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# Stuart vortices on a sphere

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The exact steady two-dimensional solutions of the Euler equations due to Stuart (1967) are generalized to the surface of a sphere. The solutions are parametrized by three parameters $N$, $\theta_0$ and $\Omega_{max}$; the integer $N > 1$ denotes the number of smooth vorticity extrema, with extremal vorticity $\Omega_{max}$, that are equally spaced in longitudinal angle around a latitude circle at latitudinal spherical polar angle $\theta_0$. The solutions have two equal point-vortex singularities at the north and south spherical poles. Like Stuart's, the solutions are exact and explicit. The solutions are expected to be useful in geophysical and astrophysical applications where curvature effects are important.

## 1. Introduction

In 1967, Stuart introduced an exact solution of the steady two-dimensional Euler equations which have since become well-known to the fluid dynamics community as the 'Stuart vortices'. Stuart studied the solution as a model of the free shear layer. In this solution, the vorticity $\omega$ is exponentially related to the streamfunction $\psi$ so that

$$\omega = -\nabla^2\psi = -\mathrm{e}^{-2\psi}. \tag{1.1}$$

Mathematically, the relation (1.1) between the vorticity and the streamfunction results in the quasi-linear elliptic partial differential equation known as the Liouville equation. Stuart (1967) devotes a portion of his paper to a discussion of some general solutions of this equation.

Stuart's solution to (1.1) consists of an infinite periodic array of vortices described by the streamfunction

$$\psi = \log(C\cosh y + \sqrt{C^2-1}\cos x) \tag{1.2}$$

where $C$ is a real parameter satisfying $1 \leqslant C < \infty$. When $C = 1$, the solution represents a homogeneous shear layer profile in which all streamlines are parallel to the $x$-axis and the horizontal velocity varies like a hyperbolic tangent function with vertical distance $y$. In the opposite limit, $C \to \infty$, the solution reduces to an infinite row of identical point vortices separated by distance $2\pi$, each of circulation $-4\pi$. For all intermediate values $1 < C < \infty$, the solution has the structure of an infinite row of (Kelvin) cat's-eyes (Lamb 1932) with a smooth vorticity distribution. The parameter $C$ governs the steepness of the vorticity profile.

Stuart's solution is one of a small group of known exact solutions of the planar Euler equations for distributed vortical equilibria. Several others are known (see, for example, Saffman 1992 or Newton 2001 for references) but the majority of these are 'weak' solutions in the sense that they involve vortex patches. Examples include

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the celebrated Kirchhoff ellipse (Lamb 1932), the Moore–Saffman vortices (Moore & Saffman 1975) and the recently derived equilibria found by the present author (Crowdy 1999, 2002). Apart from Stuart's solution, another famous solution with a smooth vorticity distribution is the Lamb dipole (Lamb 1932). Meleshko & van Heijst (1994) survey a number of related smooth-vorticity solutions.

With regard to vortex equilibria on the sphere, much less is known. By far, the study of point vortex motion on the sphere has received the most attention. Newton (2001) is a good source of references. Bogomolov (1977) was one of the first to formulate the mathematical study of point-vortex dynamics on a sphere. Subsequent work ranges from that of Kimura & Okamoto (1987), Kidambi & Newton (1998, 2000), Lim, Montaldi & Roberts (2001) to Souliere & Tokieda (2002). Hally (1980) considered streets of vortices on surfaces of revolution (such as a sphere). Regarding distributed vorticity, Dritschel & Polvani (1992) and Polvani & Dritschel (1993) have studied vortex patches on a sphere using numerical methods (e.g. contour surgery). The only exact solutions for distributed vortex equilibria on a sphere known to the author are those of Verkley (1984, 1993) (on a rotating sphere) and the recently derived solutions of Crowdy & Cloke (2003) which involve hybrid vortex-patch/point-vortex combinations modelling multipolar vortices on a non-rotating sphere.

The ubiquity of large-scale vortex structures as well as organized vortex layers and vortex streets/rows in the atmospheres of planets and in geophysical flows is well-known (e.g. Marcus & Kundu 2000). Many of these vortical structures exist on a planetary scale and so the curvature of the planet must be expected to play a non-trivial role in both the structure and stability of the vortex configurations. For theoretical purposes, it is therefore of interest to examine whether it is possible to generalize the smooth Stuart-vortex solution to the surface of a sphere. This paper presents such a generalization. It is expected that the solutions might find application in geophysical and astrophysical contexts. Here, however, attention is restricted to a detailed derivation and characterization of the new mathematical solutions.

Mathematically, in Stuart's solution (1.2), there is an accumulation point of vortices at infinity but this does not cause any problems in terms of a physical interpretation of the solution because the plane is not a closed compact surface and the point at infinity is not part of the surface. The sphere, however, is a closed compact surface. Any physically meaningful generalization of Stuart's solution might therefore be expected to have a finite collection of vorticity extrema distributed over the spherical surface. One reason to suspect this is that if there are infinitely many and if, in analogy to the planar Stuart vortex solution, there is a corresponding limit in which these extrema tend to point vortices, then, owing to the compactness of the sphere, the solution would necessarily exhibit an accumulation point of point vortices on the spherical surface. This would constitute an unphysical singularity.

Another complication in generalizing Stuart's solution to the sphere arises from the fact that the surface of a sphere is closed. If there are no solid boundaries present (which we assume in order that our results might be used as a basic model for planetary atmospheres) then any vorticity distribution on the sphere must satisfy the condition that the integral of the vorticity over the entire spherical surface is zero.

The generalizations of Stuart's solution presented here have a relation between the streamfunction and the vorticity given by

$$\omega = -\nabla^2_\Sigma \psi = c\mathrm{e}^{d\psi} + \frac{2}{d} \tag{1.3}$$

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where $c$ and $d$ are real constants. $\nabla^2_\Sigma$ is the Laplace–Beltrami operator on a unit-radius sphere defined by

$$\nabla^2_\Sigma \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{1.4}$$

where $\theta$ and $\phi$ are the usual angle variables in spherical polar coordinates. General solutions of (1.3) are derived in this paper.

Once the general solution of (1.3) is determined, the specific choice corresponding to generalized Stuart vortices is then identified. The solutions are parametrized by an integer $N$. For $N = 1$, the solution of (1.3) is globally valid everywhere on the sphere but corresponds to an uninteresting solution having streamlines which are simply latitude circles. For all integers $N > 1$, the solutions are generally non-trivial and correspond to Stuart vortex layers consisting of $N$ smooth vorticity extrema equally spaced around some (specified) latitude circle. The solutions satisfy (1.3) everywhere on the sphere except at the north and south poles where the solution exhibits point-vortex singularities with identical circulations (which, significantly, are physically acceptable steady solutions of the Euler equations provided they do not move under the influence of the non-self-induced flow).

## 2. Vortex motion on a sphere

Consider vortex motion on the surface $S$ of a sphere. The sphere is non-rotating. Without loss of generality assume it has unit radius. In terms of standard spherical polar coordinates $(r, \theta, \phi)$ with the latitude angle $\theta$ measured from the axis through the north pole, the velocity vector has the form

$$\boldsymbol{u} = (0, v, u) \tag{2.1}$$

where $u$ and $v$ are the zonal and meridional components of the velocity field respectively. The incompressible nature of the flow allows the introduction of a scalar streamfunction $\psi(\theta, \phi)$ via

$$\boldsymbol{u} = \nabla\psi \wedge \boldsymbol{e}_r \tag{2.2}$$

where $\boldsymbol{e}_r$ is the radial unit vector. It is then possible to define a scalar vorticity field $\omega(\theta, \phi)$ such that

$$\omega\, \boldsymbol{e}_r = \nabla \wedge \boldsymbol{u} \tag{2.3}$$

where

$$\omega = -\nabla^2_\Sigma\psi. \tag{2.4}$$

$\nabla^2_\Sigma$ is the spherical Laplace–Beltrami operator defined in (1.4). In terms of the streamfunction $\psi$, $u$ and $v$ are given by

$$u = -\frac{\partial\psi}{\partial\theta}, \quad v = \frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi}. \tag{2.5}$$

In the steady case, the material conservation of vorticity is expressed by

$$\left(\frac{u}{\sin\theta}\frac{\partial}{\partial\phi} + v\frac{\partial}{\partial\theta}\right)\omega = 0. \tag{2.6}$$

With use of (2.5), (2.6) becomes

$$\frac{1}{\sin\theta}\left(-\frac{\partial\psi}{\partial\theta}\frac{\partial\omega}{\partial\phi} + \frac{\partial\psi}{\partial\phi}\frac{\partial\omega}{\partial\theta}\right) = 0. \tag{2.7}$$

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Thus, as in the planar case, a solution of (2.7) is clearly obtained when the vorticity $\omega$ is purely a function of the streamfunction $\psi$, i.e.

$$\omega = h(\psi) \tag{2.8}$$

for some differentiable function $h$. The choice of $h$ determines the vorticity–streamfunction relation. The equations above are isomorphic to the steady barotropic vorticity equation for a non-rotating sphere (see, for example, Polvani & Dritschel 1993).

