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PAPERS

On fire whirls and Popov's equation

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Abstract

A fire whirl is a swirling flow with a burning core. A very simple model of a fire whirl is used, consisting of a linear second-order elliptic partial differential equation (PDE) for a stream function, extending established models for axisymmetric swirling flow of an incompressible inviscid fluid. The introduction of thermal effects means that the PDE cannot be solved by separating variables. The PDE contains a dimensionless parameter A. When A is large, parabolic approximations can be obtained, in much the same way that high-frequency approximations for diffraction problems are derived. The new PDE has similarities with Popov's equation, which arises when a whispering gallery wave encounters a boundary inflection. To solve it, two approaches are developed, both leading to first-order PDEs. The simpler assumes that a second parameter in the PDE is also large. If this assumption is not made, progress can be made by using a Laplace transform with respect to a radial coordinate, but the result is seen to be much more complicated.

1. Introduction

This paper began with a quest for solvable analytical models for fire whirls (Figure 1). The phenomenology is complicated, combining fluid flow and combustion; for an extensive review, see [18]. Mathematical models can be built, but, inevitably, they are complicated. For early models, see [5, 12]. For a much more sophisticated model, including compressibility, viscosity and unsteadiness, see [7].

We start from a scalar nonlinear partial differential equation (PDE) for a stream function $\psi(r, z)$, as derived by Battaglia et al. [3]; here, r and z are cylindrical polar coordinates. After linearisation, we obtain a second-order elliptic PDE containing three (dimensionless) parameters, A, B and C; A is related to the circulation around the whirl, B is related to the head (or Bernoulli's constant), and C is related to temperature changes. In the absence of thermal effects, $C = 0$ and then we recover models used for steady axisymmetric flows of an incompressible inviscid fluid [2, Section 7.5], [16, Section 3.13]. In what follows, we only consider the PDE with $C \neq 0$; although the PDE obtained is linear and homogeneous, we do not have any non-trivial solutions – the method of separation of variables does not succeed.

In the absence of exact solutions, we construct Frobenius-type expansions, both near the ground at $z = 0$ (Section 4.1) and near the axis at $r = 0$ (Section 4.2). These may be regarded as local expansions, but they have limited use if we want to apply boundary conditions at $z = 0$ and $z = \infty$, for example. Therefore, we are interested in asymptotic approximations. Motivated by classic approximations for high-frequency diffraction problems, we seek approximations based on the assumption that the parameter A is large (Section 5.1). As with diffraction theory (and long-distance sound propagation through an inhomogeneous ocean [4, Section 7.6]), the result is a parabolic PDE (see (5.3) below), but it is not

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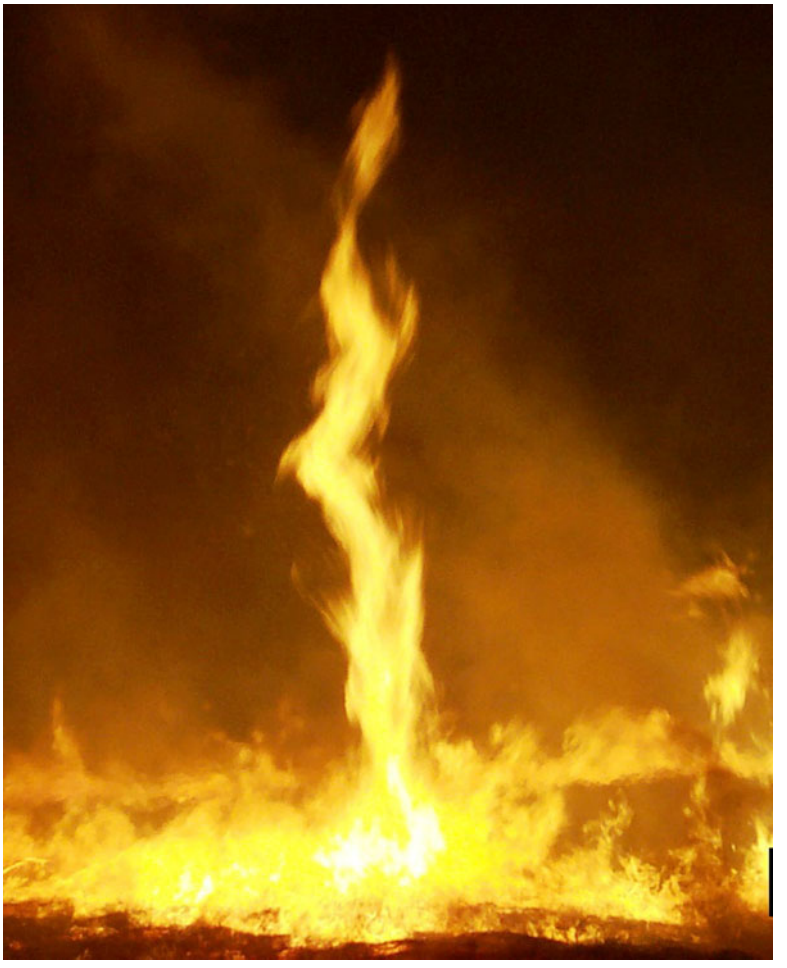


