recurrent, then for all $x \in X$ the complement of K_x is a John domain. This is an adaptation of the result of Carleson *et al.* [**CJY94**] on the characterization of polynomials whose Fatou components are John domains. The next proposition is a kind of expansivity on the Julia set.

PROPOSITION 6.5. *If P is m_0-non-recurrent, there exist $C > 0$ and $\theta > 1$ such that for all $m \in \mathbb{N}$, $x \in X$ and $\widehat{A}_m \in \widehat{\mathcal{P}}_m(x)$*

$$\operatorname{diam}(\widehat{A}_m) \leq C\theta^m.$$

We start with the following technical lemma valid for all fibered polynomials corresponding to a parameter $c \in \mathcal{M}^*_{1/2}(X, f)$.

LEMMA 6.6. *Fix (X, f) to be a basis and $c \in \mathcal{M}^*_{1/2}(X, f)$. There exists $p_0 \geq 0$ such that for all $m \geq 0$, if $A_m \in \mathcal{P}_m(x)$ and $\widehat{A}_{p_0+m} \in \widehat{\mathcal{P}}_{p_0+m}(x)$ satisfy $\widehat{A}_{p_0+m} \cap A_m \neq \emptyset$, then*

$$\widehat{A}_{p_0+m} \subset \widehat{A}_m = \pi_m(A_m).$$

Proof. Let us look carefully at the puzzle $\widehat{P}_1(x)$ at depth 1. We claim that there exists $p_0 \geq 0$ such that the lemma holds for $m = 1$. It suffices to remark that the pieces of $\widehat{P}_n(x)$ containing $\alpha(x)$ (or $-\alpha(x)$) are compactly contained in those of $\widehat{P}_1(x)$, for all $n \geq p_0$. Moreover, p_0 depends only on r_1 and r_2, the two external arguments $\frac{5}{12} < r_1 < \frac{1}{2} < r_2 < \frac{7}{12}$, such that $\mathcal{R}_{r_1}(x)$ and $\mathcal{R}_{r_2}(x)$ land at the same point and $c(x)$ belongs to the connected component of $\mathbb{C} \setminus \overline{\mathcal{R}_{r_1}(x) \cup \mathcal{R}_{r_2}(x)}$ that does not contain 0. Next, let us consider A_m and $\widehat{A}_{p_0+m}$ as in the statement. Considering $P_x^{m-1}(A_m)$ and $P_x^{m-1}(\widehat{A}_{p_0+m})$ we obtain pieces of $\mathcal{P}_1(f^{m-1}(x))$ and $\widehat{\mathcal{P}}_1(f^{m-1}(x))$ with non-empty intersection. Consequently,

$$P_x^{m-1}(\widehat{A}_{m+p_0}) \subset \pi_1(P_x^{m-1}(A_m)),$$

hence $\widehat{A}_{p_0+m} \subset \pi_m(A_m)$. □

Remark. In what follows, p_0 will be independent both of the parameter and the base point since we shall have $r_1 = \frac{11}{24}$ and $r_2 = \frac{13}{24}$.

Proof of Proposition 6.5. Let $\widehat{A}_{n+1} \in \widehat{\mathcal{P}}_{n+1}(x)$. We denote by $\widehat{A}_1, \ldots, \widehat{A}_{n-1}, \widehat{A}_n$ the sequence of pieces such that

$$\widehat{A}_{m-1} = \hat{f}_m(\widehat{A}_m).$$

From Lemma 6.6 and the hypothesis of m_0-non-recurrence, we deduce that the branched degree of $P_x^{n-1} : \widehat{A}_n \to P_x^{n-1}(\widehat{A}_n)$ is bounded from above by $2^{m_0+p_0}$. Indeed if $P_x^{n_1}(\widehat{A}_n(x))$ contains 0, then $P_x^{n_1+n_2}(\widehat{A}_n(x))$ is again critical only if $n - n_1 - n_2 < m_0 + p_0$. Thus the annulus $\widehat{A}_n \setminus \overline{\widehat{A}_{n+1}}$ is mapped under a branched covering of degree at most $2^{m_0+p_0}$ on $P_x^{n-1}(\widehat{A}_n) \setminus \overline{P_x^{n-1}(\widehat{A}_{n+1})}$. This yields

$$\operatorname{mod}(\widehat{A}_n \setminus \overline{\widehat{A}_{n+1}}) \geq \frac{1}{2^{m_0+p_0}} M_0,$$

where M_0 is the constant given by Lemma 6.3. Using the Grötzsch inequality, one has

$$\operatorname{mod}(\widehat{A}_1 \setminus \overline{\widehat{A}_n}) \geq \frac{n}{2^{m_0+p_0}} M_0.$$

Since the diameter of $\widehat{A}_1$ is bounded, this completes the proof of Proposition 6.5. □

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

Let us denote

$$\varepsilon_0 = \inf_{(x,z)\in K} \{\varepsilon > 0 : D(z,\varepsilon) \subset \pi_1(A_1(x,z))\},$$

where $A_1(x,z)$ denotes the piece of $\mathcal{P}_1(x)$ that contains z (if $z = \pm\alpha(x)$ we set $A_1(x,z) = A_{1,0}(x)$). From the proposition above we deduce the following 'semi-hyperbolic' property.

COROLLARY 6.7. *Let $z \in J_x$ and D_n be a connected component of $(P_x^n)^{-1}(D(P_x^n(z), \varepsilon_0))$. Then the branched degree of $P_x^n : D_n \to D(P_x(z), \varepsilon_0)$ is smaller than $2^{m_0+p_0}$. Moreover, there exist constants $C > 0$ and $\theta > 1$ such that* $\operatorname{diam}(D_n) \leq C\theta^n$.

Another important consequence of Proposition 6.5 is the continuity of J_x under the non-recurrent assumption.

COROLLARY 6.8. *If P is m_0-non-recurrent then $x \mapsto J_x$ is continuous (with respect to the Hausdorff metric on compact subsets of $\mathbb{C}$).*

Proof. We already know that $x \mapsto J_x$ is lower-semi-continuous (see [**Ses99**]). Thus assume that $x \mapsto J_x$ is not upper-semi-continuous or equivalently that $E := \bigcup_{x\in X}\{x\} \times J_x$ is not closed. Let $(x_0, z_0) \in \overline{E} \setminus E$, then there exists $\delta > 0$ such that $D(z_0, \delta) \subset \operatorname{int}(K_{x_0})$. Since $P_x^n : D(z_0,\delta) \to \mathbb{C}$ is bounded, the same holds for the derivatives $(P_{x_0}^n)'(z_0)$. On the other hand, if D_n is the connected component of $(P_{x_0}^n)^{-1}(D(P_{x_0}^n(z_0), \varepsilon_0))$ that contains z_0, then from Corollary 6.7 $\operatorname{diam}(D_n) \leq C\theta^n$. This implies that $(P_{x_0}^n)'(z_0)$ is unbounded as n tends to infinity and thus provides a contradiction. □

Let $\mathcal{A}_\infty(P_x)$ be the basin of infinity of P_x, i.e. $\mathcal{A}_\infty(P_x) = \mathbb{C} \setminus K_x$.