In the spherical case, there exists a global constraint on the vorticity distribution. Gauss's theorem dictates that only vorticity fields which integrate to zero over the sphere are permitted, i.e.

$$\int_S \omega \, \mathrm{d}\sigma = 0 \tag{2.9}$$

where $\mathrm{d}\sigma$ denotes the area element on a spherical surface. Equation (2.9) will be referred to as the *Gauss constraint*.

## 3. The vorticity–streamfunction relation

A natural proposal for extending the Stuart solution to the sphere is to suggest the same vorticity–streamfunction relation as chosen in the planar case, $\omega = -c\mathrm{e}^{d\psi}$, thereby leading to the generalized partial differential equation

$$\nabla^2_\Sigma \psi = c\mathrm{e}^{d\psi} \tag{3.1}$$

where $c$ and $d$ are real constants. Equation (3.1) differs from (1.1) in that $\nabla^2$ has been replaced by $\nabla^2_\Sigma$. However, unless the solution has integrable singularities at certain points on the sphere, such a solution cannot satisfy the Gauss constraint (2.9) because the integral of the right-hand side of (3.1) over the spherical surface cannot equal zero.

At this point, it is worth remarking that when generalizing the notion of a planar point vortex to a spherical surface, exactly the same obstruction occurs. If $\boldsymbol{x}$ denotes a point on $S$, there can exist no solution of the equation

$$\nabla^2_\Sigma \psi = \delta(\boldsymbol{x}) \tag{3.2}$$

satisfying the Gauss constraint (2.9). To overcome this, it is conventional (Bogomolov 1977; Kimura & Okamoto 1987) to add a constant to the right-hand side of (3.2) and instead seek solutions to the modified equation

$$\nabla^2_\Sigma \psi = \delta(\boldsymbol{x}) + g. \tag{3.3}$$

In this case, it is clear that we must take

$$g = -\frac{1}{4\pi}, \tag{3.4}$$

so that

$$\nabla^2_\Sigma \psi = \delta(\boldsymbol{x}) - \frac{1}{4\pi}. \tag{3.5}$$

The right-hand side of (3.5) then satisfies (2.9). The solution of (3.5) yields the standard solution for an isolated point vortex on a sphere – apparently first derived by Bogomolov (1977).

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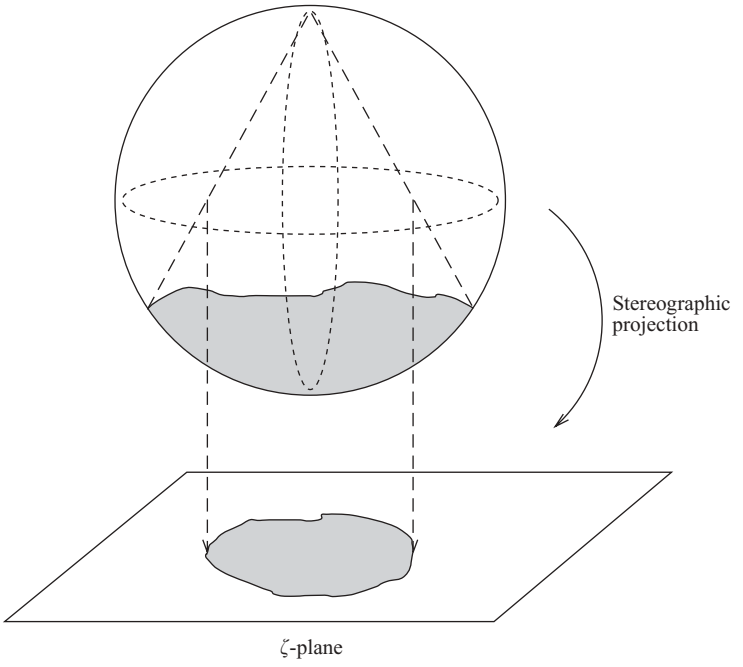



FIGURE 1. Schematic illustrating stereographic projection from the sphere to a complex $\zeta$-plane.

Motivated by this, the right-hand side of equation (3.1) is similarly modified to form the equation

$$\nabla_{\Sigma}^{2}\psi = c\mathrm{e}^{d\psi} + g \tag{3.6}$$

where $c, d$ and $g$ are real constants. Equation (3.6) will be referred to here as a *modified Liouville equation*. It will be shown in what follows that in the special case where the parameters $d$ and $g$ are related by

$$g = \frac{2}{d} \tag{3.7}$$

then the general solution of (3.6) can be written in closed form. To find these explicit solutions, we employ a combination of transformations of both independent and dependent variables. In the next section, a change of independent variables induced by stereographic projection is introduced. The angular variables $(\theta, \phi)$ are replaced by the complex variables $(\zeta, \bar{\zeta})$ in a stereographically projected plane. Then, in § 5, a special transformation of the dependent variable $\psi$ will be presented that reduces equation (3.6) to the solution of the standard Liouville equation in this plane of projection.

## 4. Stereographic projection

The surface of a sphere can be endowed with a complex analytic structure obtained by stereographic projection. A projection in which the north pole maps to infinity in a projected $\zeta$-plane is performed. Figure 1 shows a schematic illustrating this projection onto a complex $\zeta$-plane through the equator. Stereographic projection has

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been applied to problems of vortex dynamics on a sphere by several previous authors, e.g. Kidambi & Newton (2000), Newton (2001) and Crowdy & Cloke (2003).

In polar form,

$$\zeta = \tilde{r}\mathrm{e}^{\mathrm{i}\phi} \tag{4.1}$$

where

$$\tilde{r} = \cot\left(\frac{\theta}{2}\right). \tag{4.2}$$

The origin $\zeta = 0$ corresponds to the south pole of the sphere. It is convenient to observe that

$$\cos\theta = \frac{\zeta\overline{\zeta} - 1}{\zeta\overline{\zeta} + 1}, \quad \sin\theta = \frac{2\sqrt{\zeta\overline{\zeta}}}{\zeta\overline{\zeta} + 1}. \tag{4.3}$$

It can be verified that

$$\left.\frac{\partial}{\partial\theta}\right|_{\phi} = -\frac{\zeta}{\sin\theta}\left.\frac{\partial}{\partial\zeta}\right|_{\overline{\zeta}} - \frac{\overline{\zeta}}{\sin\theta}\left.\frac{\partial}{\partial\overline{\zeta}}\right|_{\zeta}, \tag{4.4}$$

and

$$\left.\frac{\partial}{\partial\phi}\right|_{\theta} = \mathrm{i}\zeta\left.\frac{\partial}{\partial\zeta}\right|_{\overline{\zeta}} - \mathrm{i}\overline{\zeta}\left.\frac{\partial}{\partial\overline{\zeta}}\right|_{\zeta}. \tag{4.5}$$

With use of (4.4) and (4.5), simple algebraic manipulations reveal that

$$\nabla^2_{\Sigma}\psi = (1 + \zeta\overline{\zeta})^2\psi_{\zeta\overline{\zeta}} \tag{4.6}$$

where subscripts denote partial differentiation.

## 5. Transformation of the dependent variable

On the stereographically projected $\zeta$-plane, (3.6) becomes

$$(1 + \zeta\bar{\zeta})^2\psi_{\zeta\bar{\zeta}} = c\mathrm{e}^{d\psi} + g. \tag{5.1}$$

Now, introduce the change of dependent variable

$$\phi(\zeta, \bar{\zeta}) \equiv \psi(\zeta, \bar{\zeta}) - \frac{2}{d}\log(1 + \zeta\bar{\zeta}). \tag{5.2}$$

Then

$$\mathrm{e}^{d\phi} = \frac{\mathrm{e}^{d\psi}}{(1 + \zeta\bar{\zeta})^2}. \tag{5.3}$$

Differentiation of (5.2) yields

$$\phi_{\zeta\bar{\zeta}} = \psi_{\zeta\bar{\zeta}} - \frac{2}{d}\frac{1}{(1 + \zeta\bar{\zeta})^2}. \tag{5.4}$$

Substitution of (5.3) and (5.4) into (5.1) implies

$$\phi_{\zeta\bar{\zeta}} + \frac{2}{d}\frac{1}{(1 + \zeta\bar{\zeta})^2} = c\mathrm{e}^{d\phi} + \frac{g}{(1 + \zeta\bar{\zeta})^2}. \tag{5.5}$$

Now choose

$$g = \frac{2}{d}, \tag{5.6}$$

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then the equation to solve for $\phi$ is

$$\phi_{\zeta\bar{\zeta}} = c\mathrm{e}^{d\phi}. \tag{5.7}$$

This is precisely the standard Liouville equation for $\phi$ in the projected plane.