Figure 1. *Image of a typical firewhirl. The image is courtesy of U.S. Department of Fish and Wildlife Services.*

separable; as such, it is similar to a PDE known as *Popov's equation* [15]. We develop two approaches for solving this equation. In the first, we assume that the parameter B is also large (Section 5.2). In the second, we solve the PDE using Laplace transforms with respect to a radial coordinate (Section 5.3), inspired by Ockendon and Tew's analysis of Popov's equation using Fourier transforms [14]. In both approaches, we have to solve a first-order PDE using standard methods (as outlined in Section B). The large-B approximation is fairly simple (see (5.7) below), but the more general solution (which does not require B to be large) is much more complicated, requiring the inverse Laplace transform of an expression involving rational combinations of Airy functions (see (5.10) below). Evidently, further work is needed before this solution can be used and tested.

2. The basic nonlinear model

Batchelor [2, Section 7.5] and Saffman [16, Section 3.13] give clear discussions of steady axisymmetric flow of an incompressible inviscid fluid. The continuity equation is satisfied using a stream function ψ, with the velocity components given by

$$u = v_r = -\frac{1}{r}\,\frac{\partial \psi}{\partial z} \quad \text{and} \quad w = v_z = \frac{1}{r}\,\frac{\partial \psi}{\partial r}, \tag{2.1}$$

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where r and z are cylindrical polar coordinates. Then, Batchelor [2, equation (7.5.11)] and Saffman [16, equation (3.13.9)] show that $\psi(r, z)$ satisfies the nonlinear PDE

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} + F(\psi) = 0 \tag{2.2a}$$

with

$$F(\psi) = \Gamma\frac{\mathrm{d}\Gamma}{\mathrm{d}\psi} - r^2\frac{\mathrm{d}H}{\mathrm{d}\psi}. \tag{2.2b}$$

Here, Γ and H are given functions of ψ. In the context of magnetohydrodynamics, (2.2) is known as the Grad–Shafranov equation [10, equation (68.20)], [1]. In [11, equation (12)], the authors have a third term in (2.2*b*), proportional to r^4.

To introduce buoyancy-driven convection, suppose that the z-axis points upward from the ground at $z = 0$. Then (2.2*b*) has a third term, $zr^2\,\mathrm{d}\Theta/\mathrm{d}\psi$, where $\Theta = (T - T_0)/T_0$, T is the temperature, and T_0 is the ambient temperature [8, equation (1.4)], [3, equation (10)]. The resulting PDE for ψ will be of interest to us.

3. The basic linear model

To make analytical progress, we assume that $F(\psi)$ is linear in ψ and take

$$F(\psi) = (A^2 - B^2r^2 + Czr^2)\psi,$$

where A, B and C are constants. Dimensionally, given a length scale L, A, B and C have dimensions L^{-1}, L^{-2} and L^{-5}, respectively. Therefore, to make (2.2*a*) dimensionless (ignoring the dimensions of ψ), we send $r/L \to r$, $z/L \to z$, $AL \to A$, $BL^2 \to B$ and $CL^5 \to C$, giving (see [8, equation (1.5)])

$$\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + (A^2 - B^2r^2 + Czr^2)\psi = 0. \tag{3.1}$$

Now, r, z, A, B and C are dimensionless.

To eliminate the term in $\partial\psi/\partial r$, put $\rho = r^2$. Hence, (3.1) becomes

$$\frac{\partial^2\psi}{\partial z^2} + 4\rho\frac{\partial^2\psi}{\partial\rho^2} + (A^2 - B^2\rho + Cz\rho)\psi = 0. \tag{3.2}$$

Although this is a linear homogeneous PDE, it cannot be solved by the method of separation of variables when $C \neq 0$. There are numerous papers on solving (3.2), when $C = 0$, leading to solutions involving various known special functions; we do not consider these further and concentrate on the non-separable PDE.

