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Threshold distribution functions for some random representable matroids

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1. *Introduction*

A random submatroid ω_r of the projective geometry $PG(r-1,q)$ is obtained from $PG(r-1,q)$ by deleting elements so that each element has, independently of all other elements, probability $1-p$ of being deleted and probability p of being retained. The properties of such random structures were studied in [5] and [6]. In the first of these papers, p was kept fixed, while in the second, motivated by Erdös and Rényi's work ([3], [4]) on random graphs, p was taken to be a function of r. A recent paper of Bollobás [2] strengthens and extends a number of the results of Erdös and Rényi. In this paper we prove matroid analogues of several of Bollobás's results thereby extending some of the results of [6].

If A is a fixed property, which a matroid may or may not have, and $P_{r,p(r)}(A)$ denotes the probability that ω_r has property A, then it was shown in [6] that there are several matroid properties for which there exists a *threshold function*, that is, a function $t(r)$ such that

$$\lim_{r\to\infty} P_{r,p(r)}(A) = \begin{cases} 0, & \text{if } \lim_{r\to\infty} \frac{p(r)}{t(r)} = 0, \\ 1, & \text{if } \lim_{r\to\infty} \frac{p(r)}{t(r)} = \infty. \end{cases}$$

In particular, [6] determined a threshold function for the existence in ω_r of certain simple matroids of fixed rank and cardinality. These matroids were defined by the condition that, for every non-empty subset T of the ground set S,

$$\frac{|T|}{\rho T} \leqslant \frac{|S|}{\rho S},$$

where ρ is the matroid rank function. By analogy with graphs, such matroids were called *balanced* matroids. They were studied in a different context by Naranyan and Vartak [7], who called them *molecular* matroids. In Section 2 of this paper, we show that, by making minor modifications to the relevant proof of [6], we can obtain a threshold function for the existence in ω_r of any fixed submatroid of $PG(r-1,q)$.

If M is a loopless matroid on the set S, let

$$d(M) = \max\left\{\frac{|T|}{\rho T} : \varnothing \neq T \subseteq S\right\}.$$

We call M *strictly balanced* if $d(M|U) < d(M)$ for all non-empty proper subsets U of S. Such matroids are called *atomic* in [7]. In Section 3, we prove several results for strictly

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balanced matroids, each of these being the analogue of a result of Bollobás for strictly balanced graphs.

In general, we shall follow Welsh [8] for all matroid terminology which is otherwise unexplained. We shall be considering projective geometries over a finite field $GF(q)$ which we take as fixed. The following notation will be useful. If f and g are functions of r, then we shall write $f = o(g)$ if $\lim_{r\to\infty} f/g = 0$, and $f \sim g$ if $\lim_{r\to\infty} f/g = 1$. For $k = 1, 2, \ldots, r$, let $[r]_k = (q^r - 1)(q^{r-1} - 1)\ldots(q^{r-k+1} - 1)$. In addition, let $[r]_0 = 1$. Then $\begin{bmatrix} r \\ k \end{bmatrix}$, the number of rank-$k$ subspaces of $PG(r-1, q)$, equals $\dfrac{[r]_k}{[k]_k}$.

It is routine to verify (see [5], Section A) that

$$q^{k(r-k+1)} \geqslant \begin{bmatrix} r \\ k \end{bmatrix} \geqslant q^{k(r-k)} \tag{1·1}$$

and

$$[r]_k = q^{kr-\binom{k}{2}}(1-q^{-r})(1-q^{-r+1})\ldots(1-q^{-r+k-1}). \tag{1·2}$$

From (1·2), we get ([5], (5)) that

$$[r]_k \sim q^{kr-\binom{k}{2}} \quad \text{if} \quad kq^{-r+k} = o(r), \tag{1·3}$$

and also that

$$q^{kr-\binom{k}{2}} \geqslant [r]_k \geqslant \beta q^{kr-\binom{k}{2}}, \tag{1·4}$$

where

$$\beta = \prod_{n=1}^{\infty}(1-q^{-n}).$$

The following lemma will be used frequently in Section 3. Its straightforward proof is omitted.

(1·5) LEMMA. *The number of rank-s subspaces of $PG(r-1, q)$ which meet a fixed rank-t subspace in a subspace of rank u is*

$$\begin{bmatrix} r-t \\ s-u \end{bmatrix}\begin{bmatrix} t \\ u \end{bmatrix} q^{(s-u)(t-u)}.$$

2. *Threshold functions for fixed matroids*

The following result, the matroid analogue of theorem 8 of [2], is an extension of theorem 1·1 of [6]. Its proof is obtained by making various slight alterations to the proof of the latter, so we shall omit some of the details and concentrate on those points where the two proofs differ.

(2·1) THEOREM. *Let k and m be fixed positive integers with $k \leqslant m$ and let $\mathscr{B}$ be a non-empty finite set of simple matroids each of which is representable over $GF(q)$. If $d(M) = m/k$ for all M in $\mathscr{B}$, then a threshold function for the property B that ω_r has a submatroid isomorphic to some member of $\mathscr{B}$ is $q^{-rk/m}$.*

Proof. As $\mathscr{B}$ is finite, the general result will follow if it can be proved in the special case that $\mathscr{B}$ consists of a single matroid M. Let T be a subset of the ground set S of M such that $|T| = nm$ and $\rho T = nk$ for some positive integer n. Then, arguing as in

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the proof of ([6], theorem 1·1), we get that if $pq^{rk/m} \to 0$, then $P(\omega_r$ has a submatroid isomorphic to $M|T) \to 0$. But if ω_r has no submatroid isomorphic to $M|T$, it certainly has no submatroid isomorphic to M. We conclude that if $\lim_{r\to\infty} pq^{rk/m} = 0$, then $\lim_{r\to\infty} P_{r,p}(B) = 0$.