It is interesting to remark that, in the stereographic plane, the streamfunction

$$\psi(\zeta, \bar{\zeta}) = \frac{2}{d}\log(1 + \zeta\bar{\zeta}) + \phi(\zeta, \bar{\zeta}) \tag{5.8}$$

is a linear superposition of a contribution corresponding to uniform vorticity (the first term on the right-hand side of (5.8)) and a 'Stuart vortex' contribution given by the $\phi$ portion of the right-hand side of (5.8) where $\phi$ satisfies the standard Liouville equation (5.7).

## 6. Mathematical solutions

These combined changes of independent and dependent variables have reduced the problem to that of solving the elliptic Liouville equation (5.7) for $\phi$ in a stereographically projected $\zeta$-plane. Once a solution for $\phi$ is found, $\psi$ can be reconstructed using (5.2). In § 6.1, we briefly discuss some general solutions of the elliptic Liouville equation; in § 6.2 the particular solutions that lead to generalized Stuart vortices on the sphere are presented.

### 6.1. *General solutions of the Liouville equation*

Many general solutions to the elliptic Liouville equation have been reported. Stuart (1967) lists several. One form of general solution to (5.7) given in Stuart (1967) (see also Liouville 1853) is

$$\phi(\zeta, \bar{\zeta}) = \frac{1}{d}\log\left(\frac{2f'(\zeta)\bar{f}'(\bar{\zeta})}{-cd(1 + f(\zeta)\bar{f}(\bar{\zeta}))^2}\right) \tag{6.1}$$

when $cd < 0$ and where $f(\zeta)$ is an analytic function in the domain $D$ in which the equation is to be solved. Equation (6.1) is the form of general solution to be exploited here; however we first make two important remarks on the nature of the analytic function $f(\zeta)$:

(*a*) The function $f(\zeta)$ need not necessarily be analytic everywhere in the domain $D$; rather, it can admit a distribution of isolated simple pole singularities and still produce a solution of the elliptic Liouville equation (5.7) that is non-singular in $D$. To see this, note that any simple pole of $f(\zeta)$ at a point $\hat{\zeta}$ would produce a double pole of $f'(\zeta)$ at the same point. However, the denominator in (6.1) contains the quantity $(1 + f(\zeta)\bar{f}(\bar{\zeta}))^2$ which will also, in general, possess a second-order pole at $\hat{\zeta}$. The second-order poles of both numerator and denominator thus conspire, in general, to produce a non-zero and non-singular argument of the logarithm, leading to a solution for $\phi$ which is non-singular at $\hat{\zeta}$.

(*b*) The derivative $f'(\zeta)$ must not vanish anywhere in $D$ if $\phi$ is required to be non-singular there. This result is clear by inspection of the formula (6.1).

The author has not found these facts stated explicitly in the literature but they are crucial in the analysis to follow. It therefore seems appropriate to emphasize them here.

Stuart (1967) records that the solution (1.2) can be derived from (6.1) by the particular choice

$$f(z) = A\tan(z/2) \tag{6.2}$$

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where the constant $A$ is related to the constant $C$ in (1.2) and $z$ is the usual complex variable with real and imaginary parts equal to the $x$ and $y$ Cartesian coordinates of the plane. Of course, in the planar Stuart vortex solution, the domain $D$ of interest is the complex plane, i.e.

$$D = \mathbb{C}. \tag{6.3}$$

Note that, consistent with remarks (*a*) and (*b*) above, the choice of $f(z)$ given in (6.2) has just simple pole singularities in $D$. Moreover, $f'(z) = (A/2)\sec^2(z/2)$ vanishes nowhere in $D$. This means that the solution $\psi$ is non-singular everywhere in $D$. It should be noted, however, that $f(z)$ has an accumulation point of simple poles at infinity which is therefore a singular point of the solution, but this point is not part of the domain $D$ and causes no problems in terms of a physical interpretation of the solution.

For a derivation of the solution (6.1), the reader is referred to Crowdy (1997) where the most general solution of the elliptic Liouville equation is derived using elementary methods. This solution subsumes all other known solutions and can easily be manipulated to retrieve the form of solution given in (6.1). The general solution presented in Crowdy (1997) depends on two arbitrary analytic functions and is closely related to the general solution described in equations (5.7)–(5.9) of Stuart (1967) and attributed there to E. Varley.

It is tempting, now that the equation for $\phi$ is just the standard Liouville equation in the projected plane, to postulate that the choice

$$f(\zeta) = A\tan(\zeta/2) \tag{6.4}$$

will yield the desired generalizations of the planar Stuart solution to the sphere. But things are not so simple. In the present case, $D$ is the extended (or compactified) complex $\zeta$-plane, i.e.

$$D = \mathbb{C} \cup \{\infty\}. \tag{6.5}$$

The choice (6.4) has a cluster point of simple pole singularities at $\zeta = \infty$ which is now part of the domain $D$ and corresponds to the spherical north pole. It is not clear how to assign any physical interpretation to such a mathematical singularity in $\phi$ (and hence in $\psi$) at the north pole which is why the initially tempting choice (6.4) is discarded.

### 6.2. *Generalized Stuart-vortex solutions*

It is clear from (6.1) that altering $d$ just rescales the strength of the vorticity everywhere on the sphere. Following Stuart (1967), we set $d = -2$ so that $g = -1$ from (5.6). The choice of constant $c$ is inconsequential; changing it corresponds to changing $\psi$ by an additive constant which does not alter the velocity or vorticity fields. However, from (6.1), it must be positive if $d$ is negative. We set $c = 1$.

Consider the rational function $f(\zeta)$ given by the $N$th-order polynomial

$$f(\zeta) = \tilde{a}\zeta^N + \tilde{b} \tag{6.6}$$

where $\tilde{a}, \tilde{b} \in \mathbb{C}$. This function has an $N$th-order pole at $\zeta = \infty$. Substituting (6.6) into (6.1), it is seen that the solution depends only on $|\tilde{a}|$, $|\tilde{b}|$ and $\arg[\tilde{a} - \tilde{b}]$. Therefore, without loss of generality, $\tilde{a}$ can be assumed real and positive, i.e. we take $\tilde{a} = a$ where $a \in \mathbb{R}$. For the time being, $\tilde{b}$ will be taken to be generally complex.

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Substitution into (6.1) implies that the streamfunction $\psi$ is then given explicitly as

$$\psi = -\frac{1}{2}\log\left(\frac{N^2a^2\zeta^{N-1}\bar{\zeta}^{N-1}(1+\zeta\bar{\zeta})^2}{(1+(a\zeta^N+\tilde{b})(a\bar{\zeta}^N+\bar{\tilde{b}}))^2}\right). \tag{6.7}$$

First, note that this choice of $\psi$ is invariant under the transformation $\zeta \mapsto \zeta e^{2\pi i/N}$ which means that it will yield an associated flow field which has an $N$-fold rotational symmetry about both the north and south poles. Recalling remarks (*a*) and (*b*) in § 6.1, when $N > 1$ (6.7) will fail to be a solution of (5.7) at $\zeta = 0$ (where there is a zero of $f'(\zeta)$) or at $\zeta = \infty$ (where $f(\zeta)$ has a pole of order $N > 1$). These points correspond to the south and north poles of the sphere respectively. However, it is crucial to note that at both of these points the streamfunction (6.7) is locally that associated with a point-vortex singularity on a spherical surface. To see this, recall that the streamfunction $\psi_{pv}$ associated with an isolated point vortex of circulation $\Gamma$ at a point on a unit-radius sphere corresponding to the projected point $\zeta_s$ is given by (see Newton 2001; Crowdy & Cloke 2003)

$$\psi_{pv} = -\frac{\Gamma}{4\pi}\log\left(\frac{(\zeta-\zeta_s)(\bar{\zeta}-\bar{\zeta}_s)}{(1+\zeta\bar{\zeta})(1+\zeta_s\bar{\zeta}_s)}\right). \tag{6.8}$$

This means that as $\zeta \to \zeta_s$,

$$\psi_{pv} \sim -\frac{\Gamma}{4\pi}\log((\zeta-\zeta_s)(\bar{\zeta}-\bar{\zeta}_s)) + \text{regular}. \tag{6.9}$$

This observation is significant because such singularities are physically admissible. Given their presence, it is a requirement of the Helmholtz laws of vortex motion (Saffman 1992) that, for a consistent steady solution of the Euler equations, both point vortices must be stationary under the effects of the local non-self-induced velocity field. The constant term in the local expansion of the non-self-induced velocity field must therefore vanish at both the north and south pole. However it should be clear, by the $N$-fold rotational symmetry of the streamfunction (6.7), that this condition is automatically satisfied. It can be verified analytically.