For a natural boundary condition, we may specify the vertical velocity $w(r, z)$ at the ground, $z = 0$. For example, take $w(r, 0) = U\,\mathrm{e}^{-\frac{1}{2}\gamma r^2}$, where U and γ are constants. Then, from (2.1), $\psi(\rho, 0) = -(U/\gamma)\,\mathrm{e}^{-\frac{1}{2}\gamma\rho}$. The choice $\gamma = B$, giving

$$\psi(\rho, 0) = -(U/B)\,\mathrm{e}^{-\frac{1}{2}B\rho}, \tag{3.3}$$

fits with later approximations (see (5.7)) and is consistent with a rough estimate from (3.2) when $\rho \gg 1$.

4. Frobenius-type expansions

We are unaware of any exact solutions of (3.2) (with $C \neq 0$). For this reason, we first attempt to construct solutions using Frobenius expansions. We start in Section 4.1, looking for solutions near the ground at $z = 0$. Solutions near the axis at $r = 0$ are constructed in Section 4.2.

As (3.2) is a second-order elliptic PDE, we anticipate applying one boundary condition at $z = 0$ (such as (3.3)), but the solutions constructed below involve two arbitrary functions. This difficulty can only be resolved by adjusting the solution so as to satisfy a desired condition for large z. A similar difficulty arises with the solutions obtained in powers of r.

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4.1. Solution near $z = 0$

We are interested in solving (3.2) for $\rho > 0$ and $z > 0$, with an appropriate boundary condition at $z = 0$. Near $z = 0$, we can try writing

$$\psi(\rho, z) = \sum_{n=0}^{\infty} z^{n+\delta} f_n(\rho) \quad \text{with} \quad f_0(\rho) \not\equiv 0. \tag{4.1}$$

We substitute in (3.2). We have

$$4\rho \frac{\partial^2 \psi}{\partial \rho^2} + (A^2 - B^2\rho)\psi = \sum_{n=0}^{\infty} z^{n+\delta} (\mathcal{P} f_n)(\rho), \quad C\rho z\psi = C\rho \sum_{n=1}^{\infty} z^{n+\delta} f_{n-1}(\rho),$$

$$\frac{\partial^2 \psi}{\partial z^2} = \sum_{n=0}^{\infty} z^{n+\delta-2}(n+\delta-1)(n+\delta) f_n(\rho)$$

$$= \delta(\delta-1) f_0 z^{\delta-2} + \delta(\delta+1) f_1 z^{\delta-1} + \sum_{n=0}^{\infty} z^{n+\delta}(n+\delta+2)(n+\delta+1) f_{n+2},$$

where $(\mathcal{P}f)(\rho) = 4\rho f''(\rho) + (A^2 - B^2\rho) f(\rho)$.

As the term in $z^{\delta-2}$ must vanish, we obtain $\delta = 0$ or $\delta = 1$. The choice $\delta = 0$ also removes the term in δ^{-1} (so that f_1 is arbitrary), followed by $2f_2 + \mathcal{P} f_0 = 0$ and then $\mathcal{P} f_n + C\rho f_{n-1} + (n+2)(n+1) f_{n+2} = 0$ for $n = 1, 2, \ldots$. The solution has the form

$$\psi(\rho, z) = f_0(\rho) + z f_1(\rho) + z^2 f_2(\rho) + \cdots, \tag{4.2}$$

in which both f_0 and f_1 are arbitrary; they determine f_n for $n \geq 2$.

Evidently, from (4.2), $\psi(\rho, 0) = f_0(\rho)$ and $(\partial\psi/\partial z)(\rho, 0) = f_1(\rho)$, and these are related to the velocity at $z = 0$; see (2.1). In particular, specifying f_0 is equivalent to specifying the vertical velocity at $z = 0$; see (3.3). Then, f_1 must be related, somehow, to the solution for large z. (It turns out that the choice $\delta = 1$ is of no interest because it requires that $f_1 \equiv 0$.)