THEOREM 6.9. *If P is a m_0-non-recurrent fibered polynomial, then for all $x \in X$, $\mathcal{A}_\infty(P_x)$ is a John domain.*

We just give an outline of the proof since it is really the same as that of Carleson *et al.* [**CJY94**]. The so-called semi-hyperbolicity condition is given by Corollary 6.7, and we are only interested in the connected case.

Notation. For all $x \in X$, ρ_x stands for the Poincaré metric on $\mathcal{A}_\infty(P_x) \cup \{\infty\}$ containing z. Let $z \in \mathcal{A}_\infty(P_x)$, and let Γ be the geodesic on $\mathcal{A}_\infty(P_x) \cup \{\infty\}$. Then, we denote by γ_z the half-geodesic included in Γ, which runs from z to J_x.

Proof of Theorem 6.9. The proof makes use of the two following ingredients. The first is a characterization of a simply connected John domain: $\mathcal{A}_\infty(P_x)$ is a John domain if and only if there exists $M > 0$ such that whenever $z \in \mathcal{A}_\infty(P_x)$, $\delta(z) \leq 1$, then for all $\omega \in \gamma_z$

$$\rho_x(\omega, z) \geq M \Longrightarrow \delta(\omega) \leq \tfrac{1}{2}\delta(z).$$

Here $\delta(z)$ denotes the Euclidean distance from z to $\partial\mathcal{A}_\infty(P_x) = \partial K_x$. In our fibered setting, the constant M will turn out to be independent of x. The second tool is a distortion lemma for p-valent functions.

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

LEMMA 6.10. *Let $\mathcal{D} \subset \mathbb{C}$ be a simply connected domain and let $F : \mathcal{D} \to \mathbb{D}$, $F(\partial\mathcal{D}) \subset \partial\mathbb{D}$ be p-valent. Then*

$$\{w \in \mathbb{D} : \rho_{\mathbb{D}}(F(z_0), w) \leq C^{-1}\} \subset F(\{z \in \mathcal{D} : \rho_{\mathcal{D}}(z_0, z) \leq 1\}) \subset \{w \in \mathbb{D} : \rho_{\mathbb{D}}(F(z_0), w) \leq 1\},$$

where C depends only on p.

Let ε_0 be the real number given by Corollary 6.7. From Corollary 6.8 and the continuity of the Green function $G : X \times \mathbb{C} \to \mathbb{C}$, we deduce that there exists $\delta > 0$, such that $\forall x \in X$, whenever $z \in \mathcal{A}_\infty(P_x)$, $G_x(z) \leq \delta$, then there exists $Z \in J_x$ such that

$$\gamma_z \subset B\left(Z, \frac{\varepsilon_0}{2}\right).$$

Let $\eta > 0$ be a real number that will be fixed later. By compactness and continuity of $x \mapsto J_x$, there exists $M > 0$ such that whenever $z \in \mathcal{A}_\infty(P_x)$ and $\delta/2 < G_x(z) \leq \delta$, then

$$z' \in \gamma_z,\ \rho_x(z, z') \geq M \Longrightarrow \delta(z') \leq \eta\delta(z).$$

Now, let us consider $\omega \in \mathcal{A}_\infty(P_x)$, $G_x(\omega) \leq \delta$ and let n be an integer such that

$$2^{-(n+1)}\delta < G_x(w) \leq 2^{-n}\delta.$$

From Lemma 6.10, we find a constant M' such that whenever $\omega' \in \delta_\omega$, $\rho_x(\omega, \omega') \geq M'$ then $\rho_{f^n(x)}(P_x^n(\omega), P_x^n(\omega')) \geq M$. Hence $\delta(P_x^n(\omega)) \leq \eta\delta(P_x^n(\omega'))$. Applying again Lemma 6.10 there exists a constant $C > 0$ such that $\delta(\omega') \leq C\eta\delta(\omega)$. Finally, if η is small enough, there exists $M(\eta)$ such that if $\rho_x(w, w') \geq M(\eta)$ then $\delta(w') \leq \frac{1}{2}\delta(w)$. □

Now, we are able to give a topological description of Julia sets associated to m_0-non-recurrent fibered polynomials.

THEOREM 6.11. *If $P : (x, z) \mapsto (f(x), z^2 + c(x))$ is a m_0-non-recurrent fibered polynomial with $c \in \mathcal{M}^*_{1/2}(X, f)$, then for all $x \in X$, J_x is holomorphically removable and K_x is locally connected.*

Recall that Λ, a compact subset of $\overline{\mathbb{C}}$, is holomorphically removable, if whenever $F : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a homeomorphism and F is holomorphic in $\overline{\mathbb{C}} \setminus K$, then F is a Möbius transformation. As a consequence of this notion of removability one has that $J_x = K_x$ and J_x has Lebesgue measure zero.

Proof of Theorem 6.11. The removability results from a Theorem of Jones [**Jon95**] which states that the boundary of a John domain is holomorphically removable. As regards the local connectivity of K_x, let $z \in K_x$ and suppose that z is not a pre-image of $\alpha(f^n(x))$. We denote by $A_n(x, z)$ the unique piece of $\mathcal{P}_m(x)$ containing z and $\widehat{A}_m(x, z) = \pi_m(A_m(x, z))$. If z is a pre-image of $\alpha(f^n(x))$, we chose $\widehat{A}_m(x, z)$ as the unique connected component of $(P_x^m)^{-1}(A_{1,0}(f^m(x)))$ containing z. In view of Proposition 6.5, the family $(\widehat{A}_m(x, z))_{n\in\mathbb{N}}$ constitutes a basis of connected neighborhoods of z. □

We want now to explain in what sense the parameter realizing a given non-recurrent configuration is unique. Let $P_1 : (x, z) \mapsto (f(x), z^2 + c_1(x))$ and $P_2 : (x, z) \mapsto$

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

$(f(x), z^2 + c_2(x))$ be two fibered polynomials with c_1 and c_2 in $\mathcal{M}^*_{1/2}(X, f)$. We denote by h_1 and $h_2 : X \to X_\infty^{1/2,1/2}$ the continuous maps given by Proposition 5.5 such that $h_1(x)$ and $h_2(x)$ are the configurations at x of P_1 and P_2 respectively. The next proposition states that if h_1 and h_2 coincide, the same holds for c_1 and c_2.