## 7. The Gauss constraint

We now verify that the solutions (6.7) satisfy the Gauss constraint (2.9). The total vorticity on the spherical surface will consist of a sum of the circulations of the two (equal) point vortices at the north and south pole plus the contribution from the smooth Stuart-type vorticity on the rest of the spherical surface associated with the streamfunction in (6.7).

Consider first the two point vortices. As $\zeta \to 0$, the streamfunction (6.7) has local behaviour

$$\psi \sim -\frac{(N-1)}{2}\log(\zeta\bar{\zeta}) + \text{regular}. \tag{7.1}$$

By comparison with (6.8) the circulation $\Gamma_S$ of the point vortex at the south pole is

$$\Gamma_S = 2\pi(N-1). \tag{7.2}$$

To compute the circulation of the point vortex at the north pole, it is necessary to rewrite the streamfunction (6.7) in terms of the local complex coordinate $\eta$, say, associated with the second chart in the atlas of the spherical surface (i.e. the coordinate obtained by stereographic projection through the south pole). It is easy to show that $\eta$

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is conformally related to $\zeta$ by $\eta = \zeta^{-1}$. Rewriting (6.7) in terms of this local coordinate reveals that, as $\eta \to 0$, $\psi$ has the local behaviour

$$\psi \sim -\frac{(N-1)}{2}\log(\eta\bar{\eta}) + \text{regular}, \tag{7.3}$$

from which we deduce that the circulation $\Gamma_N$ of the point vortex at the north pole is

$$\Gamma_N = 2\pi(N-1) = \Gamma_S. \tag{7.4}$$

Thus the circulations of the two antipodal point vortices are equal. It should be noted that in the case $N = 1$ the solution (6.7) is globally valid (i.e. there are no point vortices at the spherical poles).

It turns out to be possible to analytically compute the integral over the sphere of the smooth vorticity distribution associated with the streamfunction (6.7). Consider the quantity

$$\int\int_S \mathrm{e}^{-2\psi}\,\mathrm{d}\sigma = \int\int_S \left(\frac{f'(\zeta)\bar{f}'(\bar{\zeta})(1+\zeta\bar{\zeta})^2}{(1+f(\zeta)\bar{f}(\bar{\zeta}))^2}\right)\mathrm{d}\sigma \tag{7.5}$$

where $f(\zeta)$ is given in (6.6). If $x$ and $y$ denote the real and imaginary parts of $\zeta$, so that $\zeta = x + \mathrm{i}y$, then it can be shown that the area element $\mathrm{d}\sigma$ is given by

$$\mathrm{d}\sigma = \frac{4\mathrm{d}x\,\mathrm{d}y}{(1+\zeta\bar{\zeta})^2} \tag{7.6}$$

which implies

$$\begin{aligned}
\int\int_S \mathrm{e}^{-2\psi}\,\mathrm{d}\sigma &= -4\int\int_S \frac{\partial}{\partial\bar{\zeta}}\left(\frac{f'(\zeta)}{f(\zeta)}\frac{1}{1+f(\zeta)\bar{f}(\bar{\zeta})} - \frac{f'(\zeta)}{f(\zeta)}\right)\mathrm{d}x\,\mathrm{d}y \\
&= -\lim_{R\to\infty}\frac{4}{2\mathrm{i}}\oint_{|\zeta|=R}\left(\frac{f'(\zeta)}{f(\zeta)}\frac{1}{1+f(\zeta)\bar{f}(\bar{\zeta})} - \frac{f'(\zeta)}{f(\zeta)}\right)\mathrm{d}\zeta \\
&= \lim_{R\to\infty}\frac{4}{2\mathrm{i}}\oint_{|\zeta|=R}\frac{f'(\zeta)\bar{f}(\bar{\zeta})}{1+f(\zeta)\bar{f}(\bar{\zeta})}\,\mathrm{d}\zeta \\
&= \lim_{R\to\infty}\frac{4}{2\mathrm{i}}\int_0^{2\pi}\frac{NaR^{N-1}\mathrm{e}^{\mathrm{i}(N-1)\theta}(aR^N\mathrm{e}^{-\mathrm{i}N\theta}+\bar{\tilde{b}})\mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta}{[1+(aR^N\mathrm{e}^{\mathrm{i}N\theta}+\tilde{b})(aR^N\mathrm{e}^{-\mathrm{i}N\theta}+\bar{\tilde{b}})]^2} \\
&= 4\pi N
\end{aligned} \tag{7.7}$$

where we have used the complex form of Green's theorem and the definition (6.6) of $f(\zeta)$. The integral of the vorticity due to the streamfunction $\psi$ given in (6.7) over the surface of the sphere is

$$-\int_S (\mathrm{e}^{-2\psi} - 1)\,\mathrm{d}\sigma \tag{7.8}$$

which, using the result (7.7) and the well-known value of the surface area of a unit-radius sphere, equals

$$-4\pi(N-1). \tag{7.9}$$

The total vorticity on the spherical surface is the sum of (7.2), (7.4) and (7.9). This sum is zero. The solutions (6.7) therefore satisfy the Gauss constraint (2.9) for any choices of the parameters $a, \tilde{b}$ and $N$.

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## 8. Characterization of the solutions

In this section, the solutions are studied in more detail. The choice of integer $N \geqslant 1$ is arbitrary. Consider first the case $N = 1$. From (6.7), the streamlines, or $\psi$-contours, are given by

$$a^2\zeta\bar{\zeta} + a\overline{\tilde{b}}\zeta + a\tilde{b}\bar{\zeta} + 1 + |\tilde{b}|^2 + \alpha(1 + \zeta\bar{\zeta}) = 0 \tag{8.1}$$

where $\alpha$ is any negative constant (which labels the streamlines). These streamlines are circles. This solution is uninteresting. However, as pointed out earlier, it is non-singular everywhere on the sphere with no point vortices at the spherical poles.

The solutions for $N > 1$ are more interesting. To examine them, let $\tilde{b} = be^{i\phi}$ where $b \geqslant 0$ and $\phi$ are real parameters. It is found that the extrema of vorticity occur on the $N$ rays (in the stereographically projected plane) for which

$$\arg[\zeta^N] = \pi + \phi. \tag{8.2}$$

By the $N$-rotational symmetry of the configuration about the south (or north) pole, it is clear that $\phi$ can be taken to be zero without loss of generality in the physical solutions. From now on, we therefore take $\tilde{b} = b \in \mathbb{R}$. A solution corresponding to a $\tilde{b}$ with non-zero $\phi$ simply corresponds to a rotation of the sphere (displaying the $\phi = 0$ solution) through angle $\phi$ about an axis through the north and south poles.

Setting $\zeta = se^{i\pi/N}$ where $s \in \mathbb{R}$, we define $\Omega(s)$ to be the vorticity profile along the ray $\arg[\zeta] = \pi/N$. Substitution yields the formula

$$\Omega(s) = \left(1 - \frac{N^2a^2s^{2N-2}(1+s^2)^2}{(1 + b^2 + a^2s^{2N} - 2abs^N)^2}\right). \tag{8.3}$$

The vorticity extrema occur at that value of $s$ ($s_{max}$, say) satisfying

$$\Omega'(s_{max}) = 0. \tag{8.4}$$

After some algebra, this condition reduces to

$$\big((N+1)s_{max}^2 + (N-1)\big)\big(1 + b^2 + a^2s_{max}^{2N} - 2abs_{max}^N\big) = 2Nas_{max}^N\big(1 + s_{max}^2\big)\big(as_{max}^N - b\big). \tag{8.5}$$

To study the solutions, it is natural to fix the latitude angle $\theta_0$ at which the $N$ vorticity extrema occur; $s_{max}$ then follows from this specification because

$$s_{max} = \cot(\theta_0/2). \tag{8.6}$$

We assume $0 < \theta_0 \leqslant \pi/2$ so that the layer of vorticity extrema is situated in the northern hemisphere. This is done without loss of generality because the circulations of the two antipodal point vortices are identical. Solutions corresponding to $\pi/2 \leqslant \theta_0 < \pi$ can be obtained from those corresponding to $0 < \theta_0 \leqslant \pi/2$ simply by turning the sphere upside-down.

