Ergod. Th. & Dynam. Sys., page 1 of 47 © The Author(s), 2026. Published by Cambridge University Press.
doi:10.1017/etds.2026.10360

Keplerian shear with Rajchman property

ARTHUR BOOS† and BENOIT SAUSSOL†‡

† *I2M, France*
(e-mail: arthur.boos@univ-amu.fr)
‡ *Aix-Marseille University, France*
(e-mail: benoit.saussol@univ-amu.fr)

(*Received* 8 *January* 2026 *UTC and accepted in revised form* 9 *September* 2026 *UTC*)

Abstract. The Keplerian shear was introduced within the context of measure preserving dynamical systems by Thomine, as a version of mixing for non-ergodic systems. In this study, we provide a characterization of the Keplerian shear using Rajchman measure for some flows on tori bundles. Our work applies to dynamical systems with singularities or with non-absolutely continuous measures. We relate the speed of decay of conditional correlations with the Rajchman order of the measures. Some of these results are extended to the case of compact Lie group bundles.

Key words: classical ergodic theory, random dynamics, harmonic analysis, diophantine approximation
2020 Mathematics Subject Classification: 70H08 (Primary); 37J40 (Secondary)

Contents

1 Introduction 2
2 Settings 3
2.1 Mixing property and Keplerian shear 3
2.2 Rajchman measure 4
2.2.1 Functional equivalences 5
2.2.2 Fourier–Rajchman decay 6
2.2.3 Radon–Nikodym–Lebesgue decomposition 7
2.2.4 Example of singular Rajchman measures 7
2.3 Tori bundle 8
3 Main results in discrete dynamical systems 8
3.1 Previous results 10
3.2 Generalizations to singular measures 10
3.3 Invariant functions and σ-algebra 14
4 Main result in continuous dynamical systems 14
4.1 Invariant functions and σ-algebra 17
4.2 Convergence speed with the real Rajchman property 19

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

4.3 Speed of decay of conditional correlations for absolutely continuous measures 20
5 Keplerian shear, Rajchman property, and Diophantine approximation 26
6 Flow on compact Lie group bundle 34
6.1 Lie group bundle and compatible flow 35
6.2 Keplerian shear and Rajchman measures on G-bundle 36
6.3 Keplerian shear on G-bundle in the regular case 40
6.4 Measurable selection from the orbital group to the maximal torus 44
6.5 Main examples of compact Lie groups where the results can be applied 45
Acknowledgements 46
References 46

1. *Introduction*

In a dynamical system, the mixing property reveals that trajectories are intermingled and asymptotically distributed somewhat homogeneously. A paradigmatic example is provided by Arnold's cat map on the 2-torus $\mathbb{T}^2$, endowed with the Lebesgue measure,

$$T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

From a probabilistic perspective, mixing signifies the asymptotic independence of events; long-term evolution forgets the initial conditions. This property, along with its quantitative counterparts (e.g., decay of correlations), forms the foundation for various probabilistic limit theorems within the context of deterministic dynamics, such as the central limit theorem and Borel–Cantelli lemmas (see [**CK2**]).

However, there are many systems in which certain quantities are conserved during evolution. This occurs in integrable systems of Hamiltonian dynamics, particularly in some geodesic flows and rational billiards (see [**Tho**] and references therein). This phenomenon can also manifest in biological systems, where the preservation of a character results in different evolutionary paths. The presence of these invariants prevents the system from being ergodic and, consequently, from exhibiting mixing. The transvection

$$T := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

acting on the torus $\mathbb{T}^2$, is a paradigmatic example. This map acts on the second coordinate as a rotation on the circle. The evolution in each ergodic component (each individual system) is notably simple and predictable. Nevertheless, Kesten demonstrated that, with some randomness introduced into the rotation angle, trajectories distribute quite homogeneously in the long term. Specifically, the discrepancies, suitably normalized, converge to a Cauchy distribution [**Kes**] (see also [**DF**] for a multidimensional generalization).

In celestial mechanics, planetary rings (the motion of each dust particle) may be modeled by the flow,

$$\begin{aligned} g_t : [a, b] \times \mathbb{T} &\to [a, b] \times \mathbb{T} \\ (r, \theta) &\mapsto (r, \theta + t r^{-(3/2)}). \end{aligned}$$

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

While the distance to the center is preserved, and each trajectory is essentially a rotation, the fact that angular velocities vary allows for the aggregation of materials, potentially explaining the formation of larger bodies. Recently, Thomine [**Tho**] introduced the notion of Keplerian shear, formalizing the fact that in non-ergodic systems, trajectories may distribute homogeneously and independently of their past, provided we ignore the invariants.

The aim of our work is to pursue the study of non-ergodic dynamical systems that have the property of Keplerian shear.

The probabilistic dynamical system (X, T, μ) has Keplerian shear if for all $f \in L^2(\mu)$, we have the weak convergence

$$f \circ T^n \rightharpoonup E_\mu(f|\mathcal{I}), \tag{1.1}$$

$\mathcal{I}$ being the σ-algebra of invariant by the transformation T. The lack of ergodicity is indeed adding randomness to the dynamics (the choice of the ergodic component) and, even if the dynamics on the fiber is not mixing, the system can globally appear mixing conditionally to the fibers.

We now present the organization of the article. In §2, we define Keplerian shear, and relate it to the mixing property and the Rajchman property of a measure, the main notion in this article. We also make a description of the dynamical systems that we consider in this work, which includes locally action-angle dynamics.

In §3, we show the main result in the discrete case and we investigate the speed of shearing, using anisotropic Sobolev spaces.

Section 4 addresses in the continuous case the same questions of §3.

Section 5 provides applications of Keplerian shear to dynamical Borel–Cantelli lemmas and Diophantine approximation.

To finish, §6 generalizes the work to the case where the phase space is a connected-compact Hausdorff Lie group.

2. *Settings*

2.1. *Mixing property and Keplerian shear.*

The mixing property is well known and is satisfied in many situations. However, it makes sense only for ergodic systems. From this point of view, Keplerian shear may be a right compromise.

We will note in the article $(\Omega, \mathcal{T}, \mu, (g_t)_{t\in\mathbb{R}})$ (respectively $(\Omega, \mathcal{T}, \mu, T)$), a continuous dynamical (respectively discrete) system. A mixing system is an asymptotically independent system in the following sense.

Definition 2.1. (Mixing system) We say that $(\Omega, \mathcal{T}, \mu, (g_t)_{t\in\mathbb{R}})$ (respectively $(\Omega, \mathcal{T}, \mu, T)$) is mixing if for all $A, B \in \mathcal{T}$,

$$\mu(A \cap g_t^{-1}(B)) - \mu(A)\mu(B) \xrightarrow[t\to+\infty]{} 0 \tag{2.1}$$

$$\big(\text{respectively } \mu(A \cap (T^n)^{-1}(B)) - \mu(A)\mu(B) \xrightarrow[n\to+\infty]{} 0\big). \tag{2.2}$$

We recall that mixing systems are ergodic. Consequently the following systems are not mixing, but we will show that they exhibit Keplerian shear.

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

Example 2.1. (Non-ergodic systems) We endow these systems with the Lebesgue measure on $\mathbb{T}^2$:

$$(1)\quad \begin{aligned} T:\ \mathbb{T}^2 &\to \mathbb{T}^2 \\ (x,y) &\mapsto (x, y+x); \end{aligned}$$

$$(2)\quad \begin{aligned} g_t:\ \mathbb{T}^2 &\to \mathbb{T}^2 \\ (x,y) &\mapsto (x, y + t\cos(2\pi(x-\tfrac{1}{2}))). \end{aligned}$$

Definition 2.2. (Invariant σ-algebra) The invariant σ-algebra $\mathcal{I}$ is the σ-algebra of measurable sets invariant by continuous flow (respectively discrete):

for $(\Omega, \mathcal{T}, \mu, (g_t)_{t\in\mathbb{R}})$ (respectively $(\Omega, \mathcal{T}, \mu, T)$), a continuous dynamical system (respectively discrete),

$$\mathcal{I} := \{A \in \mathcal{T} :\ \text{for all } t \in \mathbb{R},\ \mu(A \Delta g_t^{-1}(A)) = 0\}$$
$$(\text{respectively} \{A \in \mathcal{T} : \mu(A \Delta T^{-1}(A)) = 0\}).$$

The Keplerian shear is a notion of asymptotic independence conditionally to the T-invariant algebra.

Definition 2.3. (Keplerian shear) The dynamical system $(\Omega, \mathcal{T}, \mu, (g_t)_{t\in\mathbb{R}})$ exhibits Keplerian shear if for all $f \in \mathbb{L}^2_\mu(\Omega)$,

$$f \circ g_t \underset{t\to+\infty}{\rightharpoonup} \mathbb{E}_\mu(f|\mathcal{I}) \ \big(\text{respectively } f \circ T^n \underset{n\to+\infty}{\rightharpoonup} \mathbb{E}_\mu(f|\mathcal{I})\big). \tag{2.3}$$

Definition 2.4. (Conditional correlation) We define the conditional correlation for $f_1, f_2 \in \mathbb{L}^2_\mu(\Omega)$ by

$$\begin{aligned} &\mathrm{Cov}_t(f_1, f_2|\mathcal{I}) := \mathbb{E}_\mu(\overline{f_1} \cdot (f_2 \circ g_t)|\mathcal{I}) - \overline{\mathbb{E}_\mu(f_1|\mathcal{I})} \cdot \mathbb{E}_\mu(f_2|\mathcal{I}) \\ &(\text{respectively } \mathrm{Cov}_n(f_1, f_2|\mathcal{I}) := \mathbb{E}_\mu(\overline{f_1} \cdot (f_2 \circ T^n)|\mathcal{I}) - \overline{\mathbb{E}_\mu(f_1|\mathcal{I})} \cdot \mathbb{E}_\mu(f_2|\mathcal{I})). \end{aligned} \tag{2.4}$$

PROPOSITION 2.2. [**Tho**] *The Keplerian shear property is equivalent to the convergence to* 0 *of the expectation of the conditional correlation, in other words, for all* $f_1, f_2 \in \mathbb{L}^2_\mu(\Omega)$,

$$\mathbb{E}_\mu(\mathrm{Cov}_t(f_1, f_2|\mathcal{I})) \xrightarrow[t\to+\infty]{} 0 \ \big(\textit{respectively } \mathbb{E}_\mu(\mathrm{Cov}_n(f_1, f_2|\mathcal{I})) \xrightarrow[n\to+\infty]{} 0\big). \tag{2.5}$$

In ergodic systems, Keplerian shear is equivalent to mixing. The mixing property is thus equivalent to the ergodicity and Keplerian shear.

2.2. *Rajchman measure.* The Fourier theory will be instrumental in our study of correlation decay. In particular, measures with the Riemann–Lebesgue property will play an important role in this work.

Definition 2.5. (Rajchman measure [**Lyo**]) In the continuous case, a measure ν on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is Rajchman if

$$\widehat{\nu}(t) \xrightarrow[t\to\pm\infty]{} 0\,. \tag{2.6}$$

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

In the discrete case, a measure ν on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ is Rajchman if

$$\widehat{\nu}(n) \xrightarrow[n\to\pm\infty]{} 0, \tag{2.7}$$

where $\hat{\nu}(t) = \int_{\mathbb{R}} e^{2i\pi tx}\, d\nu(x)$ (respectively $\hat{\nu}(n) = \int_{\mathbb{T}} e^{2i\pi nx}\, d\nu(x)$).

2.2.1. *Functional equivalences.* We recall an equivalent interpretation of the Rajchman property in terms of convergence in distribution or equidistribution (in probabilistic terms, this is equivalent to the convergence of the characteristic function).

PROPOSITION 2.3. [**Bre**] *In the continuous case (respectively discrete), the measure ν is Rajchman if and only if for x distributed by ν and for all $a > 0$,*

$$\frac{nx}{a} \bmod 1 \xrightarrow[n\to+\infty]{\mathscr{L}} \mathcal{U}(\mathbb{T}) \tag{2.8}$$

$$\left(\textit{respectively } nx \bmod 1 \xrightarrow[n\to+\infty]{\mathscr{L}} \mathcal{U}(\mathbb{T})\right). \tag{2.9}$$

LEMMA 2.4. (Weak-$*$ convergence and Rajchman property) *Let μ be a Borel-probability measure on $\mathbb{R}$.*

The measure μ is Rajchman if and only if

$$(x \mapsto e^{2i\pi tx}) \xrightarrow[t\to+\infty]{} 0 \tag{2.10}$$

in the weak-$*$ topology on $(\mathbb{L}^1_\mu(\mathbb{R}))^* = \mathbb{L}^\infty_\mu(\mathbb{R})$.

Proof. The reciprocal implication is obtained immediately applying the duality property (2.10) with $(x \in \mathbb{R} \mapsto 1) \in \mathbb{L}^1_\mu(\mathbb{R})$.

Now, we prove the direct implication. Let $\varphi \in \mathcal{S}(\mathbb{R})$. Writing φ with its inverse Fourier transform, we get for all $x \in \mathbb{R}$,

$$\varphi(x) = \int_{\mathbb{R}} e^{2i\pi xs}\widehat{\varphi}(s)\, d\lambda(s). \tag{2.11}$$

By Fubini theorem and (2.11), we get

$$\int_{\mathbb{R}} e^{2i\pi tx}\varphi(x)\, d\mu(x) = \int_{\mathbb{R}} \widehat{\varphi}(s)\widehat{\mu}(t+s)\, d\lambda(s).$$

Since μ is Rajchman, by Lebesgue convergence theorem,

$$\int_{\mathbb{R}} \widehat{\varphi}(s)\widehat{\mu}(t+s)\, d\lambda(s) \xrightarrow[t\to+\infty]{} 0. \qquad \square$$

LEMMA 2.5. (Weak-$*$ convergence and Rajchman property in a discrete case) *Let μ be a Borel-probability measure on $\mathbb{T}$.*

The measure μ is Rajchman if and only if

$$(x \mapsto e^{2i\pi nx}) \xrightarrow[n\to+\infty]{} 0 \tag{2.12}$$

in the weak-$*$ topology on $\mathbb{L}^2_\mu(\mathbb{T})$.

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

Proof. The reciprocal implication is obtained immediately applying the duality property (2.12) with $(x \in \mathbb{R} \mapsto 1) \in \mathbb{L}^2_\mu(\mathbb{T})$.

Now, we prove the direct implication. Let $\varphi \in \mathcal{C}^\infty(\mathbb{T})$. Expanding φ in Fourier series, we get for all $x \in \mathbb{T}$,

$$\varphi(x) = \sum_{k\in\mathbb{Z}} \widehat{\varphi}(k) e^{2i\pi kx}. \tag{2.13}$$

By Fubini theorem and (2.11), we get

$$\int_{\mathbb{T}} e^{2i\pi nx}\varphi(x)\, d\mu(x) = \sum_{k\in\mathbb{Z}} \widehat{\varphi}(k)\widehat{\mu}(k+n) \xrightarrow[n\to+\infty]{} 0$$

by Lebesgue convergence theorem, using the fact that μ is Rajchman. The conclusion follows by density. □

2.2.2. *Fourier–Rajchman decay.* The notion of Rajchman measure is only qualitative. To estimate the speed of convergence to 0 of the expectation of conditional correlations (2.5), a quantitative version of the Fourier decay will be needed.

Definition 2.6. (Fourier–Rajchman decay) A Rajchman speed of a Rajchman measure on $\mathbb{T}$ (respectively $\mathbb{R}$) is a value $r \geq 0$ such that there exists $C > 0$ satisfying for all $n \in \mathbb{Z}$ (respectively $t \in \mathbb{R}$),

$$|\widehat{\nu}(n)| \leq \frac{C}{|n|^r}, \tag{2.14}$$

$$\text{respectively } |\widehat{\nu}(t)| \leq \frac{C}{|t|^r}. \tag{2.15}$$

In other words, $\widehat{\nu}(n) = O_\infty(|n|^{-r})$ (respectively $\widehat{\nu}(t) = O_\infty(|t|^{-r})$).

Definition 2.7. (Rajchman order) The Rajchman order of a Rajchman measure is the supremum of its Rajchman speeds. We denote it by $r(\mu)$.

Remark 2.6. The Rajchman order is linked to the Fourier dimension $\dim F(\mu)$ defined in [Sol] by the relation $\dim F(\mu) = 2r(\mu)$. Additionally, as mentioned there, when the Rajchman order r verifies $r > \frac{1}{2}$, $\mu \ll \lambda$. Consequently, a Rajchman measure can be singular continuous only if its Rajchman order r satisfies $r \leq \frac{1}{2}$.

Definition 2.8. (Diophantine exponent) We define the Diophantine exponent of $x \in \mathbb{R}$ by $\mathrm{Dio}(x) = \inf A = \sup B$, where (with the convention $\inf \emptyset = +\infty$)

$$A = \left\{ s > 0 : \text{ there exists } C > 0 \text{ for all } (p,q) \in \mathbb{Z}\times\mathbb{N}^*, |qx - p| \geq \frac{C}{q^s} \right\} \tag{2.16}$$

and

$$B = \left\{ t > 0 : \text{ there exists infinitely many } (p,q) \in \mathbb{Z}\times\mathbb{N}^*, |qx - p| < \frac{1}{q^t} \right\}. \tag{2.17}$$

Definition 2.9. (s-Diophantine number) Let $s > 0$.

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

A number $x \in \mathbb{R}$ is called s-Diophantine if there exists $C > 0$ such that for all $p \in \mathbb{Z}$, for all $q \in \mathbb{N}^*$,

$$|qx - p| \geq \frac{C}{q^s}. \tag{2.18}$$

Let $\mathcal{D}\mathrm{io}(s)$ be the set of s-Diophantine numbers.

Definition 2.10. (Liouville number) A number $x \in \mathbb{R}$ is Liouville if and only if for all $s > 0$, x is not s-Diophantine, that is, $\mathrm{Dio}(x) = +\infty$. In other words,

$$x \in \mathbb{L} := \mathbb{R} \setminus \left(\bigcup_{s>0} \mathcal{D}\mathrm{io}(s) \right).$$

The Rajchman order of a measure controls the almost sure Diophantine exponent (see Proposition 5.9).

PROPOSITION 2.7. *If* $r(\mu) \leq \frac{1}{2}$, *then* μ*-almost surely (a.s.)* $\alpha \in [0, 1[$, $\mathrm{Dio}(\alpha) \leq (1/r(\mu)) - 1$. *In the absolutely continuous case,* $\mathrm{Dio}(\alpha) = 1$ *a.s.*

2.2.3. *Radon–Nikodym–Lebesgue decomposition.* We discuss the simple relations that can be seen between the Rajchman property and the Radon–Nikodym–Lebesgue decomposition

$$\nu = \nu_{ac} + \nu_{sc} + \nu_d, \tag{2.19}$$

with ν_{ac} the absolutely continuous part, ν_{sc} the singular continuous part and ν_d the discrete part.

