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A Robust Approach to Stabilization of 2-DOF Underactuated Mechanical Systems

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SUMMARY
This paper studies the stabilization problem for a class of underactuated systems in the presence of unknown disturbances. Due to less number of control inputs with respect to the degrees of freedom of the system, closed-loop asymptotic stability is a challenging issue in this field. In this paper, anti-swing controllers are designed for nominal and disturbed systems. In the case of the nominal system, the proposed two-loop controller is a combination of collocated partial feedback linearization and hierarchical sliding mode control (HSMC) theories. Then, due to the importance of robustness in control of physical systems, the proposed controller is developed for underactuated mechanical systems in the presence of additive disturbances. One of the main advantages of the proposed design method is that it does not need any switching algorithm. Finally, to illustrate the performance of the proposed controllers, they are applied to two underactuated mechanical systems: a pendubot and a Furuta pendulum. In addition, the practicality of the proposed approach is also verified experimentally using a quadrotor stand.

KEYWORDS: Underactuated mechanical system; Hierarchical sliding mode control; Two-loop control; Collocated partial feedback linearization.

1. Introduction

Studies of underactuated systems have attracted the attention of many researchers since the 1990s because of their wide application field such as robotics industries.[1–3] In underactuated systems, there is less number of control inputs than the degrees of freedom of the system.[4] Therefore, they are economical in weight and cost with respect to other systems; however, due to their complex nonlinear dynamics and nonholonomic behavior, conventional control methods may not be applied to these systems directly. As a result, control of underactuated systems, as a challenging problem, has been investigated by many researchers.[5] Wide applications of these systems in various fields such as robotics, spacecrafts, underwater vehicles, and medical science show the importance of studying these systems.[6]

One of the most commonly used methods to stabilize underactuated systems around their unstable equilibrium points is the switching algorithm.[7] In this strategy, with the help of swing-up control law, the system approaches to a neighborhood of its unstable equilibrium point from its initial position. The switching time, which depends on the control law and the system dynamics, should be determined such that the nonlinear system and its linearized model around the unstable equilibrium point be approximately equivalent. After switching, the second controller is activated that is designed based on either linear methods such as linear quadratic regulator and pole placement[7,8] or nonlinear methods such as Takagi-Sugeno fuzzy model and adaptive neural network.[8,9] In the design of the first

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control law, control theories such as partial feedback linearization,[10] sliding mode,[11] observer-based control,[12] energy-based control,[13] optimal control,[14] and backstepping control[15] have been utilized. Unfortunately, in some systems, it is difficult to predict the switching time offline or to propose an algorithm to determine it online. Therefore, the controller schemes that do not require any switching procedure are preferred, that is, control of the underactuated system at one stage.

In the one-stage control of underactuated systems, the most common method is the sliding mode control (SMC) method. For instance, in ref. [16], a higher-order sliding mode controller has been designed to stabilize a special form of 2-DOF underactuated systems (such as an inverted pendulum), where the control input only appears in one of the channels. Whereas, in many of 2-DOF underactuated systems, one control input appears in both channels. Furthermore, in 2004, a hierarchical sliding mode controller has been proposed to control second-order underactuated systems.[17] Then, incremental hierarchical structure sliding mode control (IHSSMC) and aggregated hierarchical structure sliding mode control (AHSSMC) methods have been introduced in 2007.[11] However, unfortunately, in refs. [18, 19], it was shown that there is a contradiction in the proof of theorems in the mentioned references and, therefore, results of the presented simulations are unreliable. Consequently, other papers that have used this strategy to control different underactuated systems have no admissible stability analysis.[20–23]

On the other hand, even among the switch-based methods, there are unsolved theoretical problems. For instance, in ref. [24], partial feedback linearization has been used to control the underactuated systems; however, determining the switching time to have a successful transient behavior is very difficult. Moreover, in recent theoretical developments in this category, such as ref. [10, 25, 26], because of using the zero-dynamic analysis, the guaranteed region of stability may be very limited.

On the other hand, to the best of author's knowledge, a robust method to control underactuated systems with guaranteed robustness (theoretically) has not been presented yet. For example, in ref. [27], the controller has been designed based on the linearized model of the system, which is a model-based method. Moreover, in ref. [28–33], the robustness of the control law has been only investigated in simulation results, and it has not been guaranteed theoretically. Therefore, control of underactuated mechanical systems in the presence of external disturbances is one of the topics of interest for researchers.

In this paper, a new structure, called the two-loop control method, is proposed to robust control of underactuated mechanical systems with 2 degrees of freedom without the necessity of any switching procedure. In the proposed strategy, partial feedback linearization and sliding mode theories are combined to control the nominal system with guaranteed stability. Then, the proposed strategy is developed to be robust against additive disturbances by replacing the partial feedback linearization method with another sliding mode controller. In the proposed method, there are some parameters as the degrees of freedom of the method which may be chosen properly to shape the desired transient response of the closed-loop system. The contributions of this work are as follows:

- In the proposed structure, the underactuated system is controlled at one stage. Therefore, any switching procedure is not required.
- A hierarchical sliding mode control (HSMC) is presented to control underactuated systems with guaranteed stability.
- The proposed method has been developed for disturbed underactuated systems, and its robustness has been guaranteed theoretically.

The outline of this paper is as follows: In Section 2, the formulation of a class of 2-DOF underactuated systems is reviewed. The main results of this paper are presented in Section 3. In this section, to guarantee the asymptotical stability and robustness of the proposed controllers, two theorems are given and proved. In Sections 4 and 5, simulation and experimental results are investigated to evaluate the main results. Finally, the paper is concluded in Section 6.

2. Problem Formulation and Preliminaries

One of the methods to obtain the dynamical equation of mechanical systems is the Lagrangian formulation. For Series-Robots, the matrix form of the Euler–Lagrange equation is as follows[34]:

$$D(q)\,\ddot{q} + C(q,\dot{q})\,\dot{q} + G(q) = \tau \tag{1}$$

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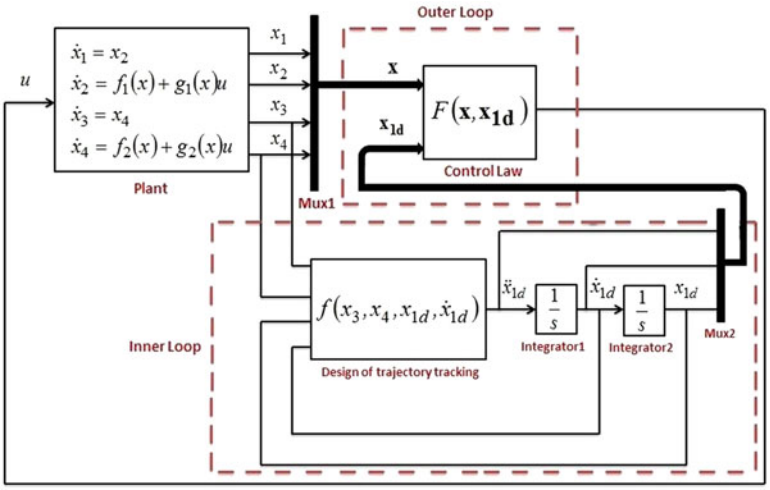


Fig. 1. Schematic of the proposed control strategy for 2-DOF underactuated systems.

where $D(q) \in R^{n\times n}$, $C(q, \dot{q}) \in R^{n\times n}$, and $G(q) \in R^n$ are the inertia matrix, Coriolis matrix, and gravitation vector, respectively. The order of the system (n) is determined by the dimension of $q \in R^n$ so-called generalized coordinates. In other words, n is the number of degree of freedom of the mechanical system. The nonlinear dynamic (1) is controlled by generalized forces ($\tau \in R^n$). For underactuated systems, only m ($m \prec n$) elements of τ are nonzero.

In the case of $n = 2$, the system (1) is called a 2-DOF system. In order to investigate the stability of 2-DOF underactuated systems, the following affine form could be considered:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f_1(x) + g_1(x)\, u \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = f_2(x) + g_2(x)\, u \end{cases} \tag{2}$$

where $\begin{pmatrix} x_1 = q_1 - q_{1\text{Desired}} \\ x_2 = \dot{q}_1 \end{pmatrix}$ and $\begin{pmatrix} x_3 = q_2 \\ x_4 = \dot{q}_2 \end{pmatrix}$ are actuated and unactuated state variables, respectively. Moreover, $f_i(x)$ and $g_i(x)$ for $i = 1, 2$ are smooth nonlinear functions, and $u \in R$ is the control input. Since in this paper, in addition to control of the nominal system (2), control of 2-DOF underactuated systems in the presence of unknown additive disturbances is done, the following dynamic is also considered:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f_1(x) + g_1(x)\, u + d_1(x) \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = f_2(x) + g_2(x)\, u + d_2(x) \end{cases} \tag{3}$$

where $d_i(x)$'s ($i = 1, 2$) are the unknown external disturbances; however, some information about them, like the upper bound on $|d_i(x)|$s, are known. Assume that

$$|d_i(x)| \le D_i \quad \forall x \in R^{2n},\ (i = 1, 2) \tag{4}$$

where $D_i{}'$s are positive constants.