Suppose now that $N, a$ and $\theta_0$ are given. This means that $N, a$ and $s_{max}$ are known. Equation (8.5) then becomes a nonlinear equation for the parameter $b$. It turns out that this nonlinear condition can be rewritten as the quadratic

$$c_2(a, s_{max}, N)b^2 + c_1(a, s_{max}, N)b + c_0(a, s_{max}, N) = 0 \tag{8.7}$$

where

$$c_2(a, s_{max}, N) = (N+1)s_{max}^2 + (N-1), \tag{8.8}$$

$$c_1(a, s_{max}, N) = 2as_{max}^N\big(1 - s_{max}^2\big), \tag{8.9}$$

$$c_0(a, s_{max}, N) = (N-1) + (N+1)s_{max}^2 - (N+1)a^2s_{max}^{2N} - (N-1)a^2s_{max}^{2N+2}, \tag{8.10}$$

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leading to explicit solutions for $b$ given by

$$b = \frac{-c_1(a, s_{max}, N) \pm \sqrt{c_1(a, s_{max}, N)^2 - 4c_2(a, s_{max}, N)c_0(a, s_{max}, N)}}{2c_2(a, s_{max}, N)}, \tag{8.11}$$

whenever these formulas yield $b$-values that are real and non-negative.

Let $\Omega_{max}$ be the extremal value of the vorticity at $s_{max}$. With $b$ given by (8.11), an explicit formula for $\Omega_{max}$ can be derived using (8.3) and the definition $\Omega_{max} = \Omega(s_{max})$. There should be no confusion in our notational choice of using $\Omega_{max}$ to denote the extremal value of $\Omega(s)$ at $s = s_{max}$ irrespective of whether it is a maximum or a minimum.

With the positive sign taken in (8.11) it is easily shown that

$$b \sim s_{max}^N a \quad \text{as} \quad a \to \infty. \tag{8.12}$$

This means that the positive-square-root branch in (8.11) always leads to a solution with real non-negative $b$ for sufficiently large $a$. For given $s_{max}$ and $N$, the vanishing of the discriminant in (8.11) gives a lower bound on the admissible values of $a$ for which a corresponding real non-negative $b$ can be found. Denote this lower bound by $a_{crit}(s_{max}, N)$. Algebraic manipulations lead to the formula

$$a_{crit}(s_{max}, N) = \frac{(N-1) + (N+1)s_{max}^2}{N s_{max}^N \left(1 + s_{max}^2\right)}. \tag{8.13}$$

Note that when $\theta_0 = \frac{1}{2}\pi$, so that $s_{max} = 1$, then $a_{crit} = 1$ for all values of $N$. In general, $a_{crit}$ is a function of both $s_{max}$ and $N$. Thus, for given $\theta_0$ and $N$, at least one real non-negative solution for $b$ is found to exist in the range

$$a \in [a_{crit}, \infty). \tag{8.14}$$

We now explore whether the negative branch of the square root in (8.11) can provide a second real positive solution for $b$. For large values of $a$, the solution (8.11) with the negative-square-root branch is negative but becomes zero at a second critical value of $a$, which will be denoted $a_*(s_{max}, N)$ since it will generally be a function of both $s_{max}$ and $N$. This occurs when $c_0(a, s_{max}, N) = 0$ so that the equation satisfied by $a_*$ is

$$c_0(a_*, s_{max}, N) = 0. \tag{8.15}$$

This leads to the explicit formula

$$a_*(s_{max}, N) = \frac{1}{s_{max}^N} \sqrt{\frac{(N-1) + (N+1)s_{max}^2}{(N+1) + (N-1)s_{max}^2}}. \tag{8.16}$$

If $a_* > a_{crit}$ then there will exist a range of $a$-values for which there are two possible real non-negative $b$-values. Using (8.13) and (8.16), it is found that

$$a_* = \left( \frac{N\left(1 + s_{max}^2\right)}{\sqrt{\left(N^2\left(1 + s_{max}^2\right)^2 - \left(s_{max}^2 - 1\right)^2\right)}} \right) a_{crit}. \tag{8.17}$$

Hence $a_*(s_{max}, N) > a_{crit}(s_{max}, N)$ for all $N$ if $s_{max} \neq 1$. In the special case $s_{max} = 1$, the vorticity extrema are situated at the equator. In this case,

$$a_{crit}(1, N) = a_*(1, N) = 1 \tag{8.18}$$

so that there is a single branch of solutions for real non-negative $b$ for $a$ in the range

$$a \in [1, \infty). \tag{8.19}$$

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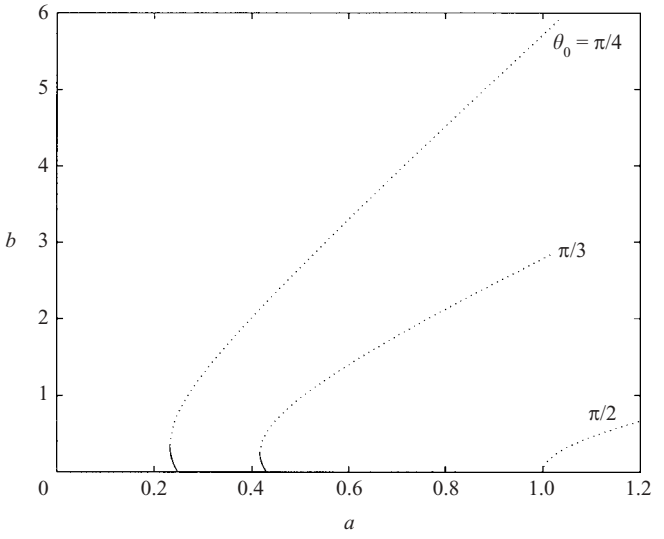



FIGURE 2. Solutions for $b$ as a function of $a$ with $N=2$ and the three choices of $\theta_0 = \pi/2, \pi/3, \pi/4$. Note that in the case $\theta_0 = \pi/2$ there is only one solution for each $a$. The solid line denotes the solution for $b$ corresponding to the negative-square-root branch in (8.11), the dotted line to the positive-square-root branch. The two solutions join smoothly when $a = a_{crit}$ forming a continuous branch of solutions.

A graph of the generic solution structure for $b$ is given in figure 2 for the case $N=2$ and $\theta_0 = \pi/4, \pi/3$ and $\pi/2$. In the two cases $\theta_0 = \pi/4$ and $\pi/3$ it is important to note that what appears at first sight to be two possible solution branches is, in fact, a single continuous branch of solutions. In the terminology of bifurcation theory, with $a$ as the order parameter, there is a saddle-node bifurcation at $a = a_{crit}$.

It is instructive to examine the graph of $\Omega_{max}$ against $a$. Figure 3 shows this graph in the case $N=2$ and for $\theta_0 = \pi/4, \pi/3$ and $\pi/2$. The behaviour shown is typical. Note that $\Omega_{max}$ is a function that is monotone-decreasing as one follows the complete branch of solutions starting at $a_*$. That is, those $a$-values having two possible values of $b$ yield solutions with different values of $\Omega_{max}$. This suggests that a more natural, and more physical, choice of parameters than $a$ and $b$ (which were introduced for purely mathematical reasons) is to characterize the solutions by the three parameters $N, \theta_0$ and $\Omega_{max}$. In this way, the solutions are characterized by the number $N$ of vorticity extrema, with extremal vorticity $\Omega_{max}$, situated on the latitude circle at angle $\theta = \theta_0$. Such parameters specify a unique solution.

The streamlines of the flow on the projected $\zeta$-plane are the contours $\psi = \text{constant}$. These are straightforward to plot using the explicit formula (6.7). Given the set of points $\zeta$ on a given $\psi$-contour in the projected $\zeta$-plane, the following equations can be inverted for $\theta$ and $\phi$ for any point on the contour:

$$\cot(\theta/2) = |\zeta|, \quad \phi = \arg[\zeta]. \tag{8.20}$$

Given $\theta$ and $\phi$, the corresponding point $(X, Y, Z)$ on the physical sphere can be calculated using

$$X = \sin\theta\cos\phi, \quad Y = \sin\theta\sin\phi, \quad Z = \cos\theta. \tag{8.21}$$

Here we make an observation. In the $\zeta$-plane of projection, it is seen from (6.7) that the streamlines are real algebraic curves. In their studies of point-vortex dynamics

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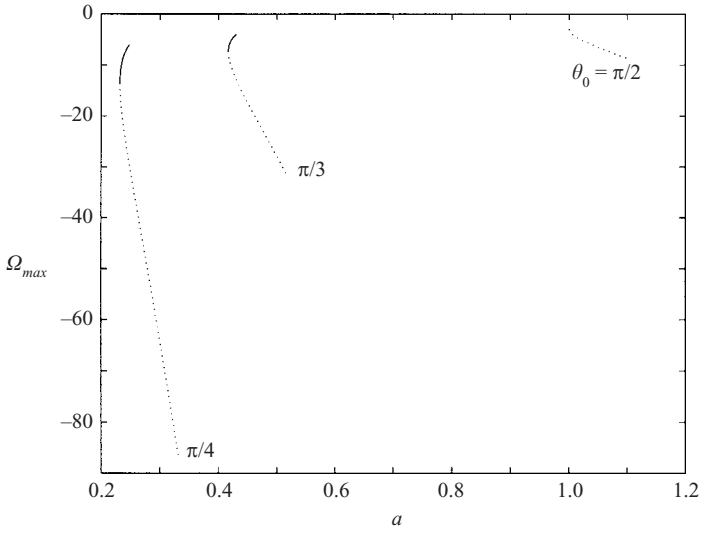



FIGURE 3. $\Omega_{max}$ as a function of $a$ with $N=2$ and the three choices of $\theta_0=\pi/2, \pi/3, \pi/4$. The solid line denotes the value of $\Omega_{max}$ derived using the solution for $b$ corresponding to the negative-square-root branch in (8.11), the dotted line to that computed using the positive-square-root branch.