4.2. Solution near the axis, $r = 0$

Let us look for a solution in the form

$$\psi(\rho, z) = \sum_{n=0}^{\infty} \rho^n g_n(z) \quad \text{with} \quad g_0(z) \not\equiv 0.$$

We substitute in (3.2). We have

$$\frac{\partial^2 \psi}{\partial z^2} + A^2 \psi = \sum_{n=0}^{\infty} \rho^n \big\{ g_n''(z) + A^2 g_n(z) \big\},$$

$$4\rho \frac{\partial^2 \psi}{\partial \rho^2} + \rho(Cz - B^2)\psi = \sum_{n=1}^{\infty} \rho^n \big\{ 4n(n+1) g_{n+1} + (Cz - B^2) g_{n-1} \big\}.$$

At the lowest order ($n = 0$), we have $g_0''(z) + A^2 g_0(z) = 0$, whence

$$\psi(0, z) = g_0(z) = \psi_0 \, \mathrm{e}^{\pm \mathrm{i} A z}, \tag{4.3}$$

where ψ_0 is a constant. (This formula is consistent with the ansatz (5.1) below). Higher orders give

$$g_n'' + A^2 g_n + 4n(n+1) g_{n+1} + (Cz - B^2) g_{n-1} = 0, \quad n = 1, 2, \ldots.$$

This can be used to construct $g_{n+1}(z)$ from $g_n(z)$ and $g_{n-1}(z)$. Thus, we choose $g_1(z)$ and then proceed; the choice $g_1 = 0$ seems to be admissible but may not be interesting.

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5. Asymptotic approximations

In this section, we seek approximations to ψ, assuming that the (dimensionless) coefficient A is large ($|A| \gg 1$). This strategy is motivated by methods coming from high-frequency diffraction, in which a (dimensionless) wavenumber k is assumed to be large, $k \gg 1$. (We shall cite relevant literature below.) The result is a PDE with some resemblance to an equation known as Popov's equation. To solve our equation, we offer two approaches. In the first, we assume further that the parameter B is also large ($|B| \gg 1$); see Section 5.2. The second approach (which does not assume that B is large) is more complicated; it uses Laplace transforms with respect to ρ (Section 5.3).

5.1. A Popov-type equation

Start by writing

$$\psi(\rho, z) = \Psi(\rho, z)\, \mathrm{e}^{\mathrm{i}Az}. \tag{5.1}$$

We have

$$\frac{\partial \psi}{\partial z} = \left(\frac{\partial \Psi}{\partial z} + \mathrm{i}A\Psi\right) \mathrm{e}^{\mathrm{i}Az}, \quad \frac{\partial^2 \psi}{\partial z^2} = \left(\frac{\partial^2 \Psi}{\partial z^2} + 2\mathrm{i}A\frac{\partial \Psi}{\partial z} - A^2\Psi\right) \mathrm{e}^{\mathrm{i}Az}.$$

Hence, (3.2) becomes (the terms in A^2 cancel)

$$\frac{\partial^2 \Psi}{\partial z^2} + 2\mathrm{i}A\frac{\partial \Psi}{\partial z} + 4\rho\frac{\partial^2 \Psi}{\partial \rho^2} - \rho(B^2 - Cz)\Psi = 0. \tag{5.2}$$

Approximate solutions of (5.2) for large z are constructed in Appendix A using a similarity variable.

If $|A| \gg 1$, we may justify discarding the first term in (5.2) when compared to the second term, leaving the approximate equation

$$2\mathrm{i}A\frac{\partial \Psi}{\partial z} + 4\rho\frac{\partial^2 \Psi}{\partial \rho^2} - \rho(B^2 - Cz)\Psi = 0. \tag{5.3}$$

This is a parabolic PDE, reminiscent of the approach introduced by Leontovich and Fock in the 1940s; see, for example, [9, equation (4)] or [13, equation (2.5)]. However, the last term $C\rho z\Psi$ means that (5.3) is closer to a PDE introduced by Mikhail Popov in 1979 [15, equation (1.8)]. Since then Popov's equation has been studied by several authors [9, 14, 17]. In fact, Kiselev and Zlobina [9, p. 101] remark that the non-separability of Popov's equation 'makes its study an extremely difficult task'!

Actually, Popov's equation is not exactly the same as (5.3); his PDE has the form

$$2\mathrm{i}A\frac{\partial U}{\partial z} + \frac{\partial^2 U}{\partial y^2} + CyzU = 0. \tag{5.4}$$

We see that (5.3) contains the additional parameter B, and this can be exploited to obtain approximate solutions. Just as the ansatz (5.1) enabled us to discard the term in $\partial^2\Psi/\partial z^2$ from (5.2) for $|A| \gg 1$, a similar ansatz for Ψ will enable us to discard the term in $\partial^2\Psi/\partial\rho^2$ from (5.3) for $|B| \gg 1$. The result of doing this will be a *first*-order PDE which we can solve explicitly; see Section 5.2.