(1) *Absolutely continuous part*. Here, ν_{ac} is an absolutely continuous measure and therefore, it is Rajchman. This is guaranteed by Riemann–Lebesgue. When we know the regularity of the density, we can estimate the Rajchman order. The work of Damien Thomine addresses the case of absolutely continuous measures (see Theorem 4.1).

(2) *Singular continuous part*. Many singular continuous measures are Rajchman and other are not. The main discussion is about the Rajchman property of ν_{sc}. We will extend Damien Thomine's results for continuous singular measures (see Theorem 4.2).

(3) *Discrete part*. Discrete measures are never Rajchman because of the non-convergence of complex exponential to 0.

Consequently, ν is Rajchman if and only if ν_{sc} is Rajchman and $\nu_d = 0$.

2.2.4. *Example of singular Rajchman measures.*

Definition 2.11. (Pisot number) A real θ is a Pisot number if and only if $\theta > 1$ is algebraic and every other roots θ_r of its minimal polynomial satisfy $|\theta_r| < 1$.

We note that Lebesgue-almost all numbers are not Pisot. Now, to highlight the Rajchman property for the singular measure, we will present examples of continuous singular measures having this property.

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

Example 2.8. (Self-similar measure) Consider for $\theta > 2$, the distribution μ_θ of $\sum_{n\in\mathbb{N}^*} \pm\theta^{-n}$, where signs are chosen with independent and identically distributed (i.i.d) probabilities $\frac{1}{2}$. Note that μ_θ has as a characteristic function $\widehat{\mu_\theta} : k \in \mathbb{Z} \mapsto \prod_{n\in\mathbb{N}} \cos(2\pi\theta^{-n}k)$.

Then, μ_θ is Rajchman if and only if θ is not a Pisot number [**Kah**].

Additionally, for Lebesgue-almost all real $\theta > 2$, these measures are singular and the Rajchman order $r(\mu_\theta) > 0$.

Example 2.9. (Rajchman measure with Liouville set as support (Dio $= +\infty$)) Bluhm [**Blu**] has built a Rajchman measure μ_∞ supported on Liouville numbers.

According to Proposition 2.7, its order is $r(\mu_\infty) = 0$.

2.3. *Tori bundle.* We study a class of dynamical systems introduced in [**Tho**], tori bundles. These systems have local action-angle coordinates.

Definition 2.12. (Tori bundle) Let $(M, \mathcal{A})$ be an $n \in \mathbb{N}^*$ dimensional $\mathcal{C}^1$ Lindelöf manifold. (Any open cover has a countable subcover.)

Let $d \in \mathbb{N}^*$, (Ω, μ) be a Borel space and π a continuous projection of Ω to M.

Here, Ω is an (n, d)-dimensional tori bundle if:

(1) locally, we have for charts U of $\mathcal{A}$ a homeomorphism $\psi_U : \pi^{-1}(U) \to U \times \mathbb{T}^d$;
(2) for all U in $\mathcal{A}$, $\pi_1 \circ \psi_U = \pi_{|U}$, where π_1 is the projection on the first coordinate.

Bundles of the form $M \times \mathbb{T}^d$ constitute already interesting examples.

Definition 2.13. (Compatible flow/map) Let $(\Omega, \mu, (g_t)_{t\in\mathbb{R}})$ (respectively (Ω, μ, T)) be a continuous time (respectively discrete time) measure preserving dynamical system.

The flow $(g_t)_{t\in\mathbb{R}}$ (respectively the map T) is compatible with the tori bundle (Ω, μ) if for all charts $U \in \mathcal{A}$ there exists $v_U \in (\mathbb{R}^d)^U$ measurable such that for $t \in \mathbb{R}$ (respectively $n \in \mathbb{N}$),

$$\psi_U \circ g_t \circ \psi_U^{-1}(x, y) = (x, y + t v_U(x))$$
$$(\text{respectively } \psi_U \circ T^n \circ \psi_U^{-1}(x, y) = (x, y + n v_U(x))).$$

We note $g_t^U = \psi_U \circ g_t \circ \psi_U^{-1}$(respectively $T_U = \psi_U \circ T \circ \psi_U^{-1}$).

Definition 2.14. (Compatible measure) A measure μ is compatible if for all charts, $U \in \mathcal{A}$, $(\mu_{|\pi^{-1}(U)})_{\psi_U} = (\mu_\pi)_{|U} \otimes \lambda$.

3. *Main results in discrete dynamical systems*

In this case, we use the Rajchman property with the help of the corresponding Fourier series. We will highlight the necessary path to show the presence of Keplerian shear in this case. We now state the main result in the discrete case.

THEOREM 3.1. *Let (Ω, μ, T) be a discrete dynamical system with Ω a tori bundle on which the map T and the measure μ are compatibles as in §2.3.*

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

The dynamical system exhibits Keplerian shear if and only if for all $0 \neq \xi \in \mathbb{Z}^d$ *and for all charts,* $U \in \mathcal{A}$, $m^{\mathbb{T}}_{\xi,U}$ *is Rajchman where*

$$m^{\mathbb{T}}_{\xi,U} = (((\mu_\pi)_{|U})_{\langle \xi | v_U(\cdot)\rangle - \lfloor \langle \xi | v_U(\cdot)\rangle \rfloor})_{|\mathbb{T}\setminus\{0\}}. \tag{3.1}$$

Proof. Let us start with the direct implication. Let $\xi \in \mathbb{Z}^d$, $\xi \neq 0$, and $U \in \mathcal{A}$. Take

$$f_1 : x \in \pi^{-1}(U) \mapsto \mathbb{1}_{(\langle \xi | v_U(\cdot)\rangle - \lfloor \langle \xi | v_U(\cdot)\rangle \rfloor)^{-1}(\mathbb{T}\setminus\{0\})}(\pi(x)) e^{2i\pi \langle \xi | \pi_2 \circ \psi_U(x)\rangle}.$$

Take $f_2 = f_1$. For all $n \in \mathbb{N}$,

$$\int_{\mathbb{T}} e^{2i\pi nz}\, dm^{\mathbb{T}}_{\xi,U}(z) = \int_{\pi^{-1}(U)} \overline{f_1} \cdot f_2 \circ T^n \, d\mu(x) \xrightarrow[n\to+\infty]{} \mathbb{E}_\mu(\mathbb{E}_\mu(\overline{f_1}\mathbb{1}_{\pi^{-1}(U)}|\mathcal{I})\mathbb{E}_\mu(f_2|\mathcal{I}))$$

by Keplerian shear. Using the explicit form of f_2 and Birkhoff theorem, we find that $\mathbb{E}_\mu(f_2|\mathcal{I}) = 0$; therefore,

$$\int_{\mathbb{T}} e^{2i\pi nz}\, dm^{\mathbb{T}}_{\xi,U}(z) \xrightarrow[n\to+\infty]{} 0.$$

Hence, $m^{\mathbb{T}}_{\xi,U}$ is Rajchman.

Let us treat the reciprocal implication. Using a partition of unity, the problem reduces, for one chart $U \in \mathcal{A}$ and two functions f_1 and f_2 in $\mathbb{L}^2(U \times \mathbb{T}^d, \mu_{\pi|U} \otimes \lambda)$, to the convergence of the covariance

$$C_n := \int_{U\times\mathbb{T}^d} \overline{f_1}(x, y) f_2(x, y + n v_U(x))\, d(\mu_\pi)_{|U} \otimes \lambda(x, y).$$

Using the Fourier decomposition on $\mathbb{T}^d$ of $f_1(x, \cdot)$ and $f_2(x, \cdot)$, we get

$$C_n = \sum_{\xi\in\mathbb{Z}^d} I_n(\xi) \quad \text{where } I_n(\xi) = \int_U \overline{\widehat{f_1(x,\cdot)}(\xi)}\, \widehat{f_2(x,\cdot)}(\xi) e^{2in\pi\langle \xi | v_U(x)\rangle}\, d(\mu_\pi)_{|U}(x). \tag{3.2}$$

For $\xi \in \mathbb{Z}^d$, writing $a^{(1)}_\xi(x) = \widehat{f_1(x,\cdot)}(\xi)$ and $a^{(2)}_\xi(x) = \widehat{f_2(x,\cdot)}(\xi)$ for all $x \in U$, we have

$$I_n(\xi) = \int_U \mathbb{E}(\overline{a^{(1)}_\xi} a^{(2)}_\xi | \langle \xi | v_U(x)\rangle) e^{2in\pi\langle \xi | v_U(x)\rangle}\, d(\mu_\pi)_{|U}(x). \tag{3.3}$$

Let g be the integrable function such that $\mathbb{E}(\overline{a^{(1)}_\xi} a^{(2)}_\xi | \langle \xi | v_U(x)\rangle) = g(\langle \xi | v_U(x)\rangle)$. By transfer theorem, for $\xi \neq 0$,

$$\begin{aligned} &\int_U \mathbb{E}(\overline{a^{(1)}_\xi} a^{(2)}_\xi | \langle \xi | v_U(x)\rangle) e^{2in\pi\langle \xi | v_U(x)\rangle} \mathbb{1}_{\langle \xi | v_U(x)\rangle \neq 0}\, d(\mu_\pi)_{|U}(x) \\ &\quad = \int_{\mathbb{T}} g(z) e^{2i\pi nz}\, dm^{\mathbb{T}}_{\xi,U}(z), \end{aligned} \tag{3.4}$$

which converges to 0 by the Rajchman hypothesis and Lemma 2.5.

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

For $\xi \in \mathbb{Z}^d$, for all $n \in \mathbb{N}$, we have

$$\left| \int_U \overline{a_\xi^{(1)}(x)} a_\xi^{(2)}(x) e^{2in\pi\langle \xi | v_U(x)\rangle} \, d(\mu_\pi)_{|U}(x) \right| \leq \|a_\xi^{(1)}\|_{\mathbb{L}^2} \|a_\xi^{(2)}\|_{\mathbb{L}^2} \tag{3.5}$$

and

$$\sum_{\xi \in \mathbb{Z}^d} \|a_\xi^{(1)}\|_{\mathbb{L}^2} \|a_\xi^{(2)}\|_{\mathbb{L}^2} \leq \|f_1\|_{\mathbb{L}^2} \|f_2\|_{\mathbb{L}^2}. \tag{3.6}$$

Therefore, using (3.3) and (3.4), we can apply the dominated convergence theorem in (3.2), which proves that C_n converges. □

3.1. *Previous results.* We recall some results obtained by Thomine on this topic in the discrete case for Lebesgue measure.

Definition 3.1. (Anisotropic Sobolev spaces [**Tho**, §2.5.1]) Let $s > 0$ and

$$h : (x, y) \in \mathbb{R}^2 \mapsto \begin{cases} (1 + (x^2/y^2))^{1/2} & \text{if } y \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

We define an anisotropic Sobolev space of order $s > 0$ as

$$H^{s,0}(\mathbb{T}^2) := \left\{ f \in \mathbb{L}^2(\mathbb{T}^2) : \sum_{\xi \in \mathbb{Z}^d} |\widehat{f}(\xi)|^2 h^{2s}(\xi) \in \mathbb{R} \right\}.$$

Provided the observables have some regularity, one can get an upper estimate of the decay of conditional correlations for the transvection, as expressed below.

PROPOSITION 3.2. [**Tho**, Proposition 2.10] *Considering the dynamical system* $(\mathbb{T}^2, \lambda \otimes \lambda, T)$ *with* $T : (x, y) \in \mathbb{T}^2 \mapsto (x, y + x)$, *we have that for* $s > 0$, *for all* $f_1, f_2 \in H^{s,0}(\mathbb{T}^2)$, $n \in \mathbb{N}^*$,

$$|\mathbb{E}_{\lambda\otimes\lambda}(\mathrm{Cov}_n(f_1, f_2|\mathcal{I}))| \leq \frac{4^s}{n^{2s}} \|f_1\|_{H^{s,0}(\mathbb{T}^2)} \|f_2\|_{H^{s,0}(\mathbb{T}^2)}. \tag{3.7}$$

3.2. *Generalizations to singular measures.* Here, we present a version of the previous result for non-absolutely continuous measures.

Definition 3.2. (Sobolev space on the torus) We will note for $s > 0$, $d \in \mathbb{N}^*$, the Sobolev space

$$H^s(\mathbb{T}^d) := \left\{ f \in \mathbb{C}^{\mathbb{T}^d} : \sum_{k \in \mathbb{Z}^d} |\widehat{f}(k)|^2 (1 + \|k\|_2^2)^s \in \mathbb{R} \right\}.$$

Definition 3.3. We will note for $s > 0$, $d \in \mathbb{N}^*$ and $f \in H^s(\mathbb{T}^d)$,

$$C_f(s) := \sup\{|\widehat{f}(\xi)|(1 + \|\xi\|_2^2)^{s/2} : \xi \in \mathbb{Z}^d\}. \tag{3.8}$$

Again with some regularity on the observables and assuming, in addition, a positive Rajchman order on the distribution of the fibers, one can bound the decay of conditional correlations.

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

PROPOSITION 3.3. *Let μ be a measure on $\mathbb{T}$ of Rajchman order equal to $r > 0$. Considering the dynamical system $(\mathbb{T}^2, \mu \otimes \lambda, T)$ with $T : (x, y) \in \mathbb{T}^2 \mapsto (x, y + x)$, we have for all $s > 2$, there exists $\gamma_s \in [\min\{s - 1, r\}, r]$ such that for all $f_1, f_2 \in H^s(\mathbb{T}^2)$, for all $\gamma < \gamma_s$, there exists $C > 0$ satisfying for all $n \in \mathbb{N}^*$,*

$$|\mathbb{E}_{\mu\otimes\lambda}(\mathrm{Cov}_n(f_1, f_2|\mathcal{I}))| \leq \frac{C}{n^\gamma}, \tag{3.9}$$

where γ_s is optimal in the sense that

$$\gamma_s = \sup\Bigg\{\alpha > 0 : \text{ there exists } C > 0 \text{ for all } f_1, f_2 \in H^s(\mathbb{T}^2), \text{ for all } n, \\ |\mathbb{E}_{\mu\otimes\lambda}(\mathrm{Cov}_n(f_1, f_2|\mathcal{I}))| \leq \frac{C}{n^\alpha}\Bigg\}. \tag{3.10}$$

Proof. Let $s > 2$, $n \in \mathbb{N}^*$ and $f_1, f_2 \in H^s(\mathbb{T}^2)$. We recall that for $i \in \{1, 2\}$, using the definition (3.8), for $q_1, q_2 \in \mathbb{Z}$,

$$|\widehat{f_i}(q_1, q_2)(1 + q_1^2 + q_2^2)^{s/2}| \leq C_{f_i}(s). \tag{3.11}$$

For $\xi_1, \xi_2 \in \mathbb{Z}$, let

$$g_{(\xi_1,\xi_2)} : x \in \mathbb{T} \mapsto e^{2i\pi\xi_1 x}\left(\int_{\mathbb{T}} \overline{f_1}(x, y)e^{2i\pi\xi_2 y}\, d\lambda(y)\right).$$

By Fourier decomposition, for $\xi_1, \xi_2 \in \mathbb{Z}^d$ and for all $x \in \mathbb{T}$,

$$g_{(\xi_1,\xi_2)}(x) = \sum_{k\in\mathbb{Z}} \overline{\widehat{f_1}(k, \xi_2)}e^{2i\pi(\xi_1-k)x}. \tag{3.12}$$

By (3.12),

$$\int_{\mathbb{T}^2} \overline{f_1}(f_2 \circ T^n)\, d(\mu \otimes \lambda) = \sum_{\xi\in\mathbb{Z}^2} \widehat{f_2}(\xi) \int_{\mathbb{T}} e^{2i\pi n\xi_2 x} g_{(\xi_1,\xi_2)}(x)\, d\mu(x) \tag{3.13}$$

and for all $k, \xi_1, \xi_2 \in \mathbb{Z}$,

$$\widehat{g_{(\xi_1,\xi_2)}}(k) = \widehat{f_1}(\xi_1 - k, \xi_2). \tag{3.14}$$

By (3.11) and (3.14),

$$|\widehat{g_{(\xi_1,\xi_2)}}(k)|(1 + k^2)^{s/2} \leq C_{f_1} \frac{(1 + k^2)^{s/2}}{(1 + \xi_2^2 + (\xi_1 - k)^2)^{s/2}}. \tag{3.15}$$

Moreover, by (3.13) and (3.14), for $n \in \mathbb{N}$,

$$\mathbb{E}_{\mu\otimes\lambda}(\mathrm{Cov}_n(f_1, f_2|\mathcal{I})) = \sum_{\xi\in\mathbb{Z}\times\mathbb{Z}^*} \widehat{f_2}(\xi) \int_{\mathbb{T}} e^{2i\pi n\xi_2 x} g_{(\xi_1,\xi_2)}(x)\, d\mu(x)$$

and

$$\mathbb{E}_{\mu\otimes\lambda}(\mathrm{Cov}_n(f_1, f_2|\mathcal{I})) = \sum_{(k,\xi)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}^*} \widehat{f_2}(\xi)\widehat{f_1}(k, \xi_2)\widehat{\mu}(\xi_1 - k + n\xi_2). \tag{3.16}$$

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

We estimate the sum above, considering three different cases. In the first case, we suppose that $k, \xi_1, \xi_2 \in \mathbb{Z}$ and $n \in \mathbb{N}$ are such that

$$|\xi_1 - k + n\xi_2| \leq \frac{n|\xi_2|}{4} \tag{3.17}$$

and

$$|n\xi_2 - k| \leq \frac{n|\xi_2|}{2}. \tag{3.18}$$

Therefore,

$$|k| \geq \frac{n|\xi_2|}{2}, \tag{3.19}$$

$$|\xi_1| \geq \frac{3}{4}n|\xi_2|, \tag{3.20}$$

and

$$|\widehat{\mu}(\xi_1 - k + n\xi_2)| \leq 1. \tag{3.21}$$

Assuming (3.17) and (3.18), by (3.15), (3.19), (3.20), and (3.21),

$$\sum_{\substack{(k,\xi_1,\xi_2)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}^* \\ |\xi_1-k+n\xi_2|\leq(n|\xi_2|)/4 \\ |n\xi_2-k|\leq(n|\xi_2|)/2}} |\widehat{f_2}(\xi_1,\xi_2)||\widehat{f_1}(k,\xi_2)|\widehat{\mu}(\xi_1 - k + n\xi_2)| \leq \frac{2^{6(s+1)}\zeta(2(s-1))C_{f_1}(s)C_{f_2}(s)}{3^s n^{2(s-1)}}. \tag{3.22}$$

In the second case, we assume (3.17) and

$$|n\xi_2 - k| > \frac{n|\xi_2|}{2}. \tag{3.23}$$

Thus, we get again (3.21) and

$$|\xi_1| > \frac{n|\xi_2|}{4}. \tag{3.24}$$

By (3.21) and (3.11), $|\widehat{f_1}(k,\xi_2)||\widehat{\mu}(\xi_1 - k + n\xi_2)| \leq (C_{f_1}(s)/(1 + k^2 + \xi_2^2)^{s/2})$. Moreover, by (3.11) and (3.24),

$$|\widehat{f_2}(\xi_1,\xi_2)| \leq \frac{4^s C_{f_2}(s)}{|\xi_2|^s n^s}. \tag{3.25}$$