(specifically, 'twisters') on a sphere, Souliere & Tokieda (2002) make some similar observations on the fact that the trajectories of point vortices are algebraic curves. Similarities between the streamline patterns shown here and those in figure 3 of Souliere & Tokieda (2002) are clear.

It is natural to examine the nature of the two limiting cases when $\Omega_{max}$ tends to its minimal and maximal value. $\Omega_{max}$ assumes its maximal value when $a=a_*$ as is seen from figure 3. There is no lower bound on the value of $\Omega_{max}$ and it becomes infinite as $a$ tends to infinity. We now examine these two cases separately.

### 8.1. *The case $a = a_*$*

When $a=a_*$, $b=0$. Inspection of (6.7) reveals that the solution for $\psi$ then depends only on $|\zeta|$ so that the streamlines in this case are all latitude circles (that is, concentric circles centred on $\zeta=0$ in the stereographically projected plane). This is analogous to the $C=1$ case of Stuart's planar solution (1.2) where it becomes a hyperbolic-tangent shear layer profile with straight streamlines aligned with the $x$-axis. Owing to the spherical geometry, the analogous streamlines here are concentric latitude circles.

### 8.2. *Point vortex limit, $a \to \infty$*

In the limit $a\to\infty$, $\Omega_{max}\to\infty$. Moreover, it is known that $b$ also tends to infinity according to the asymptotic relation (8.12). From (6.7) this means that, asymptotically,

$$\psi \sim -\frac{1}{2}\log\left(\frac{N^2}{a^2}\frac{\zeta^{N-1}\bar{\zeta}^{N-1}(1+\zeta\bar{\zeta})^2}{\left(\zeta^N+s_{max}^N\right)^2\left(\bar{\zeta}^N+s_{max}^N\right)^2}\right). \tag{8.22}$$

To within constants, (8.22) can be rewritten

$$\psi \sim -\frac{N-1}{2}\log\left(\frac{\zeta\bar{\zeta}}{1+\zeta\bar{\zeta}}\right)+\sum_{j=1}^{N}\log\left(\frac{(\zeta-\zeta_j)(\bar{\zeta}-\bar{\zeta}_j)}{(1+\zeta\bar{\zeta})(1+\zeta_j\bar{\zeta}_j)}\right)-\frac{N-1}{2}\log\frac{1}{1+\zeta\bar{\zeta}} \tag{8.23}$$

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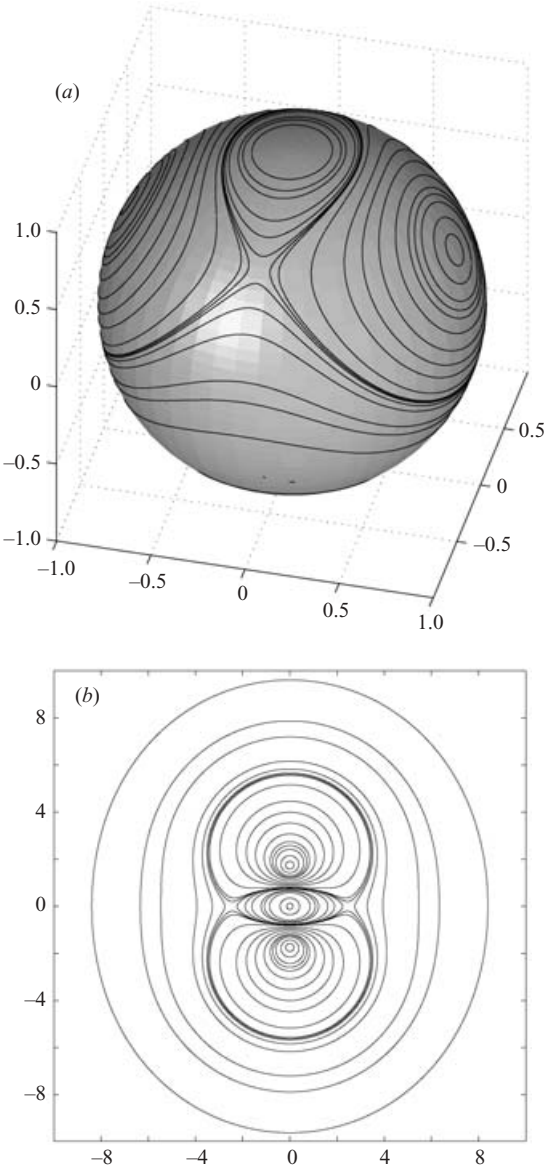



FIGURE 4. Solution for $N = 2$, $a = 1$ and $\theta_0 = \pi/3$ shown in orthographic ($a$) and stereographic projection ($b$). The corresponding values of $b$ and $\Omega_{max}$ are 2.78 and $-173.93$ respectively. There are point vortices, each of circulation $2\pi$, at both the north and south poles.

where

$$\zeta_j = s_{max} e^{i\pi(1+2(j-1))/N}, \quad j = 1, 2, .., N. \tag{8.24}$$

Comparison of (8.23) with (6.8) reveals that the first term on the right-hand side of (8.23) corresponds to a point vortex at the south pole (i.e. $\zeta = 0$), the second term to $N$ identical point vortices at positions $\{\zeta_j | j = 1, .., N\}$, while the third term corresponds to a point vortex at the north pole. The circulations of the point vortices at the north and south pole are equal and are given by $2\pi(N-1)$ as previously determined. The circulations of the point vortices at $\{\zeta_j | j = 1, 2, \ldots, N\}$ are all equal and are given

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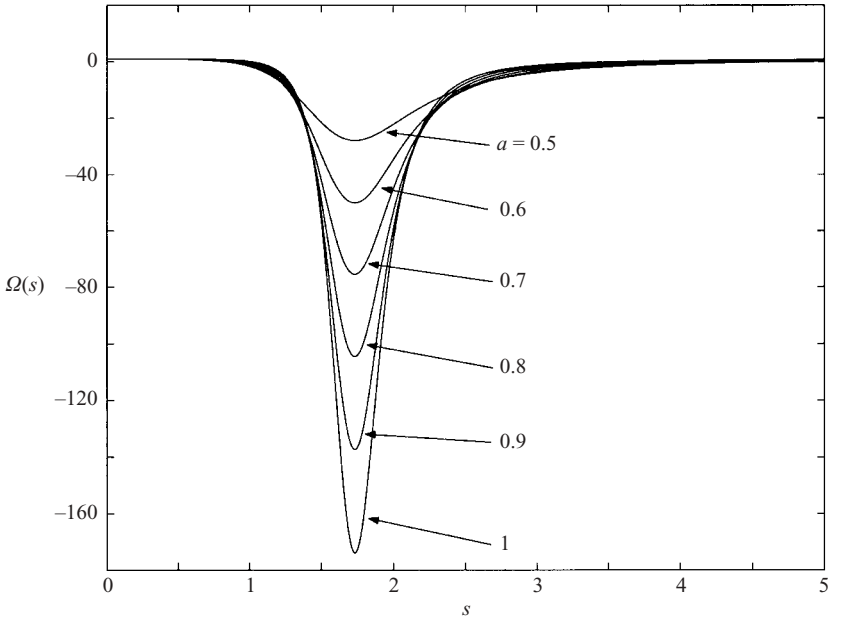



FIGURE 5. Graph of $\Omega(s)$ against $s$ for $N=2, \theta_0=\pi/3$ and $a=0.5, (0.1), 1$. The respective values of $\Omega_{max}$ are $-27.93, -50.09, -75.51, -104.55, -137.35$ and $-173.93$. As $a$ increases, the vorticity profiles become more concentrated about the extremal value $\Omega_{max}$ which itself increases in magnitude with increasing $a$.

by $-4\pi$. This limit is analogous to the point-vortex limit $C \to \infty$ of Stuart's planar solution (1.2).