3. Main Results

In this section, a two-loop control strategy is proposed to guarantee the asymptotical stability of 2-DOF underactuated systems at one stage (Fig. 1). In the proposed scheme, the purpose of the outer loop is to make the state variable x_1 to track the reference signal x_{1d} (as a result, x_2 will track $\dot{x}_{1d}$). Therefore, the controller of this loop is designed such that the actuated state variables (x_1, x_2) track the desired dynamic response. Moreover, the desired response, x_{1d}, must be designed such

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that the unactuated states have asymptotically stable behavior (control of the inner loop). Therefore, asymptotic stability of all the state variables (control of the entire system) is guaranteed.

3.1. Controller design for the nominal case

For the nominal system (2), the controllers of outer and inner loops are designed based on collocated partial feedback linearization and hierarchical sliding mode theories, respectively.

Theorem 1. *Consider the 2-DOF nominal underactuated system (2). The following control law stabilizes the closed-loop system asymptotically:*

$$u = \frac{\ddot{x}_{1d} - k_1(x_1 - x_{1d}) - k_2(x_2 - \dot{x}_{1d}) - f_1}{g_1} \tag{5}$$

where k_1 and k_2 are arbitrary positive constants, and x_{1d} is the desired dynamic response, satisfying the following differential equation:

$$\ddot{x}_{1d} = \frac{-kS - \eta \mathrm{sgn}(S) - \alpha c_1 x_4 - \alpha f_2 + \alpha\left(\frac{f_1 g_2}{g_1}\right) - c_2 \dot{x}_{1d}}{\alpha\left(\frac{g_2}{g_1}\right) + 1} \tag{6}$$

where k and η are arbitrary positive constants, and the sliding surface S is defined as follows:

$$S = \alpha s_1 + s_2 \tag{7}$$

where s_1 and s_2 are defined as,

$$s_1 = c_1 x_3 + x_4 \tag{8}$$

$$s_2 = c_2 x_{1d} + \dot{x}_{1d} \tag{9}$$

and the coefficients α, c_1, and c_2 are control parameters that should be designed such that the following matrix to be Hurwitz:

$$A_{\text{nominal}} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\partial (F|_{S=0})}{\partial x_3} & \frac{\partial (F|_{S=0})}{\partial x_4} & \frac{\partial (F|_{S=0})}{\partial x_5} \\ -\alpha c_1 & -\alpha & -c_2 \end{pmatrix} \tag{10}$$

where the function F is given by,

$$F = f_2 - \frac{g_2}{g_1} f_1 + \frac{g_2}{g_1} \ddot{x}_{1d} \tag{11}$$

Proof. Dynamical equations of the actuated state variable with the control law (5) are as follows:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \ddot{x}_{1d} - k_1(x_1 - x_{1d}) - k_2(x_2 - \dot{x}_{1d}) \end{cases} \tag{12}$$

If the error is defined as $e = x_1 - x_{1d}$, it is easy to show that the dynamical equation of error will be as,

$$\ddot{e} + k_2 \dot{e} + k_1 e = 0 \tag{13}$$

Therefore, if k_1 and k_2 are chosen as positive constants, asymptotical stability of error for this subsystem is guaranteed. Applying the control law (5) to the system (2) and considering asymptotical stability of error leads to the following dynamical equations of the unactuated state variables:

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$$\begin{cases} \dot{x}_3 = x_4 \\ \dot{x}_4 = f_2 - \dfrac{g_2}{g_1} f_1 + \dfrac{g_2}{g_1} \ddot{x}_{1d} \end{cases} \tag{14}$$

Since the goal is to guaranty the asymptotical stability of the entire system, $\ddot{x}_{1d}$ should be designed such that in addition to x_3 and x_4, x_{1d} and $\dot{x}_{1d}$ also converge to zero. Now, the following new state variables will be defined:

$$x_5 \triangleq x_{1d},\ x_6 \triangleq \dot{x}_{1d} \tag{15}$$

Therefore, the dynamical equations (14) may be rewritten as follows:

$$\begin{cases} \dot{x}_3 = x_4 \\ \dot{x}_4 = f_2 - \dfrac{g_2}{g_1} f_1 + \dfrac{g_2}{g_1} \zeta \\ \dot{x}_5 = x_6 \\ \dot{x}_6 = \zeta \end{cases} \tag{16}$$

where $\zeta = \ddot{x}_{1d}$ is the control input of the state-space equations. Now, the sliding mode theory is utilized to design ζ as the control input of the underactuated subsystem (16). First, consider the system (16) in the form of the following subsystems:

$$\begin{cases} \dot{x}_3 = x_4 \\ \dot{x}_4 = f_2 - \dfrac{g_2}{g_1} f_1 + \dfrac{g_2}{g_1} \zeta \end{cases} \quad \begin{cases} \dot{x}_5 = x_6 \\ \dot{x}_6 = \zeta \end{cases} \tag{17}$$

Sliding surfaces, s_1 and s_2, for the subsystems (17) should be designed such that the control objective of each subsystem is satisfied. This is fulfilled by choosing the positive coefficients c_1 and c_2 in Eqs. (8) and (9). Then, the final sliding surface, S, is chosen as a linear combination of s_1 and s_2. To continue the proof procedure, the following candidate Lyapunov function is chosen:

$$V(x) = \frac{S^2}{2} \tag{18}$$

The time derivative of the Lyapunov function (18) is as,

$$\dot{V}(x) = S\left[\alpha c_1 x_4 + \alpha f_2 - \alpha\left(\frac{g_2}{g_1} f_1\right) + \alpha\left(\frac{g_2}{g_1}\right)\zeta + c_2 x_6 + \xi\right] \tag{19}$$

Suppose that the control input ζ is designed to have the following negative definite $\dot{V}(x)$:

$$\dot{V}(x) = -kS^2 - \eta\,|S| \tag{20}$$

when k and η are positive constants. Thus, according to (15), (19), and (20), ζ will be obtained as (6). From (18) to (20), it can be concluded that S will converge to zero. Now, the dynamic equations of the system on the surface should be investigated. According to (7)–(9), the relation between state variables on the surface (when $S = 0$) is as follows:

$$x_6 = -\alpha c_1 x_3 - \alpha x_4 - c_2 x_5 \tag{21}$$

Thus, reduced-order dynamic equations of system (16) on the surface are achieved as,

$$\begin{cases} \dot{x}_3 = x_4 \\ \dot{x}_4 = f_2^* - \dfrac{g_2^*}{g_1^*} f_1^* + \dfrac{g_2^*}{g_1^*} \zeta^* \\ \dot{x}_5 = -\alpha c_1 x_3 - \alpha x_4 - c_2 x_5 \end{cases} \tag{22}$$

where f_i^*, g_i^*, and ζ^* are, respectively, the functions f_i, g_i, and ζ with considering $S = 0$ ($i = 1, 2$). The Jacobian matrix of the system (22) will be as presented in (10). By appropriate design of α, c_1,

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and c_2, the matrix (10) will be Hurwitz, and therefore, the asymptotical stability of system (16) is guaranteed. Now, by considering the asymptotical stability of error, it is concluded that the system (2) with the proposed controller is asymptotically stable and the proof is completed. □

3.2. Controller design in the presence of additive disturbances

In the previous section, the controller of the outer loop was designed based on collocated partial feedback linearization. This method is model based and is only applicable to the nominal system.[6] On the other hand, in many underactuated systems, the physical parameters (such as the length of the link, the mass, and the moment of inertia) can be measured with acceptable accuracy. However, the measurement of the friction force is more challenging, and it could be considered as an unknown additive disturbance in the input channel. Therefore, the disturbance could be considered as a time-invariant function depending on the state variables. In this regard, the strategy of Theorem 1 should be developed in the presence of unknown additive disturbances. Since the partial feedback linearization is a model-based method, in this section, control of the outer loop is changed to a more robust method such as SMC. Furthermore, the schematic of the implementation is kept similar to the previous case (Fig. 1), except that the plant is exposed to external disturbances.

Theorem 2. *Consider the 2-DOF underactuated system (3). The following control law stabilizes the system (3) asymptotically:*

$$u = \frac{-k_1 s - (\eta_1 + D_1)\, sgn(s) - c_1 x_2 - f_1 + c_1 \dot{x}_{1d} + \ddot{x}_{1d}}{g_1} \tag{23}$$

where k_1, η_1, and c_1 are positive constants and according to (4), D_1 is an upper bound for the additive disturbance in the channel of actuated state variables. Also, the sliding surface s is calculated from the following equation:

$$s = c_1(x_1 - x_{1d}) + (x_2 - \dot{x}_{1d}) \tag{24}$$

where x_{1d} is the desired dynamic response, satisfying the following equation:

$$\ddot{x}_{1d} = \frac{-k_2 S - (\eta_2 + \alpha D_2)\, sgn(S) - \alpha f_2 + \alpha \left(\frac{g_2}{g_1}\right) f_1 - \alpha c_2 x_4 - c_3 \dot{x}_{1d}}{\alpha \left(\frac{g_2}{g_1}\right) + 1} \tag{25}$$

where the sliding surface S is defined as (7). The appropriate parameters α, c_1, and c_2 should be designed such that the following inequalities to be held:

$$\begin{cases} a_2 + a_6 \prec 0 \\ a_2 a_6 - a_3 a_5 - a_1 \succ 0 \\ a_1 a_6 - a_3 a_4 \succ 0 \\ (a_2 + a_6)\,(-a_2 a_6 + a_3 a_5 + a_1) - a_1 a_6 + a_3 a_4 \succ 0 \end{cases} \tag{26}$$

where a_j ($j = 1, .., 6$) are elements of the Jacobian matrix of the closed-loop system.