PROPOSITION 6.12. *Fix $x \in X$. If $h_1(x) = h_2(x)$ then $c_1(x) = c_2(x)$.*

Proof. Recall that if $h_1(x) = h_2(x)$, we have actually the equality $h_1(f^i(x)) = h_2(f^i(x))$ for all $i \in \mathbb{N}$. We denote by $K_{1,x}$ and $K_{2,x}$ the filled-in Julia set over x of P_1 and P_2 respectively. Let $\varphi_{i,x}$ be the Böttcher coordinates of $\mathbb{C} \setminus K_{i,x}$ for $i = 1, 2$. The map $\varphi_{2,x} \circ \varphi_{1,x}^{-1} : \mathbb{C} \setminus K_{1,x} \to \mathbb{C} \setminus K_{2,x}$ is a homeomorphism tangent to the identity at infinity. Since $K_{1,x}$ and $K_{2,x}$ are locally connected $\varphi_{2,x} \circ \varphi_{1,x}^{-1}$ extends continuously to the whole plane. Moreover, the pieces of $\widehat{\mathcal{P}}_m(x, c_1)$ are mapped bijectively to those of $\widehat{\mathcal{P}}_m(x, c_2)$. Hence $\varphi_{2,x} \circ \varphi_{1,x}^{-1}$ extends to a homeomorphism of the plane such that $\varphi_{2,x} \circ \varphi_{1,x}^{-1}(0) = 0$. Thanks to the removability of $K_{1,x}$, the map $\varphi_{2,x} \circ \varphi_{1,x}^{-1}$ is a Möbius transformation. Hence $\varphi_{2,x} \circ \varphi_{1,x}^{-1} = \mathrm{id}$ and $c_1(x) = c_2(x)$. □

Fix X to be a compact subset of $X_\infty^{1/2,1/2}$ which consists of m_0-non-recurrent configurations. Another consequence of Theorem 6.11 is that every bounded function $c : X \to \mathbb{C}$, such that the corresponding fibered polynomial realizes X, is automatically continuous. According to Remark 5.6, if $c : X \to \mathbb{C}$ is just a bounded function, we can nevertheless define $h_c : X \to X_\infty^{1/2,1/2}$ and thus the associated configurations. To that purpose, it suffices to consider $\widehat{X}$, the Stone–Cech compactification of X, and $\hat{c}$ and $\hat{f}$, the continuations of c and f to $\widehat{X}$.

PROPOSITION 6.13. *Let $X \subset X_\infty^{1/2,1/2}$ be an invariant compact set of m_0-non-recurrent configurations. Let $P(x, z) = (\mathcal{F}_{|X}(x), z^2 + c(x))$ be a fibered polynomial over $(X, \mathcal{F}_{|X})$ with $c : X \to \mathbb{C}$ a bounded map. Assume that $h_c : X \to X_\infty^{1/2,1/2}$ is well defined and that it coincides with the canonical injection, then c is continuous.*

Proof. We reason by contradiction. If x is a point of discontinuity of c, we can find two distinct values $c(x)$ and c_0 (a limit of $c(x_n)$ where (x_n) converges to x) such that the associated configurations are the same. This contradicts Proposition 6.12. □

7. *Realizability of primitive configurations*

7.1. *Introduction.* This section is devoted to the realizability of some non-recurrent configurations associated to the $\frac{1}{2}$-satellite of the real limb. Recall that $J_3(x)$ stands for the critical piece of the puzzle $\mathcal{P}_3(x)$ whose boundary consists of the Green line and external rays of arguments $\frac{5}{24}$, $\frac{7}{24}$, $\frac{17}{24}$ and $\frac{19}{24}$. The configurations we are dealing with describe the most simple non-recurrent behavior in the following sense.

Definition 7.1. A fibered polynomial $P : (x, z) \mapsto (f(x), z^2 + c(x))$ is said to be primitive (non-recurrent) if $c \in \mathcal{M}^*_{1/2}(X, f)$ and

$$P_x^k(0) \notin J_3(f^k(x)), \quad \forall x \in X, \ \forall k \in \mathbb{N}^*.$$

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

A (finite) configuration $\gamma_n \in \Gamma_n$ is primitive if each associated fibered polynomial is primitive. Finally, a configuration $\gamma_\infty = (\gamma_n)_{n\in\mathbb{N}} \in X_\infty^{1/2,1/2}$ is said to be primitive if each γ_n is.

We can also give a purely abstract definition. Recall that $Q : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is the doubling modulo one. Let Λ_3 be the compact set of points $x \in \mathbb{R}/\mathbb{Z}$ such that for all $n \in \mathbb{N}$ the image $Q^n(x)$ does not belong to

$$]\tfrac{5}{24}, \tfrac{7}{24}[\cup]\tfrac{17}{24}, \tfrac{19}{24}[.$$

We call $\gamma \in X_\infty^{1/2,1/2}$ a primitive configuration if $\forall n \in \mathbb{N}$, $c_\infty(\mathcal{F}^n(\gamma)) \subset \Lambda_3$. The set of these configurations, denoted X_3, is obviously a compact subset of $X_\infty^{1/2,1/2}$, forward invariant under $\mathcal{F}$.

The purpose of this section is to prove Theorem 1.3 which says that X_3 is realizable and rigid. Note that the condition $c \in \mathcal{M}_{1/2}^*(X, f)$ implies $\widehat{J_4}(x) \subset J_3(x)$, $\forall x \in X$, thus a primitive fibered polynomial is also 4-non-recurrent. Therefore the results of the previous section apply to the primitive fibered polynomials. Hence, we just need to determine a function $c : X_3 \longrightarrow \mathbb{C}$ such that the fibered polynomial associated to c realizes X_3. By virtue of the removability of the Julia set, the function c will automatically be unique and continuous.

Example 7.2. For all $n \in \mathbb{N}$ we know that the two external rays of arguments

$$\frac{3 \times 2^{n-1} - 1}{3 \times 2^n} \quad \text{and} \quad \frac{3 \times 2^{n-1} + 1}{3 \times 2^n}$$

always land at the same point. Thus we can define a configuration of X_3, $\gamma_\infty = (\gamma_n)_{n\in\mathbb{N}}$, by setting

$$c(\mathcal{F}^n(\gamma_\infty)) = \left\{ \frac{3 \times 2^{n-1} - 1}{3 \times 2^n}, \frac{3 \times 2^{n-1} + 1}{3 \times 2^n} \right\}.$$

This configuration does not appear in the non-fibered case since it is not invariant under $\mathcal{F}$, but Theorem 1.3 guarantees that it is realizable by a fibered polynomial.