To check this limiting solution, in the Appendix a pure point-vortex problem on the sphere is examined. In general, if two equal point vortices of strength $\gamma_0$ are placed at the north and south spherical poles while $N$ equal point vortices of circulation $\gamma_s$ are equally spaced around some latitude circle, the configuration will constitute a relative equilibrium of the Euler equations in which the point vortices off the spherical poles rotate about the axis through the poles at constant angular velocity while the vortices at the spherical poles remain stationary. At a critical ratio of circulations the entire configuration is stationary (i.e. the angular velocity of the off-pole vortices is zero). In the Appendix, this ratio is found to be

$$\frac{\gamma_0}{\gamma_s} = -\frac{(N-1)}{2} \tag{8.25}$$

which is precisely the ratio of circulations in the point-vortex limit of the generalized Stuart vortices.

### 8.3. *General case: $a_* < a < \infty$*

Like the planar solutions (1.2) of Stuart, all intermediate cases $a_* < a < \infty$ are found to correspond to smooth vorticity distributions displaying a distinctive cat's-eye pattern. In figure 4, a typical streamline distribution both on the sphere and on the stereographically projection $\zeta$-plane is shown in the case $N=2$ with $\theta_0=\pi/3$ and $a=1$. The corresponding $b$ is calculated to be 2.78 (correct to 2 decimal places) while the corresponding $\Omega_{max}$ is $-173.93$. Figure 5 features a graph of $\Omega(s)$ as a function of distance $s$ along the rays in the $\zeta$-plane on which the vorticity extrema are

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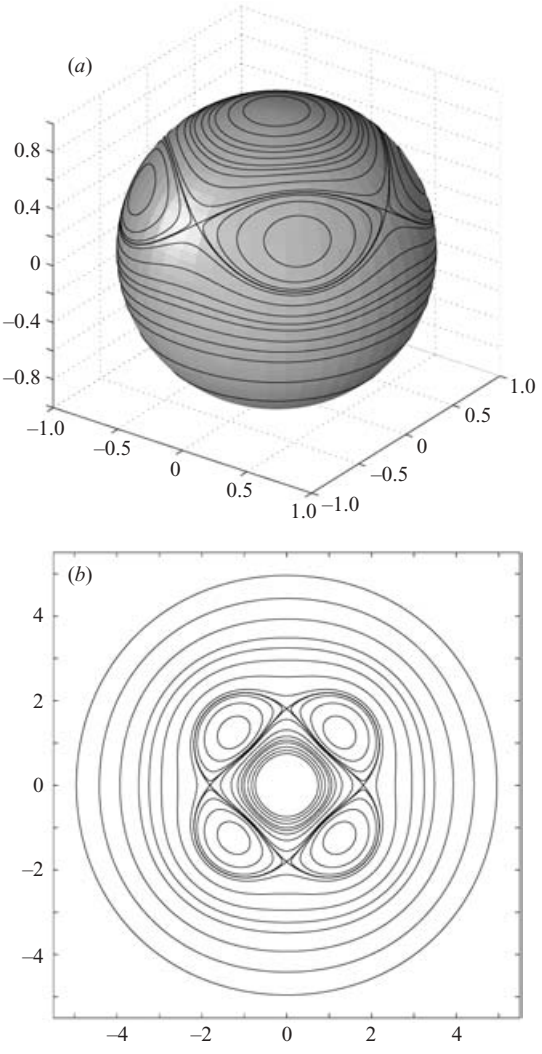



FIGURE 6. Solution for $N=4, a=0.15$ and $\theta_0=\pi/3$ shown in orthographic ($a$) and stereographic projection ($b$). The corresponding values of $b$ and $\Omega_{max}$ are 0.81 and $-92.74$ respectively. There are point vortices, each of circulation $6\pi$, at both the north and south poles.

located. In this case where $N=2$, this corresponds to the vorticity distribution along the imaginary $\zeta$-axis. Graphs are shown for several different values of $a$ in order to illustrate that, as $a$ increases, the distribution of vorticity becomes more concentrated about the extremal (negative) value $\Omega_{max}$ which also increases in magnitude with increasing $a$. As mentioned above, as $a\to\infty$, the vorticity distribution tends to a $\delta$-function distribution with the off-pole vortices having negative circulation. Figure 5 is consistent with this.

Figure 6 shows a typical streamline distribution in the case $N=4$, $\theta_0=\pi/3$ and $a=0.15$. The corresponding value of $b$ is 0.81 while the value of $\Omega_{max}$ is $-92.74$. Figure 7 shows superposed graphs of $\Omega(s)$ against $s$ for several values of $a$. Again, the profiles steepen with increasing $a$. Figure 8 shows the streamline distribution with $N=10$ and $\theta_0=\pi/3$. Now the distinctive cat's-eye patterns typically associated with

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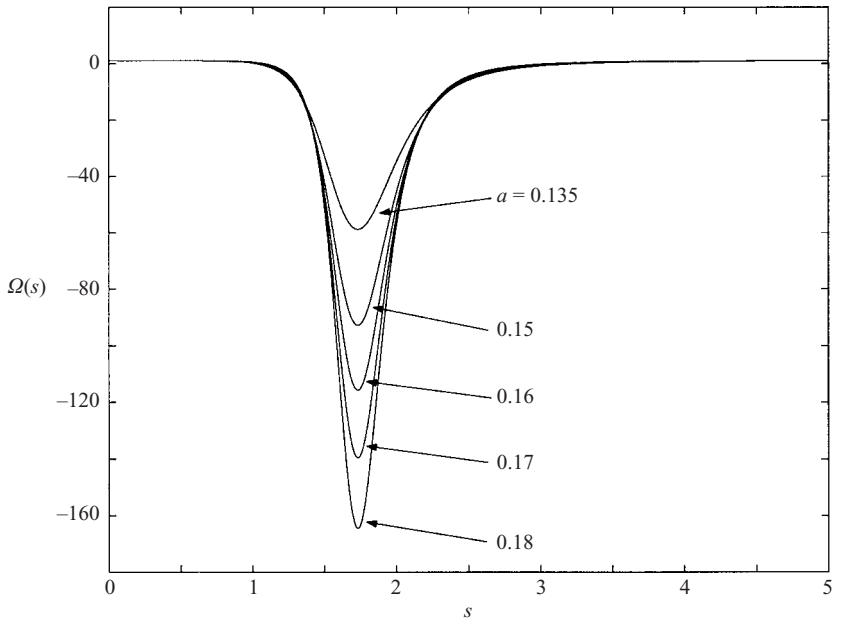



FIGURE 7. Graph of $\Omega(s)$ against $s$ for $N=4, \theta_0=\pi/3$ and $a=0.135, 0.15, 0.16, 0.17$ and 0.18. The respective values of $\Omega_{max}$ are $-58.77, -92.74, -115.70, -139.56$ and $-164.53$. As $a$ increases, the vorticity profiles become more concentrated about the extremal value $\Omega_{max}$ which itself increases in magnitude with increasing $a$.

the planar Stuart vortex solution become more apparent. As the number of vortices $N$ is taken even larger, greater numbers of cat's-eyes congregate around the given latitude circle. Figure 9 shows a street of sixteen smooth vortices equally spaced around the equator.

## 9. Discussion

The general solution to a particular modified Liouville equation on a sphere has been found. This equation is (3.6) with $g=2/d$. A particular class of solutions to this equation has then been identified sharing all the features of the well-known planar Stuart vortex solution. For this reason, the solutions have been interpreted as generalizations of the planar Stuart vortex solution to the sphere.

It is worth pointing out that the planar analogue of (5.1) is

$$\nabla^2\psi = c\mathrm{e}^{d\psi} + g \tag{9.1}$$

which is precisely the Poisson–Boltzmann equation. It is not known whether the general solution to this equation in the plane can be written in closed form. It is interesting that the Poisson–Boltzmann equation, generalized to a sphere by replacing the planar Laplacian by the Laplace–Beltrami operator, does admit a closed-form general solution, as we have shown.