Proof. The following sliding surface is designed for the first subsystem:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f_1(x) + g_1(x)\, u + d_1(x) \end{cases} \rightarrow s = c_1(x_1 - x_{1d}) + (x_2 - \dot{x}_{1d}) \tag{27}$$

To investigate the stability of this subsystem, the candidate Lyapunov function is as,

$$V_1(x) = \frac{s^2}{2} \tag{28}$$

Then, $\dot{V}_1(x)$ in the line of the first subsystem (27) is as follows:

$$\dot{V}_1(x) = s\dot{s} = s(c_1 x_2 - c_1 \dot{x}_{1d} + f_1 + g_1 u + d_1 - \ddot{x}_{1d}) \tag{29}$$

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Considering the inequality (4) $\dot{V}_1(x)$ satisfies the following inequality:

$$\dot{V}_1(x) \leq s(c_1 x_2 - c_1 \dot{x}_{1d} + f_1 + g_1 u - \ddot{x}_{1d}) + |s|\, D_1 \tag{30}$$

Therefore, replacing the control law (23) leads to $\dot{V}_1(x) \leq -k_1 s^2 - \eta_1 |s|$. By choosing the parameters k_1, η_1, and c_1 as positive scalars, the sliding surface s, and consequently, $e = x_1 - x_{1d}$ will converge to zero. Moreover, considering the asymptotical stability of e and applying the control law (23) to the system (3) lead to the following dynamical equations:

$$\begin{cases} \dot{x}_3 = x_4 \\ \dot{x}_4 = f_2 + g_2 \dfrac{\ddot{x}_{1d} - f_1}{g_1} + d_2(x) \end{cases} \tag{31}$$

Similar to the proof of Theorem 1, new state variables should be defined according to (15). Therefore, the Eq. (31) will be rewritten as,

$$\begin{cases} \dot{x}_3 = x_4 \\ \dot{x}_4 = f_2 - \dfrac{g_2}{g_1} f_1 + \dfrac{g_2}{g_1}\zeta + d_2(x) \\ \dot{x}_5 = x_6 \\ \dot{x}_6 = \zeta \end{cases} \tag{32}$$

To stabilize the system (32) in the presence of external disturbance $d_2(x)$, a HSMC method is utilized. Therefore, candidate Lyapunov function for this subsystem is as,

$$V_2(x) = \frac{S^2}{2} \tag{33}$$

According to relations (7)–(9) and (32), (33), the time derivative of Lyapunov function is as,

$$\dot{V}_2(x) = S\dot{S} = S\left(\alpha c_2 x_4 + \alpha f_2 - \alpha\left(\frac{g_2}{g_1}\right) f_1 + c_3 x_6 + \left(\alpha\frac{g_2}{g_1} + 1\right)\zeta + \alpha d_2(x)\right) \tag{34}$$

Now, considering the upper bound of d_2 leads to the following inequality for $\dot{V}_2(x)$:

$$\dot{V}_2(x) \leq S\left(\alpha c_2 x_4 + \alpha f_2 - \alpha\left(\frac{g_2}{g_1}\right) f_1 + c_3 x_6 + \left(\alpha\frac{g_2}{g_1} + 1\right)\zeta\right) + \alpha D_2\, |S| \tag{35}$$

Now, using the definitions (15) and replacing the control input (25) ($\zeta = \ddot{x}_{1d}$) in (35) leads to,

$$\dot{V}_2(x) \leq -k_2 S^2 - \eta_2\, |S| \tag{36}$$

For any positive constants, k_2 and η_2, the sliding surface S will converge to zero. Therefore, the order of dynamical equations (32) will be reduced on the surface ($S = 0$). The reduced-order equation is similar to the (22), except that the disturbance $d_2(x)$ should be added to the dynamical equation of $\dot{x}_4$. The characteristic equation of nominal reduced-order system (for $d_2(x) = 0$) is as follows:

$$\Delta\,(\lambda) = |\lambda_{1\times 1} I_{3\times 3} - A_{\text{nominal}}| = \lambda^3 - (a_2 + a_6)\,\lambda^2 + (a_2 a_6 - a_3 a_5 - a_1)\,\lambda + (a_1 a_6 - a_3 a_4) \tag{37}$$

where $a_{j_1} = \frac{\partial F}{\partial x_{j_1+2}}$ (for $j_1 = 1, 2, 3$) and,

$$F = f_2^* - \frac{g_2^*}{g_1^*} f_1^* + \frac{g_2^*}{g_1^*}\zeta^* \tag{38}$$

Also,

$$a_4 = -\alpha c_1,\ a_5 = -\alpha,\ a_6 = -c_2 \tag{39}$$

According to Routh–Hurwitz stability criteria, the inequalities (26) must be satisfied. Now consider the case that $d_2\,(x) \neq 0$, and there is not any knowledge about $d_2\,(x)$ except its upper bound. In the presence of $d_2\,(x)$, the admissible region of convergence in the state space will be decreased with respect to the case $d_2\,(x) = 0$. The proof is completed. □

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Corollary 1. *Suppose that the assumptions of Theorem 2 are held. If there exist some other information about the disturbance $d_2(x)$, the conservatism of Theorem 2 may be decreased. For instance, assume that the following assumption is satisfied:*

Assumption 1. *The disturbance term $d_2(x)$ is only a function of the state variable x_3, that is, $d_2(x) = d_2(x_3)$. Moreover, the term $\frac{\partial d_2(x)}{\partial x}$ exists and $\left\| \frac{\partial d_2(x)}{\partial x} \right\|_\infty \leq \mu_2$.*

Considering the assumption, the Jacobian matrix of the entire system will be as follows:

$$A = A_{\text{nominal}} + \begin{pmatrix} 0 & 0 & 0 \\ \dfrac{\partial d_2(x)}{\partial x_3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{40}$$

According to the Assumption, the following inequality is satisfied:

$$A_{\text{nominal}} - \begin{pmatrix} 0 & 0 & 0 \\ \mu_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \leq A \leq A_{\text{nominal}} + \begin{pmatrix} 0 & 0 & 0 \\ \mu_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{41}$$

It should be noted that the meaning of the above inequality, is inequality in peer-to-peer elements. Now, two cases are considered:
Case 1: $0 \prec \frac{\partial d_2(x)}{\partial x_3} = \varepsilon \prec \mu_2$. In this case, the following matrix A is obtained as,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ a_1 + \varepsilon & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} \tag{42}$$

Therefore, the necessary and sufficient condition to guarantee the stability of the closed-loop system is as follows:

$$\begin{cases} a_2 + a_6 \prec 0 \\ a_2 a_6 - a_3 a_5 - a_1 \succ \varepsilon \\ a_1 a_6 - a_3 a_4 \succ -a_6 \varepsilon \\ (a_2 + a_6)\left(-a_2 a_6 + a_3 a_5 + a_1\right) - a_1 a_6 + a_3 a_4 \succ -a_2 \varepsilon \end{cases} \tag{43}$$

Case 2: In this case, $\frac{\partial d_2(x)}{\partial x_3}$ is negative. Therefore, by considering $\frac{\partial d_2(x)}{\partial x_3} = -\varepsilon$; $0 \prec \varepsilon \prec \mu_2$, the following inequalities should be satisfied:

$$\begin{cases} a_2 + a_6 \prec 0 \\ a_2 a_6 - a_3 a_5 - a_1 \succ -\varepsilon \\ a_1 a_6 - a_3 a_4 \succ a_6 \varepsilon \\ (a_2 + a_6)\left(-a_2 a_6 + a_3 a_5 + a_1\right) - a_1 a_6 + a_3 a_4 \succ a_2 \varepsilon \end{cases} \tag{44}$$

It is clear that to guarantee the stability of the entire system (32), all three conditions ((40), (43), and (44)) should be fulfilled. Therefore, the appropriate values of α, c_1, and c_2 should be chosen to satisfy the following inequalities:

$$\begin{cases} a_2 + a_6 \prec 0 \\ a_2 a_6 - a_3 a_5 - a_1 \succ \mu_2 \\ a_1 a_6 - a_3 a_4 \succ -a_6 \mu_2 \\ (a_2 + a_6)\left(-a_2 a_6 + a_3 a_5 + a_1\right) - a_1 a_6 + a_3 a_4 \succ |a_2|\, \mu_2 \end{cases} \tag{45}$$

Remark 1. In Theorem 2, it has been considered that only the upper bounds of disturbances are known. Then, the control law and the desired dynamic response are designed based on the Lyapunov theory. Indeed, to overcome the effect of the disturbance d_1, by considering the upper bound D_1

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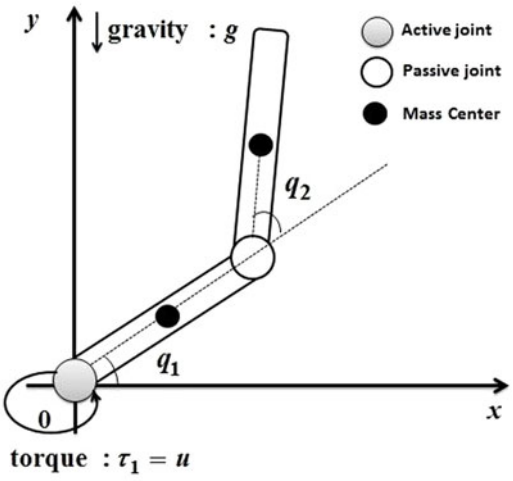


Fig. 2. Schematic of a pendubot system.

in the relation (29), the control law (23) is designed. Similarly, the term $D_2\text{sgn}(S)$ in the proposed dynamic response (25) reduces the effect of disturbance d_2 in the design procedure.