Now suppose that $\lim_{r\to\infty} pq^{rk/m} = \infty$. Let $|S| = \hat{m}$, $\rho S = \hat{k}$ and denote by X and ν the numbers of submatroids of ω_r and $PG(\hat{k}-1, q)$ respectively which are isomorphic to M. If α_i denotes the number of ordered pairs (A_1, A_2) such that $PG(\hat{k}-1,q)|A_j \cong M$ for $j = 1, 2$ and $|A_1 \cap A_2| = \hat{m} - i$, then $EX^2 = \sum_{i=0}^{\hat{m}} p^{\hat{m}+i}\alpha_i$, and so

$$EX^2 \leqslant p^{2\hat{m}}\left[\left(\nu\begin{bmatrix} r \\ \hat{k} \end{bmatrix}\right)^2 + \sum_{i=0}^{\hat{m}-1} \alpha_i\, p^{i-\hat{m}}\right]. \tag{2·2}$$

In order to obtain upper bounds on the numbers $\alpha_0, \alpha_1, \ldots, \alpha_{\hat{m}-1}$, suppose that $|A_1 \cap A_2| = \hat{m} - i$ where $i \in \{0, 1, 2, \ldots, \hat{m}-1\}$, and $PG(\hat{k}-1, q)|A_j \cong M$ for $j = 1, 2$. Then as $d(M) = m/k$,

$$\frac{|A_1 \cap A_2|}{\rho(A_1 \cap A_2)} \leqslant \frac{m}{k}$$

and so

$$\rho(A_1 \cap A_2) \geqslant \frac{k(\hat{m}-i)}{m}. \tag{2·3}$$

Now $\alpha_i = \begin{bmatrix} r \\ \hat{k} \end{bmatrix} \nu\gamma_i$ where $\begin{bmatrix} r \\ \hat{k} \end{bmatrix}\nu$ is the number of ways to choose A_1 and γ_i is the number of ways to choose A_2 so that $|A_1 \cap A_2| = \hat{m} - i$, A_1 having already been chosen.

Evidently there are at most $\binom{\hat{m}}{\hat{m}-i}$ choices for $A_1 \cap A_2$. Moreover, once $A_1 \cap A_2$ has been chosen, A_2 must be contained in some rank-$\hat{k}$ subspace W of $PG(r-1, q)$ which contains $A_1 \cap A_2$. By Lemma 1·5, the number δ of such subspaces W equals $\begin{bmatrix} r-s \\ \hat{k}-s \end{bmatrix}$ where $s = \rho(A_1 \cap A_2)$. But $\begin{bmatrix} r-s \\ \hat{k}-s \end{bmatrix} \leqslant q^{r(\hat{k}-s)}$. and so, by (2·3), $\delta \leqslant q^{r\hat{k}-(rk/m)(\hat{m}-i)}$. Hence

$$\gamma_i \leqslant \binom{\hat{m}}{\hat{m}-i} \nu q^{r\hat{k}-(rk/m)(\hat{m}-i)},$$

and so

$$\alpha_i \leqslant \begin{bmatrix} r \\ \hat{k} \end{bmatrix} \nu^2 \binom{\hat{m}}{\hat{m}-i} q^{r\hat{k}-(rk/m)(\hat{m}-i)}.$$

Using this and the fact that $EX = \begin{bmatrix} r \\ \hat{k} \end{bmatrix} \nu p^{\hat{m}}$, we obtain from (2·2) and (1·1) that

$$\begin{aligned} \frac{EX^2}{(EX)^2} &\leqslant 1 + \sum_{i=0}^{\hat{m}-1} \binom{\hat{m}}{\hat{m}-i} p^{i-\hat{m}} q^{r\hat{k}-(rk/m)(\hat{m}-i)} \left(\begin{bmatrix} r \\ \hat{k} \end{bmatrix}\right)^{-1} \\ &\leqslant 1 + \sum_{i=0}^{\hat{m}-1} \binom{\hat{m}}{\hat{m}-i} p^{i-\hat{m}} q^{-(rk/m)(\hat{m}-i)} q^{\hat{k}^2}. \end{aligned}$$

It now follows easily, since $\lim_{r\to\infty} pq^{rk/m} = \infty$, that

$$\limsup_{r\to\infty} \frac{EX^2}{(EX)^2} \leqslant 1.$$

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Hence,

$$\lim_{r\to\infty} \frac{EX^2}{(EX)^2} = 1$$

and so, by Chebyshev's Inequality, $\lim_{r\to\infty} P(X \neq 0) = 1$. We conclude that $q^{-rk/m}$ is a threshold function for the property B. |

3. *Strictly balanced matroids*

In this section we shall prove four theorems on strictly balanced matroids. These results are the analogues of theorems 1–4 of [2] and are presented here in such a way as to draw attention to this fact. The main result is the first theorem; the other results are essentially extensions and refinements of it.