By (3.25) and series-integral criterion with $s > 2$, there exists $C_s > 0$ such that

$$\sum_{\substack{(k,\xi_1,\xi_2)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}^* \\ |\xi_1-k+n\xi_2|\leq(n|\xi_2|)/4 \\ |n\xi_2-k|>(n|\xi_2|)/2}} \frac{C_{f_1}(s)|\widehat{f_2}(\xi_1,\xi_2)|}{(1 + k^2 + \xi_2^2)^{s/2}} \leq \frac{C_s 4^s C_{f_1}(s)C_{f_2}(s)}{n^{(s-1)}}.$$

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

Finally, in the third case, we assume

$$|\xi_1 - k + n\xi_2| > \frac{n|\xi_2|}{4}. \tag{3.26}$$

Here, μ is a Rajchman measure and $r > 0$ its order, by the definition (2.14), there exists $C_\mu > 0$ such that for $q \in \mathbb{Z}$,

$$|\widehat{\mu}(q)| \leq \frac{C_\mu}{|q|^r}. \tag{3.27}$$

Applying (3.27) to $q = \xi_1 - k + n\xi_2$ and by assumption (3.26),

$$|\widehat{\mu}(\xi_1 - k + n\xi_2)| \leq \frac{4^r C_\mu}{(n|\xi_2|)^r}. \tag{3.28}$$

By (3.28), (3.11), and the series-integral criterion, for $s > 2$, there exists a constant C'_s such that

$$\sum_{\substack{(k,\xi_1,\xi_2)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}^* \\ |\xi_1-k+n\xi_2|>(n|\xi_2|)/4}} \frac{C_{f_1}(s)|\widehat{f_2}(\xi_1,\xi_2)||\widehat{\mu}(\xi_1-k+n\xi_2)|}{(1+k^2+\xi_2^2)^{s/2}} \leq \frac{4^r(1+3\zeta(s/2))C_\mu C'_s C_{f_1}(s)C_{f_2}(s)}{n^r}. \tag{3.29}$$

Let us pose

$$M(r,s) = \sup\left\{4^r\left(1+\zeta\left(\frac{s}{2}\right)\right)C_\mu C'_s, 4^s C_s, 2^{6(s+1)}\zeta(2(s-1))\right\}.$$

We get

$$|\mathbb{E}_{\mu\otimes\lambda}(\mathrm{Cov}_n(f_1, f_2|\mathcal{I}))| \leq M(r,s)\left(\frac{C_{f_1}C_{f_2}}{n^{\min\{s-1,r\}}}\right). \tag{3.30}$$

We now prove the optimality of the exponent. The observable f defined by $f(x,y) = e^{2i\pi y}$ belongs to $H^s(\mathbb{T}^2)$. Moreover,

$$\int_{\mathbb{T}^2} \overline{f}(x,y) f(T^n(x,y))\, d\mu\otimes\lambda(x,y) = \widehat{\mu}(n). \tag{3.31}$$

Thus, for $\gamma < \gamma_s$, there exists $C > 0$ such that for all $n \in \mathbb{N}^*$,

$$|\widehat{\mu}(n)| \leq \frac{C}{n^\gamma}. \tag{3.32}$$

By definition of the Rajchman order r, $r \geq \gamma$. Therefore, $r \geq \gamma_s$. □

Remark 3.4. The convergence speed is bounded by the order of Rajchman of the measure μ; after some threshold, increasing the regularity will not accelerate the decay of correlation. However, this is consistent with the result obtained with the Lebesgue measure because its Rajchman order is $+\infty$, which removes the maximum imposed by r for the speed of convergence and allows the observation of ever greater speeds of convergence when the regularity of the functions used increases.

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

3.3. *Invariant functions and σ-algebra.* We can identify the σ-algebra of invariant sets for the discrete flow using Fourier series and orthogonality.

THEOREM 3.5. *Under the setting of Theorem 3.1, a function $f \in \mathbb{L}^2_\mu(\Omega)$ is invariant according to T if and only if for all $U \in \mathcal{A}$, there exists $(a_k)_{k\in\mathbb{Z}^d} \in \mathbb{L}^2_{(\mu_\pi)_{|U}}(U)^{\mathbb{Z}^d}$ such that $(\mu_\pi)_{|U} \otimes \lambda$-almost every (a.e.) $(x, y) \in U \times \mathbb{T}^d$,*

$$f_{|\pi^{-1}(U)}(\psi_U^{-1}(x, y)) = \sum_{k\in\mathbb{Z}^d, \langle k|v_U(x)\rangle\in\mathbb{Z}} a_k(x)e^{2i\pi\langle k|y\rangle}.$$

Proof. Let $f \in \mathbb{L}^2_\mu(\Omega)$ and $U \in \mathcal{A}$.

Let us prove the direct assertion. Suppose that $f \circ T = f$ μ-a.s. Let $(x, y) \in U \times \mathbb{T}^d$. So,

$$f(T(\psi_U^{-1}(x, y))) = f(\psi_U^{-1}(x, y)).$$

Namely, $f(\psi_U^{-1}(x, y + v_U(x))) = f(\psi_U^{-1}(x, y))$. By Fourier series decomposition:

$$\sum_{k\in\mathbb{Z}^d} f \circ \widehat{\psi_U(x, \cdot)}(k)e^{2i\pi\langle k|y+v_U(x)\rangle} = \sum_{k\in\mathbb{Z}^d} f \circ \widehat{\psi_U(x, \cdot)}(k)e^{2i\pi\langle k|y\rangle}. \tag{3.33}$$

By uniqueness of the coefficients of a Fourier series in (3.33), for μ-a.s. $x \in U$, for all $k \in \mathbb{Z}^d$,

$$f \circ \widehat{\psi_U(x, \cdot)}(k)(e^{2i\pi\langle k|v_U(x)\rangle} - 1) = 0. \tag{3.34}$$

Thus, $\langle k|v_U(x)\rangle \in \mathbb{Z}$ or $f \circ \widehat{\psi_U(x, \cdot)}(k) = 0$.

Let us prove the reciprocal. Suppose that f satisfies for all charts $U \in \mathcal{A}$, $(\mu_\pi)_{|U} \otimes \lambda$-a.s. $(x, y) \in U \times \mathbb{T}^d$,

$$f(\psi_U^{-1}(x, y)) = \sum_{k\in\mathbb{Z}^d, \langle k|v_U(x)\rangle\in\mathbb{Z}} a_k(x)e^{2i\pi\langle k|y\rangle}. \tag{3.35}$$

Let $z \in \Omega$. There exists $U \in \mathcal{A}$ such that $z \in \pi^{-1}(U)$. We have

$$f(T(z)) = f(T(\psi_U^{-1}(\psi_U(z)))) = f(\psi_U^{-1}(\pi(z), \pi_2 \circ \psi_U(z) + v_U(\pi(z)))).$$

Applying (3.35) with the above computation,

$$\begin{aligned} f(T(z)) &= \sum_{k\in\mathbb{Z}^d, \langle k|v_U(x)\rangle\in\mathbb{Z}} a_k(\pi(z))e^{2i\pi\langle k|\pi_2\circ\psi_U(z)+v_U(\pi(z))\rangle} \\ &= \sum_{k\in\mathbb{Z}^d, \langle k|v_U(x)\rangle\in\mathbb{Z}} a_k(\pi(z))e^{2i\pi\langle k|\pi_2\circ\psi_U(z)\rangle} \\ &= f(z). \end{aligned}$$

□

4. *Main result in continuous dynamical systems*

As mentioned in the introduction, Damien Thomine gave many results which guarantee Keplerian shear in [**Tho**], in particular, the following one.

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

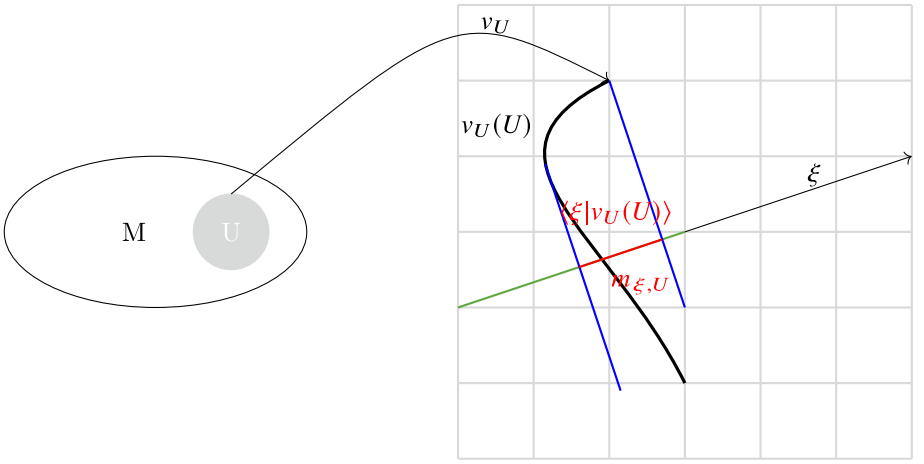


FIGURE 1. Push forward of measure $((\mu)_\pi)_{|U}$ on $\mathbb{R}$ by $\langle \xi | v_U \rangle$ (color online).

THEOREM 4.1. (Result in the regular case, [**Tho**, Theorem 3.3]) *Let* $(\Omega, \mu, (g_t)_{t\in\mathbb{R}})$ *be a compatible flow with a compatible measure on a tori (affine) bundle as in §2.3.*

Assume that:

(1) $\mu \ll \lambda$*;*
(2) *all velocity vectors* v_U *are* $\mathcal{C}^1$*;*
(3) *for all* $U \in \mathcal{A}$, $\mu(\bigcup_{\xi\in\mathbb{Z}^d\setminus\{0\}}\{x \in U : d\langle \xi | v_U(x)\rangle = 0\}) = 0$.

Then, the system exhibits Keplerian shear. Moreover,

$$\mathcal{I} = \{B \in \mathcal{B}(\Omega) : \text{ there exists } A \in \mathcal{B}(M), \mu(B \Delta \pi^{-1}(A)) = 0\}.$$

The proof of this theorem uses a Fourier transform and relies on methods of differential geometry; in particular, the normal form of submersions. We can see that the push forward measures $m_{\xi,U} := ((\mu_\pi)_{|U})_{\langle \xi | v_U(\cdot)\rangle}$ (see Figure 1) are absolutely continuous and verify the Riemann–Lebesgue lemma. This last property, in other words, Rajchman property is enough to get Keplerian shear, it is even an equivalence. We have successfully abstracted this property and obtained the Keplerian shear under the Rajchman property. Using tools from measure theory (conditional expectation) allows to some extent to get rid of the regularity of the velocity. The next theorem is the main result in the continuous case and refers to the real Rajchman property.

THEOREM 4.2. (The main result) *Let* $(\Omega, \mu, (g_t)_{t\in\mathbb{R}})$ *be a compatible flow with a compatible measure on a tori bundle as in §2.3.*

The dynamical system exhibits Keplerian shear if and only if for all charts $U \in \mathcal{A}$*, for all non-zero* $\xi \in \mathbb{Z}^d$*, the push forward measures* $(m_{\xi,U})_{|\mathbb{R}^*}$ *are Rajchman.*

Proof. Let us prove the direct implication. Let $U \in \mathcal{A}$ and non-zero $\xi \in \mathbb{Z}^d$. Take

$$f_1 : z \in \pi^{-1}(U) \mapsto \mathbb{1}_{\langle \xi | v_U\rangle \neq 0}(\pi(z)) e^{2i\pi \langle \xi | \pi_2 \circ \psi_U(z)\rangle}.$$

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

We have, thanks to the compatibility of μ,

$$\int_\Omega \overline{f_1}(z)\cdot(f_1\circ g_t(z))\,d\mu(z)=\int_{\mathbb{R}}\mathbb{1}_{\mathbb{R}^*}(s)e^{2i\pi ts}\,dm_{\xi,U}(s)=\int_{\mathbb{R}}e^{2i\pi ts}\,d(m_{\xi,U})_{|\mathbb{R}^*}(s)$$

by push forward theorem. Moreover, with Keplerian shear,

$$\int_\Omega \overline{f_1}(z)\cdot(f_1\circ g_t(z))\,d\mu(z)\xrightarrow[t\to+\infty]{}\int_\Omega \overline{\mathbb{E}_\mu(f_1|\mathcal{I})}\mathbb{E}_\mu(f_1|\mathcal{I})\,d\mu=0$$

because by Birkhoff's ergodic theorem,

$$\begin{aligned}&\mathbb{E}_\mu(f_1|\mathcal{I})(\psi_U^{-1}(x,y))\\&\overset{(\mu_\pi)_{|U}\otimes\lambda\text{-a.s.}}{=}e^{2i\pi\langle\xi|y\rangle}\mathbb{1}_{\langle\xi|v_U\rangle\neq0}(x)\lim_{T\to+\infty}\frac{1}{T}\int_0^T e^{2i\pi t\langle\xi|v_U(x)\rangle}\,dt=0.\end{aligned}\tag{4.1}$$

So, $(m_{\xi,U})_{|\mathbb{R}^*}$ is Rajchman.

Let us prove the reciprocal implication. Let $(U_i)_{i\in I}\in\mathcal{A}^I$ be a countable partition of M modulo μ_π. Let

$$Y:=\bigcup_{(j,\xi)\in I\times\mathbb{Z}^d\setminus\{0\}}\{(a\circ\pi)\cdot e^{2i\pi\langle\xi|\pi_2\circ\psi_{U_j}\rangle}\in L^2_\mu(\pi^{-1}(U_j)):a\in\mathbb{L}^\infty_\mu(U_j)\}.$$

Let $(i,j)\in I^2$, $(a_1,a_2)\in\mathbb{L}^\infty_\mu(U_i)\times\mathbb{L}^\infty_\mu(U_j)$ and $(\xi_1,\xi_2)\in(\mathbb{Z}^d)^2$. Take

$$f_l=(a_l\circ\pi)\cdot e^{2i\pi\langle\xi|\pi_2\circ\psi_{U_j}\rangle}$$

for $l\in\{1,2\}$. When $i\neq j$, the supports are disjoint and, in this case,

$$\mathbb{E}_\mu(\mathrm{Cov}_t(f_1,f_2|\mathcal{I}))=0.$$

The same happens when $\xi_1\neq\xi_2$ by periodicity of complex exponentials. Finally, when $\xi_1=\xi_2=0$, $\mathbb{E}_\mu(\mathrm{Cov}_t(f_1,f_2|\mathcal{I}))$ is constant.

Suppose without loss of generality that $i=j$ and $\xi_1=\xi_2=\xi\neq0$. Set $W_i^\xi=U_i\setminus(\langle\xi|v_{U_i}\rangle=0)$. We have

$$\int_\Omega\overline{f_1}\cdot(f_2\circ g_t)\cdot\mathbb{1}_{\langle\xi|v_{U_i}\rangle\neq0}\circ\pi\,d\mu=\int_{\mathbb{R}}f(s)e^{2i\pi ts}d(m_{\xi,U_i})_{|\mathbb{R}^*}(s)\xrightarrow[t\to\pm\infty]{}0$$

because $(m_{\xi,U_i})_{|\mathbb{R}^*}$ is Rajchman and thus converges weakly-$*$ according to Lemma 2.4.

Moreover,

$$\int_\Omega\overline{f_1}\cdot(f_2\circ g_t)\cdot\mathbb{1}_{\langle\xi|v_{U_i}\rangle=0}\circ\pi\,d\mu=\int_\Omega\overline{f_1}\cdot f_2\cdot\mathbb{1}_{\langle\xi|v_{U_i}\rangle=0}\circ\pi\,d\mu.$$

By [**Tho**, Proposition 2.4], we have that

$$f_2\cdot\mathbb{1}_{\langle\xi|v_{U_i}\rangle=0}\circ\pi=\mathbb{E}_\mu(f_2|\mathcal{I}).$$

We obtain by totality of Y that for all $f_1,f_2\in\mathbb{L}^2_\mu(\Omega)$,

$$\mathbb{E}(\mathrm{Cov}_t(f_1,f_2|\mathcal{I}))\xrightarrow[t\to\pm\infty]{}0.$$

□

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

Remark 4.3. We can also note that the result generalizes to infinite dimensional tori bundles $\mathbb{T}^{\mathbb{N}}$ with product topology.

Remark 4.4. When Keplerian shear property holds, the decomposition of Radon–Nikodym–Lebesgue of the image measures $m_{\xi,U}$ in Theorem 4.2 reads as $(m_{\xi,U})_{ac} + (m_{\xi,U})_{sc} + (m_{\xi,U})_d$ with $(m_{\xi,U})_d = \alpha\delta_0$, and $\alpha \geq 0$. Recall that the absolutely continuous part satisfies the Rajchman property. The discrete part is concentrated on 0 because otherwise, a non-trivial periodicity would break the Keplerian shear. The behavior of the singular part ν is not obvious, since it may be Rajchman or not, see §2.2.4.

Remark 4.5. When all the $m_{\xi,U}$ are of the form $(m_{\xi,U})_{ac} + \alpha_\xi\delta_0$, Theorem 4.2 immediately gives Keplerian shear.

4.1. *Invariant functions and σ-algebra.* Under the conditions of Theorem 4.1, in the regular case, the invariant σ-algebra is just $\pi^{-1}(\mathcal{B}(M))$ modulo zero measure set. On the functional aspect, it amounts to say that a measurable function f is invariant by the flow if and only if it is μ-a.s. independent on the second variable on every chart U. However, we will first highlight the invariant functions and then the σ-algebra of invariant sets to compare with the regular case mentioned above. In the following theorem, we will use the Fourier series to identify a characterization of invariance by an orthogonality property.

PROPOSITION 4.6. (Orthogonality and Fourier characterization of invariant functions) *Under the conditions of Theorem 4.2, a function $f \in \mathbb{L}^2_\mu(\Omega)$ is invariant according to $(g_t)_{t\in\mathbb{R}}$ if and only if for all $U \in \mathcal{A}$, there exists $(a_k)_{k\in\mathbb{Z}^d} \in \mathbb{L}^2_{(\mu_\pi)_{|U}}(U)^{\mathbb{Z}^d}$ such that $(\mu_\pi)_{|U} \otimes \lambda$-a.e. $(x, y) \in U \times \mathbb{T}^d$,*

$$f_{|\pi^{-1}(U)}(\psi_U^{-1}(x, y)) = \sum_{k\in\mathbb{Z}^d\cap\{v_U(x)\}^\perp} a_k(x)e^{2i\pi\langle k|y\rangle}.$$

In other words, for a chart $U \in \mathcal{A}$ and for $x \in U$ fixed, non-zero Fourier coefficients have indices orthogonal with the vector $v_U(x)$.