It is intriguing that the (apparently special) choice $g=2/d$ has led both to the possibility of finding an explicit representation for the general solution of (3.6) and to a special class of those solutions satisfying the Gauss constraint (2.9). While this might be coincidence, it is easier to believe that the condition $g=2/d$ is a 'solvability condition' for finding solutions of (1.3) on a sphere, in the same way that the constant

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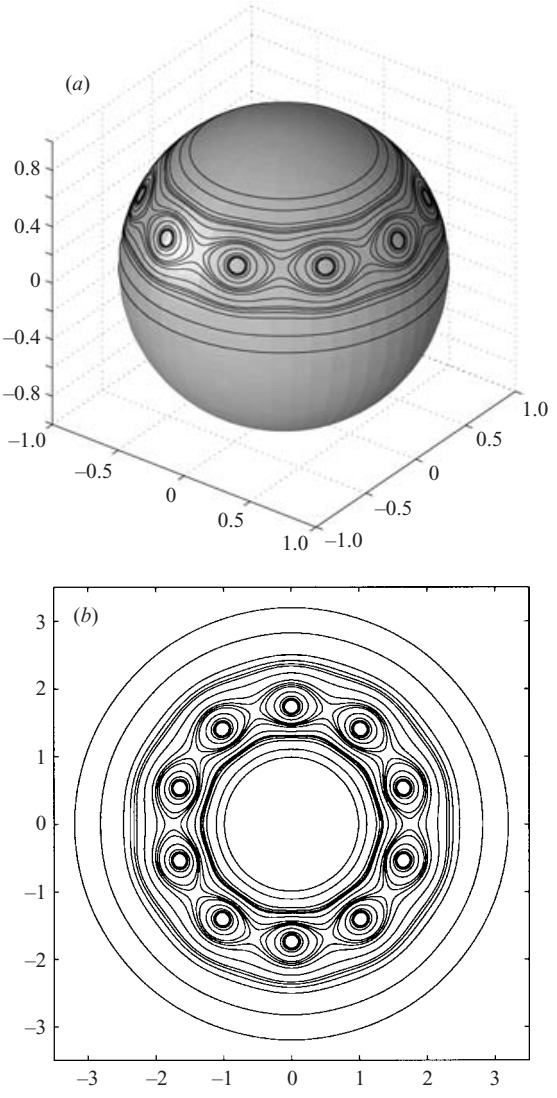



FIGURE 8. A row of 10 Stuart vortices at latitude $\pi/3$ shown in orthographic ($a$) and stereographic projection ($b$). The solution parameters are $N = 10, a = 0.01,\ b = 2.203$ and $\theta_0 = \pi/3$. There are point vortices, each of circulation $18\pi$, at both the north and south poles.

on the right-hand side of (3.3) cannot be arbitrary but must equal $-1/4\pi$ if the sphere has unit radius and the coefficient of the $\delta$-function is unity. In the case of the modified Liouville equation, however, it is not obvious why the choice $g = 2/d$ is relevant. This remains an open mathematical issue. Will other choices of $g$ also lead to solutions satisfying the Gauss constraint?

Only those solutions deemed to be the most natural generalizations of the planar Stuart solution have been presented in this paper. These correspond to the special choice (6.6). It is clear that other choices of rational function $f(\zeta)$ in our scheme will potentially lead to other classes of physically admissible solutions consisting of a smooth Stuart-type vorticity distribution punctuated by a finite distribution of point vortices. However, in constructing more general solutions, care must be taken; the

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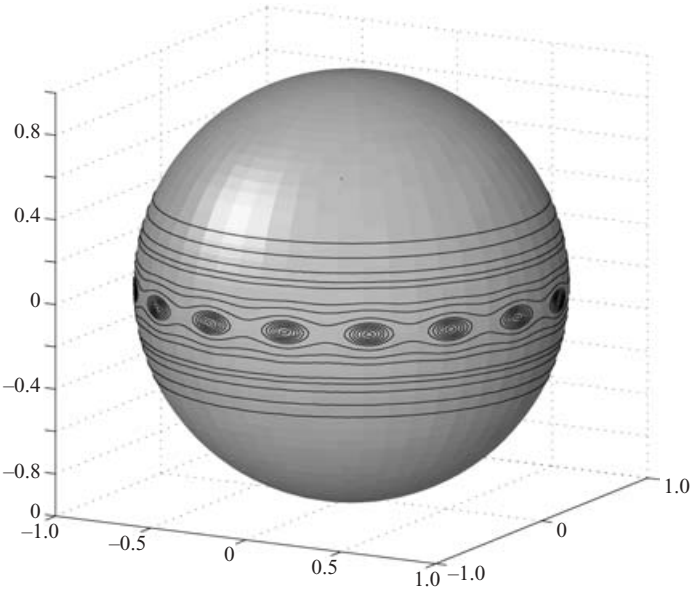



FIGURE 9. A row of 16 Stuart vortices at the equator. The solution parameters are $N=16, a=1.05,\ b=0.320$ and $\theta_0=\pi/2$. There are point vortices, each of circulation $30\pi$, at both the north and south poles.

zeros of $f'(\zeta)$ and the poles of $f(\zeta)$ that are of order two or higher will, in general, induce point-vortex singularities in the solution for $\psi$ which, while being physically admissible, must be stationary according to the Helmholtz laws of vortex motion if the solution is to be a consistent equilibrium of the Euler equations. This will place constraints on the choice of rational function $f(\zeta)$. Nevertheless, the problem of finding equilibria can be reduced to the solution of a finite set of purely algebraic nonlinear equations.

The Stuart vortices have been generalized to a row of counter-rotating vortices by Mallier & Maslowe (1993). It would be of interest to examine whether the methods presented here, i.e. the combination of stereographic projection with a strategic change of dependent variable, might similarly lead to explicit solutions of a modified sinh–Gordon equation on the sphere, i.e.

$$\nabla^2_\Sigma \psi = c \sinh d\psi + g, \tag{9.2}$$

perhaps with some special choice of $g$. It is also worth mentioning that Haslam & Mallier (2003*a*, *b*) have considered various types of smooth vorticity distributions on other surfaces, including the cylinder and the sphere, although their approach is quite different from that presented here.

Pierrehumbert & Widnall (1982) have analysed the stability properties of the planar Stuart vortex solution. The stability properties of the generalized spherical solutions remain to be investigated. Such a study is likely to be interesting: Dritschel & Polvani (1992) have found that, owing to curvature effects absent in the planar case, distributions of vorticity on the sphere can have very different stability properties to their planar counterparts.

The simplicity and physical importance of the planar Stuart model has made it the basis of many theoretical investigations where, for example, additional physical

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effects (such as viscosity, rotation) have been added perturbatively or numerically. For a small sample of such investigations see Mallier (1995), Godeford, Cambon & Leblanc (2001), Meiron, Moore & Pullin (2000) and Tio *et al.* (1993). It is hoped that the new solutions herein will similarly constitute a basic model for inviscid vortex layers when curvature effects are important, as is the case for many planetary-scale geophysical and astrophysical flows.

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## Appendix. A point-vortex problem

Consider a configuration of point vortices on the surface of a sphere where there are two equal point vortices of circulation $\gamma_0$ at the north and south poles and $N$ equal point vortices of circulation $\gamma_s$ at points corresponding to the stereographically projected points

$$\zeta_k = r\mathrm{e}^{2\pi\mathrm{i}(k-1)/N}, \quad k = 1, 2, .., N, \tag{A 1}$$

where $r$ is some real number. The instantaneous streamfunction is given by

$$\psi = -\frac{\gamma_0}{4\pi}\log\left(\frac{\zeta\bar{\zeta}}{1+\zeta\bar{\zeta}}\right) - \frac{\gamma_s}{4\pi}\sum_{k=1}^{N}\log\left(\frac{(\zeta-\zeta_k)(\bar{\zeta}-\bar{\zeta}_k)}{(1+\zeta\bar{\zeta})(1+\zeta_k\bar{\zeta}_k)}\right) - \frac{\gamma_0}{4\pi}\log\left(\frac{1}{1+\zeta\bar{\zeta}}\right) \tag{A 2}$$

where the first and third terms on the right-hand-side correspond to point vortices at the south and north pole respectively. For general values of $\gamma_0$ and $\gamma_s$, the vortices off the spherical poles will rotate with constant angular velocity about the axis through the poles. However, using (A 2) and applying the condition that the point vortex at $\zeta_1$ is stationary under the effects of the non-self-induced components of the local velocity field, it can be shown that the condition for this point vortex to be steady requires that

$$\frac{\gamma_0}{\gamma_s} = -\frac{(N-1)}{2}. \tag{A 3}$$

This is done using the fact (derivable from (2.5)) that

$$u - \mathrm{i}v = \frac{2\zeta}{\sin\theta}\psi_\zeta. \tag{A 4}$$

By the rotational symmetry of the configuration about the axis through the poles, it is clear that if (A 3) is satisfied then the other $N-1$ vortices at (projected) positions $\{\zeta_j | j = 2, .., N\}$ will also be steady. The steadiness of the point vortices at the north and south poles is also obvious from symmetry considerations.

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