(3·1) THEOREM. *Let M be a fixed strictly balanced simple matroid having rank k and m elements and suppose that M is representable over $GF(q)$. Denote by X and ν the numbers of submatroids of ω_r and $PG(k-1,q)$ respectively which are isomorphic to M. Let $p \sim cq^{-rk/m}$ where c is an arbitrary positive constant. Then, for all natural numbers n,*

$$P(X = n) \sim e^{-\lambda}\lambda^n/n!,$$

where

$$\lambda = \nu c^m([k]_k q^{\binom{k}{2}})^{-1}.$$

Proof. Let $\mathscr{D}$ denote the set of subsets A of $PG(r-1,q)$ such that $PG(r-1,q)|A \cong M$. The basic structure of this proof is the same as that of the proof of theorem 1 of [2]. In particular, we put

$$X_n = X(X-1)\dots(X-n+1) \quad \text{for} \quad n = 1, 2, \dots,$$

and let $E_nX = EX_n$. Then E_nX is the expected number of ordered n-tuples $(A_1, A_2, \dots, A_n)$ of distinct members of $\mathscr{D}$ such that, for all i, all elements of A_i are in ω_r. We shall show that, for all positive integers n,

$$E_nX \sim \lambda^n.$$

Then, as the Poisson distribution has a well-defined moment generating function and is therefore uniquely determined by its moments, the theorem will follow (see Billingsley ([1], section 30)).

Now it is clear that

$$EX = \begin{bmatrix} r \\ k \end{bmatrix} \nu p^m$$

$$\sim \begin{bmatrix} r \\ k \end{bmatrix} \nu c^m q^{-rk}.$$

But, since k is fixed, we get from (1·3) that

$$\begin{bmatrix} r \\ k \end{bmatrix} \sim q^{rk-\binom{k}{2}}([k]_k)^{-1},$$

and so

$$EX \sim \lambda.$$

For $n = 1, 2, \dots$, let E_n' denote the expected number of ordered n-tuples

$$(A_1, A_2, \dots, A_n)$$

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of members of $\mathscr{D}$ such that, for all i, all elements of A_i are in ω_r and the subspaces of $PG(r-1,q)$ spanned by $\bigcup_{j=1}^{i-1} A_j$ and A_i are disjoint. We shall show that $E'_n \sim \lambda^n$ and $E_n - E'_n = o(1)$. By Lemma 1·5,

$$E'_n = \begin{bmatrix} r \\ k \end{bmatrix} \nu \begin{bmatrix} r-k \\ k \end{bmatrix} \nu q^{k^2} \begin{bmatrix} r-2k \\ k \end{bmatrix} \nu q^{2k^2} \dots \begin{bmatrix} r-(n-1)k \\ k \end{bmatrix} \nu q^{(n-1)k^2} p^{nm}.$$

Now

$$\begin{bmatrix} r-jk \\ k \end{bmatrix} q^{jk^2} = \frac{[r-jk]_k}{[k]_k} q^{jk^2}$$

$$\sim q^{rk} \left([k]_k q^{\binom{k}{2}} \right)^{-1} \quad \text{by (1·3).}$$

Thus

$$E'_n \sim \left(\nu q^{rk} p^m \left([k]_k q^{\binom{k}{2}} \right)^{-1} \right)^n \sim \lambda^n.$$

Next we note that $E_n - E'_n$ equals the expected number of ordered n-tuples $(A_1, A_2, \dots, A_n)$ of distinct members of $\mathscr{D}$ such that, for all i, all elements of A_i are in ω_r but for some t,

$$\bar{A}_t \cap \left(\overline{\bigcup_{j=1}^{t-1} A_j} \right) \neq \varnothing,$$

where $\bar{Y}$ denotes the closure of the set Y in $PG(r-1,q)$. Among such integers t, let s be the smallest. Now $\overline{\bigcup_{j=1}^{s-1} A_j}$ can be chosen in $\begin{bmatrix} r \\ (s-1)k \end{bmatrix}$ ways, and, once this subspace is chosen, it follows using Lemma 1·5 that the sets $A_1, A_2, \dots, A_{s-1}$ can be chosen in τ_{s-1} ways where

$$\tau_{s-1} = \begin{bmatrix} (s-1)k \\ k \end{bmatrix} \nu \begin{bmatrix} (s-2)k \\ k \end{bmatrix} \nu q^{k^2} \begin{bmatrix} (s-3)k \\ k \end{bmatrix} \nu q^{2k^2} \dots \begin{bmatrix} k \\ k \end{bmatrix} \nu q^{(s-2)k^2}.$$

If $\overline{\bigcup_{j=1}^{s-1} A_j}$ meets $\bar{A}_s$ in a rank-i subspace, then, by Lemma 1·5 and (1·4), the number of choices for A_s is at most

$$\nu \begin{bmatrix} r-(s-1)k \\ k-i \end{bmatrix} \begin{bmatrix} (s-1)k \\ i \end{bmatrix} q^{(k-i)((s-1)k-i)} \leqslant \nu\beta^{-1} \begin{bmatrix} (s-1)k \\ i \end{bmatrix} q^{(r-k)(k-i)}.$$

Thus the number of choices for $A_1, A_2, \dots, A_s$ subject to the above restrictions is bounded above by

$$\begin{bmatrix} r \\ (s-1)k \end{bmatrix} \begin{bmatrix} (s-1)k \\ i \end{bmatrix} q^{(r-k)(k-i)} \beta^{-1} \nu \tau_{s-1}. \tag{3·2}$$

Next we consider

$$\left| \bigcup_{j=1}^{s} A_j \right|.$$

Evidently

$$\left| \bigcup_{j=1}^{s} A_j \right| = \sum_{j=1}^{s} \left| A_j \right| - \left| A_s \cap \left(\bigcup_{j=1}^{s-1} A_j \right) \right|,$$

and so

$$\left| \bigcup_{j=1}^{s} A_j \right| = sm - \left| A_s \cap \left(\bigcup_{j=1}^{s-1} A_j \right) \right|. \tag{3·3}$$