Proof. Let $f \in \mathbb{L}^2_\mu(\Omega)$. Let us start with the direct implication. Suppose that for all $t \in \mathbb{R}$,

$$f \circ g_t = f \quad \mu\text{-almost everywhere.}$$

Let $U \in \mathcal{A}$, $t \in \mathbb{R}$, and $(x, y) \in U \times \mathbb{T}^d$. Since

$$f(g_t(\psi_U^{-1}(x, y))) = f(\psi_U^{-1}(x, y)),$$

we have

$$f(\psi_U^{-1}(x, y + tv_U(x))) = f(\psi_U^{-1}(x, y)).$$

By Fourier series decomposition, this gives

$$\sum_{k\in\mathbb{Z}^d} \widehat{f \circ \psi_U(x, \cdot)}(k)e^{2i\pi\langle k|y+tv_U(x)\rangle} = \sum_{k\in\mathbb{Z}^d} \widehat{f \circ \psi_U(x, \cdot)}(k)e^{2i\pi\langle k|y\rangle}.$$

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

Hence, by uniqueness of Fourier series coefficients, for all $k \in \mathbb{Z}^d$,

$$f \circ \widehat{\psi_U(x, \cdot)}(k)(e^{2i\pi t\langle k|v_U(x)\rangle} - 1) = 0.$$

When k is such that $f \circ \widehat{\psi_U(x, \cdot)}(k) \neq 0$, we get that for all $t \in \mathbb{R}$, $t\langle k|v_U(x)\rangle \in \mathbb{Z}$. So, $\langle k|v_U(x)\rangle = 0$ and then $k \in \{v_U(x)\}^\perp$. Hence,

$$(\mu_\pi)_{|U} \otimes \lambda\text{-a.e. } (x, y) \in U \times \mathbb{T}^d,$$
$$f(\psi_U^{-1}(x, y)) = \sum_{k \in \mathbb{Z}^d \cap \{v_U(x)\}^\perp} f \circ \widehat{\psi_U(x, \cdot)}(k) e^{2i\pi\langle k|y\rangle}.$$

Let us see the reciprocal implication. Let $t \in \mathbb{R}$. Suppose that f satisfies for all charts $U \in \mathcal{A}$,

$$(\mu_\pi)_{|U} \otimes \lambda\text{-a.e. } (x, y) \in U \times \mathbb{T}^d, \quad f(\psi_U^{-1}(x, y)) = \sum_{k \in \mathbb{Z}^d \cap \{v_U(x)\}^\perp} a_k(x) e^{2i\pi\langle k|y\rangle}.$$

Let $z \in \Omega$ and take $U \in \mathcal{A}$ such that $z \in \pi^{-1}(U)$. We have

$$f(g_t(z)) = f(g_t(\psi_U^{-1}(\psi_U(z)))) = f(\psi_U^{-1}(\pi(z), \pi_2 \circ \psi_U(z) + t v_U(\pi(z)))).$$

Thus,

$$\begin{aligned} f(g_t(z)) &= \sum_{k \in \mathbb{Z}^d \cap \{v_U(\pi(z))\}^\perp} a_k(\pi(z)) e^{2i\pi\langle k|\pi_2 \circ \psi_U(z) + t v_U(\pi(z))\rangle} \\ &= \sum_{k \in \mathbb{Z}^d \cap \{v_U(\pi(z))\}^\perp} a_k(\pi(z)) e^{2i\pi\langle k|\pi_2 \circ \psi_U(z)\rangle} \end{aligned}$$

by orthogonality. Hence,

$$f(g_t(z)) = f(\psi_U^{-1}(\psi_U(z))) = f(z). \qquad \square$$

Remark 4.7. As a byproduct, the proposition gives an explicit form to the conditional expectation, with respect to the invariant σ-algebra, of a function $f \in \mathbb{L}^2_\mu(\Omega)$ locally by the next formula for $U \in \mathcal{A}$, $(\mu_\pi)_{|U} \otimes \lambda$-a.e. $(x, y) \in U \times \mathbb{T}^d$,

$$(\mathbb{E}_\mu(f|\mathcal{I}) \circ \psi_U^{-1})(x, y) = \sum_{k \in \{v_U(x)\}^\perp \cap \mathbb{Z}^d} \widehat{(f \circ \psi_U^{-1})(x, \cdot)}(k) e^{2i\pi\langle k|y\rangle}.$$

We obviously have $\pi^{-1}(\mathcal{B}(M)) \subset \mathcal{I}$ the invariant σ-algebra; however, the inclusion is generally strict as shown in the following example.

Example 4.8. Consider $g_t = \mathrm{Id}_{\mathbb{T}^2}$, $\Omega = \mathbb{T}^2$, $M = \mathbb{T}$.

Clearly, $\mathcal{I} = \mathcal{B}(\mathbb{T}^2)$, while $\pi^{-1}(\mathcal{B}(\mathbb{T})) = \{A \times \mathbb{T} \in \mathcal{B}(\mathbb{T}^2) : A \in \mathcal{B}(\mathbb{T})\} \subsetneq \mathcal{I}$.

PROPOSITION 4.9. *The invariant σ-algebra is*

$$\mathcal{I} = \left\{ \bigcup_{U \in \mathcal{A}} B_U \in \mathcal{B}(\Omega) : (B_U)_{U \in \mathcal{A}} \in \prod_{U \in \mathcal{A}} \psi_U^{-1}(\mathcal{I}_U) \right\},$$

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

where for $U \in \mathcal{A}$, the local invariant σ-algebra on U is

$$\mathcal{I}_U := \{C \in \mathcal{B}(U \times \mathbb{T}^d) : \mu_{\pi_{|U}}\text{-a.e. } x \in U, \widehat{\mathbb{1}_{C(x,\cdot)}}^{-1}(\mathbb{C}^*) \subset \{v_U(x)\}^\perp\}. \tag{4.2}$$

Proof. We prove the direct inclusion. Let $A \in \mathcal{I}$. Let $U \in \mathcal{A}$ and set $C = \psi_U(A \cap \pi^{-1}(U))$. By Proposition 4.6,

$$\mathbb{1}_C(x, y) = \sum_{k \in \{v_U(x)\}^\perp \cap \mathbb{Z}^d} a_k(x) e^{2i\pi\langle k|y\rangle} \ (\mu_\pi)_{|U} \otimes \lambda\text{-a.s.},$$

which shows that $C \in \mathcal{I}_U$. Hence, $A \cap \pi^{-1}(U) \in \psi_U^{-1}(\mathcal{I}_U)$ and, finally,

$$A = \bigcup_{U \in \mathcal{A}} A \cap \pi^{-1}(U).$$

Now, we prove the reciprocal inclusion.

Let $A \in \mathcal{B}(\Omega)$ such that there exists $(B_U)_{U\in\mathcal{A}} \in \prod_{U\in\mathcal{A}} \psi_U^{-1}(\mathcal{I}_U)$, $A = \bigcup_{U\in\mathcal{A}} B_U$. Then, for all $t \in \mathbb{R}$,

$$g_t^{-1}(A) = \bigcup_{U \in \mathcal{A}} g_t^{-1}(B_U).$$

Let $U \in \mathcal{A}$. By bijectivity of ψ_U, $\psi_U(B_U) \in \mathcal{I}_U$. Hence,

$$(\mu_\pi)_{|U} \otimes \lambda\text{-a.e. } (x, y) \in U \times \mathbb{T}^d,$$
$$\mathbb{1}_{\psi_U(B_U)}(x, y) = \sum_{k \in \{v_U(x)\}^\perp \cap \mathbb{Z}^d} \widehat{\mathbb{1}_{\psi_U(B_U)}(x, \cdot)}(k) e^{2i\pi\langle k|y\rangle},$$

which proves that $A \in \mathcal{I}$ by Proposition 4.6. □

Remark 4.10. We can see that local invariant sets $A \in \mathcal{I}_U$ satisfy

$$\mathbb{1}_A(x, y) = \sum_{k \in \mathbb{Z}^d \cap \{v_u(x)\}^\perp} \widehat{\mathbb{1}_A(x, \cdot)}(k) e^{2i\pi\langle \xi|y\rangle} (\mu_\pi)_{|U} \otimes \lambda\text{-a.s.} \tag{4.3}$$

4.2. *Convergence speed with the real Rajchman property.* We are now interested in the speed of convergence of the conditional correlations. The next result shows that even in C^∞ regularity, the order of convergence is limited by the Rajchman order.

PROPOSITION 4.11. (Speed roof) *Let $0 \neq \xi \in \mathbb{Z}^d$ and $U \in \mathcal{A}$. For all $\gamma > r((m_{\xi,U})_{|\mathbb{R}^*})$, there exists $(f_1, f_2) \in (\mathcal{C}^\infty(\pi^{-1}(U)) \cap \mathbb{L}^2_\mu(\Omega))^2$ such that the decay of conditional correlations is not faster than $t^{-\gamma}$, that is, there exists $C > 0$ such that*

$$\limsup_{t\to+\infty} |\mathbb{E}_\mu(\mathrm{Cov}_t(f_1 \cdot \mathbb{1}_U \circ \pi, f_2|\mathcal{I}))| t^\gamma > C.$$

Proof. Let $U \in \mathcal{A}$, $\xi \in \mathbb{Z}^d \setminus \{0\}$. Let $\gamma > r((m_{\xi,U})_{|\mathbb{R}^*})$. Consider

$$f : z \in \Omega \mapsto e^{2i\pi\langle \xi|(\pi_2 \circ \psi_U^{-1})(z)\rangle} \mathbb{1}_{\pi^{-1}(U)}(z) \in \mathcal{C}^\infty(\pi^{-1}(U)) \cap \mathbb{L}^2_\mu(\Omega).$$

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

We get

$$\int_\Omega \overline{f}(f\circ g_t)\,d\mu = \int_U e^{2i\pi t\langle \xi | v_U(x)\rangle}\,d(\mu_\pi)(x) = \int_{\mathbb{R}} e^{2i\pi tz}\,dm_{\xi,U}(z).$$

By optimality of $r((m_{\xi,U})_{|\mathbb{R}^*})$, for all $C>0$, for all $T>0$, there exists $t>T$ such that

$$\left|\int_{\mathbb{R}} e^{2i\pi tz}\,d(m_{\xi,U})_{|\mathbb{R}^*}\right| > \frac{C}{t^\gamma}.$$

The result follows by definition of conditional correlations. □

Remark 4.12. When $\Omega \simeq M\times\mathbb{T}^d$, we can take f_1, f_2 globally defined of class $\mathcal{C}^\infty$ in the Proposition 4.11.

4.3. *Speed of decay of conditional correlations for absolutely continuous measures.* We assume that $\Omega = M\times\mathbb{T}^d$ is the trivial bundle, endowed with an absolutely continuous measure. The speed of decay will depend on the regularity properties of the velocity vector v. We will use the stationary phase method [**Zwo**] that allows us to evaluate in an optimal way oscillating integrals. The regularity of the velocity vector and the presence of critical points influences the convergence order.

Before this study, we recall that by an immediate consequence of the Morse lemma, the set of critical points of the velocity vector is discrete.

LEMMA 4.13. (Isolation of non-degenerate critical points) *Let M be a finite dimensional $\mathcal{C}^2$ manifold and $v\in\mathcal{C}^2(M,\mathbb{R})$. Then, every non-degenerate critical point of v is isolated.*

THEOREM 4.14. (Stationary phase, e.g. [**Zwo**]) *Let $\varphi\in\mathcal{C}^2(\mathbb{R}^n,\mathbb{R})$ with a unique critical point x_c. We suppose that x_c is non-degenerate, in other words,* $\det(\mathrm{Hess}(x_c))\neq 0$.

Therefore, for any $a\in\mathcal{C}^1_0(\mathbb{R}^n,\mathbb{R})$, for all $t>0$,

$$\int_{\mathbb{R}^n} e^{2i\pi t\varphi(x)}a(x)\,d\lambda(x) = \frac{a(x_c)e^{i(\pi/4)(\sum_{\lambda\in\sigma(\mathrm{Hess}(x_c))}\mathrm{sgn}(\lambda))}}{t^{n/2}\sqrt{|\det(\mathrm{Hess}(x_c))|}} + O_{+\infty}\left(\frac{1}{t^n}\right).$$

In the following, we say that a is a critical point of order q of a function f if for $1\le m\le q$, $f^{(m)}(a)=0$ and $f^{(q+1)}(a)\neq 0$.

We recall that for a probability measure $\mathbb{P}$, we note $\widehat{\mathbb{P}}$ its characteristic function.

LEMMA 4.15. (Convergence order with singular points in dimension $(1,d)$ with $d\ge 2$) *Suppose that* $\dim(M)=1\le d$ *and M is a compact manifold of class $\mathcal{C}^\infty$.*

Let $\xi\neq 0_{\mathbb{Z}^d}$ and $\ell\ge 2$.

We suppose that v is of class $\mathcal{C}^\ell$, that there exists a unique critical point of order $\ell-1$ for the function $\langle\xi|v(\cdot)\rangle$, and that all the other critical points are of strictly smaller order.

We have

$$r(\lambda_{\langle\xi|v(\cdot)\rangle}) = \frac{1}{\ell}.$$

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

Proof. Here, M is compact Hausdorff, so the number of critical points for each function $\langle \xi | v \rangle$ is finite. Let $(a_k)_{k\in[\![1,m]\!]}$ be the family of critical point of $\langle \xi | v(\cdot)\rangle$ with respective orders $l_k - 1 \geq 1$, $1 \leq m < l_k$,

$$\langle \xi | v^{(m)}(a_k)\rangle = 0 \quad \text{and} \quad \langle \xi | v^{(l_k)}(a_k)\rangle \neq 0.$$

Let $(U_k)_{k\in[\![1,m]\!]}$ be charts such that for all $k \in [\![1, m]\!]$, $a_k \in U_k$. We have

$$\begin{aligned}\langle \xi | v(\varphi_{U_k}^{-1}(x))\rangle = \langle \xi | v(a_k)\rangle + \frac{(x-\varphi_{U_k}(a_k))^{l_k}}{l_k!}\langle \xi | v^{(l_k)}(a_k)\rangle \\ + (x-\varphi_{U_k}(a_k))^{l_k} h(x-\varphi_{U_k}(a_k))\end{aligned}$$

with $h : \mathbb{R} \to \mathbb{R}$ $\mathcal{C}^1$ such that $h(x) \xrightarrow[x\to 0]{} h(0) = 0$. Let us pose for $k \in [\![1, m]\!]$,

$$w_k : x \in \mathbb{R} \mapsto (x-\varphi_{U_k}(a_k))\left(\frac{\langle \xi | v^{(l_k)}(a_k)\rangle}{l_k!} + h(x-\varphi_{U_k}(a_k))\right)^{1/l_k}.$$

Therefore,

$$w_k' : x \in \mathbb{R} \mapsto \left(\frac{\langle \xi | v^{(l_k)}(a_k)\rangle}{l_k!} + h(x-\varphi_{U_k}(a_k))\right)^{1/l_k} + (x-\varphi_{U_k}(a_k))\kappa(x-\varphi_{U_k}(a_k)) \tag{4.4}$$

with $\kappa_k(x) = h'(x)/l_k(\langle \xi | v^{(l)}(a_k)\rangle + h(x))^{(l_k-1)/l_k}$. Applying (4.4) to $x = \varphi_{U_k}(a_k)$,

$$w_k'(\varphi_{U_k}(a_k)) = \left(\frac{\langle \xi | v^{(l_k)}(a)\rangle}{l_k!}\right)^{1/l_k} \neq 0.$$

We get a local inverse of w_k on a neighborhood W_k of 0. Let $(V_k)_{k\in[\![1,m+m']\!]}$ be a family of charts on M such that, for all $k \in [\![1, m]\!]$,

$$(V_k = w_k^{-1}(W_k) \subset U_k \text{ and for all } j \in [\![m+1, m']\!], a_k \notin V_k). \tag{4.5}$$

Let $(\psi_k)_{k\in[\![1,m+m']\!]}$ be a partition of unity subordinated to $(V_k)_{k\in[\![1,m+m']\!]}$. We get immediately that for $k \in [\![1, m]\!]$, $\psi_k(a_k) = 1$. By inverse function theorem, for $k \in [\![1, m]\!]$, we get that on W_k, for all $x \in W_k$,

$$\langle \xi | v(\varphi_{U_k}^{-1}(w_k^{-1}(x)))\rangle = \langle \xi | v^{(l_k)}(a_k)\rangle + x^{l_k}. \tag{4.6}$$

Writing $\check{\lambda}$ as the push forward measure by charts of Lebesgue measure on the manifold M, for $k \in [\![1, m]\!]$, we get

$$\begin{aligned}\int_{\varphi_{U_k}^{-1}(w^{-1}(W_k))} e^{2i\pi t\langle \xi | v(x)\rangle}\psi_k(x)\, d\check{\lambda}(x) \\ = e^{2i\pi t\langle \xi | v(\varphi_{W_k}^{-1}(w^{-1}(a_k)))\rangle}\int_{W_k} e^{2i\pi t x^{l_k}}\check{\psi_k}(x)\, d\lambda(x)\end{aligned} \tag{4.7}$$

with

$$\check{\psi_k} : x \in W_k \mapsto \psi(\varphi_{U_k}^{-1}(w_k^{-1}(x)))J(\varphi_{U_k}^{-1})(w_k^{-1}(x))J(w_k^{-1})(x)$$

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

and $J(f)(x)$ denotes the Jacobian of the function f at the point x. We know that $\check{\psi}_k$ is subordinated to $(V_j)_{j\in[\![1,m+m']\!]}$. Moreover, by (4.5), the support of $\check{\psi}_k$ is included in U_k. Thus, by (4.7), for all $t \neq 0$,

$$\int_{\varphi_{U_k}^{-1}(w_k^{-1}(W_k))} e^{2i\pi t\langle \xi | v(x)\rangle}\psi(x)\, d\check{\lambda}(x)$$
$$= e^{2i\pi t\langle \xi | v(\varphi_{U_k}^{-1}(w_k^{-1}(a_k)))\rangle}\frac{1}{t^{1/l_k}}\int_{\mathbb{R}} e^{2i\pi x^{l_k}}\check{\psi}_k\left(\frac{x}{t^{1/l_k}}\right) d\lambda(x).$$

Integrating by parts $u(x)v'(x)$ with $u(x) = (\check{\psi}_k(x/t^{1/l_k})/(2il_k\pi x^{l_k-1}))$ and $v(x) = e^{2i\pi x^{l_k}} - 1$,

$$\int_{\mathbb{R}} e^{2i\pi x^{l_k}}\check{\psi}_k\left(\frac{x}{t^{1/l_k}}\right) d\lambda(x)$$
$$= \frac{1}{2i\pi l_k}\left(\int_{\mathbb{R}} \frac{(l_k-1)(e^{2i\pi x^{l_k}}-1)}{x^{l_k}}\check{\psi}_k\left(\frac{x}{t^{1/l_k}}\right) d\lambda(x)\right.$$
$$\left. - \frac{1}{t^{1/l_k}}\int_{\mathbb{R}} \frac{(e^{2i\pi x^{l_k}}-1)}{x^{l_k-1}}\check{\psi}_k{}'\left(\frac{x}{t^{1/l_k}}\right) d\lambda(x)\right).$$