In [14, Section 4.1], the authors obtain a first-order PDE from Popov's equation by taking a Fourier transform with respect to y. In Section 5.3, we show that we can do the same for (5.3) (without assuming $|B| \gg 1$) using a Laplace transform with respect to ρ. The resulting PDE can be solved exactly in terms of Airy functions.

5.2. A further approximation assuming $|B| \gg 1$

Writing

$$\Psi(\rho, z) = \Phi(\rho, z)\, \mathrm{e}^{-\frac{1}{2}B\rho}, \tag{5.5}$$

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we have

$$\frac{\partial\Psi}{\partial\rho}=\left(\frac{\partial\Phi}{\partial\rho}-\frac{B}{2}\Phi\right)\mathrm{e}^{-\frac{1}{2}B\rho},\quad \frac{\partial^2\Psi}{\partial\rho^2}=\left(\frac{\partial^2\Phi}{\partial\rho^2}-B\frac{\partial\Phi}{\partial\rho}+\frac{B^2}{4}\Phi\right)\mathrm{e}^{-\frac{1}{2}B\rho}.$$

Hence, (5.3) becomes (the terms in B^2 cancel)

$$2\mathrm{i}A\frac{\partial\Phi}{\partial z}+4\rho\left(\frac{\partial^2\Phi}{\partial\rho^2}-B\frac{\partial\Phi}{\partial\rho}\right)+C\rho z\Phi=0.$$

Then, for $|B|\gg1$, we may justify discarding the second-derivative term in comparison with the next term, leaving the approximate equation

$$2\mathrm{i}A\frac{\partial\Phi}{\partial z}-4B\rho\frac{\partial\Phi}{\partial\rho}+C\rho z\Phi=0. \tag{5.6}$$

This is a first-order linear PDE for Φ. It can be solved explicitly (see Appendix B.1 for details). Having done this, we obtain the approximation (assuming that $|A|\gg1$ and $|B|\gg1$)

$$\psi(r,z)=\mathrm{e}^{\mathrm{i}Az-\frac{1}{2}Br^2}\,\Phi \tag{5.7a}$$

where

$$\Phi(r,z)=\mathcal{F}(\xi)\exp\left\{\frac{Cr^2}{4B}\left(z+\frac{\mathrm{i}A}{2B}\right)\right\}, \tag{5.7b}$$

$\xi=r^2\,\mathrm{e}^{-2\mathrm{i}Bz/A}$ and $\mathcal{F}$ is an arbitrary smooth function. In particular,

$$\psi(r,0)=\mathcal{F}(r^2)\exp\left\{\left(-\frac{1}{2}B+\frac{\mathrm{i}AC}{8B^2}\right)r^2\right\}\quad\text{and}\quad\psi(0,z)=\mathcal{F}(0)\,\mathrm{e}^{\mathrm{i}Az}.$$

Matching with the boundary condition (3.3) gives

$$\mathcal{F}(\rho)=-\frac{U}{B}\exp\left(-\frac{\mathrm{i}AC}{8B^2}\,\rho\right). \tag{5.8}$$

This determines the function $\mathcal{F}$ for positive real arguments.

If we accept the approximation (5.7), we can then calculate $\partial\psi/\partial z$ at $z=0$, and this can be used to define $f_1(\rho)$ in the Frobenius expansion (4.2). Differentiating (5.7) and making use of (5.8), we obtain the simple result $f_1(\rho)=\mathrm{i}Af_0(\rho)$.

5.3. Solving (5.3) using Laplace transforms

Consider (5.3), derived assuming that $|A|\gg1$. In order to obtain a first-order PDE, we compute the Laplace transform of $\Psi(\rho,z)$ with respect to ρ, defining

$$\widetilde{\Psi}(s,z)=\int_0^\infty\Psi(\rho,z)\,\mathrm{e}^{-s\rho}\,\mathrm{d}\rho.$$

Now, for a given function f, its Laplace transform is $F(s)=\mathcal{L}\{f\}=\int_0^\infty f(t)\,\mathrm{e}^{-st}\,\mathrm{d}t$. Then

$$\mathcal{L}\{f''\}=s^2F(s)-sf(0)-f'(0),\ \mathcal{L}\{tf(t)\}=-F'(s),\ \mathcal{L}\{tf''(t)\}=-s^2F'(s)-2sF(s)+f(0).$$