Remark 2. In Theorem 1, a method was proposed to control the nominal underactuated systems ($d_i(x)$=0; $i = 1, 2$). By choosing proper coefficients, the controller designed in Theorem 2 can also be used to control the nominal system. Simulation results and comparison of the performance between these two methods will be presented in Section 4.

Remark 3. In order to eliminate the chattering of control signals (5) and (23), the sign function ($sgn(S)$) can be replaced by the saturation function ($sat\left(\frac{S}{\varepsilon}\right)$), where ε is a small positive constant.[35,36] Therefore, a smooth control signal can be obtained; however, the convergence rate of the error signal will be decreased.

4. Simulation Results

In this section, the performance of the proposed nominal and robust controllers is investigated through numerical simulations.

Example 1. In this example, an underactuated robot called pendubot is considered (Fig. 2).

4.1. Dynamical equations of the pendubot system

The pendubot is an underactuated revolute–revolute robot. Indeed, the underactuation has arisen from this fact that the first link is actuated by a DC motor (active joint). Whereas, the second link is not actuated directly (passive joint).[4] The angles q_1 and q_2 are two degrees of freedom of the system, as shown in Fig 2. The equation of motion of this system with no friction is given in a matrix form (1), where the system matrices are as follows:[4]

$$
\begin{aligned}
D(q) &= \begin{pmatrix} \theta_1 + \theta_2 + 2\theta_3 \cos(q_2) & \theta_2 + \theta_3 \cos(q_2) \\ \theta_2 + \theta_3 \cos(q_2) & \theta_2 \end{pmatrix}, \; q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \\
C(q, \dot{q}) &= \begin{pmatrix} -\theta_3 \sin(q_2)\, \dot{q}_2 & -\theta_3 \sin(q_2)\, \dot{q}_1 - \theta_3 \sin(q_2)\, \dot{q}_2 \\ \theta_3 \sin(q_2)\, \dot{q}_1 & 0 \end{pmatrix} \\
G(q) &= \begin{pmatrix} \theta_4 g \cos(q_1) + \theta_5 g \cos(q_1 + q_2) \\ \theta_5 g \cos(q_1 + q_2) \end{pmatrix}, \quad \tau = \begin{pmatrix} \tau_1 \\ 0 \end{pmatrix}
\end{aligned}
\tag{46}
$$

where $\theta_i (i = 1, .., 5)$ are constant parameters. Table I contains the values of these parameters which have been used in simulations:

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Table I. Parameters of the pendubot used in numerical simulations.[4]

g	θ_1	θ_2	θ_3	θ_4	θ_5
9.81	0.034	0.0125	0.01	0.215	0.073

For this system, nonlinear functions in the affine form (2) are as follows:[4]

$$\begin{aligned}
f_1 &= \frac{1}{\theta_1\theta_2 - \theta_3^2\, cos^2(x_3)} \left[\theta_2\theta_3 \;\; sin(x_3)\,(x_2+x_4)^2 + sin(x_3)\, cos(x_3)\,\theta_3^2 x_2^2 \right.\\
&\quad \left. +\theta_2\theta_4 g\, sin(x_1) - \theta_3\theta_5 g\, cos(x_3)\, sin(x_1+x_3)\right],\\
f_2 &= \frac{1}{\theta_1\theta_2 - \theta_3^2\, cos^2(x_3)} \left[-\theta_3(\theta_2+\theta_3\, cos(x_3))\, sin(x_3)\,(x_2+x_4)^2 \right.\\
&\quad -\theta_3 x_2^2\, sin(x_3)\,(\theta_1+\theta_3\, cos(x_3)) - \theta_4 g\, sin(x_1)\,(\theta_2+\theta_3\, cos(x_3))\\
&\quad \left. +\theta_5 g\, sin(x_1+x_3)\,(\theta_1+\theta_3\, cos(x_3))\right],\\
g_1 &= \frac{\theta_2}{\theta_1\theta_2 - \theta_3^2\, cos^2(x_3)},\\
g_2 &= \frac{-\theta_2 - \theta_2\, cos(x_3)}{\theta_1\theta_2 - \theta_3^2\, cos^2(x_3)}.
\end{aligned} \tag{47}$$

4.2. Controller design for nominal pendubot system

In this section, the pendubot model (without any additive disturbances) is controlled using the proposed strategy in Theorem 1. If the parameters of the outer loop are chosen as $k_1 = 190$ and $k_2 = 19$, the error signal will converge to zero. Therefore, using the control law (5), the subsystem (14) will have the following dynamics:

$$\begin{cases} \dot{x}_3 = x_4 \\ \dot{x}_4 = -\dfrac{\theta_3}{\theta_2}\sin(x_3)\,\dot{x}_{1d}^2 + \dfrac{\theta_5}{\theta_2} g\sin(x_3+x_{1d}) - \left(1+\dfrac{\theta_3}{\theta_2}\cos(x_3)\right)\ddot{x}_{1d} \end{cases} \tag{48}$$

In order to satisfy condition (20), the parameters k and η can be chosen as,

$$k = 2.5,\ \eta = 0.1 \tag{49}$$

Furthermore, the coefficients of the inner loop should be chosen such that (a) the desired eigenvalues of the closed-loop system to be in the left half-plane, (b) the appropriate transient response will appear, and (c) the admissible initial value of the system to be increased as far as possible. For eigenvalues $\lambda_i = -6.9942, -7.9998, -9.0158$, $(i = 1, 2, 3)$, all of these conditions will be satisfied; thus,

$$\alpha = 0.619,\ c_1 = 7.5676,\ c_2 = 5.6899 \tag{50}$$

Remark 4. If the desired eigenvalues are chosen more negative with respect to the above values, the transient response will be improved; however, the admissible range of the initial conditions of the system will be decreased. Therefore, there is a trade-off between an appropriate transient response and a wide range for initial conditions. These considerations also exist in the case of the robust controller.

Furthermore, as mentioned in Remark 2, the performance of two controllers (presented in theorems 1, 2) could be compared in the case of the nominal pendubot system. State trajectories of the nominal system using the initial conditions $x_1(0) = 0.4$ (rad), $x_2(0) = 0.2\left(\frac{\text{rad}}{\text{s}}\right)$, $x_3(0) = -0.55(\text{rad})$, $x_4(0) = 0.2\left(\frac{\text{rad}}{\text{s}}\right)$, are shown in Fig. 3.

As it is seen in Fig. 3, the stability of the system is guaranteed in both methods; however, in PFL-SMC (Partial Feedback Linearization-Sliding Mode Control) method, transient response of the closed-loop system is better. The comparison of control signals is shown in Fig. 4.

The amplitude of the control law in SMC-SMC (Sliding Mode Control-Sliding Mode Control) method is less than another method (Fig. 4).

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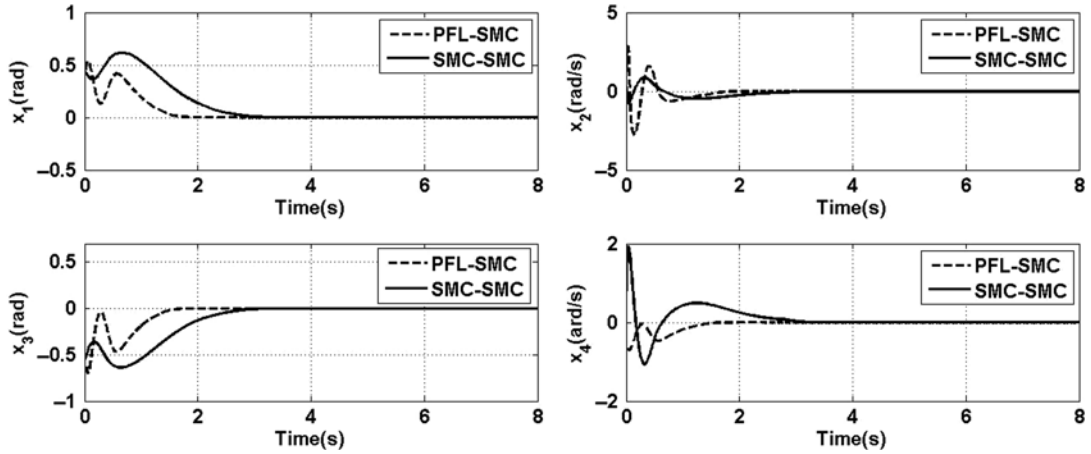


Fig. 3. State trajectories of the closed-loop pendubot system with two methods: PFL-SMC and SMC-SMC (nominal case).