In order to obtain a bound on

$$\left| A_s \cap \left(\bigcup_{j=1}^{s-1} A_j \right) \right|,$$

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we now show that $A_s \not\subseteq \bigcup_{j=1}^{s-1} A_j$. As $PG(r-1,q)|A_s$ is strictly balanced, it is straightforward to check that it is a connected matroid. Suppose that $A_s \subseteq \bigcup_{j=1}^{s-1} A_j$. Then, since the subspaces $\overline{A_{s-1}}$ and $\overline{\bigcup_{j=1}^{s-2} A_j}$ of $PG(r-1,q)$ are disjoint and $PG(r-1,q)|A_s$ is connected, either $A_s \subseteq \overline{A_{s-1}}$ or $A_s \subseteq \overline{\bigcup_{j=1}^{s-2} A_j}$. In the latter case, since $\overline{A_{s-2}}$ and $\overline{\bigcup_{j=1}^{s-3} A_j}$ are disjoint, either $A_s \subseteq \overline{A_{s-2}}$ or $A_s \subseteq \overline{\bigcup_{j=1}^{s-3} A_j}$. Repeating this argument, we obtain eventually that $A_s \subseteq \overline{A_g}$ for some g in $\{1,2,\ldots,s-1\}$. But $A_s \subseteq \bigcup_{j=1}^{s-1} A_j$ and so $A_s \subseteq A_g$; a contradiction. We conclude that

$$A_s \cap \left(\bigcup_{j=1}^{s-1} A_j\right) \neq A_s.$$

Thus, as $PG(r-1,q)|A_s$ is strictly balanced,

$$\frac{\left|A_s \cap \left(\bigcup_{j=1}^{s-1} A_j\right)\right|}{\rho\left(A_s \cap \left(\bigcup_{j=1}^{s-1} A_j\right)\right)} < \frac{m}{k}$$

provided

$$A_s \cap \left(\bigcup_{j=1}^{s-1} A_j\right)$$

is non-empty. Since

$$\rho\left(A_s \cap \left(\bigcup_{j=1}^{s-1} A_j\right)\right) \leqslant \rho\left(\bar{A}_s \cap \left(\overline{\bigcup_{j=1}^{s-1} A_j}\right)\right) = i,$$

it follows that, provided

$$A_s \cap \left(\bigcup_{j=1}^{s-1} A_j\right) \neq \varnothing,$$

$$\left|A_s \cap \left(\bigcup_{j=1}^{s-1} A_j\right)\right| \leqslant (im-1)/k. \tag{3·4}$$

But, as $i > 0$, (3·4) also holds if

$$A_s \cap \left(\bigcup_{j=1}^{s-1} A_j\right) = \varnothing.$$

It therefore holds in general and we may substitute into (3·3) to get that

$$\left|\bigcup_{j=1}^{s} A_j\right| \geqslant sm - (im-1)/k. \tag{3·5}$$

To complete the determination of an upper bound on $E_n - E'_n$, we now consider the number of choices for $A_{s+1}, A_{s+2}, \ldots, A_n$. This number is the sum over all $(n-s)$-tuples $(t_1, t_2, \ldots, t_{n-s})$ of the number of choices for $A_{s+1}, A_{s+2}, \ldots, A_n$ with

$$\left|A_{s+h} \cap \left(\bigcup_{j=1}^{s+h-1} A_j\right)\right| = t_h$$

for all h in $\{1,2,\ldots,n-s\}$. Evidently $0 \leqslant t_h \leqslant m$ for all h. Now, as $PG(r-1,q)|A_{s+h}$ is strictly balanced, if $t_h > 0$,

$$\frac{\left|A_{s+h} \cap \left(\bigcup_{j=1}^{s+h-1} A_j\right)\right|}{\rho\left(A_{s+h} \cap \left(\bigcup_{j=1}^{s+h-1} A_j\right)\right)} \leqslant \frac{m}{k}.$$

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Thus, for $t_h > 0$ and also trivially for $t_h = 0$, we have

$$\rho\left(A_{s+h} \cap \left(\bigcup_{j=1}^{s+h-1} A_j\right)\right) \geqslant kt_h/m. \tag{3·6}$$

To bound the number of possibilities for A_{s+h} subject to the above restrictions, we observe that, since

$$\left|\bigcup_{j=1}^{s+h-1} A_j\right| \leqslant (s+h-1)m,$$

$$A_{s+h} \cap \left(\bigcup_{j=1}^{s+h-1} A_j\right)$$

can be chosen in at most $\binom{(s+h-1)m}{t_h}$ ways once $A_1, A_2, \ldots, A_{s+h-1}$ have been fixed. Now, by Lemma 1·5, once

$$A_{s+h} \cap \left(\bigcup_{j=1}^{s+h-1} A_j\right)$$

has been chosen, there are $\begin{bmatrix} r-\mu \\ k-\mu \end{bmatrix}$ choices for $\overline{A_{s+h}}$, where

$$\mu = \rho\left(A_{s+h} \cap \left(\bigcup_{j=1}^{s+h-1} A_j\right)\right).$$

Moreover, by (1·1) and (3·6),

$$\begin{bmatrix} r-\mu \\ k-\mu \end{bmatrix} \leqslant q^{r(k-\mu)} \leqslant q^{r(k-(kt_h/m))}$$

and hence the number of choices for A_{s+h} is at most

$$\binom{(s+h-1)m}{t_h} \nu q^{r(k-(kt_h/m))}. \tag{3·7}$$

Now let Δ denote the expected number of ordered n-tuples $(A_1, A_2, \ldots, A_n)$ of distinct members of $\mathscr{D}$ subject to the above restrictions, that is,

(i) every element of $\bigcup_{j=1}^{n} A_j$ is in ω_r;

(ii) $\bar{A}_t \cap \left(\overline{\bigcup_{j=1}^{t-1} A_j}\right) = \varnothing$ for all t in $\{2, 3, \ldots, s-1\}$;

(iii) $\rho\left(\bar{A}_s \cap \left(\overline{\bigcup_{j=1}^{s-1} A_j}\right)\right) = i > 0$;

and

(iv) $\left|A_{s+h} \cap \left(\bigcup_{j=1}^{s+h-1} A_j\right)\right| = t_h$ for all h in $\{1, 2, \ldots, n-s\}$.