We know that for all $\alpha > 0$,

$$x \in \mathbb{R} \mapsto \check{\psi}_k{}'(x)(1+|x|^\alpha),\ x \in \mathbb{R} \mapsto \check{\psi}_k(x)(1+|x|^\alpha) \text{ are bounded by a constant } C > 0. \tag{4.8}$$

Using the development at $x = 0$ of $x \in \mathbb{R} \mapsto ((e^{2i\pi x^{l_k}} - 1)/x^{l_k-1})$, we get that

$$x \in \mathbb{R} \mapsto \frac{(e^{2i\pi x^{l_k}}-1)}{x^{l_k-1}}(1+|x|^{l_k-1}) \quad \text{is bounded by a constant } C_2 > 0. \tag{4.9}$$

For all $t \neq 0$ such that $|t| > 1$, by (4.8), for all $\alpha > 0$,

$$x \in \mathbb{R} \mapsto \check{\psi}_k{}'\left(\frac{x}{t^{1/l_k}}\right)\frac{(1+|x|^\alpha)}{t^{1/l_k}} \quad \text{is bounded by a constant } C_3 > 0. \tag{4.10}$$

Thus, by (4.8), there exists $C_4 > 0$ such that for all $x \in \mathbb{R}$, for all $t \neq 0$,

$$\left|\frac{(l_k-1)(e^{2i\pi x^{l_k}}-1)}{x^{l_k}}\check{\psi}_k\left(\frac{x}{t^{1/l_k}}\right)\right|(1+|x|^{l_k}) \leq C_4\|\psi\|_\infty. \tag{4.11}$$

By Lebesgue convergence theorem,(4.9), (4.10), and (4.11),

$$\int_{\mathbb{R}} e^{2i\pi x^{l_k}}\check{\psi}_k\left(\frac{x}{t^{1/l_k}}\right) d\lambda(x) \xrightarrow[t\to\pm\infty]{} \frac{\check{\psi}_k(0)}{2i\pi l_k}\int_{\mathbb{R}} \frac{(l_k-1)(e^{2i\pi x^{l_k}}-1)}{x^{l_k}}\, d\lambda(x).$$

For $k \in [\![m+1, m+m']\!]$, by Green's theorem, we get

$$\int_{\varphi_{U_k}^{-1}(V_k)} e^{2i\pi t\langle \xi | v(x)\rangle}\psi_k(x)\, d\check{\lambda}(x) \in O_{\pm\infty}\left(\frac{1}{t}\right).$$

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

Thus,

$$\int_M e^{2i\pi t\langle \xi | v(x)\rangle}\, d\check{\lambda}(x) = \sum_{k=1}^{m} \frac{1}{t^{1/l_k}} \left(J_{\varphi_{U_k}^{-1}}(\varphi_{U_k}(a_k)) \left(\frac{l_k!}{\langle \xi | v^{(l_k)}(a_k)\rangle} \right)^{1/l_k} I_{l_k} + o_{\pm\infty}(1) \right) + O_{\pm\infty}\left(\frac{1}{t}\right)$$

with $I_l := (1/(2i\pi l)) \int_{\mathbb{R}} ((e^{2i\pi x^l} - 1)/x^l)\, d\lambda(x)$. A simple analysis shows that I_l does not vanish. (For l even, $I_l \in (-\infty, 0)$ trivially, while for l odd, we have $I_l \in i(0, \infty)$, using a decomposition of the integral on intervals $[2n, 2n + 2]$.)

By hypothesis, there exists just one critical a point with maximal order ℓ; hence,

$$\int_M e^{2i\pi t\langle \xi | v(x)\rangle}\, d\check{\lambda}(x) = \frac{1}{t^{1/\ell}} \left(J_{\varphi_{U_j}^{-1}}(\varphi_{U_j}(a)) \left(\frac{\ell!}{\langle \xi | v^{(\ell)}(a)\rangle} \right)^{1/\ell} I_\ell + o_{\pm\infty}(1) \right).$$

Thus,

$$r(\lambda_{\langle \xi | v(\cdot)\rangle}) = \frac{1}{\ell}. \qquad \square$$

We then get the following proposition.

PROPOSITION 4.16. *Under the hypothesis of Lemma 4.15, we get that the order of decay of correlation γ with f_1, $f_2 \in \mathcal{C}^\infty(\Omega)$ satisfies*

$$\gamma \leq \frac{1}{\ell}.$$

Proof. In the proof of Lemma 4.15, you can take $f_1 = e^{2i\pi\langle \xi | v(\cdot)\rangle}$ and $f_2 = \psi_j$ with j the index of the chart containing the critical point a with a maximal order ℓ. $\square$

We can now treat the general case.

The following lemma is obtained with [**AGZV**, Theorem 7.5, p. 226].

LEMMA 4.17. (Convergence order around a singular point with analytic functions) *We will use the notation and results of* [**AGZV**].

We place ourselves in $\mathbb{R}^n$.

Let $\xi \neq 0_{\mathbb{Z}^d}$.

Let v be such that $\langle \xi | v\rangle$ is analytic with 0 a critical point of multiplicity $m \in \mathbb{N}^*$.

Then, there exist $\alpha \in \mathbb{Q}_+^*$ and $j \in \mathbb{N}$ in a compact neighborhood V of 0 such that for all $\varphi \in \mathcal{D}_V(\Omega)$,

$$\text{there exists } C \in \mathbb{R}^*, \quad \int_{\mathbb{R}^n} e^{2i\pi t\langle \xi | v(x)\rangle} \varphi(x)\, d\lambda(x) \sim \frac{C(\ln(t))^j}{t^\alpha}.$$

Definition 4.1. (Logarithmic convergence order) Consider a probability measure ν on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

We call the logarithmic convergence order of v the value

$$rl(v) := \inf\left\{\beta > 0 : \text{ there exists } C > 0 \text{ for all } t > 0, |\widehat{\mu(t)}| \leq \frac{C|\ln(t)|^\beta}{t^{r(v)}}\right\}.$$

The next result may be obtained by a partition of unity and the previous lemma.

PROPOSITION 4.18. (Convergence order with singular points in dimension (n, d) with $d \geq 2$) *Suppose that M is analytic. Let $\xi \neq 0_{\mathbb{Z}^d}$. Suppose that v is such that $\langle \xi | v\rangle$ is analytic. We have that*

$$r(\lambda_{\langle \xi | v(\cdot)\rangle}) \in \mathbb{Q}_+^* \quad \text{and} \quad rl(\lambda_{\langle \xi | v(\cdot)\rangle}) \in \mathbb{N}.$$

COROLLARY 4.19. *Let v be a $\mathcal{C}^2$ function. If there exists only one critical point $a \in M$ and it satisfies*

$$\text{there exists } \xi \neq 0_{\mathbb{Z}^d}, \quad (\nabla(\langle \xi | v(\cdot)\rangle))(a) = 0 \text{ and } \det(\mathrm{Hess}\langle \xi | v(\cdot)\rangle(a)) \neq 0),$$

then $r(\lambda_{\langle \xi | v(\cdot)\rangle}) = n/2$.

Proof. Let $\xi \neq 0_{\mathbb{Z}^d}$. Applying the stationary phase theorem on respective charts (U, φ_U) of $\mathcal{A}$ containing at most just one critical point $a \in U$ and test functions ψ on U, we get

$$\begin{aligned}\int_M e^{2i\pi t\langle \xi | v(x)\rangle}\psi(x)\, d\lambda(x) &= \int_{\mathbb{R}^n} e^{2i\pi t\langle \xi | v(\varphi_U^{-1}(x))\rangle}\psi(\varphi_U^{-1}(x))J(\varphi_U^{-1})(x)\, d\lambda(x)\\ &= \frac{\psi(\varphi_U^{-1}(a))e^{i(\pi/4)(\sum_{\lambda\in\sigma(\mathrm{Hess}(a))}\mathrm{sgn}(\lambda))}}{t^{n/2}\sqrt{|\det(\mathrm{Hess}(a))|}} + O_{+\infty}\left(\frac{1}{t^n}\right).\end{aligned}$$

By partition of unity, we obtain

$$\text{there exists } C \in \mathbb{R}, \quad \int_M e^{2i\pi\langle \xi | v(x)\rangle}\, d\lambda(x) = \frac{C}{t^{n/2}} + O_{+\infty}\left(\frac{1}{t^n}\right). \qquad \square$$

Remark 4.20. Note that the result is consistent with the case $n = 1$.

We recall the definition of Thomine [**Tho**].

Definition 4.2. (Anisotropic Sobolev space on $\mathbb{R}\times\mathbb{T}$) Let $s \geq 0$. Let $h : x \in \mathbb{R} \mapsto \sqrt{1+x^2}$. Let

$$H^{s,0}(\mathbb{R}\times\mathbb{T}) := \left\{f \in \mathbb{L}^2(\mathbb{R}\times\mathbb{T}) : \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} |\widehat{f}(x,k)|^2 h^{2s}(x)\, d\lambda(x) \in \mathbb{R}\right\}.$$

PROPOSITION 4.21. *Consider $(\mathbb{R}\times\mathbb{T}, \mu\otimes\lambda, (g_t)_{t\in\mathbb{R}})$ such that $r = r(\mu) > 0$ and for $(x, y) \in \mathbb{R}\times\mathbb{T}$, $g_t(x, y) = (x, y + tx)$. Let $s > \frac{1}{2}$. For $\varepsilon \in\ ]0, 1/2s[$, let $q_\varepsilon := \min\{s(1-\varepsilon), r(\mu)\}$. Then, we get for all $\varepsilon \in\ ]0, 1/2s[$, there exists $C_{s,\varepsilon} > 0$ such that for all $t > 0$,*

$$|\mathbb{E}_{\mu\otimes\lambda}(\mathrm{Cov}_t(f_1, f_2|\mathcal{I}))| \leq \frac{C_{s,\varepsilon}\|f_1\|_{H^{s,0}(\mathbb{R}\times\mathbb{T})}\|f_2\|_{H^{s,0}(\mathbb{R}\times\mathbb{T})}}{t^{q_\varepsilon}}$$

and if $\gamma > 0$ denotes the convergence order on $H^{s,0}(\mathbb{R}\times\mathbb{T})$, we get $\min\{s - \frac{1}{2}, r(\mu)\} \leq \gamma$. Moreover, if $\mathrm{supp}(\mu)$ is compact, then $\gamma \leq r(\mu)$.

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

Proof. Let $s > \frac{1}{2}$ and $(f_1, f_2) \in (H^{s,0}(\mathbb{R} \times \mathbb{T}))^2$. Then, by Cauchy–Schwarz inequality and Parseval,

$$|\mathbb{E}_{\mu\otimes\lambda}(\mathrm{Cov}_t(f_1, f_2|\mathcal{I}))| = \left|\sum_{k\in\mathbb{Z}^*} \int_{\mathbb{R}^2} \overline{\widehat{f_1}}(u,k)\widehat{f_2}(v,k)\widehat{\mu}(kt-(u-v))\, d\lambda(u,v)\right|$$
$$\leq S(t)\|f_1\|_{H^{s,0}(\mathbb{R}\times\mathbb{T})}\|f_2\|_{H^{s,0}(\mathbb{R}\times\mathbb{T})} \tag{4.12}$$

with

$$S(t) := \sup_{k\in\mathbb{Z}^*} C_\mu \left(\int_{\mathbb{R}^2} \frac{1}{(1+|kt-(u-v)|)^{2r(\mu)}h^{2s}(u)h^{2s}(v)}\, d\lambda(u,v)\right)^{1/2}.$$

We can consider $k = 1$ considering kt instead of t. Let $D_1(t) := \{(u,v) \in \mathbb{R}^2 : |t-(u-v)| \leq 2t \text{ and } |v| \leq (t/4)\}$, $D_2(t) := \{(u,v) \in \mathbb{R}^2 : |t-(u-v)| \leq 2t \text{ and } |v| > (t/4)\}$, and $D_3(t) := \{(u,v) \in \mathbb{R}^2 : |t-(u-v)| > 2t\}$.

We get that for $(u,v) \in D_1(t)$,

$$|u| \geq \frac{3t}{4} \quad \text{and} \quad h^{2s}(u) \geq \frac{16^s}{9^s t^{2s}} \tag{4.13}$$

and for $(u,v) \in D_2(t)$,

$$h^{2s}(v) > \frac{16^s}{t^{2s}}. \tag{4.14}$$

Using that $s > \frac{1}{2}$, for all $\varepsilon \in]0, 1/2s[$,

$$\int_{\mathbb{R}} h^{-2s\varepsilon}(u)\, d\lambda(u) < +\infty \tag{4.15}$$

and by (4.13) and (4.14), and for

$$\int_{D_1(t)} \frac{1}{(1+|t-(u-v)|)^{2r(\mu)}h^{2s}(u)h^{2s}(v)}\, d\lambda(u,v)$$
$$\leq \frac{16^{s(1-\varepsilon)}}{(3t)^{2s(1-\varepsilon)}} \int_{\mathbb{R}^2} h^{-2s\varepsilon}(u)h^{-2s}(v)\, d\lambda(u,v), \tag{4.16}$$
$$\int_{D_2(t)} \frac{1}{(1+|t-(u-v)|)^{2r(\mu)}h^{2s}(u)h^{2s}(v)}\, d\lambda(u,v)$$
$$\leq \frac{16^{s(1-\varepsilon)}}{t^{2s(1-\varepsilon)}} \int_{\mathbb{R}^2} h^{-2s}(u)h^{-2s\varepsilon}(v)\, d\lambda(u,v). \tag{4.17}$$

Therefore, by definition of $D_3(t)$, (4.16), and (4.17),

$$\int_{\mathbb{R}^2} \frac{1}{(1+|t-(u-v)|)^{2r(\mu)}h^{2s}(u)h^{2s}(v)}\, d\lambda(u,v) \leq \frac{D_{s,\varepsilon}^2}{t^{2q_\varepsilon}} \int_{\mathbb{R}^2} h^{-2s}(u)h^{-2s\varepsilon}(v)\, d\lambda(u,v) \tag{4.18}$$

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

with $D_{s,\varepsilon} = \max\{1/(2^{r(\mu)}), 4^{s(1-\varepsilon)}\sqrt{2}\}$ and $q_\varepsilon = \min\{r(\mu), s(1-\varepsilon)\}$. Thus,

$$\text{for all } \varepsilon \in \left]0, \frac{1}{2s}\right[, \quad S(t) \le \frac{C_\mu C_{s,\varepsilon}}{t^{q_\varepsilon}} \left(\int_{\mathbb{R}} h^{-2s}(u)\, d\lambda(u) \int_{\mathbb{R}} h^{-2s\varepsilon}(v)\, d\lambda(v) \right)^{1/2}. \tag{4.19}$$

By (4.12), (4.15), and (4.19),

$$|\mathbb{E}_{\mu\otimes\lambda}(\mathrm{Cov}_t(f_1, f_2|\mathcal{I}))| \le \frac{C_{s,\varepsilon}}{t^{q_\varepsilon}} \|f_1\|_{H^{s,0}(\mathbb{R}\times\mathbb{T})} \|f_2\|_{H^{s,0}(\mathbb{R}\times\mathbb{T})}.$$

Moreover, for $\gamma > 0$, the convergence order on $H^{s,0}(\mathbb{R}\times\mathbb{T})$, we get $\min\{s - \frac{1}{2}, r(\mu)\} \le \gamma$ and when $\mathrm{supp}(\mu)$ is compact, $\gamma \le r(\mu)$. □

Remark 4.22. When $(f_1, f_2) \in (H^{s,0}(\mathbb{R}\times\mathbb{T}))^2$ with $s \ge ((r(\mu)+1)/2)$, we get that $\gamma = r(\mu)$ as the optimal order of convergence when $\mathrm{supp}(\mu)$ is compact.

Remark 4.23. For $s > \frac{1}{2}$, for all integer $k > s + \frac{3}{2}$, $\mathcal{C}^k_c(\mathbb{R}\times\mathbb{T}) \subset H^{s,0}(\mathbb{R}\times\mathbb{T})$, and then, for $k > s + \frac{3}{2}$, the convergence order $\gamma > 0$ for $\mathcal{C}^k_c(\mathbb{R}\times\mathbb{T})$ is such that $\gamma \in [\min\{s - \frac{1}{2}, r(\mu)\}, r(\mu)]$ when $\mathrm{supp}(\mu)$ is compact.

Remark 4.24. When $\mu = \lambda_v$ with $v \in \mathcal{C}^\infty$ with a unique critical point of order $l - 1$ with $l \ge 3$, then $\gamma = r(\mu) = 1/l$.

5. *Keplerian shear, Rajchman property, and Diophantine approximation*

In this section, we present an application of Keplerian shear (with speed estimates) to dynamical Borel–Cantelli lemmas. The latter is linked to Diophantine approximation and we discuss the relations between the Rajchman property and Diophantine properties.

Let (Ω, μ, T) be a measure preserving dynamical system. For a sequence of measurable sets (A_n), the aim of the dynamical Borel–Cantelli lemma is to assert that the events $T^n(x) \in A_n$ appear infinitely often almost surely, if and only if $\sum \mu(A_n) = +\infty$. The independence condition of the classical (non-dynamical) case is replaced by some decorrelation condition (see e.g. [**CK2**]).

Our main application consists here in readjusting the dynamical Borel–Cantelli lemma in the non-ergodic case. We provide an adaptation of Sprindzuk's theorem inspired by the thesis of Xing [**Xin**]. The Keplerian shear will ensure that for almost any x, there exists an infinity of integers n satisfying $T^n(x) \in A_n$, where the measurable set A_n is a shrinking target. The dynamical Borel–Cantelli lemma allows us to study the behavior of the sum $\sum_{k=0}^N \mathbb{1}_{A_k} \circ T^k$ using the Keplerian shear.

THEOREM 5.1. (Variable Sprindzuk) *Let $(f_k)_{k\in\mathbb{N}^*}$ and $(g_k)_{k\in\mathbb{N}^*}$ be two sequences of measurable and non-negative functions. Let $(\varphi_k)_{k\in\mathbb{N}^*}$ be a real sequence such that for all $k \in \mathbb{N}^*$, $0 \le g_k \le \varphi_k \le 1$ μ-almost everywhere. Let $\delta > 1$ and $C > 0$. Suppose that for all $(m, n) \in (\mathbb{N}^*)^2$ satisfying $n \ge m$,*

$$\int_\Omega \left(\sum_{k=m}^n (f_k(x) - g_k(x)) \right)^2 d\mu(x) \le C \left(\sum_{k=m}^n \varphi_k \right)^\delta.$$

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

Then, for all $n \in \mathbb{N}$, for all $\varepsilon > 0$,

$$\sum_{k=1}^{n} f_k = \sum_{k=1}^{n} g_k + O(\phi(n)^{\delta/2}(\log(\phi(n)))^{1+\varepsilon}) \quad \mu\text{-almost everywhere},$$

with $\phi(n) = \sum_{k=1}^{n} \varphi_k$.