Let $\gamma \in X_3$ and $\sim_\gamma$ be the associated closed equivalence relation on $\mathbb{R}/\mathbb{Z}$. The fundamental observation is that all compact sets $\Lambda_3/\sim_\gamma$ are homeomorphic to the same Cantor set K_3. Furthermore, this compact set is a sub-shift of finite-type of $\{-3, -2, -1, 1, 2, 3\}^{\mathbb{N}}$ where the admissible transitions are:

$$\begin{aligned} \pm 3 &\longrightarrow 3, 2 \\ \pm 2 &\longrightarrow -1, 1 \\ \pm 1 &\longrightarrow -2. \end{aligned}$$

It can also be viewed as the set of points which may be iterated infinitely many times by the map φ in Figure 9. Let L_3 be the cylinder $-3, 3$ of K_3, i.e. the set of sequences of K_3 beginning with the two symbols -3 and 3. Thus, the space X_3 is identified with the Cartesian product $L_3^{\mathbb{N}}$. Indeed, given $\gamma \in X_3$, the sequence $(c_\infty(\mathcal{F}^i(\gamma)))_{i\in\mathbb{N}}$ defines a unique element of $L_3^{\mathbb{N}}$.

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

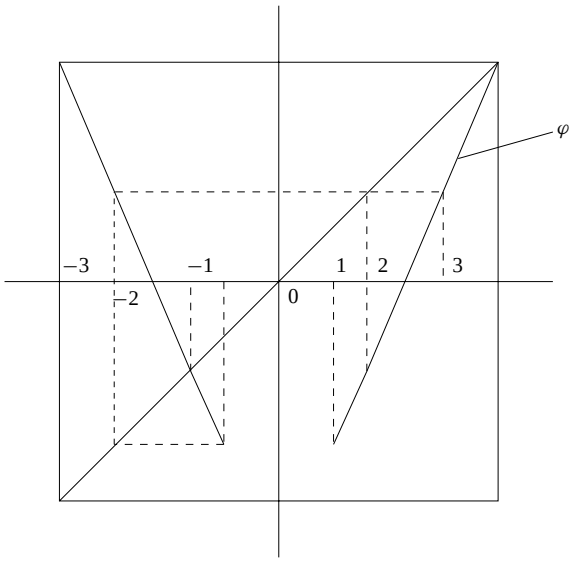


FIGURE 9. Cantor set.

The proof of Theorem 1.3 proceeds in outline as follows. The map φ provides for every $x \in X_3$, a critical value $c(x) \in L_3 \subset \mathbb{R}$. Extending φ to the whole plane, we shall construct by hand a topological model $\widehat{P}_x$ which turns out to be a degree two branched covering with 0 as its critical point and $c(x)$ as its critical value. Finally, the resolution of the Beltrami equation associated to $\widehat{P}_x$ will lead us to a quadratic fibered polynomial realizing the configurations of X_3.

7.2. *Topological model.* Our starting point is an explicit formula for the affine map φ. This formula depends on a parameter $\lambda > 2$ which is assumed to be close to 2 and will be fixed later. First consider the real numbers

$$\alpha = -\frac{1}{2}; \quad a_0 = \alpha(1 - 2\lambda^{-2});$$
$$\varepsilon = \frac{1}{2}\frac{\lambda - 2}{\lambda^2(\lambda - 1)} \quad \text{and} \quad \beta = -\alpha\left(\frac{1+\lambda}{\lambda - 1}\right).$$

We also denote by T^+ the triangle whose vertices are $-a_0 + ia_0$, $-a_0 - ia_0$ and $\beta + \varepsilon$; T^- is the symmetric triangle of T^+ with respect to zero. We define φ_λ in $T^- \cup T^+$ by the following formula:

$$\varphi_\lambda(z) = \begin{cases} -\lambda(z - \alpha) + \alpha, & \text{if } z \in T^+ \\ \lambda(z + \alpha) + \alpha, & \text{if } z \in T^-. \end{cases}$$

Thus we set $T := \varphi_\lambda(T^-) = \varphi_\lambda(T^+)$. Let us make some comments about this notation. The number α is a fixed point of our model and $-\lambda$ is its multiplier. The number a_0 has been chosen to be a pre-image of α under φ_λ of order two. The second fixed point of φ_λ is

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

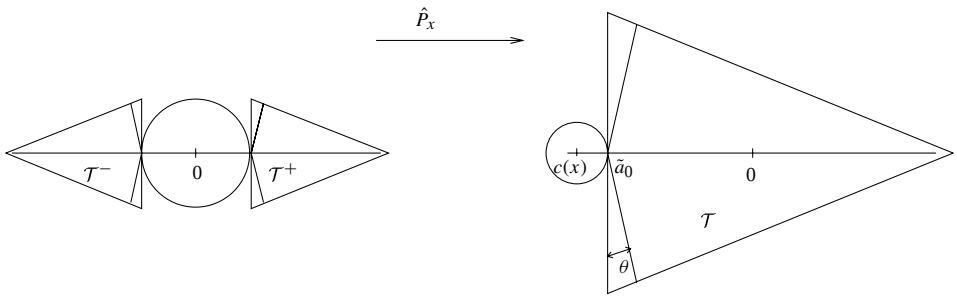


FIGURE 10. Geometry of the domain of extension.

just β. We denote by $\tilde{a}_0 = \alpha - \lambda^{-1}$ and $\tilde{a}_1 = \alpha - \lambda^{-1} - \lambda^{-2}$ two pre-images of α of orders two and three, respectively. The point $\tilde{a}_0$ is thought of as the landing point of the external rays of arguments $\frac{5}{12}$ and $\frac{7}{12}$; $\tilde{a}_1$ is thought of as the landing point of the external rays of arguments $\frac{11}{24}$ and $\frac{13}{24}$. On the other hand, we may consider K_λ as the dynamical Cantor set induced by φ_λ, i.e.

$$K_\lambda = \{z \in [-\beta; \beta] : \forall n \in \mathbb{N}, \varphi_\lambda^n(z) \notin]a_0, -a_0[\},$$

and $L_\lambda = K_\lambda \cap [-\beta, \tilde{a}_1]$.

As already mentioned above, the primitive configurations are completely characterized by the data of critical values $c(x) \in L_\lambda$, $\forall x \in X_3$. Nevertheless, our construction works for every continuous function $c : X_3 \to [-\beta, \tilde{a}_1]$. Thus, until §5, all we require on c is that it is a continuous function with values in $[-\beta, \tilde{a}_1]$.