Then, on combining (3·2), (3·5) and (3·7), we have that

$$\Delta \leqslant \begin{bmatrix} r \\ (s-1)k \end{bmatrix} \begin{bmatrix} (s-1)k \\ i \end{bmatrix} q^{(r-k)(k-i)} \beta^{-1} \nu \tau_{s-1} p^{sm-(im-1)k} \prod_{h=1}^{n-s} \gamma_h,$$

where

$$\gamma_h = p^{m-t_h} \binom{(s+h-1)m}{t_h} \nu q^{r(k-(kt_h/m))}.$$

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Since $E_n - E'_n$ is the sum of finitely many terms of this type and $\begin{bmatrix}(s-1)k\\ i\end{bmatrix}\beta^{-1}\nu\tau_{s-1}$ is bounded, we may establish that $E_n - E'_n = o(1)$ by showing that

$$\begin{bmatrix} r \\ (s-1)k \end{bmatrix} q^{(r-k)(k-i)} p^{sm-(im-1)k} \prod_{h=1}^{n-s} \gamma_h = o(1).$$

As $p \sim cq^{-rk/m}$, for sufficiently large r,

$$p \leqslant 2cq^{-rk/m}. \tag{3·8}$$

Hence, for such r, on using (1·4) we get that

$$\gamma_h \leqslant \nu(2c)^{m-t_h}\begin{pmatrix}(s+h-1)m\\ t_h\end{pmatrix} q^{-rk(m-t_h)/m} q^{r(k-(kt_h/m))}.$$

Thus

$$\gamma_h \leqslant \nu(2c)^{m-t_h}\begin{pmatrix}(s+h-1)m\\ t_h\end{pmatrix}. \tag{3·9}$$

The last quantity is certainly bounded, hence to complete the proof it suffices to show that

$$\begin{bmatrix} r \\ (s-1)k \end{bmatrix} q^{(r-k)(k-i)} p^{sm-(im-1)/k} = o(1). \tag{3·10}$$

By (3·8) and (1·4), for sufficiently large r,

$$\begin{aligned}\begin{bmatrix} r \\ (s-1)k \end{bmatrix} & q^{(r-k)(k-i)} p^{sm-(im-1)/k} \\ &\leqslant \beta^{-1} q^{(s-1)k(r-(s-1)k)} q^{(r-k)(k-i)} (2c)^{sm-(im-1)/k} q^{-srk+ir-r/m} \\ &\leqslant \beta^{-1}(2c)^{sm-(im-1)/k} q^{-r/m+ik-((s-1)^2+1)k^2}.\end{aligned} \tag{3·11}$$

As c, k, m and β are constants, we conclude that (3·10) holds and the theorem follows immediately. ∎

Under certain conditions the preceding theorem may be extended beyond the case when M is a fixed matroid.

(3·12) THEOREM. *The conclusion of Theorem* 3·1 *holds if* M, k, m, ν *and* c *depend on* r, *provided that*

$$\nu c^m([k]_k q^{\binom{k}{2}})^{-1} \sim \lambda$$

for some positive constant λ, and $km^2 = o(r)$.

Proof. We need to check that, for all n, we still have

$$E'_n \sim \lambda^n \tag{3·13}$$

and

$$E_n - E'_n = o(1). \tag{3·14}$$

To verify the first of these, we observe that, for all j in $\{0, 1, 2, \ldots, n-1\}$,

$$\lim_{r\to\infty} kq^{-r+(j+1)k} = 0$$

since $k = o(r)$. Therefore, for all such j we have by (1·3) that

$$\begin{bmatrix} r-jk \\ k \end{bmatrix} q^{jk^2} \sim q^{rk}([k]_k q^{\binom{k}{2}})^{-1}$$

and (3·13) now follows as before.

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To check (3·14), we recall firstly that

$$\tau_{s-1} = \nu^{s-1}\frac{[(s-1)\,k]_k\,[(s-2)\,k]_k \dots [k]_k}{([k]_k)^{s-1}}\,q^{k^2}q^{2k^2}\dots q^{(s-2)\,k^2}.$$

Therefore, by (1·4),

$$\tau_{s-1} \leqslant \nu^{s-1}\frac{q^{k(s-1)\,k-\binom{k}{2}}\,q^{k(s-2)\,k-\binom{k}{2}}\dots q^{k^2-\binom{k}{2}}}{([k]_k)^{s-1}}\,q^{\binom{s-1}{2}k^2}$$

and so

$$\tau_{s-1} \leqslant \nu^{s-1}q^{k^2}((s-1)^2+1)\,([k]_k\,q^{\binom{k}{2}})^{-s}. \tag{3·15}$$

Furthermore, by (1·4),

$$\begin{bmatrix}(s-1)\,k\\ i\end{bmatrix} \leqslant \beta^{-1}q^{i((s-1)\,k-i)}. \tag{3·16}$$