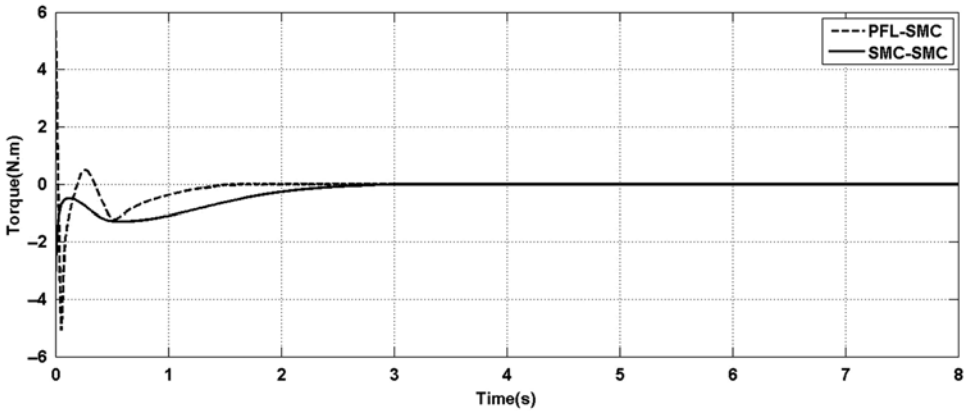


Fig. 4. Control law of the closed-loop pendubot system with two methods, PFL-SMC and SMC-SMC (nominal case).

4.3. Design of robust controller for pendubot system

In this case, it is considered that the pendubot is exposed to the external disturbances. The external disturbances could be a model of some practical considerations such as friction. The disturbed pendubot could be controlled using the proposed strategy in Theorem 2. In this regard, the results of the proposed method are compared with ref. [28] in the presence of two kinds of external disturbances.

Remark 5. As mentioned in the introduction section, a robust method to control the underactuated systems with guaranteed robustness (theoretically) has not been presented yet. Therefore, in order to compare the results with a robust controller, the nominal controller of ref. [28] should be extended to handle additive disturbances. In this regard, the Lyapunov redesign method is suggested.[37]
First, the external disturbance is considered as $d_1(x) = d_2(x) = 0.5\sin(x_3)$. The parameters have been designed as follows:

$$\begin{aligned} &c_1 = 2,\ k_1 = k_2 = 2.5,\ \eta_1 = \eta_2 = 0.1,\ D_1 = D_2 = 0.5, \\ &\alpha = 0.619,\ c_2 = 7.5676,\ c_3 = 5.6899 \end{aligned} \tag{51}$$

The state trajectories of the closed-loop system and the control signal using the initial conditions $x_1(0) = 0.4(\text{rad})$, $x_2(0) = 0.3\left(\frac{\text{rad}}{\text{s}}\right)$, $x_3(0) = -0.65(\text{rad})$, $x_4(0) = 0.31\left(\frac{\text{rad}}{\text{s}}\right)$, are compared with ref. [28] in Figs. 5 and 6, respectively.

As shown in Fig. 5, the proposed controller leads to a closed-loop system with a proper transient response, even in the presence of external disturbances. However, in the second method which is based on the switching algorithm, the variation of the state trajectories is inappropriate particularly

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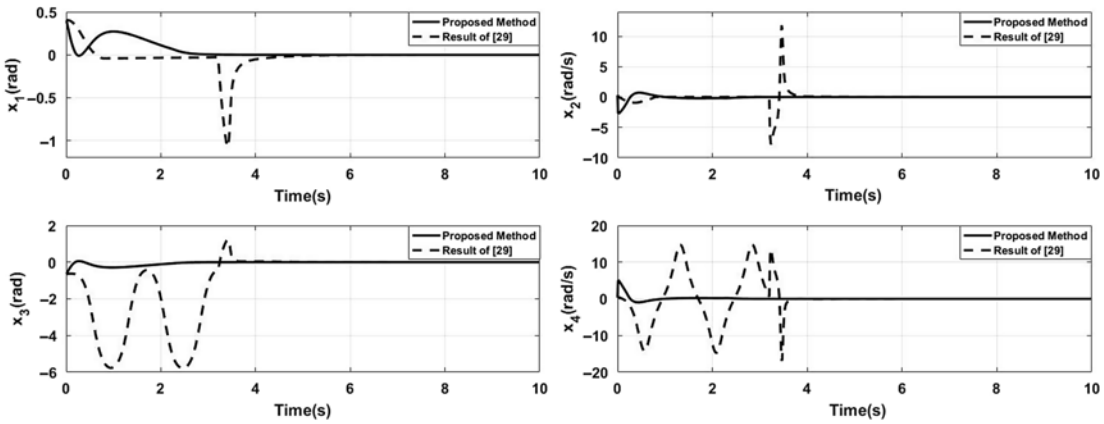


Fig. 5. State trajectories of the closed-loop pendubot system under the first external disturbances.

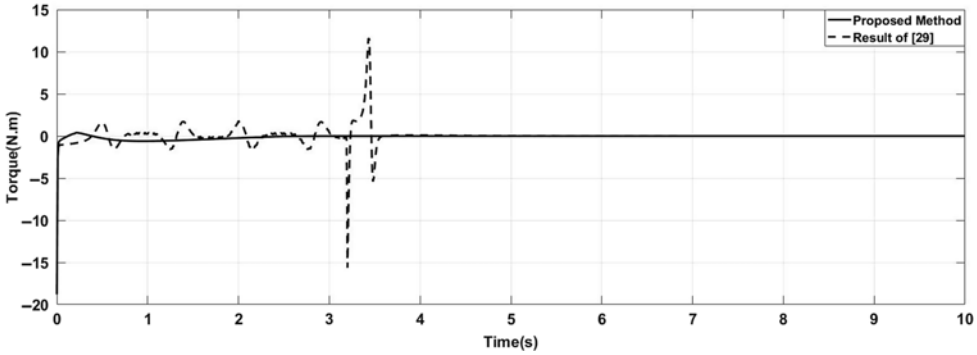


Fig. 6. Control law of the closed-loop pendubot system under first external disturbances.

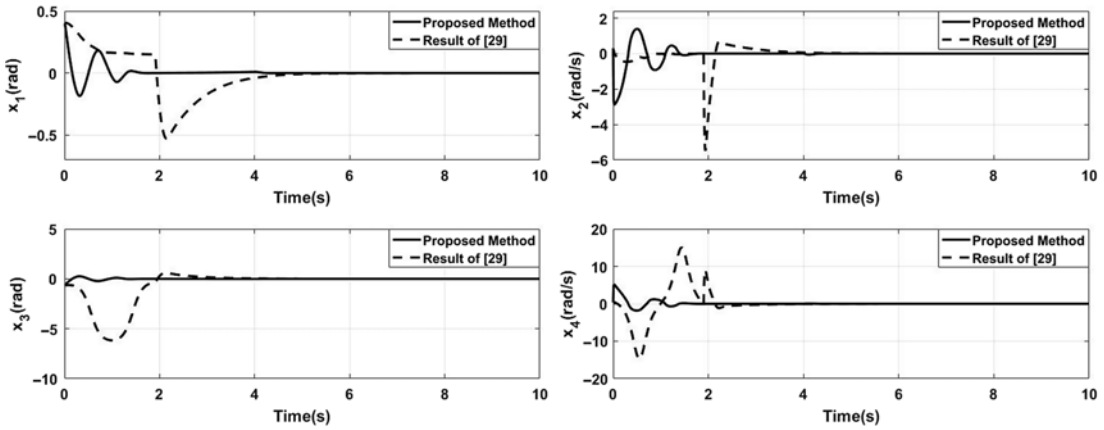


Fig. 7. State trajectories of the closed-loop pendubot system under second external disturbances.

at the switching time ($t_{\text{switch}} = 3.2(s)$). Moreover, the variation of the control signal in ref. [28] is very huge at the switching time (see Fig. 6).

Now, another external disturbance is considered as a sinusoidal function with variable amplitude between -5 and 5. The state trajectories of the closed-loop system and the control signal are compared with the controller of ref. [28] in Figs. 7 and 8, respectively.

It is concluded that the performance of the proposed controller is better than the result of ref. [28] with switching time, $t_{\text{switch}} = 1.9(s)$ (Figs. 7 and 8).