Now, we make precise some geometric features of the local extension domain. In fact, this domain does not depend on $x \in X_3$. Let θ be a small angle that we shall fix later. We denote by $\mathcal{T}$ the quadrangle included in T whose vertices are

$$\tilde{a}_0, \quad \tilde{a}_0 + r_0 e^{i(\pi/2-\theta)}, \quad \beta + \lambda\varepsilon, \quad \tilde{a}_0 + r_0 e^{-i(\pi/2-\theta)},$$

where r_0 denotes the length of the line segments included in T making an angle θ with the vertical line issuing from $\tilde{a}_0$ (see Figure 10). We denote by $\mathcal{T}^+$ and $\mathcal{T}^-$ the two inverse images of $\mathcal{T}$ under φ_λ which are included in T^+ and T^-, respectively. Observe that the real number ε has been chosen so that the vertex of $\mathcal{T}^-$ is always included in the disk $D(c(x), |c(x) - \tilde{a}_0|)$. We set $k(x) = [\tilde{a}_0 - c(x)]/a_0^2$ and define $\widehat{P}_x$ by the formulas

$$\widehat{P}_x(z) = \begin{cases} k(x)z^2 + c(x), & \text{if } z \in D(0, |a_0|) \\ \varphi_\lambda(z), & \text{if } z \in \mathcal{T}^+ \cup \mathcal{T}^-. \end{cases}$$

At this point of the construction, we notice that $\widehat{P}_x(D(0, |a_0|)) = D(c(x), |c(x) - \tilde{a}_0|)$. On the other hand, the domain of definition of $\widehat{P}_x$ is not a Jordan domain.

We shall extend $\widehat{P}_x$ to the whole plane in two steps. First, we define a Jordan domain $\mathcal{D}$ and an extension of $\widehat{P}_x$ to this domain satisfying, in particular, $\mathcal{D} \Subset \widehat{P}_x(\mathcal{D})$. Next, we extend $\widehat{P}_x$ globally and make it coincide with $z \mapsto z^2$ in a neighborhood of infinity.

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

7.3. *Local extension.* For all $a \in \mathbb{R}$ and $0 < \rho < \pi/2$, we denote by $\mathcal{C}(a, \rho)$ the open cone of vertex a and angle ρ:

$$\mathcal{C}(a, \rho) = \{z \in \mathbb{C} : \pi/2 - \rho < |\arg(z - a)| < \pi/2 + \rho\}.$$

With the map $\widehat{P}_x$ being defined on $D(0, |a_0|) \cup T^- \cup T^+$, the local extension of $\widehat{P}_x$ with the required properties is given by the following proposition.

PROPOSITION 7.3. *Assume that λ and θ are close enough to 2 and 0, respectively. Then there exist constants $d_0, \rho_0 > 0$ and a Jordan domain $\mathcal{D}$ such that $\forall x \in X_3$ there exists $\widetilde{P}_x$, an extension of $\widehat{P}_x$, satisfying:*

(1) $d(\mathcal{D}, \partial \widetilde{P}_x(\mathcal{D})) \geq d_0$;

(2) $\widetilde{P}_x : \mathcal{D} \to \widetilde{P}_x(\mathcal{D})$ *is a degree-two branched covering. It is analytic in $\mathcal{D} \setminus \Sigma$ where Σ satisfies*

- $\Sigma \cap \mathcal{C}(\alpha, \rho_0) = \emptyset$ *and* $\Sigma \cap \mathcal{C}(-\alpha, \rho_0) = \emptyset$;
- $\widetilde{P}_x(\Sigma) \cap \mathcal{D} \subset \mathcal{C}(\tilde{a}_0, \rho_0)$.

We shall construct the extension of $\widehat{P}_x$ to a neighborhood of a_0 and $-a_0$ in a symmetric way with respect both to zero and to the x-axis. Therefore we focus on the continuation of $\widehat{P}_x$ near $-a_0$ and we restrict ourselves to the upper half-plane. The final extension will be deduced from this continuation by symmetry and parity.

The first step is to glue together the map $\psi_x : z \mapsto k(x)z^2 + c(x)$ defined on the disk $D(0, |a_0|)$ and $\zeta_x : z \mapsto \lambda(x)(z + a_0) + \tilde{a}_0$ defined on the vertical line coming from $-a_0$. The coefficient $\lambda(x)$ is chosen so that the final extension of $\widehat{P}_x$ is differentiable in the vertical direction. Hence, we must fix $\lambda(x) = 2k(x)|a_0| = 2[\tilde{a}_0 - c(x)]/|a_0|$.

LEMMA 7.4. *There exist a constant $\delta_1 > 0$, a sector s_1 and a gluing $f_x : s_1 \to f_x(s_1)$ of ψ_x and ζ_x such that f_x has bounded C^1-norm and $d(\mathcal{T}^-, \partial f_x(s_1)) > \delta_1$.*

Proof. We introduce a new system of coordinates which enables us to give an explicit formula for the continuation of $\widehat{P}_x$. We consider h_{-a_0} and $h_{\tilde{a}_0}$, the inverse functions defined in a neighborhood of $-a_0$ and $\tilde{a}_0$, respectively, by

$$h_{-a_0}(z) = \frac{-1}{z + a_0} \quad \text{and} \quad h_{\tilde{a}_0}(z) = \frac{-1}{z - \tilde{a}_0}.$$

The function h_{-a_0} maps the vertical line coming from $-a_0$ onto the y-axis and the circle $S(0, |a_0|)$ onto the line $D_1 = \{(1/2|a_0|)(1 + it),\ t \in \mathbb{R}\}$. Indeed, if $z = |a_0|e^{i\theta}$ then

$$h_{-a_0}(z) = \frac{1}{2|a_0|}\left(1 + i \cot\left(\frac{\theta}{2}\right)\right).$$

Similarly, $h_{\tilde{a}_0}$ maps the vertical line coming from $\tilde{a}_0$ onto the y-axis and the circle $S(c(x), |c(x) - \tilde{a}_0|)$ onto the line

$$D_2 = \left\{\frac{1}{2|c(x) - \tilde{a}_0|}(1 + it),\ t \in \mathbb{R}\right\}.$$

We denote by $g_x^1 = h_{\tilde{a}_0} \circ \zeta_x \circ h_{-a_0}^{-1}$ the map ζ_x viewed in the new system of coordinates. A brief calculation yields: $g_x^1(z) = z/\lambda(x)$. Concerning ψ_x, we denote $g_x^2 = h_{\tilde{a}_0} \circ \psi_x \circ h_{-a_0}^{-1}$

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

and we obtain a map $g_x^2 : D_1 \to D_2$ such that

$$g_x^2\left(\frac{1+it}{2|a_0|}\right) = \frac{1+i(t-1/t)/2}{2|\tilde{a}_0 - c(x)|}, \quad \text{for all } t \in \mathbb{R}^*.$$

Let $a \in \mathbb{R}^+$. We denote by $B(a) = \{z; 0 \le \mathrm{Re}(z) \le a\}$ the vertical strip and I_1 and I_2 the horizontal segments included in $B(1/2|a_0|)$ and $B(1/2(\tilde{a}_0 - c(x)))$, respectively, such that

$$I_1 = \left\{\frac{s+2i}{2|a_0|},\ s \in [0;1]\right\} \quad \text{and} \quad I_2 = \left\{\frac{s+i}{2(\tilde{a}_0 - c(x))},\ s \in [0;1]\right\}.$$