Now let

$$\psi_{s,i} = \begin{bmatrix}r\\ (s-1)\,k\end{bmatrix}\begin{bmatrix}(s-1)\,k\\ i\end{bmatrix}q^{(r-k)\,(k-i)}\beta^{-1}\nu\tau_{s-1}\,p^{sm-(im-1)/k}\Gamma_{n-s},$$

where

$$\Gamma_{n-s} = \Sigma\left(\prod_{h=1}^{n-s}\gamma_h\right),$$

the sum being taken over all $(n-s)$-tuples $(t_1, t_2, \dots, t_{n-s})$ such that $t_h \in \{0, 1, 2, \dots, m\}$ for all h. To bound Γ_{n-s}, we observe firstly that, from (3·9),

$$\begin{aligned}\gamma_h &\leqslant \nu(2c)^{m-t_h}\binom{(s+h-1)\,m}{t_h}\\ &\leqslant \nu 2^m c^{m-t_h}\,2^{(s+h-1)\,m}\\ &\leqslant \nu 2^{nm}C,\end{aligned}$$

where

$$C = \begin{cases}c^m & \text{if} \quad c \geqslant 1,\\ 1 & \text{if} \quad c < 1.\end{cases}$$

Hence

$$\prod_{h=1}^{n-s}\gamma_h \leqslant (\nu 2^{nm}C)^{n-s}.$$

But the number of $(n-s)$-tuples $(t_1, t_2, \dots, t_{n-s})$ is $(m+1)^{n-s}$, and so

$$\begin{aligned}\Gamma_{n-s} &\leqslant (\nu 2^{nm}C(m+1))^{n-s}\\ &\leqslant \nu^{n-s}C^{n-s}2^{(n+1)\,nm}.\end{aligned}$$

On combining this with (3·11), (3·15) and (3·16), we get that, for sufficiently large r,

$$\begin{aligned}\psi_{s,i} &\leqslant (\nu c^m([k]_k\,q^{\binom{k}{2}})^{-1})^s\beta^{-3}c^{-(im-1)/k}\,2^{sm}q^{ski-i^2-r/m}\nu^{n-s}C^{n-s}2^{(n+1)nm}\\ &\leqslant (\nu c^m([k]_k\,q^{\binom{k}{2}})^{-1})^s\beta^{-3}2^{(n+2)nm}q^{sk^2-r/m}c^{-(im-1)/k}C^{n-s}\nu^{n-s}.\end{aligned}$$

We now distinguish two cases, according to whether $c \geqslant 1$ or $c < 1$. In the first case,

$$\begin{aligned}c^{-(im-1)/k}C^{n-s}\nu^{n-s} &\leqslant c^{m(n-s)}\nu^{n-s}\\ &= (\nu c^m([k]_k\,q^{\binom{k}{2}})^{-1})^{n-s}\,([k]_k\,q^{\binom{k}{2}})^{n-s}\\ &\leqslant (\nu c^m([k]_k\,q^{\binom{k}{2}})^{-1})^{n-s}\,q^{k^2(n-s)}.\end{aligned}$$

If $c < 1$, then

$$\begin{aligned}c^{-(im-1)/k}C^{n-s}\nu^{n-s} &\leqslant c^{-(im/k)}\nu^{n-s}\\ &\leqslant c^{-m}\nu^{n-s}.\end{aligned}$$

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But

$$c^{-m} = (\nu c^m([k]_k q^{\binom{k}{2}})^{-1})^{-1}\,\nu([k]_k q^{\binom{k}{2}})^{-1}$$

and

$$\nu \leqslant \binom{(q^k-1)/(q-1)}{m}.$$

Therefore,

$$\nu \leqslant q^{km}.$$

It follows that

$$\psi_{s,i} \leqslant \begin{cases} (\nu c^m([k]_k q^{\binom{k}{2}})^{-1})^n \beta^{-3} 2^{(n+2)\,nm} q^{nk^2-r/m} & \text{if} \quad c \geqslant 1, \\ (\nu c^m([k]_k q^{\binom{k}{2}})^{-1})^{s-1} \beta^{-3} 2^{(n+2)\,nm} q^{sk^2+km(n-s+1)-r/m} & \text{if} \quad c < 1. \end{cases}$$

Now $k \leqslant m$, $km^2 = o(r)$ and $\nu c^m([k]_k q^{\binom{k}{2}})^{-1} \sim \lambda$. Using these facts, the proof of (3·14) may now be completed as before.

The next theorem corresponds to a result of Bollobás ([2], theorem 3) on the number of occurrences of a graph as an *induced* subgraph of a random graph.

(3·17) THEOREM. *Let M, k, m, ν, c, p and λ be as in Theorem 3·12 and let W_d denote the maximal number of pairwise disjoint flats of ω_r each of which is isomorphic to M. Then, for every natural number n,*

$$P(W_d = n\} \sim e^{-\lambda}\lambda^n/n!.$$

Proof. Let Y_n denote the number of ordered n-tuples $(A_1, A_2, \ldots, A_n)$ of members of $\mathscr{D}$ such that all elements of $\bigcup_{j=1}^n A_i$ are in ω_r and the subspaces of $PG(r-1, q)$ spanned by $A_1, A_2, \ldots, A_n$ are pairwise disjoint. If, in addition, each of $A_1, A_2, \ldots, A_n$ is a flat in ω_r, the number of such n-tuples will be denoted by Z_n. We shall show that, for all natural numbers n,