Using these results, we take the Laplace transform of (5.3) giving

$$2\mathrm{i}A\frac{\partial\widetilde{\Psi}}{\partial z}+(B^2-Cz-4s^2)\frac{\partial\widetilde{\Psi}}{\partial s}=8s\widetilde{\Psi}-4\psi_0, \tag{5.9}$$

where $\psi_0=\Psi(0,z)$, a constant, after combining (4.3) with (5.1). This is a first-order linear PDE for $\widetilde{\Psi}$. It can be solved explicitly (see Appendix B.2 for details). Having done this, we obtain the approximation

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(assuming that $|A| \gg 1$) $\psi(r,z) = \mathrm{e}^{\mathrm{i}Az}\Psi(\rho,z)$ where the Laplace transform $\widetilde{\Psi}(s,z)$ of the stream function $\Psi(\rho,z)$ is given by

$$\widetilde{\Psi}(s,z) = \frac{2\mathrm{i}}{A}\psi_0\,[v(z)]^2\int_0^z \frac{\mathrm{d}\zeta}{[v(\zeta)]^2} + \mathcal{F}(\xi)[v(z)]^2, \tag{5.10}$$

with $v(z) = \mathrm{Ai}(\varphi(z)) + \xi(z,s)\,\mathrm{Bi}(\varphi(z))$,

$$\xi(z,s) = \frac{s\,\mathrm{Ai}(\varphi(z)) - E\,\mathrm{Ai}'(\varphi(z))}{E\,\mathrm{Bi}'(\varphi(z)) - s\,\mathrm{Bi}(\varphi(z))}$$

$E = \frac{1}{2}\mathrm{i}(AC)^{1/3}$ and $\varphi(z) = (AC)^{-2/3}(zC - B^2)$. As before, $\mathcal{F}$ is an arbitrary smooth function.

6. Discussion and conclusions

In this paper, we have studied a very simple model of fire whirls. The model consists of a linear second-order elliptic PDE for a stream function $\psi(r,z)$, containing three dimensionless parameters, A, B and C, where r and z are positive cylindrical polar coordinates. When $C = 0$ (no thermal effects), the PDE can be solved by the method of separation of variables (as has been known for many years). This method fails when $C \neq 0$, and this is the case of interest here.

Two Frobenius-type solutions are constructed, one for small z (near the ground) and one for small r (near the axis of rotation). Approximations for large z are also obtained, using a similarity variable.

Then, motivated by strategies coming from high-frequency diffraction, we assume that the parameter A is large. Doing this leads to a PDE that has similarities with Popov's equation but also some differences. One difference involves the parameter B: if we also assume that B is large, we obtain a first-order PDE that we solve by standard methods. When B is not assumed to be large, we obtain a more complicated but solvable first-order PDE, using a Laplace transform with respect to a radial coordinate. The solution of this PDE is found; it involves Airy functions. We have not attempted to invert $\widetilde{\Psi}$ in order to obtain Ψ itself.

The calculations presented herein constitute a first step. Clearly, the basic model PDE is very simple and could be improved. On the other hand, encountering a non-separable PDE of Popov type presents mathematical difficulties and challenges.

Competing interests. None.

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Appendix A. Solving (5.2) approximately for large z

For large z, we can introduce a similarity variable $\tau = \rho z^\beta$ [3, Section II.D]. We will choose the parameter β below. Eliminate ρ in favour of τ using

$$\frac{\partial \Psi}{\partial \rho} = z^\beta \frac{\partial \Psi}{\partial \tau} \quad \text{and} \quad \frac{\partial^2 \Psi}{\partial \rho^2} = z^{2\beta} \frac{\partial^2 \Psi}{\partial \tau^2}.$$

Substituting in (5.2) gives

$$\frac{\partial^2 \Psi}{\partial z^2} + 2\mathrm{i}A \frac{\partial \Psi}{\partial z} + 4\tau z^\beta \frac{\partial^2 \Psi}{\partial \tau^2} - B^2 \tau z^{-\beta} \Psi + C\tau z^{1-\beta} \Psi = 0. \tag{A.1}$$

We are interested in approximations for large z. To obtain a non-trivial result as $z \to \infty$, we balance the third and last terms in (A.1). Thus, take $\beta = 1 - \beta$ giving $\beta = \frac{1}{2}$ and $\tau = \rho\sqrt{z}$. Then, dividing by $\tau\sqrt{z}$,

$$4 \frac{\partial^2 \Psi}{\partial \tau^2} + \frac{1}{\tau\sqrt{z}} \frac{\partial^2 \Psi}{\partial z^2} + \frac{2\mathrm{i}A}{\tau\sqrt{z}} \frac{\partial \Psi}{\partial z} + C\Psi - \frac{B^2}{z} \Psi = 0. \tag{A.2}$$