For the proof, we will elaborate on the proof of the original theorem in the book by Sprindzuk [**Spr**, (68), p. 45].

Proof. Let $\delta > 1$ and let us denote for $I \subset \mathbb{N}^*$, $\phi(I) = \sum_{k\in I} \varphi_k$. By the fact that for all $k \in \mathbb{N}^*$, $0 \leq \varphi_k \leq 1$, we have that there exists $(n_v)_{v\in\mathbb{N}^*} \in \mathbb{N}^{*\mathbb{N}^*}$ such that for all $v \in \mathbb{N}^*$,

$$\phi(n_v) < v \leq \phi(n_v + 1).$$

We have also for all $v \in \mathbb{N}^*$,

$$n_{v+1} \geq n_v + 1$$

and for all $v \in \mathbb{N}^*$,

$$\phi(n_v) + 1 < v + 1 \leq \phi(n_{v+1} + 1).$$

Thus, for all $u, v \in \mathbb{N}^*$,

$$(u < v \Longrightarrow [\![n_u + 1, n_v]\!] \neq \emptyset).$$

Let

$$\begin{aligned} \sigma : \mathcal{P}(\mathbb{N}^*) &\to \mathcal{P}(\mathbb{N}^*) \\ I &\mapsto \{n_w \in \mathbb{N} : w \in I\}. \end{aligned}$$

Let for $r \in \mathbb{N}^*$, $s \in [\![0, r]\!]$ sets of parts,

$$J_{r,s} := \{[\![i2^s + 1, (i + 1)2^s]\!] \in \mathcal{P}(\mathbb{N}^*) : i \in [\![0, 2^{r-s} - 1]\!]\}.$$

Let $r \in \mathbb{N}^*$ and $s \in [\![0, r]\!]$. We notice that

$$\bigcup_{I\in J_{r,s}} \sigma(I) = [\![1, n_{2^r}]\!].$$

Let $i \in [\![0, 2^{r-s} - 1]\!]$. We have that

$$\phi(n_{i2^s}) < i2^s \leq \phi(n_{i2^s} + 1) \leq \phi(n_{i2^s}) + 1.$$

Therefore,

$$\begin{gathered} i2^s - 1 \leq \phi(n_{i2^s}) < i2^s, \\ \phi(n_{(i+1)2^s}) < (i + 1)2^s \leq \phi(n_{(i+1)2^s} + 1) \leq \phi(n_{(i+1)2^s}) + 1, \end{gathered}$$

and

$$(i + 1)2^s - 1 \leq \phi(n_{(i+1)2^s}) < (i + 1)2^s.$$

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

Thus,

$$\phi(\sigma([\![i2^s+1,(i+1)2^s]\!])) = \phi(n_{(i+1)2^s}) - \phi(n_{i2^s}) \le 2^s + 1 \le 2^{s+1}.$$

Therefore,

$$\sum_{I\in J_{r,s}} (\phi(\sigma(I)))^\delta \le 2^\delta 2^{r+s(\delta-1)}.$$

Denote

$$J_r := \bigcup_{s\in[\![0,r]\!]} J_{r,s}.$$

Thus, we have

$$\sum_{I\in J_r} (\phi(\sigma(I)))^\delta = \sum_{s=0}^{r} \sum_{I\in J_{r,s}} (\phi(\sigma(I)))^\delta \le \frac{2^{2\delta-1}2^{r\delta}}{2^{\delta-1}-1}.$$

Let

$$\begin{aligned} h : \mathbb{N}^* \times \Omega &\to \mathbb{R} \\ (l,x) &\mapsto \sum_{I\in J_l} \Big(\sum_{k\in I} (f_k(x) - g_k(x))\Big)^2. \end{aligned}$$

Therefore,

$$\int_\Omega h(r,x)\, d\mu(x) \le C\left(\sum_{I\in J_r} \phi(\sigma(I))\right) \le C_1 2^{r\delta}$$

for some constant C_1. Let $\varepsilon > 0$. By the Markov inequality,

$$\mu(h(r,X) \ge C_1 r^{1+\varepsilon} 2^{r\delta}) \le r^{-(1+\varepsilon)}.$$

By the Borel–Cantelli lemma,

$$\mu\text{-a.e. } x \in \Omega, \text{ there exists } r_x \in \mathbb{N}^* \text{ for all } r \ge r_x,\ h(r,x) \le C_1 r^{1+\varepsilon} 2^{r\delta}.$$

For $v \in \mathbb{N}^*$, we have that $[\![1,v]\!]$ can be aligned into a finite number r_v of intervals J_r such that

$$r_v \le \lfloor \log_2(v) \rfloor + 1.$$

Let $J(v)$ denote the set of these intervals. We then get

$$\sum_{k=1}^{n_v} (f_k - g_k) = \sum_{I\in J(v)} \left(\sum_{k\in I} (f_k - g_k)\right).$$

By the Cauchy–Schwarz inequality,

$$\mu\text{-a.e. } x \in \Omega, \left(\sum_{k=1}^{n_v} (f_k - g_k)\right)^2 \le r_v \sum_{I\in J(v)} \left(\sum_{k\in I} (f_k - g_k)\right)^2 = r_v h(r_v, x).$$

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

Thus,

$$\mu\text{-}a.e.\ x \in \Omega,\ \text{there exists } r_x \in \mathbb{N}^*, \text{for all } r \geq r_x,$$
$$\left(\sum_{k=1}^{n_v}(f_k(x) - g_k(x))\right)^2 \leq C_1 r^{1+\varepsilon} 2^{r\delta} \leq C_1 \log_2^{2+\varepsilon}(v) v^{\delta}.$$

Therefore,

$$\mu\text{-}a.e.\ x \in \Omega, \text{there exists } r_x \in \mathbb{N}^* \text{ for all } r \geq r_x,$$
$$\left|\sum_{k=1}^{n_v}(f_k(x) - g_k(x))\right| \leq C_1^{1/2} \log_2^{1+(\varepsilon/2)}(v) v^{\delta/2}.$$

Let $n \in \mathbb{N}^*$. Taking $v \in \mathbb{N}^*$ such that $n_v \leq n \leq n_{v+1}$, we get

$$\mu\text{-}a.e.\ x \in \Omega, \quad \sum_{k=1}^{n_v} f_k(x) \leq \sum_{k=1}^{n} f_k(x) \leq \sum_{k=1}^{n_{v+1}} f_k(x).$$

Moreover,

$$\mu\text{-}a.e.\ x \in \Omega, \quad 0 \leq \sum_{k=n_v}^{n_{v+1}} g_k(x) \leq \phi(n_{v+1}) - \phi(n_v).$$

In addition,

$$v - 1 \leq \phi(n_v) < v \leq \phi(n_v + 1) \leq \phi(n_{v+1}) < v + 1.$$

Then,

$$0 < \phi(n_{v+1}) - \phi(n_v) < 2$$

and

$$\mu\text{-}a.e.\ x \in \Omega, 0 \leq \sum_{k=n_v+1}^{n_{v+1}} g_k(x) < 2.$$

Therefore,

$$\mu\text{-}a.e.\ x \in \Omega, \quad \sum_{k=1}^{n} f_k(x) = \sum_{k=1}^{n} g_k(x) + O(\log_2^{1+(\varepsilon/2)}(\phi(n))(\phi(n))^{\delta/2}). \qquad \square$$

THEOREM 5.2. (Dynamical Borel–Cantelli lemma by Keplerian shear) *Suppose that* (Ω, T, μ) *is a discrete measure preserving dynamical system with Keplerian shear. Let* (A_n) *be a sequence of measurable sets and note* $S_{M,N} = \sum_{k=M}^{N} \mathbb{1}_{A_k} \circ T^k$, $S_N = S_{0,N}$.

We suppose that Condition (1a) or (1b) holds and so does Condition (2).

(1) (a) *There exists* $\gamma > 0, C > 0$ *such that for all* $(j, k) \in \mathbb{N}^2$ *satisfying* $k \neq j$,

$$|\mathbb{E}(\mathrm{Cov}_{k-j}(\mathbb{1}_{A_j}, \mathbb{1}_{A_k}|\mathcal{I}))| \leq \frac{C}{|k-j|^{\gamma}}.$$

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

(b) *There exists* $\gamma > 0$, $D > 0$, *and* $\mathcal{B} \subset \mathbb{L}^2_\mu(\Omega)$, *a Banach space such that there exists* $C > 0$ *such that for all* $f, g \in \mathcal{B}$, *for all* $n \in \mathbb{N}$, $|\mathbb{E}(\mathrm{Cov}_n(f, g|\mathcal{I}))| \leq ((C\|f\|_{\mathcal{B}}\|g\|_{\mathcal{B}})/n^\gamma)$ *and for all* $j \in \mathbb{N}$, $\|\mathbb{1}_{A_j}\|_{\mathcal{B}} \leq D$.

(2) *There exists* $\beta > \max(\frac{1}{2}, 1 - \gamma/2)$ *such that* $\liminf_{N\to\infty}(\mathbb{E}(S_N|\mathcal{I})/N^\beta) > 0$ μ-*a.s.*

We have

$$\frac{S_N}{\mathbb{E}(S_N|\mathcal{I})} \xrightarrow[N\to+\infty]{\mu\text{-a.s.}} 1.$$

Remark 5.3. We emphasize that in contrast to usual dynamical Borel–Cantelli lemmas, the evolution of the sum S_N is only compared with the conditional expectation $\mathbb{E}(S_N|\mathcal{I})$, instead of the classical expectation $E(S_N)$.

Proof. Without loss of generality, we assume that $\gamma \in (0, 1)$. Note that Condition (1b) implies Condition (1a), so Condition (1a) holds in both cases.

By expansion of the square as a double sum and the series-integral comparison criterion, we get for all $(M, N) \in \mathbb{N}^2$ such that $M < N$,

$$\begin{aligned}\|S_{M,N} - \mathbb{E}(S_{M,N}|\mathcal{I})\|_2^2 &\leq \mathbb{E}(S_{M,N}) + 2C\sum_{l=1}^{N-M+1}\frac{(N-M+1)}{l^\gamma}\\ &\leq \mathbb{E}(S_{M,N}) + 2C(N-M+1)^{2-\gamma}.\end{aligned}$$

By setting $\delta = 2 - \gamma$ and noticing that $E(S_{M,N}) \leq N - M$, we get that there exists $D > 0$ satisfying

$$\text{for all } N \in \mathbb{N}, \quad \|S_{M,N} - \mathbb{E}(S_{M,N}|\mathcal{I})\|_2^2 \leq D(N-M+1)^\delta.$$

By variable Sprindzuk Theorem 5.1 with $\varphi_k = 1$ for all k, we get

$$S_N - \mathbb{E}(S_N|\mathcal{I}) \overset{a.s.}{=} O(N^{\delta/2}\log^{1+\varepsilon}(N)).$$

Then,

$$\frac{S_N}{\mathbb{E}(S_N|\mathcal{I})} = 1 + O\left(\frac{N^{\delta/2}\log^{1+\varepsilon}(N)}{N^\beta}\right) \xrightarrow[N\to+\infty]{a.s.} 1. \qquad \square$$

Example 5.4. Let us place ourselves in the case of $(\mathbb{T}^2, \lambda \otimes \lambda, T)$ with

$$\mathrm{Mat}(T) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Consider a sequence $(b_n) \in \mathbb{T}^{\mathbb{N}}$ and let

$$A_n = \mathbb{T} \times \left[b_n - \frac{1}{n^p}, b_n + \frac{1}{n^p}\right]$$

with $0 < p < \frac{1}{2}$. We get that for all $N \in \mathbb{N}^*$,

$$\mathbb{E}(S_N|\mathcal{I}) = E(S_N) = 2\sum_{k=1}^{N}\frac{1}{k^p} \sim \frac{2}{1-p}N^{1-p}.$$

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

Let $s := \frac{1}{2}$ and $n \in \mathbb{N}^*$. By direct computation of the Fourier coefficients of 1_{A_n}, we see that

$$\mathbb{1}_{A_n} \in H^{s,0}(\mathbb{T}^2)$$

since

$$\|\mathbb{1}_{A_n}\|^2_{H^{s,0}(\mathbb{T}^2)} \leq \frac{1}{\pi^2}\zeta(2) + \frac{2}{n^p} \leq 3.$$

By the decorrelation estimates in [**Tho**], for all $(k, j) \in \mathbb{N}^2$,

$$\mathbb{E}(\mathrm{Cov}_{k-j}(\mathbb{1}_{A_k}, \mathbb{1}_{A_j}|\mathcal{I})) \leq \frac{4^s 3^2}{|k-j|^{2s}}.$$

The hypotheses are then brought together, applying Theorem 5.2 with $\gamma = 1$ and $\beta \in (\frac{1}{2}, 1-p)$, so we get

$$S_N \sim \frac{2}{1-p} N^{1-p} \quad \mu\text{-a.s.}$$

In the case of any measure of Rajchman, we have as an application a proposition based on a work of Athreya [**Ath**, Theorem 4.2, p. 551] and also the theorem of Kurzweil [**Kur**] and [**Tse**, Theorem 1.3, p. 3] concerning Diophantine numbers. To introduce this theorem, we will also have to recall the shrinking target property (STP) and its monotone version (MSTP) of a dynamical system.

Definition 5.1. (Shrinking target properties) A discrete dynamical system (Ω, μ, T) is called STP if for all sequences of balls $(B_n)_{n\in\mathbb{N}}$ of radius $r_n > 0$ tending to 0 and satisfying

$$\sum_{n\in\mathbb{N}} \mu(B_n) = +\infty \quad \text{we have } \mu\Big(\overline{\lim_{n\to+\infty}} T^{-n}(B_n)\Big) = 1.$$

The MSTP property is defined in the same way, assuming in addition that the sequence r_n is monotone.

Definition 5.2. (s-Shrinking target properties) A discrete dynamical system (Ω, μ, T) is called STP if for all sequences of balls $(B_n)_{n\in\mathbb{N}}$ of radius $r_n > 0$ tending to 0 and satisfying

$$\sum_{n\in\mathbb{N}} \mu(B_n)^s = +\infty \quad \text{we have } \mu\Big(\overline{\lim_{n\to+\infty}} T^{-n}(B_n)\Big) = 1.$$

The s-MSTP property is defined in the same way, assuming in addition that the sequence r_n is monotone.

Let us now recall Kurzweil's theorem in dimension 1.

THEOREM 5.5. (Kurzweil and Tseng) *Consider the dynamical system $(\mathbb{T}, \lambda, T_\alpha)$ with $\alpha \in \mathbb{T}$ and $T_\alpha : x \mapsto x + \alpha$. Then, the dynamical system is s-MSTP if and only if α is s-Diophantine (that is, $\alpha \in \mathcal{D}\mathrm{io}(s)$, see Definition 2.9).*

We also deduce with the work of Chaika and Constantine [**CK**] the following theorem.

THEOREM 5.6. (Kurzweil) *For λ-a.e. $\alpha \in [0, 1[$, $(\mathbb{T}, \lambda, T_\alpha)$ is MSTP.*

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

In the case of singular continuous Rajchman measures, we will have a weakening of Theorem 5.6 of Kurzweil. First, we present a dynamical Borel–Cantelli lemma for Rajchman measures. As in Theorem 5.2, it is based on covariance estimates. Here, they do not rely on the Keplerian shear but are obtained directly from the Rajchman property.

THEOREM 5.7. (Dynamical Borel–Cantelli lemma for Rajchman measures) *Consider a Rajchman measure μ on $\mathbb{T}$ of order $0 \le r(\mu) \le \frac{1}{2}$. Let T be the transvection on $\mathbb{T}^2$ defined by $T(x, y) = (x, x + y)$. Let $C > 0$ and $s > (1/(r(\mu))) - 1$. We have*

$$\frac{S_N}{\mathbb{E}(S_N)} \xrightarrow[N\to+\infty]{\mu\otimes\lambda\text{-a.s.}} 1,$$

where

$$S_N = \sum_{k=1}^{N} \mathbb{1}_{A_k} \circ T^k \quad \text{and} \quad A_n = \mathbb{T} \times \left]z_n - \frac{C}{n^{1/s}}, z_n + \frac{C}{n^{1/s}}\right[.$$

Proof. Let $p = 1/s$. Let for $(m, n) \in (\mathbb{N}^*)^2$ such that $m > n$ the sum

$$S_{(m,n)} = \sum_{k=m}^{n} \mathbb{1}_{A_k} \circ T^k.$$

By definition and using the invariance of the measure, we get

$$\begin{aligned}
&\int_{\mathbb{T}^2} (S_{(m,n)} - \mathbb{E}(S_{(m,n)}))^2 \, d\mu \otimes \lambda \\
&\quad = \sum_{k=m}^{n} \sum_{j=m}^{n} \int_\Omega \mathbb{1}_{A_k} \circ T^{k-j} \mathbb{1}_{A_j} - \mathbb{E}_\mu(\mathbb{1}_{A_k}|\mathcal{I})\mathbb{E}_\mu(\mathbb{1}_{A_j}|\mathcal{I}) \, d\mu \le \mathbb{E}_\mu(S_{(m,n)}) \\
&\qquad + 2\sum_{j=m}^{n} \sum_{l=1}^{n-m} \int_\Omega \mathbb{1}_{A_{j+l}} \circ T^l \mathbb{1}_{A_j} - \mathbb{E}_\mu(\mathbb{1}_{A_{j+l}}|\mathcal{I})\mathbb{E}_\mu(\mathbb{1}_{A_j}|\mathcal{I}) \, d\mu.
\end{aligned}$$

Using the Fourier series of the indicator function of an interval, we obtain

$$\begin{aligned}
&\left| \int_\Omega \mathbb{1}_{A_{j+l}} \circ T^l \mathbb{1}_{A_j} - \mathbb{E}_\mu(\mathbb{1}_{A_{j+l}}|\mathcal{I})\mathbb{E}_\mu(\mathbb{1}_{A_j}|\mathcal{I}) \, d\mu \right| \\
&\quad = \left| \sum_{k\in\mathbb{Z}^*} \frac{\sin(2k\pi C/j^p) \sin(2k\pi C/(j+l)^p)}{k^2\pi^2} \widehat{\mu}(kl) \right| \\
&\quad \le 4C_\mu C \sum_{k\in\mathbb{N}^*} \frac{1}{l^{r(\mu)}} \frac{|\sin(2\pi kC/j^p)|}{k^{1+r}(j+l)^p\pi} \\
&\quad \simeq \frac{2^{2+r} C_\mu C^{1+r} \pi^{1-r}}{j^{pr(\mu)}(j+l)^p l^{r(\mu)}} \int_{\mathbb{R}_+^*} \frac{|\sin(x)|}{x^{1+r}} \, d\lambda(x).
\end{aligned}$$

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

Combining this upper bound with the previous estimate gives

$$\begin{aligned}\int_{\mathbb{T}^2} (S_{(m,n)} - \mathbb{E}(S_{(m,n)}))^2 \, d\mu \otimes \lambda \\ \ll \mathbb{E}_\mu(S_{(m,n)}) + \int_{[0,n-m]} \int_{[0,n-m]} \frac{1}{x^{r(\mu)}} \frac{1}{(x+y)^p} \frac{1}{y^{pr(\mu)}} \, d\lambda(y) \, d\lambda(x) \\ \ll \int_{[0,\pi/2]} \frac{(n-m)^{2-r(\mu)-p(1+r(\mu))}}{(\cos(\theta))^{r(\mu)}(\cos(\theta)+\sin(\theta))^p(\sin(\theta))^{pr(\mu)}} \, d\lambda(\theta).\end{aligned}$$

Let $\delta = 2 - r(\mu) - p(1 + r(\mu))$. Since $p < (r(\mu)/(1 - r(\mu)))$, we have $(\delta/2) < 1 - p$. Therefore, by Sprindzuk's theorem,

$$S_N \overset{\mu\otimes\lambda\text{-a.s.}}{=} \mathbb{E}(S_N) + O(N^{\delta/2} \log_2^{1+\varepsilon}(N)).$$

Hence,

$$\frac{S_N}{\mathbb{E}(S_N)} \xrightarrow[N\to+\infty]{\mu\otimes\lambda\text{-a.s.}} 1. \qquad \square$$

Remark 5.8. We observe that this 'loss of power' in the Borel–Cantelli lemma above increases as the Rajchman order decreases.