$$EX_n = E_n \sim EY_n \sim EZ_n \sim \lambda^n. \tag{3·18}$$

We begin by noting that $E_n' \leqslant EY_n \leqslant E_n$. Therefore, as $E_n' \sim \lambda^n \sim E_n$, it follows that $EY_n \sim \lambda^n$. To complete the proof of (3·18) we now show that $EY_n - EZ_n = o(1)$. Evidently $EY_n \geqslant EZ_n$. Moreover, given that every element of $\bigcup_{j=1}^n A_j$ is in ω_r, the probability that at least one of $A_1, A_2, \ldots, A_n$ is not closed in ω_r is

$$1-(1-p)^{([i]-m)\,n} \leqslant 1-(1-p)^{q^k n}.$$

Thus

$$EY_n - EZ_n \leqslant EY_n(1-2cq^{-rk/m})^{q^k n}$$

and so if we can show that

$$\lim_{r\to\infty}(1-p)^{q^k n} = 1, \tag{3·19}$$

then (3·18) will follow.

Since $p \sim cq^{-rk/m}$,

$$(1-p)^{q^k n} \geqslant (1-2cq^{-rk/m})^{q^k n} \tag{3·20}$$

for sufficiently large r. But

$$\nu c^m([k]_k q^{\binom{k}{2}})^{-1} \sim \lambda$$

for some constant λ so

$$c \sim (\lambda[k]_k q^{\binom{k}{2}}\nu^{-1})^{1/m} \leqslant (\lambda q^{k^2})^{1/m}$$

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by (1·4) and the fact that $\nu \geqslant 1$. Hence, for sufficiently large r,

$$c \leqslant 2\lambda^{1/m} q^{k^2/m}. \tag{3·21}$$

On combining (3·20) and (3·21), we get that, for sufficiently large r,

$$\begin{aligned}(1-p)^{q^k n} &\geqslant (1-4\lambda^{1/m} q^{k^2/m - rk/m})^{q^k n} \\ &\geqslant 1 - q^k n 4\lambda^{1/m} q^{k^2/m - rk/m} \\ &= 1 - 4n\lambda^{1/m} q^{k + k^2/m - rk/m}.\end{aligned}$$

But, since $km^2 = o(r)$, we have that $4n\lambda^{1/m} q^{k+k^2/m-rk/m} = o(1)$ and (3·19) follows easily. We conclude that (3·18) holds and we now use this to show that Z_n converges to X_n in distribution. As $X_n - Z_n$ is a non-negative integer-valued random variable,

$$\begin{aligned}P(X_n \neq Z_n) &= P(X_n - Z_n \neq 0) \\ &\leqslant E(X_n - Z_n) \quad ([5], \text{lemma C}) \\ &= EX_n - EZ_n.\end{aligned}$$

But, by (3·18), $EX_n \sim EZ_n$ and so, for all positive integers n,

$$\lim_{r \to \infty} P(X_n \neq Z_n) = 0.$$

To complete the proof of the theorem we note that, for all positive integers n,

$$\begin{aligned}P(W_d \geqslant n) &= P(Z_n \geqslant 1) \\ &\sim P(X_n \geqslant 1) \\ &= P(X(X-1)\dots(X-n+1) \geqslant 1) \\ &= P(X \geqslant n).\end{aligned}$$

The result now follows from Theorem 3·12. ∎

The only free matroid which is strictly balanced is that on a single element. Hence the preceding results do not give the distribution of the number of k-element independent sets in ω_r except when $k = 1$. However, it is routine to check that, for example, circuits, projective and affine spaces and cycle matroids of complete graphs are strictly balanced matroids. Moreover, by ([7], theorem 4), the duals of these matroids are also strictly balanced. We now apply the preceding result to the first two of these examples. A circuit of rank k has $k+1$ elements and, so that we may use the last theorem, we want to choose k so that $k(k+1)^2 = o(r)$. This will certainly hold if, for example, $k = r^{\frac{1}{4}}$. It was shown in ([5], (8)) that the number of $(k+1)$-element circuits in $PG(k-1, q)$ equals

$$\frac{1}{(q-1)(k+1)!} q^{\binom{k}{2}} [k]_k,$$

that is,

$$\nu = \frac{1}{(q-1)(k+1)!} q^{\binom{k}{2}} [k]_k.$$

Now if λ is a positive constant, then c must be chosen so that

$$\nu c^{k+1} ([k]_k q^{\binom{k}{2}})^{-1} \sim \lambda,$$

that is, so that

$$\frac{c^{k+1}}{(q-1)(k+1)!} \sim \lambda.$$

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Thus we choose c so that

$$\log_q c = \frac{\log_q \lambda + \log_q(q-1) + \log_q((k+1)!) + o(1)}{k+1}.$$

Then with $p \sim c(r)\, q^{-rk/(k+1)}$, the probability that the number of $(k+1)$-element circuits in ω_r equals n tends to $e^{-\lambda}\lambda^n/n!$ as r tends to ∞. The same result holds if we insist that these circuits are pairwise disjoint, or that each is a flat in ω_r, or both.

Next suppose that $M = PG(k-1, q)$. Then $m = (q^k-1)/(q-1)$. Now if $0 < \epsilon < 1$, then clearly, for all $k \leqslant \frac{1-\epsilon}{2}\log_q r$, we have $km^2 = o(r)$. In this case $\nu = 1$ and we choose c so that

$$c^{(q^k-1)/(q-1)} \sim \lambda [k]_k q^{\binom{k}{2}},$$

that is,

$$\log_q c = \frac{(q-1)\left(\log_q \lambda + \log_q([k]_k) + \binom{k}{2} + o(1)\right)}{q^k - 1}.$$

Then, with $p \sim cq^{-rk(q-1)/(q^k-1)}$, the distribution of the number of submatroids of ω_r isomorphic to $PG(k-1, q)$ tends to the Poisson distribution with mean λ.