To solve this equation, try $\Psi(\tau, z) = \Psi_0(\tau) + z^{-\gamma} \Psi_1(\tau) + \cdots$, where $\gamma > 0$ will be chosen later. Substituting in (A.2) gives, at leading order, $4\Psi_0'' + C\Psi_0 = 0$, whence

$$\Psi_0(\tau) = \mathrm{e}^{\pm \mu \tau} \quad \text{with} \quad \mu^2 = -C/4 \quad \text{and} \quad \tau = r^2\sqrt{z}. \tag{A.3}$$

Thus, we obtain the large-z approximation $\psi(r, z) \simeq \exp\{\mathrm{i}Az \pm \mu r^2 \sqrt{z}\}$. At next order, the dominant terms are $z^{-\gamma}$ and $z^{-1/2}$; balancing these gives $\gamma = \frac{1}{2}$, and then $\Psi_1(\tau)$ can be calculated, if desired.

Appendix B. Solving first-order linear PDEs

In Section 5.2, we obtained (5.6) whereas we obtained (5.9) in Section 5.3. These first-order PDEs can be written in standard form as

$$a(x, y) \frac{\partial u}{\partial x} + b(x, y) \frac{\partial u}{\partial y} = q(x, y, u), \tag{B.1}$$

and they can be solved using a standard method. The first step is to find the characteristics by solving $\mathrm{d}y/\mathrm{d}x = b/a$; the solution can be written as $\xi(x, y) = c$, where c is an arbitrary constant. We note that ξ solves the homogeneous form of (B.1),

$$a(x, y) \frac{\partial \xi}{\partial x} + b(x, y) \frac{\partial \xi}{\partial y} = 0. \tag{B.2}$$

Next, change the independent variables in (B.1) from x and y to ξ and η, where η is unspecified at present. Application of the chain rule, together with (B.2), yields

$$\left(a \frac{\partial \eta}{\partial x} + b \frac{\partial \eta}{\partial y} \right) \frac{\partial u}{\partial \eta} = q. \tag{B.3}$$

Finally, a judicious choice for η may allow us to integrate this equation with respect to η.

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B.1. Solving (5.6)

Let us start with (5.6). We write it as (B.1), in which $x = z$, $y = \rho$, $u = \Phi$, $q = u$, $a = -2\mathrm{i}A/(Cxy)$ and $b = 4B/(Cx)$. The characteristics are determined by solving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b}{a} = \mathrm{i}\lambda y \quad \text{with} \quad \lambda = \frac{2B}{A}.$$

The general solution is $y = c\,\mathrm{e}^{\mathrm{i}\lambda x}$, that is, $\xi(x, y) = c$ with $\xi = y\,\mathrm{e}^{-\mathrm{i}\lambda x}$. So, the general solution of (B.1) with $q \equiv 0$ is $u = f(\xi)$, where f is an arbitrary smooth function.

Next, choose $\eta = x^2$ in (B.3); as $q = u$, we obtain

$$\frac{\partial u}{\partial \eta} = \Delta\, u \quad \text{with} \quad \Delta = \frac{1}{2xa} = \frac{\mathrm{i}Cy}{4A} = \frac{\mathrm{i}C\xi}{4A}\,\mathrm{e}^{\mathrm{i}\lambda\sqrt{\eta}}.$$

Integrating with respect to η,

$$\begin{aligned} u &= f(\xi)\exp\left\{\frac{\mathrm{i}C\xi}{4A}\int^{\eta}\mathrm{e}^{\mathrm{i}\lambda\sqrt{t}}\,\mathrm{d}t\right\} = f(\xi)\exp\left\{\frac{C\xi\mathrm{e}^{\mathrm{i}\lambda x}}{2A\lambda}\left(x + \frac{\mathrm{i}}{\lambda}\right)\right\} \\ &= f(\xi)\exp\left\{\frac{Cy}{4B}\left(x + \frac{\mathrm{i}}{\lambda}\right)\right\}. \end{aligned} \tag{B.4}$$

One can check that this expression for u does solve (B.1).