Remark 6. In pendubot, the friction of the passive joint can affect the underactuation. In this paper, this friction has been modeled by an additive external disturbance d_2 in the channel of unactuated

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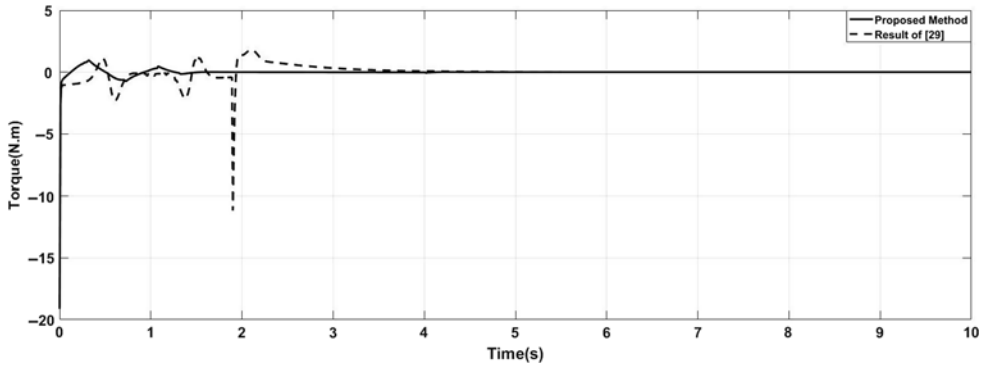


Fig. 8. Control law of the closed-loop pendubot system under second external disturbances.

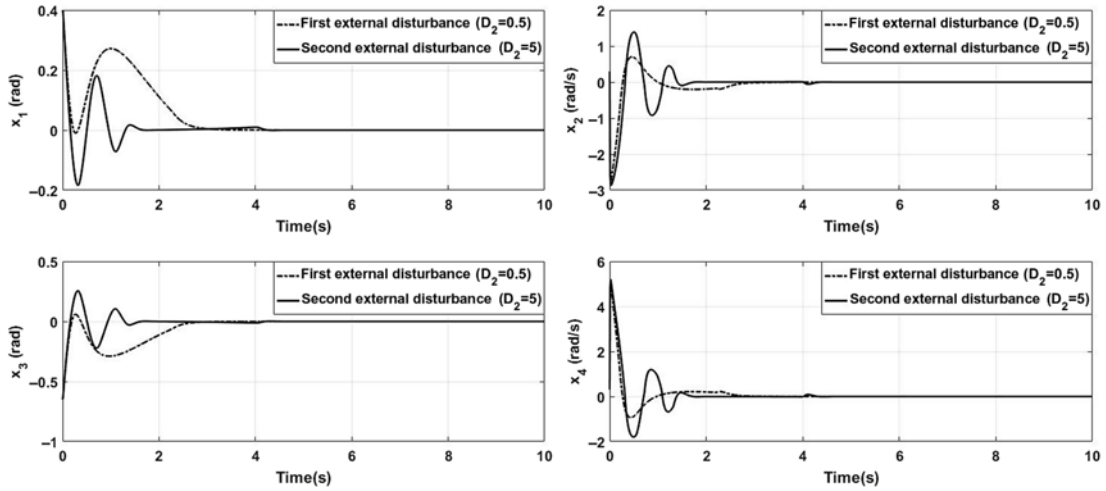


Fig. 9. Comparing the response of the closed-loop pendubot system under the first and second external disturbances.

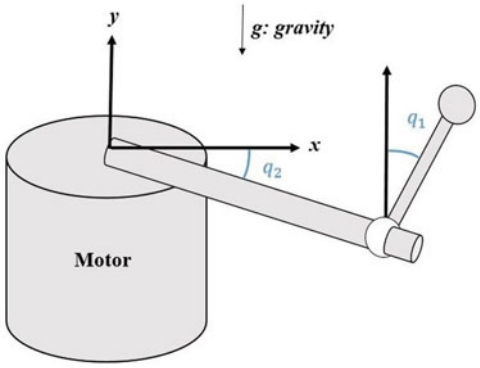


Fig. 10. Schematic of a Furuta pendulum system.

state variables. Furthermore, according to relations (23) and (25), the upper bound of $d_2(D_2)$ affects the accuracy of the control law (23). Figure 9 shows the effect of D_2 on the response of the closed-loop system.

It is obvious that the response of the closed-loop system for large values of D_2 is better.

Example 2. In this example, a Furuta pendulum which is also known as a benchmark in underactuated systems is considered (Fig. 10).

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Table II. Parameters of the Furuta pendulum used in numerical simulations.

$\boldsymbol{\theta_1}$	$\boldsymbol{\theta_2}$	$\boldsymbol{\theta_3}$	$\boldsymbol{\theta_4}$
0.0065	0.0301	0.58	−0.2953

4.4. Dynamical equations of the Furuta pendulum system

The Furuta pendulum is an inverted pendulum, connecting to a rotating arm; which the inverted pendulum is free to rotate in the vertical plane and the arm rotates in the horizontal plane.[4] Also, the underactuation mechanism of this system is similar to the pendubot. The equation of motion of this system with no friction is given in a matrix form (1), where the system matrices are as follows:[4]

$$\begin{aligned} D(q) &= \begin{pmatrix} \theta_1 & \theta_2 \cos(q_1) \\ \theta_2 \cos(q_1) & \theta_3 + \theta_1 \sin^2(q_1) \end{pmatrix}, q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \\ C(q, \dot{q}) &= \begin{pmatrix} 0 & -\theta_1 \sin(q_1)\cos(q_1)\,\dot{q}_2 \\ -\theta_2 \sin(q_1)\,\dot{q}_1 + \theta_1 \sin(q_1)\cos(q_1)\,\dot{q}_2 & \theta_1 \sin(q_1)\cos(q_1)\,\dot{q}_1 \end{pmatrix} \\ G(q) &= \begin{pmatrix} \theta_4 \sin(q_1) \\ 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 \\ \tau_1 \end{pmatrix}. \end{aligned} \tag{52}$$

where $\theta_i (i = 1, .., 4)$ are constant parameters. Table II contains the values of these parameters which have been used in simulations:

For this system, nonlinear functions in the affine form (2) are as follows:[4]

$$\begin{aligned} f_1 &= \frac{1}{\theta_1\theta_3 + \theta_1^2 \sin^2(x_1) - \theta_2^2 \cos^2(x_1)} \left[\theta_1\theta_2 \sin(2x_1)\cos(x_1)\, x_2 x_4 - 0.5\theta_2^2 \sin(2x_1)\, x_2^2 \right. \\ &\quad \left. + \left(\theta_3 + \theta_1 \sin^2(x_1)\right) \sin(x_1) \left(\theta_1 \cos(x_1)\, x_4^2 - \theta_4\right)\right], \\ f_2 &= \frac{1}{\theta_1\theta_3 + \theta_1^2 \sin^2(x_1) - \theta_2^2 \cos^2(x_1)} \left[-\theta_1^2 \sin(2x_1)\, x_2 x_4 - 0.5\theta_1\theta_2 \sin(2x_1)\cos(x_1)\, x_4^2 \right. \\ &\quad \left. + \theta_1\theta_2 \sin(x_1)\, x_2^2 + \theta_2\theta_4 \sin(x_1)\cos(x_1)\right], \\ g_1 &= \frac{-\theta_2 \cos(x_1)}{\theta_1\theta_3 + \theta_1^2 \sin^2(x_1) - \theta_2^2 \cos^2(x_1)}, \\ g_2 &= \frac{\theta_1}{\theta_1\theta_3 + \theta_1^2 \sin^2(x_1) - \theta_2^2 \cos^2(x_1)}. \end{aligned} \tag{53}$$

4.5. Design of robust controller for Furuta pendulum system

Similarly, as discussed in Sections 4.2 and 4.3, it could be shown that using the following parameters, the stability of the inner loop and outer loop of the Furuta pendulum system is satisfied.

$$\begin{aligned} &c_1 = 2, k_1 = k_2 = 2.5, \eta_1 = \eta_2 = 0.1, D_1 = D_2 = 0.5, \\ &\alpha = 3.741, c_2 = 2.6387, c_3 = 6.7434, k_2 = 2.5, \eta_2 = 0.1 \end{aligned} \tag{54}$$

The initial conditions, in this case, are as follows:

$$x_1(0) = 0.6(\text{rad}), x_2(0) = 0.3\left(\frac{\text{rad}}{\text{s}}\right), x_3(0) = -0.9\ (\text{rad}), x_4(0) = 0.3\left(\frac{\text{rad}}{\text{s}}\right) \tag{55}$$

The state trajectories of the closed-loop system and the control signal in the presence of $d_1(x) = d_2(x) = 0.5 \sin(x_3)$ are shown in Figs. 11 and 12, respectively.

Figure 11 shows the appropriate transient response of state variables convergence to zero.

5. Experimental Results

In this section, a quadrotor stand is used to implement the proposed controller (Fig. 13). In the experimental setup, using RTWT (Real-Time Windows Target) toolbox, which allows synchronizing

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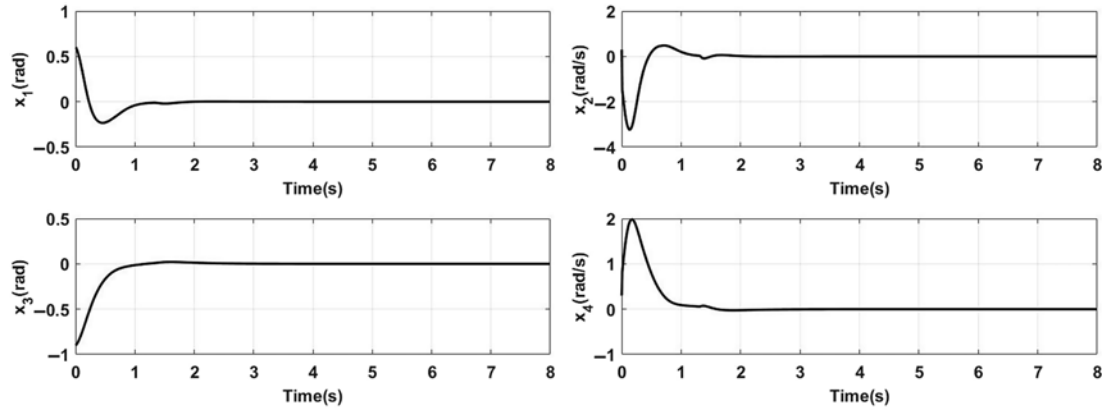


Fig. 11. State trajectories of the closed-loop Furuta pendulum system under first external disturbances.