The final result extends Theorem 3·1 to the joint distribution of several strictly balanced matroids. It is proved by modifying Bollobás's proof ([2], theorem 4) of the corresponding graph-theoretic result along the lines of the proof of Theorem 3·12. We omit the details.

(3·22) THEOREM. *Let $M_1, M_2, \ldots$ be fixed non-isomorphic strictly balanced simple matroids each of which is representable over $GF(q)$. Suppose that M_i has rank k_i and has m_i elements and that, for all i, m_i/k_i equals a fixed constant d. Let X_i and ν_i denote the numbers of submatroids of ω_r and $PG(k_i-1, q)$ respectively which are isomorphic to M_i. Let $p \sim cq^{-r/d}$. Then, whenever $t, n_1, n_2, \ldots, n_t$ are natural numbers,*

$$P(X_i = n_i, i = 1, 2, \ldots, t) \sim \prod_{i=1}^{t} e^{-\lambda_i}\lambda_i^{n_i}/n_i!$$

where

$$\lambda_i = \nu_i c^{m_i}([k_i]_{k_i} q^{\binom{k_i}{2}})^{-1}.$$

Moreover, if

$$X = \sum_{i=1}^{t} X_i \quad \text{and} \quad \lambda = \sum_{i=1}^{t} \lambda_i,$$

then, for every natural number n,

$$P(X = n) \sim e^{-\lambda}\lambda^n/n!.$$

This theorem can be extended in the same way that Theorems 3·12 and 3·17 extend Theorem 3·1 but we shall not give the details.

Two classes of matroids which are of fundamental importance in the theory of 3-connectedness are the cycle matroids of wheels and the whirls. Let $\mathscr{W}_k$ denote the cycle matroid of the wheel graph on $k+1$ vertices and $\mathscr{W}^k$ denote the corresponding whirl ([8], pp. 80–81). We now apply the last theorem letting $M_1 = \mathscr{W}_k$ and $M_2 = \mathscr{W}^k$. Since M_1 is graphic, it is representable over all finite fields. On the other hand, it is not difficult to check that M_2 is representable over all finite fields except $GF(2)$. To determine ν_i here, we consider the number of choices for a subset A of $PG(k-1, q)$ such

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that $PG(k-1,q)|A \cong M$ where $M = M_i$. Once a basis of $PG(k-1,q)$ has been chosen to equal the set of spokes of M, the remaining elements of A can be chosen in $(q-1)^{k-1}$ ways if $M = M_1$, and $(q-1)^{k-1}(q-2)$ ways if $M = M_2$. Now, there is the possibility that the same set A may arise in more than one way by this process. However, it is straightforward to check that this will only happen if $M = \mathscr{W}_3$ in which case each subset of $PG(2,q)$ which is isomorphic to $\mathscr{W}_3$ arises in precisely four ways. It follows that

$$\nu_1 = \begin{cases} [k]_k q^{\binom{k}{2}}/(4k!\,(q-1)) & \text{if} \quad k = 3, \\ [k]_k q^{\binom{k}{2}}/(k!\,(q-1)) & \text{if} \quad k > 3, \end{cases}$$

and $\nu_2 = [k]_k q^{\binom{k}{2}}(q-2)/(k!\,(q-1))$. Thus for $k \geqslant 4$,

$$\lambda_1 = \frac{c^{2k}}{k!\,(q-1)} \quad \text{and} \quad \lambda_2 = \frac{c^{2k}(q-2)}{k!\,(q-1)}$$

and so, if $p \sim cq^{-r/2}$, then

$$P(X_1 = n_1, X_2 = n_2) \sim \frac{e^{-(\lambda_1+\lambda_2)}\lambda_1^{n_1}\lambda_2^{n_2}}{n_1!\,n_2!}$$

$$= e^{-c^{2k}/k!}\left(\frac{c^{2k}}{k!\,(q-1)}\right)^{n_1+n_2}\frac{(q-2)^{n_2}}{n_1!n_2!}.$$

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REFERENCES

[1] P. Billingsley. *Probability and Measure* (Wiley, 1979).
[2] B. Bollobás. Threshold functions for small subgraphs. *Math. Proc. Cambridge Philos. Soc.* **90** (1981), 197–206.
[3] P. Erdös and A. Rényi. On random graphs: I. *Publ. Math. Debrecen* **6** (1959), 290–297.
[4] P. Erdös and A. Rényi. On the evolution of random graphs. *Publ. Math. Inst. Hungar. Acad. Sci.* **5** (1960), 17–61.
[5] D. G. Kelly and J. G. Oxley. Asymptotic properties of random subsets of projective spaces. *Math. Proc. Cambridge Philos. Soc.* **91** (1982), 119–130.
[6] D. G. Kelly and J. G. Oxley. Threshold functions for some properties of random subsets of projective spaces. *Quart. J. Math. Oxford Ser.* (2) **33** (1982), 463–469.
[7] H. Naranyan and M. N. Vartak. On molecular and atomic matroids. *Combinatorics and Graph Theory* (ed. S. B. Rao). Lecture Notes in Math. vol. 885 (Springer-Verlag, 1981), 358–364.
[8] D. J. A. Welsh. *Matroid Theory*. London Math. Soc. Monographs no. 8 (Academic Press, 1976).

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