B.2. Solving (5.9)

Let us write (5.9) in standard form as (B.1), in which $x = z$, $y = s$, $u = \widetilde{\Psi}$, $a = 2\mathrm{i}A$, $b = B^2 - Cx - 4y^2$ and $q = 8yu - 4\psi_0$. The characteristics are determined by solving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b}{a} = \frac{2\mathrm{i}}{A}\,y^2 + \frac{B^2 - Cx}{2\mathrm{i}A}.$$

This is a Riccati equation [6, p. 59]: it is of the form $y'(x) + yQ(x) + y^2R(x) = P(x)$ with $Q = 0$, $R(x) = R_0$ and $P(x) = P_0 + P_1x$, where $P_0 = B^2/(2\mathrm{i}A)$, $P_1 = -C/(2\mathrm{i}A)$ and $R_0 = -2\mathrm{i}/A$ are constants. Making the substitution $y(x) = v'(x)/(Rv)$ results in a linear second-order equation for $v(x)$,

$$Rv'' + (QR - R')v' - PR^2v = 0,$$

which simplifies to $v''(x) - R_0(P_0 + P_1x)v(x) = 0$; this can be solved in terms of Airy functions, Ai and Bi. Thus, for example, write $v(x) = \mathrm{Ai}(\varphi(x))$ with $\varphi(x) = q_0 + q_1x$. Hence $v''(x) = q_1^2\mathrm{Ai}''(\varphi) = q_1^2\varphi\mathrm{Ai}(\varphi)$, using $\mathrm{Ai}''(w) = w\mathrm{Ai}(w)$, and we obtain

$$0 = v''(x) - R_0(P_0 + P_1x)v(x) = \left\{q_1^2(q_0 + q_1x) - R_0(P_0 + P_1x)\right\}\mathrm{Ai}(\varphi).$$

Matching powers of x gives $q_1^3 = R_0P_1$ and $q_0q_1^2 = R_0P_0$, whence

$$q_1 = (R_0P_1)^{1/3} = (C/A^2)^{1/3} \quad \text{and} \quad q_0 = R_0P_0/q_1^2 = -B^2(AC)^{-2/3}.$$

Therefore, $v(x) = c_1\mathrm{Ai}(\varphi(x)) + c_2\mathrm{Bi}(\varphi(x))$, $v' = q_1\{c_1\mathrm{Ai}'(\varphi) + c_2\mathrm{Bi}'(\varphi)\}$ and

$$y(x) = \frac{v'(x)}{R_0v(x)} = E\,\frac{\mathrm{Ai}'(\varphi) + c\,\mathrm{Bi}'(\varphi)}{\mathrm{Ai}(\varphi) + c\,\mathrm{Bi}(\varphi)} \tag{B.5}$$

with $E = q_1/R_0 = \frac{1}{2}\mathrm{i}(AC)^{1/3}$ and $c = c_2/c_1$. Solving for c, we obtain $\xi(x, y) = c$ where

$$\xi(x, y) = \frac{y\,\mathrm{Ai}(\varphi) - E\,\mathrm{Ai}'(\varphi)}{E\,\mathrm{Bi}'(\varphi) - y\,\mathrm{Bi}(\varphi)}.$$

A tedious calculation verifies that $\xi(x, y)$ satisfies (B.2), as expected.

As before, change variables to ξ and η, but with $\eta = x$, giving (B.3), which reduces to

$$\frac{\partial u}{\partial \eta} = \frac{q}{a} = \frac{1}{2\mathrm{i}A}(8yu - 4\psi_0). \tag{B.6}$$

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Now, from (B.5), $y = \frac{1}{2}\mathrm{i}Av'(\eta)/v(\eta)$ and c can be replaced by ξ. Hence (B.6) becomes

$$\frac{\partial u}{\partial \eta} - 2\frac{v'(\eta)}{v(\eta)}u = \frac{2\mathrm{i}}{A}\psi_0,$$

a first-order linear equation with integrating factor $[v(\eta)]^{-2}$; its use gives

$$\frac{\partial}{\partial \eta}\left\{\frac{u}{[v(\eta)]^2}\right\} = \frac{2\mathrm{i}\,\psi_0}{A[v(\eta)]^2}.$$

Integrating with respect to η,

$$u = \frac{2\mathrm{i}}{A}\psi_0\,[v(\eta)]^2\int_0^\eta \frac{\mathrm{d}\zeta}{[v(\zeta)]^2} + f(\xi)[v(\eta)]^2 \tag{B.7}$$

where f is an arbitrary function.

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