The next result relates the order of a Rajchman measure to Diophantine properties of its support and, thus, the MSTP property. This result will therefore also show Proposition 2.7.

PROPOSITION 5.9. (General Kurzweil for Rajchman measures) *Consider a Rajchman measure μ on $\mathbb{T}$ of order $0 \le r(\mu) \le \frac{1}{2}$. Let $s > (1/(r(\mu))) - 1$. We have $\mu(\mathcal{D}\text{io}(s)) = 1$, or equivalently,*

$$\mu\text{-a.e. } \alpha \in [0,1[, (\mathbb{T}, \lambda, T_\alpha) \text{ is } s\text{-MSTP}.$$

Proof. Let $C > 0$. For $n \in \mathbb{N}^*$, set $A_n =](p/n) - (C/n^{s+1}), (p/n) + (C/n^{s+1})[$. Using an elementary computation based on the Fourier series of the indicator function of an interval and the periodicity of the intervals, we get

$$\mu(A_n) = \sum_{p=0}^{n-1} \int_{\mathbb{T}} \mathbb{1}_{](p/n)-(C/n^{s+1}),(p/n)+(C/n^{s+1})[} \, d\mu = 2\frac{C}{n^s} + \sum_{k\in\mathbb{Z}^*} \frac{\sin(2\pi k(C/n^s))}{k\pi} \widehat{\mu}(kn).$$

Take ε and r such that $1/(s+1) < \varepsilon < r < r(\mu)$. Since $|\sin(t)| \le |t|^\varepsilon$ for all real t, we obtain

$$\mu(A_n) \le 2\frac{C}{n^s} + \frac{4C_\mu C^\varepsilon}{\pi} n^{-r-s\varepsilon} \zeta(1 + r - \varepsilon).$$

Therefore, $\sum_n \mu(A_n) = +\infty$ since $r + s\varepsilon > 1$. By the Borel–Cantelli lemma,

$$\mu\text{-a.e. } \alpha \in [0,1[, \exists n \in \mathbb{N}^*, \text{ for all } k \ge n, d(k\alpha, \mathbb{Z}) \ge \frac{C}{k^s}.$$

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

This proves that $\mu(\mathcal{D}\mathrm{io}(s)) = 1$. By Kurzweil–Tseng Theorem 5.5,

$$(\mathbb{T}, \lambda, T_\alpha) \text{ is } s\text{-MSTP.} \qquad \square$$

Recall that when we have $r(\mu) > \frac{1}{2}$, the measure μ is absolutely continuous and therefore by Kurzweil theorem cited above, μ-almost all $\alpha \in [0, 1[$, the system $(\mathbb{T}, \lambda, T_\alpha)$ is 1-MSTP.

Remark 5.10. We can note that the result obtained in Proposition 5.9 remains consistent with that obtained with the Keplerian shear in Theorem 5.7 because

$$\sum_{k=1}^{n} \lambda\left(\left]z_k - \frac{C}{k^{1/s}}, z_k + \frac{C}{k^{1/s}}\right[\right)^s = \sum_{k=1}^{n} \frac{2^s C^s}{k} \xrightarrow[n\to+\infty]{} +\infty.$$

Remark 5.11. For a Rajchman measure with positive order $r(\mu) > 0$, μ-*a.e.* $\alpha \in [0, 1[$ is Diophantine, that is, $\mathrm{Dio}(\alpha) < +\infty$ μ-almost surely.

The following result shows that Proposition 5.9 is optimal.

PROPOSITION 5.12. (Optimality of the Rajchman order [**Kau**]) *Let* $0 < r < \frac{1}{2}$. *Then, for all* $1 < s < (1/r) - 1$, *there exists* μ *a Rajchman measure such that* $r(\mu) = r$ *and* $\mu(\mathcal{D}\mathrm{io}(s)) = 0$.

Proof. Let $\varepsilon = (1/r) - (s + 1) > 0$. We can consider μ_α with $\alpha = s - 1 + \varepsilon$ constructed in the document of Kaufman [**Kau**] with its support in $E(\alpha) \subset [0, 1] \setminus \mathcal{D}\mathrm{io}(s)$. $\square$

Remark 5.13. With convex combination, for any $s > 1$, we can construct a Rajchman measure μ for which $\mu(\mathcal{D}\mathrm{io}(s))$ is equal to any value in $[0, 1]$.

Example 5.14. (Diophantine property with self-similar measures) The self-similar measure μ_θ with $\theta > 1$ not being Pisot, according to the work of Pablo Shmerkin and Kahane [**Kah**], has an order of Rajchman $r(\mu_\theta) > 0$. In particular, μ_θ-a.e. $\alpha \in [0, 1]$ is Diophantine.

Example 5.15. (Liouville property with Rajchman measures on S_∞) The measure of Rajchman μ_∞ mentioned in Example 2.9 is Rajchman, but since its support S_∞ almost surely contains Liouville numbers, then we deduce by contrapositive of Proposition 5.9 that μ_∞ does not have a strictly positive Rajchman order, that is, that $r(\mu_\infty) = 0$.

6. *Flow on compact Lie group bundle*

In this section, we extend the results on Keplerian shear for tori bundles from §4 to non-abelian bundles. The setting is essentially the same except that we replace the torus with a compact connected Lie group. It is worth mentioning that the core of the proofs will not differ too much. The fundamental reason being that even if the group is not abelian, each individual orbit defines a one-parameter subgroup that is therefore abelian; hence a torus.

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

6.1. *Lie group bundle and compatible flow.* Let G be a compact connected Lie group.

Definition 6.1. Let $(M, \mathcal{A})$ be a Lindelöf manifold of dimension $n \in \mathbb{N}^*$ of class $\mathcal{C}^1$.

Let (Ω, μ) be a Borelian space, G a connected compact Lie group, and π a continuous map from Ω into M.

Then, Ω is a G-bundle if:

(1) locally, we have for the charts U of $\mathcal{A}$ a homeomorphism $\psi_U : \pi^{-1}(U) \to U \times G$;
(2) for all U in $\mathcal{A}$, $\pi_1 \circ \psi_U = \pi_{|U}$.

To define the continuous flow, we are going to use the exponential defined on the corresponding Lie algebra $\mathrm{Lie}(G)$ of the group G. Here, the connectedness and the compactness of the group provide that the exponential $\exp\colon \mathrm{Lie}(G) \to G$ is surjective, and injective from a neighborhood of the origin $B(0, \delta_{\mathrm{inj}})$ to a neighborhood of the neutral element of e of G. The latter implies that any continuous flow satisfying for all $y, z \in G$, $\phi_t(yz) = \phi_t(y)z$ is of the form $\phi_t(y) = \exp(tv)y$ for some $v \in \mathrm{Lie}(G)$. This motivates the notion of compatible flow.

Definition 6.2. (Compatible flow) Let $(g_t)_{t\in\mathbb{R}}$ be a flow on (Ω, μ).

Then, $(g_t)_{t\in\mathbb{R}}$ is a compatible flow if and only if:

(1) there exists for all $U \in \mathcal{A}$, a measurable function $v_U : U \to \mathrm{Lie}(G)$;
(2) for all $U \in \mathcal{A}$, for all $(x, y) \in U \times G$, $\psi_U \circ g_t \circ \psi_U^{-1}(x, y) = (x, \exp(t v_U(x))y)$.

Definition 6.3. (Compatible measure) Consider the notation previously established with the connected compact Lie group bundle.

Let $U \in \mathcal{A}$. Let us pose $\mu^U := \mu_{\psi_U}$ and $\mu' := \mu_\pi$. Then, μ on Ω is compatible if and only if

$$\mu^U_{|\pi^{-1}(U)} = \mu'_{|U} \otimes \mathcal{H} \text{ with } \mathcal{H} \text{ the Haar measure on } G.$$

Next, results by Delzant [**Del**] will allow us to transfer the dynamics from G to the torus.

PROPOSITION 6.1. *Let G be a compact and connected Lie group.*

(i) *Any compact, connected, and abelian Lie group is isomorphic to a torus.*
(ii) *The group G admits a maximal torus T,*

$$G = \bigcup_{g\in G} gTg^{-1}$$

and for all maximal torus T', there exists $g \in G$ such that

$$T' = gTg^{-1}.$$

(iii) *Any closed subgroup H of G admits a group structure of Lie and* $\mathrm{Lie}(H)$ *is a linear subspace of* $\mathrm{Lie}(G)$.
(iv) *The orbital group of a direction $v \in \mathrm{Lie}(G)$,*

$$H_v := \overline{\{\exp(tv) \in G : t \in \mathbb{R}\}},$$

is an abelian Lie group, compact and connected, and so is isomorphic to a torus.

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

(v) *For all* $v \in \mathrm{Lie}(G)$*, for all maximal torus* T *of* G*, there exists* $g \in G$ *such that*

$$H_v \subset gTg^{-1}.$$

Now, we fix T a maximal torus of G and an isomorphism $\chi : T \to \mathbb{T}^R$, where R is the dimension of T.

The combination of these results shows that each individual orbit can be embedded in the same torus T, a situation similar to the case of a tori bundle, which is the setting of §4. More precisely, by Proposition 6.1(v), for $v \in \mathrm{Lie}(G)$, we get a $g \in G$ such that $H_v \subset gTg^{-1}$. To exploit this property, we need a measurable selection of the element g, as a function of v.

Let $h : \mathrm{Lie}(G) \to G$ be a measurable map such that for all $v \in \mathrm{Lie}(G)$

$$H_v \subset h(v)T(h(v))^{-1}. \tag{6.1}$$

(A construction of such a measurable function h is done in §6.4, but other choices are possible.) Let us pose for $U \in \mathcal{A}$ and $x \in U$,

$$w_U(x) = D\chi(e)(\mathrm{Ad}_{h(v_U(x))}(v_U(x))) \tag{6.2}$$

with Ad the adjoint operator defined for $g \in G$ and $v \in \mathrm{Lie}(G)$ by

$$\mathrm{Ad}_g(v) = \frac{d}{dt} g \exp(tv) g^{-1} \Big|_{t=0}. \tag{6.3}$$

Let us pose for non-zero $\xi \in \mathbb{Z}^d$ and $U \in \mathcal{A}$, $\nu^{(\xi,U)} := \mu'_{|U}\langle \xi | w_U(\cdot)\rangle$.

6.2. *Keplerian shear and Rajchman measures on G-bundle.*

THEOREM 6.2. *The dynamical system* $(\Omega, \mu, (g_t)_{t\in\mathbb{R}})$ *exhibits Keplerian shear if and only if for all non-zero* $\xi \in \mathbb{Z}^R$*, and for all* $U \in \mathcal{A}$*,* $\nu^{(\xi,U)}_{|\mathbb{R}^*}$ *is Rajchman.*

The plan of the demonstration will initially be to come back to a situation very similar to that of the torus bundle by relying on the maximal torus. To do this, we are going to use the orbital group $H_{v_U(x)}$ that is isomorphic to a torus. The second step will be for a chart $U \in \mathcal{A}$ and $x \in U$ fixed to conjugate the action of $\exp(t v_U(x))$ on G to a translation by $t w_U(x)$ on the torus. Once immersed in this configuration provided by Lemma 6.3, we will be able to carry out the same reasoning as in the torus to arrive at the property of Rajchman of $\nu^{(\xi,U)}_{|\mathbb{R}^*}$.

Consider the neighborhood V of the local coordinate system of the second kind associated with $\mathrm{Lie}(T)$ and $(\mathrm{Lie}(T))^{\perp}$.

We cover G with a finite family of elements $(g_k)_{k\in[\![1,l]\!]} \in G^{[\![1,l]\!]}$ such that

$$G = \bigcup_{k=1}^{l} V g_k. \tag{6.4}$$

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

Let $(W_k)_{k\in[\![1,l]\!]}$ such that $W_1 = Vg_1$ and

$$\text{for all } k \in [\![2, l]\!], \quad W_k = Vg_k \setminus \left(\bigcup_{j\in[\![1,k-1]\!]} Vg_j \right). \tag{6.5}$$

Let $\varphi : G^2 \to G$ be defined by $\varphi(x, y) = xy$ and $\varphi^v : G^2 \to G$ be defined by $\varphi^v(x, y) = \varphi(h(v)x, y)$. By [**Bou**], we can define

$$\begin{aligned} \psi_k : Vg_k &\to \mathrm{Lie}(T) \times G \\ x &\mapsto (x_T, \exp(x_T^{\perp})g_k). \end{aligned}$$

Let $\eta : \mathrm{Lie}(G) \times G \to G^2$ be defined by $\eta(\theta, y) = (\exp(\theta), y)$. We define for $v \in \mathrm{Lie}(G)$, $k \in [\![1, l]\!]$, and $x \in Vg_k$,

$$\psi_k^v(x) = \psi_k((h(v))^{-1}x), \tag{6.6}$$

$$\varphi^v(x) = \varphi(h(v)x, y), \tag{6.7}$$

for $(\theta, y) \in \mathrm{Lie}(G) \times G$ and $t \in \mathbb{R}$,

$$\phi_t^v(\theta, y) = (\theta + t\mathrm{Ad}_{h(v)}(v), y), \tag{6.8}$$

and for $z \in G$,

$$\chi_v(z) = \chi((h(v))^{-1}zh(v)). \tag{6.9}$$

Thus, for all $k \in [\![1, l]\!]$, for all $x \in h(v)Vg_k$, we get

$$(\varphi^v \circ \eta \circ \phi_t^v \circ \psi_k^v)(x) = \exp(tv)x. \tag{6.10}$$

LEMMA 6.3. *Let $U \in \mathcal{A}$ and $(f_1, f_2) \in (\mathbb{L}^2_{\mu'_{|U}\otimes\mathcal{H}}(U \times G))^2$. We define for $t \in \mathbb{R}$,*

$$b_t(f_1, f_2) := \int_{U\times G} \overline{f_1}(x, \exp(tv_U(x))y) f_2(x, y)\, d\mu'_{|U} \otimes \mathcal{H}(x, y).$$

Then,

$$\begin{aligned} &b_t(f_1, f_2) \\ &= \sum_{k\in[\![1,l]\!]} \int_U \left(\int_{\mathbb{T}^R \times h(v_U(x))W_k} \overline{\check{f}_1}(x, z + tw_U(x), r))\, \check{f}_2(x, z, r))\, dm'_{(x,k)}(z, r) \right) d\mu'_{|U}(x) \end{aligned} \tag{6.11}$$

with

$$\begin{aligned} \check{f}_1 : (x, z, r) &\mapsto f_1(x, \varphi^{v_U(x)}(\chi^{-1}_{v_U(x)}(z), r)) \\ \check{f}_2 : (x, z, r) &\mapsto f_2(x, \varphi^{v_U(x)}(\chi^{-1}_{v_U(x)}(z), r)), \end{aligned}$$

and $m'_{(x,k)} = (\chi_{v_U(x)} \circ \exp)_* m_{(x,k)}$ is the push forward measure obtained by transfer theorem, where $m_{(x,k)} = (\psi_k^{v_U(x)})_*\mathcal{H}$ is the associated push forward measure. We recall that $D\chi_{v_U(x)}(e) = D\chi(e) \circ \mathrm{Ad}_{h(v_U(x))}$.

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

Proof. Let non-zero $\xi \in \mathbb{Z}^R$. Let $U \in \mathcal{A}$ and $(f_1, f_2) \in (\mathbb{L}^2_{\mu'_{|U} \otimes \mathcal{H}}(U \times G))^2$. We have the following expression for $b_t(f_1, f_2)$:

$$
\begin{aligned}
&= \int_{U\times G} \overline{f_1}(x, \exp(t v_U(x))y)\, f_2(x, y)\, d\mu'_{|U} \otimes \mathcal{H}(x, y) \\
&= \int_U \int_G \overline{f_1}(x, \exp(t v_U(x))y)\, f_2(x, y)\, d\mathcal{H}(y)\, d\mu'_{|U}(x) \\
&= \sum_{k=1}^{l} \int_U \Bigg(\int_{h(v_U(x))W_k} \overline{f_1}(x, \varphi^{\check{v_U}(x)} \circ \phi_t^{v_U(x)} \circ \psi_k^{v_U(x)}(y)) \\
&\qquad\qquad \times f_2(x, \varphi^{\check{v_U}(x)} \circ \psi_k^{v_U(x)}(y)) d\mathcal{H}(y) \Bigg)\, d\mu'_{|U}(x) \\
&= \sum_{k=1}^{l} \int_U \Bigg(\int_{\mathrm{Lie}(T)\times h(v_U(x))W_k} \overline{f_1}(x, \varphi^{\check{v_U}(x)} \circ \phi_t^{v_U(x)}(\theta, r)) \\
&\qquad\qquad \times f_2(x, \varphi^{\check{v_U}(x)}(\theta, r)) dm_{(x,k)}(\theta, r) \Bigg)\, d\mu'_{|U}(x).
\end{aligned}
$$

We have completed the first step by placing ourselves in the Lie algebra of the maximal torus T, which is abelian, so

$$\check{\varphi} \circ \phi_t^{v_U(x)}(\theta, r) = \check{\varphi}(\theta + t v_U(x), r) = \varphi(\exp(\theta + t v_U(x)), r) = \varphi(\exp(t v_U(x)) \exp(\theta), r).$$

We are now at the second step placing ourselves in the sections of the Lie group G by T. So,

$$b_t(f_1, f_2) = \sum_{k=1}^{l} \int_U I(k, U, x)\, d\mu'_{|U}(x),$$

where the integrand $I(k, U, x)$ is equal to

$$
\begin{aligned}
&\int_{\mathrm{Lie}(T)\times h(v_U(x))W_k} \overline{f_1}(x, \varphi(\exp(t v_U(x))\chi^{-1}_{v_U(x)}(\chi_{v_U(x)}(e^\theta)), r)) \\
&\quad \times f_2(x, \varphi(e^\theta), r))\, dm_{(x,k)}(\theta, r) \\
&= \int_{\mathbb{T}^R \times h(v_U(x))W_k} \overline{f_1}(x, \varphi(\exp(t v_U(x))\chi^{-1}_{v_U(x)}(z), r)) f_2(x, \varphi(\chi^{-1}_{v_U(x)}(z), r))\, dm'_{(x,k)}(z, r).
\end{aligned}
$$

We pass here in the torus isomorphic to T. Since

$$\exp(t v_U(x))\chi^{-1}_{v_U(x)}(z) = \chi^{-1}_{v_U(x)}(\chi_{v_U(x)}(\exp(t v_U(x))) + z)$$

and

$$\chi_{v_U(x)}(\exp(t v_U(x))) + z = z + t D\chi_{v_U(x)}(e) = z + t w_U(x),$$

we get

$$
\begin{aligned}
&I(k, U, x) \\
&= \int_{\mathbb{T}^R \times h(v_U(x))W_k} \overline{f_1}(x, \varphi(\chi^{-1}_{v_U(x)}(z + t w_U(x)), r)) f_2(x, \varphi(\chi^{-1}_{v_U(x)}(z), r))\, dm'_{(x,k)}(\theta, r).
\end{aligned}
$$

This proves (6.11). □

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

We are then in the case of a torus foliation by the variable x, which will then allow us to apply the same reasoning as in the case of the torus.