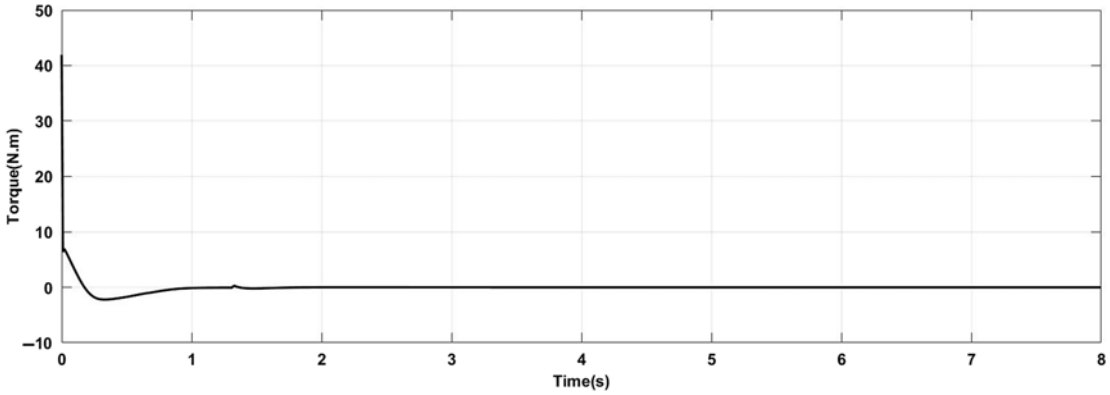


Fig. 12. Control law of closed-loop Furuta pendulum system under first external disturbances.

the Matlab/Simulink model with the real-time clock, the designed controller is translated to C/C++ code. Then, the controller is programmed in an Arduino mega 2560 board.

By defining the state vector as $x = [\phi, \dot{\phi}, \theta, \dot{\theta}, \psi, \dot{\psi}]$, the affine form of dynamical equations of the quadrotor stand, with 3-DOF and three control inputs, is as follows:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{I_{xx}} \left\{ u_1 + x_4 \Omega_r I_{\text{rotor}} - (I_{zz} - I_{yy}) x_4 x_6 \right\} \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{1}{I_{yy}} \left\{ u_2 - x_2 \Omega_r I_{\text{rotor}} - (I_{xx} - I_{zz}) x_2 x_6 \right\} \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= \frac{1}{I_{zz}} \left\{ u_3 - (I_{yy} - I_{xx}) x_2 x_4 \right\}
\end{aligned} \tag{56}$$

where I_{xx}, I_{yy}, and I_{zz} are the moment of inertia quad about the x-axis, y-axis, and i-axis, respectively. Moreover, I_{rotor} and Ω_r are the moment of inertia of the rotors and their mean of angular velocities, respectively. In order to apply the proposed method to the system (56), the physical system is locked in the direction of the yaw axis. To minimize the pressure on the lock, the same control signal has been applied to roll and pitch axes, that is, $u_1 = u_2 = u$. Therefore, the system (56) can be described by the 2-DOF underactuated system (2). Moreover, the control law u is as follows:

$$u = I_{xx} \left[\ddot{x}_{1d} - k_1 (x_1 - x_{1d}) - k_2 (x_2 - \dot{x}_{1d}) \right] \tag{57}$$

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Table III. Parameters of the quadrotor used in the experiment.

I_{xx}	I_{yy}	I_{zz}	I_{rotor}
0.0283	0.0306	0.0439	8.3e − 5

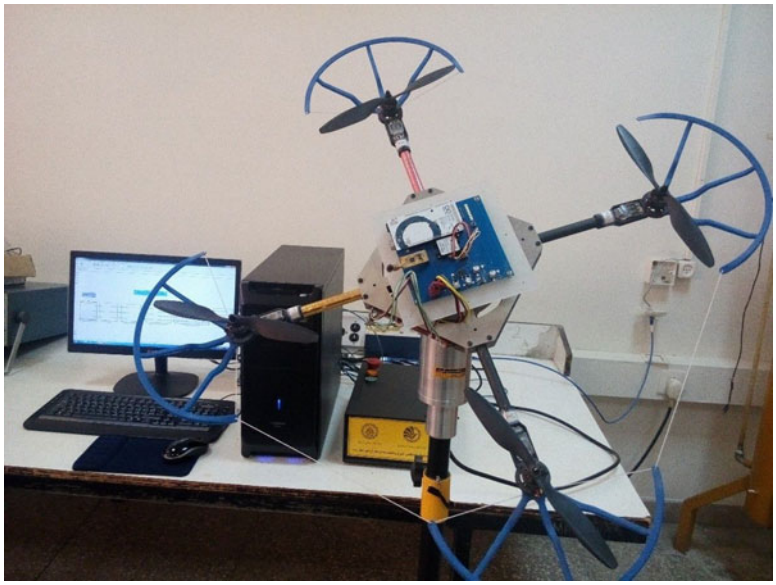


Fig. 13. The experimental setup of the quadrotor stand.

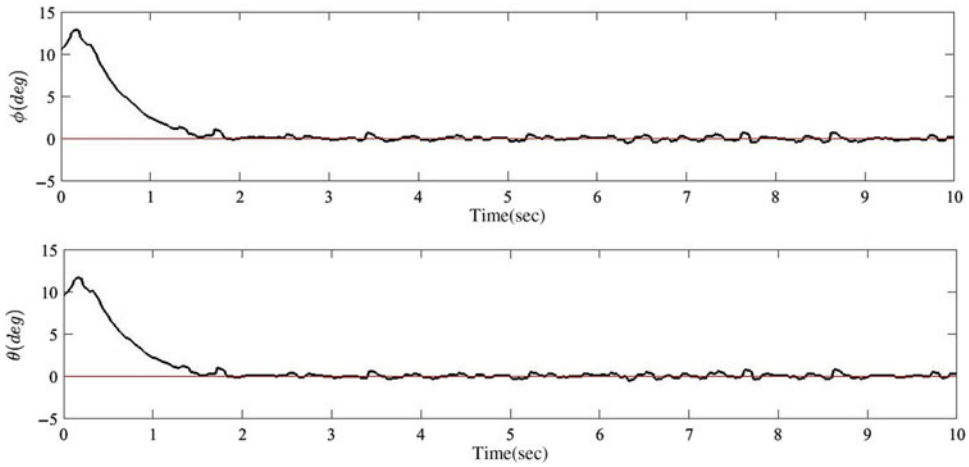


Fig. 14. State trajectories of the closed-loop quadrotor stand.

where the signal x_{1d} is obtained from the following dynamic:

$$\ddot{x}_{1d} = \frac{-kI_{yy}\left[\alpha(c_1x_3 + x_4) + c_2x_{1d} + \dot{x}_{1d}\right] - \eta I_{yy}\,\text{sgn}\left[\alpha(c_1x_3 + x_4) + c_2x_{1d} + \dot{x}_{1d}\right] - I_{yy}\alpha c_1x_4 - I_{yy}c_2\dot{x}_{1d}}{\alpha I_{xx} + I_{yy}} \tag{58}$$

Table III gives the parameters of the quadrotor.

Moreover, the parameters of the controllers (5) are chosen as $\alpha = 1$, $c_1 = c_2 = 5$, $\eta = 2$, $k = 2.5$, $k_1 = 190$, $k_2 = 19$. The state trajectories of the closed-loop system and the control signal using the initial conditions $x_1(0) = x_3(0) = 10(\text{deg})$, $x_2(0) = x_4(0) = 0(\frac{\text{deg}}{\text{s}})$ are shown in Figs. 14 and 15, respectively.

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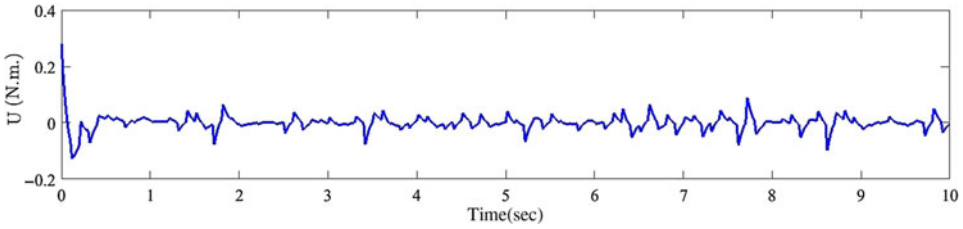


Fig. 15. Control signal in the closed-loop quadrotor stand.