The upper connected component of $B(1/2|a_0|)\setminus I_1$ is labeled $\tilde{s}_1$. Hence, the local extension of $\widehat{P}_x$ near $-a_0$ reduces, after change of coordinates, to the gluing of g_x^1 and g_x^2 in $\tilde{s}_1$. To that purpose, $\forall t \ge 2$ and $\forall s \in [0,1]$ we set

$$g_x\left(\frac{s+it}{2|a_0|}\right) = \frac{s+i(t-s/t)/2}{2(\tilde{a}_0 - c(x))}.$$

Therefore, we obtain a diffeomorphism $g_x : \tilde{s}_1 \to B(1/2|c(x) - \tilde{a}_0|)$ which coincides with g_x^1 and g_x^2 on the y-axis and D_1, respectively. Moreover, g_x has bounded C^1 norm and is tangent to $z \mapsto z/\lambda(x)$ at infinity. It is straightforward to see that the image of I_1 under g_x is situated between the segment I_2 and the x-axis. We denote

$$s_1 = h_{-a_0}^{-1}(\tilde{s}_1), \quad \gamma_1 = h_{-a_0}^{-1}(I_1) \quad \text{and} \quad \gamma_2 = h_{\tilde{a}_0}^{-1}(I_2).$$

Finally, $f_x = h_{\tilde{a}_0}^{-1} \circ g_x \circ h_{-a_0}$ defines the expected extension of $\widehat{P}_x$ to the sector s_1. In view of the relative positions of $g_x(I_1)$ and I_2, $f_x(s_1)$ is a small sector near $\tilde{a}_0$ which contains the arc γ_2. In particular, if $\lambda > 2$ is close enough to 2, there exists $\delta_1 > 0$ such that $\forall x \in X_3$, $d(\mathcal{T}^-, \partial f_x(s_1)) > \delta_1$. □

We focus now on the extension of $\zeta_x : z \mapsto \lambda(x)(z + a_0) + \tilde{a}_0$ to the other side of the vertical line near $-a_0$. We wish to make it coincide with φ_λ defined on $\mathcal{T}$. After an adapted change of coordinates, we shall again be able to write an explicit formula for the desired continuation of $\widehat{P}_x$.

LEMMA 7.5. *There exist a constant $\delta_2 > 0$, a sector σ_1 and a gluing $f_x : \sigma_1 \to \mathbb{C}$ of φ_λ and ζ_x such that f_x has bounded C^1-norm and $d(\mathcal{T}^-, \partial f_x(\sigma_1)) > \delta_2$.*

Proof. Recall that θ denotes the small angle between the vertical line and the side of $\mathcal{T}$. Again, we assume that θ is small enough and that λ is close to 2. Consider the maps

$$l_{-a_0}(z) = \log(z + a_0) \quad \text{and} \quad l_{\tilde{a}_0}(z) = \log(z - \tilde{a}_0)$$

which are defined without ambiguity on $\mathcal{C}(a_0, \theta)$ and $\mathcal{C}(\tilde{a}_0, \theta)$, respectively. We set

$$B_1 = \{z : \pi/2 - \theta \le \mathrm{Im}\, z \le \pi/2,\ \mathrm{Re}(z) \le \log r_0\}$$

and we denote by σ_1 the inverse image under l_{-a_0} of B_1. A simple calculation yields that $l_{\tilde{a}_0} \circ \zeta_x \circ l_{-a_0}^{-1}(Z) = Z + \log\lambda(x)$, for all Z such that $\mathrm{Im}(Z) = \pi/2$. In the same way, $l_{\tilde{a}_0} \circ \varphi_\lambda \circ l_{-a_0}^{-1}(Z) = Z + \log\lambda$ for all Z such that $\mathrm{Im}(Z) = \pi/2 - \theta$. Thus we may glue these translations together by setting for all $u = s + it \in B_1$,

$$h_x(u) = \frac{1}{\theta}((t - \pi/2 + \theta)\log\lambda(x) + (\pi/2 - t)\log\lambda) + u.$$

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

The map h_x has the required properties and its C^1-norm depends only on θ and λ. Let $f_x = l_{\tilde{a}_0}^{-1} \circ h_x \circ l_{-a_0}$. It is a continuation of $\widehat{P}_x$ to the sector σ_1. Furthermore, the vertical segment I_3 which is included in B_1

$$I_3 = \{z = \log r + \theta i : \theta \in [\pi/2 - \theta\pi/2]\}$$

is mapped under h_x to a curve situated in the right of the vertical segment

$$I_4 = \{z = \log r_0 + \log \gamma + \theta i : \theta \in [\pi/2 - \theta; \pi/2]\},$$

where $\gamma = 4/(\lambda^2 - 2) = 2|\tilde{a}_0 - \tilde{a}_1|/|a_0|$. This value satisfies $\gamma < \lambda$ and $\gamma \leq \lambda(x)$ with equality only if $c(x) = \tilde{a}_1$. These inequalities guarantee that $f_x(\sigma_1)$ is a Jordan domain containing an arc of circle of radius γr_0. In particular, this ensures the lower bound, $\forall x \in X_3$, $d(\mathcal{T}^-, \partial f_x(\sigma_1)) > \delta_2$. □

Proof of Proposition 7.3. By symmetry with respect both to zero and the x-axis, we obtain eight sectors s_i, σ_i for $i = 1, 2, 3, 4$ and an extension $\widetilde{P}_x$ of $\widehat{P}_x$. Setting

$$\Sigma = \bigcup_{i=1}^{4} (\sigma_i \cup s_i) \quad \text{and} \quad \mathcal{D} = D(0, |a_0|) \cup \mathcal{T}^+ \cup \mathcal{T}^- \cup \Sigma,$$

we derive the desired Jordan domain of the proposition.

Now it remains to show that the parameters $\lambda > 2$ and $0 < \theta < \pi/2$ may be adapted so that property (2) of Proposition 7.3 holds. Our first observation is that there exists $\rho_0 = (\pi/2) - \delta$ such that for all $2 < \lambda < 2.5$ and $\forall x \in X_3$,

$$\mathcal{D} \cap \widetilde{P}_x(\Sigma) \subset \mathcal{C}(\tilde{a}_0, \rho_0).$$

Fixing ρ_0, we look carefully at the behavior of Σ when λ tends to 2. In the case $\lambda = 2$, $a_0 = -1/4$ and there exists $\theta > 0$ such that

$$\overline{\mathcal{C}(\tilde{a}_0, \rho_0)} \cap \overline{\Sigma(2, \theta)} = \emptyset.$$

By continuity of the 'picture', the above intersection remains empty $\forall \lambda < 2 + \varepsilon_0$ and provided that θ is replaced by $\theta/2$. This completes the proof of Proposition 7.3. □

7.4. *Global extension.* The last part of the construction is to extend our topological model to the whole plane. Let $d_0 > 0$ be the real number given by Proposition 7.3, and let W be the domain

$$W = \{z \in \mathbb{C} : d(z, \mathcal{D}) \leq d_0/2\}.$$

Clearly, W does not depend on $x \in X_3$ and satisfies $\overline{\mathcal{D}} \Subset W \Subset \widetilde{P}_x(\mathcal{D})$. Fix $R > 3$ as a real number such that $\widetilde{P}_x(\mathcal{D}) \Subset D(0, R)$.

PROPOSITION 7.6. *There exists $K > 0$ and for all $x \in X_3$ there exists $\check{P}_x$, a degree two branched covering of the plane, such that:*

- *$\check{P}_x$ is K-quasi-regular;*
- *$\check{P}_x(z) = \widetilde{P}_x(z)$ for all $z \in \mathcal{D}$ and $\check{P}_x(z) = z^2$ for all $z \in \mathbb{C} \setminus D(0, R^2)$;*
- *$\mathbb{C} \setminus \mathcal{D} \subset \mathcal{A}^\infty(\check{P}_x)$.*

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

Here $\mathcal{A}^\infty(\check{P}_x)$ denotes the basin of infinity of $\check{P}_x$, that is

$$\mathcal{A}^\infty(\check{P}_x) = \left\{ z \in \mathbb{C} : \lim_{n \to +\infty} |\check{P}_{\mathcal{F}^n(x)} \circ \cdots \circ \check{P}_x(z)| = +\infty \right\}.$$

Proof. The proof consists first in extending $\widetilde{P}_x$ to $\mathbb{C}$ in a naive way. Then we compose this extension with a smooth homeomorphism to ensure that $\mathbb{C} \setminus \mathcal{D} \subset \mathcal{A}^\infty(\check{P}_x)$. Therefore, we set $\widetilde{P}_x(z) = z^2$, $\forall z \in \mathbb{C} \setminus D(0, R)$. Then we consider a K'-quasi-regular extension of $\widetilde{P}_x$ that we still denote by $\widetilde{P}_x$. The constant K' is greater than $K(\theta, \lambda)$ but again it can be chosen independently of $x \in X_3$ since the domain of definition of $\widetilde{P}_x$ does not depend on x.

On the other hand, let us introduce a homeomorphism of the plane, η, which is continuously differentiable everywhere except on $\partial\mathcal{D}$ and ∂W, such that

$$\begin{cases} \eta = \mathrm{id} \ \text{on} \ \mathcal{D} \\ \eta = \mathrm{id} \ \text{on} \ \mathbb{C} \setminus D(0, R^2) \\ \eta(W \setminus \mathcal{D}) = D(0, R) \setminus \mathcal{D} \\ \eta(D(0, R^2) \setminus W) = D(0, R^2) \setminus D(0, R). \end{cases}$$

Finally, $\check{P}_x = \widetilde{P}_x \circ \eta$ is a K''-quasiregular branched covering of $\mathbb{C}$. Combining Proposition 7.3 and the above properties of η, we obtain the inclusion $\mathbb{C} \setminus \mathcal{D} \subset \mathcal{A}^\infty(\check{P}_x)$. □

7.5. *Quasiconformal conjugacy.* In this subsection we introduce a family of Beltrami differentials which is fibered wise invariant under $\check{P}_x$. These forms have uniformly bounded L^∞ norm away from 1 and, after resolution of the associated Beltrami equations, they will enable us to conjugate our topological model to a quadratic fibered polynomial.

PROPOSITION 7.7. *There exists $\rho < 1$ such that $\forall x \in X_3$, there exists a Beltrami differential μ_x^∞ satisfying:*

- $\mu_x^\infty = (\check{P}_x)^*(\mu_{f(x)}^\infty)$;
- *the support of μ_x^∞ is included in $D(0, R^2)$;*
- $\sup_{z \in \mathbb{C}} |\mu_x^\infty(z)| < \rho$.

Proof. Consider the sequence of Jordan domains:

$$\Delta^0 = \Delta_x^0 = \mathbb{C} \setminus D(0, R^2),$$
$$\Delta_x^n = (\check{P}_x^n)^{-1}(\Delta^0), \quad \forall n \in \mathbb{N}.$$

We set $\mu_x^0(z) = 0$ for all $z \in \Delta^0$ and $\mu_x^n = (\check{P}_x^n)^*(\mu_x^0)$ for all $z \in \Delta_x^n$. Since, whenever $z \in \Delta_x^n$, $\check{P}_{f^n(x)}$ is holomorphic in a neighborhood of $\check{P}_x^n(z)$, one has $\mu_x^{n+1}(z) = \mu_x^n(z)$. Consequently, we may define μ_x^∞ by $\mu_x^\infty(z) = \mu_x^n(z)$ if $z \in \Delta_x^n$ and $\mu_x^\infty \equiv 0$ on any remaining part of $\mathbb{C}$. In this way, μ_x^∞ is by construction fibered-invariant under $\check{P}_x$.

Concerning the third claim of the proposition. In view of Proposition 7.6, there exists $\rho < 1$ such that $\forall z \in D(0, R^2)$

$$\left| \frac{\bar{\partial} \check{P}_x(z)}{\partial \check{P}_x(z)} \right| \leq \rho < 1.$$

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

Moreover, according to Proposition 7.3 there are at most three non-conformal maps in the compositions $\check{P}_x^n$ (the first to escape Σ and the two others to escape $D(0, R^2)$). Hence $\forall n \in \mathbb{N}$, $|\mu_x^n(z)| \leq \rho^3$, therefore $|\mu_x^\infty(z)| \leq \rho^3$. □

Let h_x be the solution of the Beltrami equation $\bar{\partial} h_x(z) = \mu_x^\infty(z) \partial h_x(z)$ such that $h_x(0) = 0$ and h_x is tangent to the identity at infinity. Then $h_{f(x)} \circ \check{P}_x \circ h_x^{-1}$ is by construction a two-to-one holomorphic mapping of $\mathbb{C}$ with 0 as critical point. Hence $h_{f(x)} \circ \check{P}_x \circ h_x^{-1}(z) = z^2 + c^*(x)$, where $c^*(x) = h_{f(x)}(c(x))$. We claim that

$$\begin{aligned} Q : X_3 \times \mathbb{C} &\longrightarrow X_3 \times \mathbb{C} \\ (x, z) &\longmapsto (\mathcal{F}(x), Q_x(z) = z^2 + c^*(x)) \end{aligned}$$

realizes the primitive configurations. First Q admits $\tilde{\alpha} : X_3 \to \mathbb{C}$, $\tilde{\alpha}(x) := h_x(\alpha)$ as invariant repelling section. Indeed, α is a fixed point of $\check{P}_x$ with multiplier $-\lambda$, hence for sufficiently small δ; $\check{P}_x(D(\alpha, \delta)) = D(\alpha, \lambda\delta)$. Setting $U_x = h_x(D(\alpha, \delta))$, we obtain a collection of open sets such that $U_{\mathcal{F}(x)} \Subset Q_x(U_x)$ and