Indeed, both variables of the quadrotor (i.e., ϕ, θ), have been controlled by one control variable (the challenging issue in underactuated systems), which is illustrated in the following figure.

6. Conclusion

In this paper, a two-loop control strategy was proposed to stabilization of 2-DOF underactuated mechanical systems in the presence of unknown additive disturbances. In this regard, using the Lyapunov approach, the asymptotical stability of the nominal and disturbed systems was guaranteed without any switching algorithm. Determination of the switching time is a challenging issue in switched-based methods. The proposed anti-swing controllers were a combination of collocated partial feedback linearization and HSMC theories. To evaluate the proposed strategy, the designed controllers were applied to a pendubot and a Furuta pendulum, and the results were compared with a robust switched-based method. Moreover, in the presence of two types of additive uncertainties, it was shown that by choosing appropriate design parameters, a desired transient response of the closed-loop system and an admissible control signal will be obtained. Furthermore, the experimental results were presented to validate the proposed method.

References

1. M. Nikkhah, H. Ashrafiuon and F. Fahimi, "Robust control of underactuated bipeds using sliding modes," *Robotica* **25**(3), 367–374 (2007).
2. A. Khanpoor, A. K. Khalaji and S. A. A. Moosavian, "Modeling and control of an underactuated tractor–trailer wheeled mobile robot," *Robotica* **35**(12), 2297–2318 (2017).
3. M. Karimi and T. Binazadeh, "Energy-based Hamiltonian approach in H∞ controller design for n-degree of freedom mechanical systems," *Syst. Sci. Contr. Eng.* **7**(1), 264–275 (2019).
4. I. Fantoni, R. Lozano and SC. Sinha, *Non-linear Control for Underactuated Mechanical Systems* (Springer Science & Business Media, London, 2002).
5. S. Gupta and A. Kumar, "A brief review of dynamics and control of underactuated biped robots," *Adv. Robotics* **31**(12), 607–623 (2017).
6. Y. Liu and H. Yu, "A survey of underactuated mechanical systems," *IET Control Theory Appl*. **7**(7), 921–935 (2013).
7. P. Ordaz, E. S. Spinoza and F. Muñoz, "Research on swing up control based on energy for the pendubot system," *J. Dyn. Syst. Meas. Control*. **136**(4), 041018 (2014).
8. X. Yang and X. Zheng, "Swing up and stabilization control design for an underactuated rotary inverted pendulum system: Theory and experiments," *IEEE Trans. Ind. Electron*. **65**(9), 7229–7238 (2018).
9. E. Aranda-Escolástico, M. Guinaldo, M. Santos and S. Dormido, "Control of a chain pendulum: A fuzzy logic approach," *Int. J. Comput. Intell. Syst.* **9**(2), 281–295 (2016).
10. L. A. Tuan and S. G. Lee, *Nonlinear Systems – Design, Analysis, Estimation and Control* (InTech Publishers, Rijeka-Croatia, 2016).
11. W. Wang, X. D. Liu and J. Q. Yi, "Structure design of two types of sliding-mode controllers for a class of under-actuated mechanical systems," *IET Control Theory Appl*. **1**(1), 163–172 (2007).
12. H. Hai, J. Li, P. Yong-jie, S. Shi-cai, T. Qi-rong, Y. Da-peng and L. Hong, "Observer-based dynamic control of an underactuated hand," *Adv. Robotics*. **24**(1–2), 123–137 (2010).
13. X. Xin, "On simultaneous control of the energy and actuated variables of underactuated mechanical systems–example of the acrobot with counterweight," *Adv. Robotics*. **27**(12), 959–969 (2013).
14. B. Sanchez, P. Ordaz, A. Garcia-Barrientos and E. Vera, "Nonlinear Suboptimal Control for a Class of Underactuated Mechanical Systems," *12th International Conference on Electrical Engineering, Computing Science and Automatic Control (CCE)*, Mexico (2015) pp. 1–6.
15. H. Kazemi, V. J. Majd and M. M. Moghaddam, "Modeling and robust backstepping control of an underactuated quadruped robot in bounding motion," *Robotica* **31**(3), 423–439 (2013).
16. S. U. Din, Q. Khan, F. U. Rehman and R. Akmeliawati, "Robust control of underactuated systems: Higher order integral sliding mode approach," *Math. Probl. Eng.* **2016**, 1–11. DOI: 10.1155/2016/5641478 (2016).
17. W. Wang, J. Yi and D. Liu, "Design of a stable sliding-mode controller for a class of second-order underactuated systems," *IEE Proc. Contr. Theor. Appl.* **151**(6), 683–690 (2004).

https://doi.org/10.1017/S0263574720000053 Published online by Cambridge University Press

18. B. L. Ma, "Comment: Design of a stable sliding-mode controller for a class of second-order underactuated systems," *IET Contr. Theor. Appl.* **1**(4), 1186–1187 (2007).
19. B. L. Ma, "Comments on Structure design of two types of sliding-mode controllers for a class of underactuated mechanical systems," *arXiv preprint arXiv*, 1208.6144, (Aug 2012).
20. D. Qian, J. Yi, D. Zhao and Y. Hao, "Hierarchical sliding mode control for series double inverted pendulums system," *2006 IEEE/RSJ International Conference on Intelligent Robots and Systems* (Oct 2006) pp. 4977–4982.
21. M. Yue and X. Wei, "Dynamic balance and motion control for wheeled inverted pendulum vehicle via hierarchical sliding mode approach," Proc. Inst. Mech. Eng. Pt. I: J. Syst. Contr. Eng. **228**(6), 351–358 (2014).
22. D. Zehar and K. Benmahammed, "Optimal sliding mode control of the pendubot," *Int. Res. J. Comp. Sci. Inf. Syst.* **2**(3), 45–51 (2014).
23. D. B. Pham and S. G. Lee, "Hierarchical sliding mode control for a two-dimensional ball segway that is a class of a second-order underactuated system," *J. Vib. Control* **15**(1), 49–55 (2018).
24. M. W. Spong, "The swing-up control problem for the Acrobot," *IEEE Control Syst.* **15**(1), 49–55 (1995).
25. L. A. Tuan and S. G. Lee, "Partial feedback linearization control of a three-dimensional overhead crane," *Int. J. Control Autom. Syst.* **11**(4), 718–727 (2013).
26. T. A. Le, S. G. Lee and S. C. Moon, "Partial feedback linearization and sliding mode techniques for 2D crane control," *Trans. Inst. Meas. Contr.* **36**(1), 78–87 (2014).
27. N. D. That, N. K. Quang, P. T. Thanh and Q. P. Ha, "Robust Exponential Stabilization of Underactuated Mechanical Systems in the Presence of Bounded Disturbances Using Sliding Mode Control," *2013 International Conference on Control, Automation and Information Sciences (ICCAIS)*, Nha Trang, Vietnam, (2013) pp. 206–211.
28. X. Xin, S. Tanaka, J. She and T. Yamasaki, "New analytical results of energy-based swing-up control for the Pendubot," *Int. J. Non-Lin Mech.* **52**, 110–118 (2013).
29. Y. Aoustin, A. Formal'skii and Y. Martynenko, "Pendubot: combining of energy and intuitive approaches to swing up, stabilization in erected pose," *Multibody Syst. Dyn.* **25**(1), 65–80 (2011).
30. R. W. O'Flaherty, R. G. Sanfelice and A. R. Teel, "Hybrid Control Strategy for Robust Global Swing-Up of the Pendubot," 2008 *American Control Conference*, Seattle, Washington, USA (2008) pp. 1424–1429.
31. N. Q. Hoang, S. G. Lee, J. J. Kim and B. S. Kim, "Simple energy-based controller for a class of underactuated mechanical systems," *Int. J. Precis. Eng. Manuf.* **15**(8), 1529–1536 (2014).
32. J. Montoya–Cháirez, V. Santibáñez and J. Moreno–Valenzuela, "Adaptive control schemes applied to a control moment gyroscope of 2 degrees of freedom," *Mechatronics* **57**, 73–85 (2019).
33. B. Sánchez, P. Ordaz and A. Poznyak, "Pendubot Robust Stabilization Based on Attractive Ellipsoid Method Using Electromechanical Model," *2018 15th International Conference on Electrical Engineering, Computing Science and Automatic Control (CCE)* (2018) pp. 1–6.
34. M. W. Spong, S. Hutchinson and M. Vidyasagar, *Robot Modeling and Control* (John Wiley & Sons, New York, 2006).
35. H. Chenarani, T. Binazadeh and M. H. Shafiei, "Velocity and body rate control of a spacecraft using robust passivity-base control," J. Space Sci. Technol. **11**(2), 41–46 (2018).
36. T. Binazadeh and M. Yousefi, "Designing a cascade-control structure using fractional-order controllers: Time-delay fractional-order proportional-derivative controller and fractional-order sliding-mode controller," *J. Eng. Mech.* **143**(7), 04017037 (2017).
37. H. K Khalil and J. W. Grizzle, *Nonlinear Systems* (New Jersey: Prentice Hall, 1996).

https://doi.org/10.1017/S0263574720000053 Published online by Cambridge University Press