$$\mathrm{mod}(Q_x(U_x) \setminus \overline{U_{\mathcal{F}(x)}}) \geq m_0.$$

From Proposition 2.5, we deduce that the invariant section $\tilde{\alpha}$ is repelling. On the other hand, the two 'external rays' of $\check{P}_x$

$$\begin{aligned} \check{\mathcal{R}}_x(\tfrac{1}{3}) &:= h_x^{-1} \circ \varphi_x^{-1}(\{\rho \exp(\tfrac{2}{3} i\pi) : \rho \in]1, +\infty[\}) \\ \check{\mathcal{R}}_x(\tfrac{2}{3}) &:= h_x^{-1} \circ \varphi_x^{-1}(\{\rho \exp(\tfrac{4}{3} i\pi) : \rho \in]1, +\infty[\}) \end{aligned}$$

land at α. Here, φ_x is the Böttcher coordinate associated to the fibered polynômial Q. Indeed, there exists a constant $C > 0$ which does not depend on x such that $\forall z \in \check{\mathcal{R}}_x(\frac{1}{3}) \cap D(0, R)$ and $\forall z' \in \check{\mathcal{R}}_x(\frac{2}{3}) \cap D(0, R)$

$$\begin{aligned} d_{\mathcal{A}^\infty(\check{P}_x)}(z, \{\alpha + it,\ t \in]0, \delta]\}) &< C \\ d_{\mathcal{A}^\infty(\check{P}_x)}(z', \{\alpha + it,\ t \in [-\delta, 0[\}) &< C, \end{aligned}$$

where $d_{\mathcal{A}^\infty(\check{P}_x)}$ denotes the hyperbolic distance in $\mathcal{A}^\infty(\check{P}_x)$. Since $\{\alpha + it,\ t \in]0, \delta]\}$ and $\{\alpha + it,\ t \in [-\delta, 0[\}$ land at α, the same hold for $\check{\mathcal{R}}_x(\frac{1}{3})$ and $\check{\mathcal{R}}_x(\frac{2}{3})$. This property guarantees that $\mathcal{R}_x(\frac{1}{3})$ and $\mathcal{R}_x(\frac{2}{3})$ both land at $\tilde{\alpha}(x)$.

Finally, if we identify L_3 with L_λ and X_3 with $L_3^{\mathbb{N}}$ through the map $\gamma \mapsto (c_\infty(\mathcal{F}^i(\gamma)))_{i \in \mathbb{N}}$, our construction works with the continuous function $c : X_3 \to L_\lambda$ such that $c(\gamma) = c_\infty(\gamma)$. Thus, we obtain a fibered polynomial Q such that the combinatorial configurations of Q at $x \in X_3$ is exactly x. □

Acknowledgements. I wish to thank J.-C. Yoccoz for numerous and significant contributions to this article. This work was prepared during the fall of '98 at the University of Geneva. I am especially thankful to V. Baladi for inviting me to Geneva, to R. Curtis for reading a preliminary version and to the Fonds National Suisse de la Recherche Scientifique for support. Finally, I wish to thank the referee for his comments and remarks that made this paper more readable.

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press

REFERENCES

[Alv98] J.-F. Alves. SRB measures for nonhyperbolic systems with multidimensional expansion. *Ann. Sci. École Norm. Sup.* **33**(1) (2000), 1–32.

[BH88] B. Branner and J. H. Hubbard. The iteration of cubic polynomials, part I: The global topology of parameter space. *Acta Math.* **160** (1988), 143–206.

[BL99] A. Blokh and G. Levin. Growing trees, laminations and the dynamics of the Julia set. *Preprint*, IHES, 1999.

[BM96] V. Baladi and M. Viana. Strong stochastic stability and rate of mixing for unimodal maps. *Ann. Scie. l'École Normale Supérieure* **29** (1996), 483–517.

[Bou50] N Bourbaki. *Eléments de topologie*. Hermann, 1950.

[CJY94] L. Carleson, P. Jones and J.C. Yoccoz. Julia and John. *Bol. Soc. Brasil. Math.* **25**(1) (1994), 1–30.

[Dug66] J. Dugundji. *Topology*. Allyn and Bacon, 1966.

[Gol92] L. R. Goldberg. Fixed points of polynomial maps, part I: Rotations subsets of the circle. *Ann. Scie. l'Ecole Normale Supérieure*, **25** (1992), 679–685.

[Hei96] S.-M. Heinemann. Julia sets for holomorphic endomorphisms of $\mathbb{C}^n$. *Ergod. Th. & Dynam. Sys.* **16** (1996), 1275–1296.

[Hub93] J. H. Hubbard. Local connectivity of Julia sets and bifurcation loci: three theorems of J. C. Yoccoz. *Topological Methods in Modern Mathematics.* Publish or Perish, 1993, pp. 467–511.

[Jon95] P. Jones. On removable sets for Sobolev spaces in the plane. *Essay on Fourier Analysis in Honor of Elias M. Stein (Princeton Math. Series, 42)*. Princeton University Press, 1995, pp. 250–267.

[Jon97] M. Jonsson. Dynamical studies in several complex variables. *PhD Thesis*, Royal Institute of Technology, Stockholm, September 1997.

[Kif88] Y. Kifer. *Random Perturbations of Dynamical Systems*. Birkhäuser, 1988.

[McM94] C. T. McMullen. *Complex Dynamics and Renormalization (Annals of Mathematics Studies, 135)*. Princeton University Press, 1994.

[Mil92] J. Milnor. Local connectivity of Julia set: expository lectures. *Preprint*, Stony Brook IMS, 1992.

[Mil98] J. Milnor. Periodic orbits, external rays and the Mandelbrot set: an expository account. *Géométric complex et systémes dynamiques (Orsay 1995). Astérisque* **261** (2000), 277–333.

[Mun84] J. R. Munkres. *Elements of Algebraic Topology*. Addison-Wesley, 1984.

[Ses97] O. Sester. Etude dynamique des polynômes fibrés. *PhD thesis*, Université de Paris-Sud, Mai 1997.

[Ses99] O. Sester. Hyperbolicité des polynômes fibrés. *Bull. Soc. Math. France* **127** (1999), 393–428.

[Ses00] O. Sester. Suite infinie de chirurgies quasi-conformes. *Preprint*, ENS Lyon, 266, 2000.

[Thu85] W. P. Thurston. On the combinatorics and dynamics of iterated rational maps. *Preprint*, 1985. ftp://math.princeton.edu/pub/wpt/ratmaps/.

[Via97] M. Viana. Multidimensional nonhyperbolic attractors. *Pub. Math. IHES* **85** (1997), 63–96.

[Yoc] J.-C. Yoccoz. Locale connexité des ensembles de Julia. Communication personnelle, 1995.

https://doi.org/10.1017/S0143385701001456 Published online by Cambridge University Press