Proof of Theorem 6.2. Let us begin with the direct implication. Suppose that the dynamical system $(\Omega, \mu, (g_t)_{t\in\mathbb{R}})$ exhibits Keplerian shear. Let $U \in \mathcal{A}$ and let

$$\xi \in \mathbb{Z}^R \setminus \{0\}.$$

Let $(f_1, f_2) \in (\mathbb{L}^2_{\mu'_{|U}\otimes\mathcal{H}}(U \times G))^2$ such that

$$\check{f}_1 : (x, z, r) \mapsto e^{2i\pi\langle\xi|z\rangle}$$

and

$$\check{f}_2 : (x, z, r) \mapsto \mathbb{1}_{\langle\xi|w_U(\cdot)\rangle\neq 0}(x)e^{2i\pi\langle\xi|z\rangle}$$

taking the notation of Lemma 6.3. Then,

$$\int_{\mathbb{R}} e^{-2i\pi tz}\, d\nu^{\xi,U}(z) = \int_U e^{-2i\pi t\langle\xi|w_U(x)\rangle} g(x)\, d\mu'(x)$$

with

$$g : x \in U \mapsto \mathbb{1}_{\langle\xi|w_U(\cdot)\rangle\neq 0}(x).$$

Thus, by Lemma 6.3,

$$\int_U e^{-2i\pi t\langle\xi|w_U(x)\rangle} g(x)\, d\mu'_{|U}(x) = \int_{U\times G} \overline{f_1}(x, \exp(tv_U(x))y) f_2(x, y)\, d\mu'_{|U} \otimes \mathcal{H}(x, y).$$

By the Keplerian shear property, the right-hand term converges and as in (4.1), the limit is zero. So,

$$\int_{\mathbb{R}} e^{-2i\pi tz}\, d\nu^{\xi,U}(z) \xrightarrow[t\to\pm\infty]{} 0.$$

Let us continue with the reciprocal implication. Let R be the dimension of the maximal torus of G.

Consider $k \in [\![1, l]\!]$ and define the measure ω^k on $U \times \mathbb{T}^R \times W_k$ by

$$\text{for all } A \in \mathcal{B}(U \times \mathbb{T}^R \times h(v_U(x))W_k),$$
$$\omega^k(A) = \int_U \int_{\mathbb{T}^R\times h(v_U(x))W_k} \mathbb{1}_A(x, y, z)\, dm'_{(x,k)}(y, z)\, d\mu'_{|U}(x).$$

Let $U \in \mathcal{A}$ and $(\xi_1, \xi_2) \in (\mathbb{Z}^R)^2$. Let $(f_1, f_2) \in (\mathbb{L}^2_{\mu'_{|U}\otimes\mathcal{H}}(U \times G))^2$ such that,

$$\check{f}_1 : (x, z, r) \mapsto a_1(x, r)e^{2i\pi\langle\xi_1|z\rangle}$$

and

$$\check{f}_2 : (x, z, r) \mapsto a_2(x, r)e^{2i\pi\langle\xi_2|z\rangle}$$

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

with a_1 and a_2 square summable functions in the appropriate Lebesgue space. If $m \neq m'$, the scalar product between f_1, f_2 is zero. Therefore, we assume that $m = m'$. So, by Lemma 6.3,

$$\int_{U\times G} \overline{f_1}(x, \exp(t v_U(x))y) f_2(x, y)\, d\mu'_{|U} \otimes \mathcal{H}(x, y) = \int_U e^{-2i\pi t\langle \xi_1 | w_U(x)\rangle} g(x)\, d\mu'_{|U}(x) \tag{6.12}$$

with

$$g : x \in U \mapsto \sum_{k=1}^{l} \left(\int_{h(v_U(x))W_k} \overline{a_1}(x, r) a_2(x, r) e^{2i\pi\langle \xi_2 - \xi_1 | z\rangle}\, dm'_{(x,k)}(z, r) \right).$$

By hypothesis, when $\xi_1 \neq 0$,

$$\int_{\mathbb{R}} e^{-2i\pi tz}\, d\nu^{\xi_1,U}(z) \xrightarrow[t\to\pm\infty]{} 0.$$

Then, by Lemma 2.4, we have that $\exp(2i\pi t\cdot)$ converge weakly-$*$ in $\mathbb{L}^{\infty}_{\nu^{\xi_1},U}(\mathbb{R})$ to 0. Thus,

$$\int_U e^{-2i\pi t\langle \xi_1 | w_U(x)\rangle} g(x)\, d\mu'_{|U}(x) \xrightarrow[t\to\pm\infty]{} \int_{\{\langle \xi_1 | w_U(\cdot)\rangle = 0\}\times G} \overline{f_1}(x, y) f_2(x, y)\, d\mu'_{|U} \otimes \mathcal{H}(x, y).$$

Then, remembering (6.12), the integral

$$\int_{U\times G} \overline{f_1}(x, \exp(t v_U(x))y) f_2(x, y)\, d\mu'_{|U} \otimes \mathcal{H}(x, y)$$

converges as $t \to \pm\infty$ towards

$$\int_{\{\langle \xi_1 | w_U(\cdot)\rangle = 0\}\times G} \overline{f_1}(x, y) f_2(x, y)\, d\mu'_{|U} \otimes \mathcal{H}(x, y).$$

By totality obtained by the Fourier series expansion, this gives that for all functions f_1, $f_2 \in \mathbb{L}^2_{\mu'_{|U}\otimes\mathcal{H}}(U \times G)$, the integral

$$\int_{U\times G} \overline{f_1}(x, \exp(t v_U(x))y) f_2(x, y)\, d\mu'_{|U} \otimes \mathcal{H}(x, y)$$

converges as $t \to \pm\infty$ towards

$$\int_{U\times G} \overline{E_{\mu'_{|U}\otimes\mathcal{H}}(f_1|\mathcal{I}_U)} E_{\mu'_{|U}\otimes\mathcal{H}}(f_2|\mathcal{I}_U)\, d\mu'_{|U} \otimes \mathcal{H}. \tag{6.13}$$

□

6.3. *Keplerian shear on G-bundle in the regular case.* Now, the aim here is to give an analog of [**Tho**, Theorem 3.3], which guarantees Keplerian shear for flows with regular velocities and the negligible critical points, for absolutely continuous measures.

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

THEOREM 6.4. *If for all $U \in \mathcal{A}$, w_U is $\mathcal{C}^1$, $\mu'_{|U} \ll \lambda$ and*

$$\mu\Big(\bigcup_{\xi\in\mathbb{Z}^d\setminus\{0\}} \{x \in U : d\langle\xi|w_U(x)\rangle = 0\} \setminus \{x \in U : \langle\xi|w_U(x)\rangle = 0\}\Big) = 0,$$

then the dynamical system $(\Omega, \mu, (g_t)_{t\in\mathbb{R}})$ has Keplerian shear.

LEMMA 6.5. *Let $V \subset \mathbb{R}^d$ be an open set and $f : V \to \mathbb{R}$ a $\mathcal{C}^1$ map. If $\{d_x f = 0\} \setminus \{f = 0\}$ has zero Lebesgue measure, then for any $a \in \mathbb{R}^*$, $\{f = a\}$ has zero Lebesgue measure.*

Proof. Let (e_i) be the canonical basis of $\mathbb{R}^d$ and denote by $\mathcal{C}_i$ the cone of vectors $h \neq 0$ such that the angle between h and e_i is less than or equal to $\pi/8$.

Let $a \in \mathbb{R}^*$, and set $A = \{f = a\}$ and A' the set of Lebesgue density points of A, that is, $\lambda(A \cap B(x, r))/\lambda(B(x, r)) \to 1$ as $r \to 0$ for each $x \in A'$.

Suppose that $x \in A'$. For each $i = 1, \ldots, d$, there exists a sequence $h_n^i \in \mathcal{C}_i$ such that $x + h_n^i \in A$ and $h_n^i \to 0$. Without loss of generality, we may assume that $h_n^i/\|h_n^i\|$ converges to some $h^i \in \mathcal{C}_i$. Since $f(x + h_n^i) = a = f(x)$, a first-order expansion shows that $d_x f(h^i) = 0$. Since the h_i form a basis of $\mathbb{R}^d$, this implies that $d_x f = 0$.

By assumption, we get that A', and therefore A, has zero measure. □

Proof of Theorem 6.4. Let $U \in \mathcal{A}$. Let us show that the measures $\nu^{(\xi,U)}$ have the Rajchman property.

Let $\xi \in \mathbb{Z}^d \setminus \{0\}$. By Radon–Nikodym,

$$\int_U e^{-2i\pi t\langle\xi|w_U(x)\rangle}\, d\mu'_{|U}(x) = \int_U e^{-2i\pi t\langle\xi|w_U(x)\rangle} \frac{d\mu'_{|U}}{d\lambda}(x)\, d\lambda(x).$$

However,

$$\mu\Big(\bigcup_{\xi\in\mathbb{Z}^d\setminus\{0\}} \{x \in U : d\langle\xi|w_U(x)\rangle = 0\} \setminus \{x \in U : \langle\xi|w_U(x)\rangle = 0\}\Big) = 0.$$

Thus, by Lemma 6.5,

$$\text{for all } a \in \mathbb{R}^*, \quad \mu\Big(\bigcup_{\xi\in\mathbb{Z}^d\setminus\{0\}} \{x \in U : \langle\xi|w_U(x)\rangle = a\}\Big) = 0.$$

By local normal form of submersions, we can choose a countable family $(\varphi_k, V_k)_{k\in\mathbb{N}}$ of open charts on M which are pairwise disjoints, cover $U \setminus (\bigcup_{\xi\in\mathbb{Z}^d\setminus\{0\}}\{x \in U : d\langle\xi|w_U(x)\rangle = 0\})$ up to a Lebesgue-negligible set, and for all $k \in \mathbb{N}$ and all $x \in \varphi_k(V_k)$,

$$\langle\xi|w_U(\varphi_k^{-1}(x))\rangle = x_1.$$

Therefore, for $k \in \mathbb{N}$,

$$\int_{V_k} e^{-2i\pi t\langle\xi|w_U(x)\rangle} \frac{d\mu'_{|U}}{d\lambda}(x)\, d\lambda(x) = \int_{\varphi_k(V_k)} e^{-2i\pi tz} \frac{d\mu'_{|U}}{d\lambda}(\varphi_k^{-1}(z))|\det(J_{\varphi_k^{-1}})(z)|\, d\lambda(z).$$

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

However, we have

$$\int_U e^{-2i\pi t\langle \xi | w_U(x)\rangle} \frac{d\mu'_{|U}}{d\lambda}(x)\, d\lambda(x) = \int_{\langle \xi | w_U\rangle = 0} \frac{d\mu'_{|U}}{d\lambda}(x)\, d\lambda(x) + \sum_{k\in\mathbb{N}} \int_{V_k} e^{-2i\pi t\langle \xi | w_U(x)\rangle} \frac{d\mu'_{|U}}{d\lambda}(x)\, d\lambda(x).$$

Thus, by Riemann–Lebesgue,

$$\int_U e^{-2i\pi t\langle \xi | w_U(x)\rangle} \frac{d\mu'_{|U}}{d\lambda}(x)\, d\lambda(x) \xrightarrow[t\to+\infty]{} \int_{\langle \xi | w_U\rangle = 0} \frac{d\mu'_{|U}}{d\lambda}(x)\, d\lambda(x).$$

Therefore,

$$\int_U e^{-2i\pi t\langle \xi | w_U(x)\rangle}\, d\mu'_{|U}(x) \xrightarrow[t\to+\infty]{} \mu'_{|U}(\{\langle \xi | w_U\rangle = 0\}).$$

By Theorem 6.2, we get that $(\Omega, \mu, (g_t)_{t\in\mathbb{R}})$ exhibits Keplerian shear. □

We can develop a corollary of Theorem 6.4. We define

$$\bar{h} : v \in \mathrm{Lie}(G) \setminus \{0\} \mapsto h\left(\frac{1}{\|v\|_2} v\right) \tag{6.14}$$

and we assume that $\bar{h}$ is $\mathcal{C}^1$ on a full measure open set.

COROLLARY 6.6. *We suppose that for a countable family $(U_j)_{j\in\mathbb{N}}$ of charts which cover M up to a Lebesgue-negligible set, $\bar{h}$ is $\mathcal{C}^1$ on V_j such that $v_{U_j}(U_j) \subset V_j$, $\mu'_{|U_j} \ll \lambda$, v_{U_j} is a $\mathcal{C}^1$-subimmersion, and for $\mu'_{|U_j}$-a.e. $x \in U_j$, $\mathrm{Rank}(Dv_{U_j}(x)) \geq 1$.*

We get that the dynamical system $(\Omega, \mu, (g_t)_{t\in\mathbb{R}})$ exhibits Keplerian shear.

First, for $R \in \mathbb{N}^*$, $0 \neq \xi \in \mathbb{Z}^R$, we define the function

$$S_\xi : v \in \mathrm{Lie}(G) \mapsto \langle \xi | D\chi(e)(\mathrm{Ad}_{h(v)}(v))\rangle. \tag{6.15}$$

We will need for $v \in \mathrm{Lie}(G)$ the differential of $A^v : g \in G \mapsto \mathrm{Ad}_g(v)$.

LEMMA 6.7. *For $v, w \in \mathrm{Lie}(G)$, noting $[\cdot,\cdot]$ the Lie bracket, we have*

$$DA^v(g)(w) = [\mathrm{Ad}_g(w), \mathrm{Ad}_g(v)].$$

By (6.1), for all $v \in \mathrm{Lie}(G) \setminus \{0\}$ and all $t \in \mathbb{R}$,

$$(\bar{h}(v))^{-1} \exp(tv)\bar{h}(v) \in T. \tag{6.16}$$

To ensure the regularity of S_ξ, we need the next lemmas.

LEMMA 6.8. *For all $v \in \mathrm{Lie}(G) \setminus \{0\}$,*

$$D\bar{h}(v)(v) = 0.$$

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

Proof. Let $t \in]-1, 1[$ and $v \in Lie(G) \setminus \{0\}$. Thus, $\bar{h}(v + tv) = h(1 + t/(1+t)\|v\|v) = \bar{h}(v)$. Therefore,

$$D\bar{h}(v)(v) = 0. \qquad \square$$

LEMMA 6.9. *Let $V \subset \mathrm{Lie}(G) \setminus \{0\}$ and $W \subset \mathbb{R}^d$ be open sets. We suppose that the restriction of h on V is $\mathcal{C}^1$. Let $\psi : V \to W$ be a $\mathcal{C}^1$-diffeomorphism. We have for $w \in W$,*

$$S_\xi(\psi^{-1}(w)) \neq 0 \implies D(S_\xi \circ \psi^{-1})(w)((D\psi^{-1}(w))^{-1} \circ \psi^{-1}(w)) \neq 0.$$

Proof. Let $w \in W$ such that $S_\xi(\psi^{-1}(w)) \neq 0$. By Lemma 6.7, for $u \in \mathbb{R}^d$, we get

$$D(S_\xi \circ \psi^{-1})(w)(u) = \begin{pmatrix} \langle \xi | D\chi(e)(\mathrm{Ad}_{\bar{h}(\psi^{-1}(w))} \circ D\psi^{-1}(w)(u))\rangle \\ +\langle \xi | D\chi(e) \circ \mathrm{Ad}_{\bar{h}(\psi^{-1}(w))} \circ [D\bar{h}(\psi^{-1}(w)) \circ D\psi^{-1}(w)(u),\, \psi^{-1}(w)]\rangle \end{pmatrix}.$$

Taking $u = (D\psi^{-1}(w))^{-1} \circ \psi^{-1}(w)$, by Lemma 6.8,

$$D(S_\xi \circ \psi^{-1})(w)(u) = S_\xi(\psi^{-1}(w)) \neq 0. \qquad \square$$

This intermediate result helps us to connect Lemmas 6.9 and 6.11.

LEMMA 6.10. *Let $U \subset M$ and $x \in U$. Let $v : U \to V \subset \mathrm{Lie}(G)$ be a $\mathcal{C}^1$ function and there be subimmersion around x such that $k := \mathrm{rank}(Dv(x)) \geq 1$. We have that there exists a $\mathcal{C}^1$-diffeomorphism $\psi : V \to V' \subset \mathrm{Lie}(G)$ and a neighborhood $W \subset U$ of x such that for all $a \in W$,*

$$\psi(v(a)) \in \mathrm{Im}(D(\psi \circ v)(a)).$$

Proof. Let $U \subset M$ and $x \in U$. Let $v : U \to V \subset \mathrm{Lie}(G)$ be a $\mathcal{C}^1$ function and there be subimmersion around x such that $k = \mathrm{rank}(Dv(x)) \geq 1$. Using the local normal form of subimmersions, there exists $V' \subset \mathbb{R}^d$, U' open sets and $\mathcal{C}^1$-diffeomorphisms $\psi : V \to V'$ and $\varphi : U \to U'$ such that for all $z \in U'$,

$$\psi(v(\varphi^{-1}(z))) = (z_1, \ldots, z_k, 0, \ldots, 0).$$

Let $a \in U$. Therefore,

$$D(\psi \circ v)(a)(u) = ((D\varphi(a)(u))_1, \ldots, (D\varphi(a)(u))_k, 0, \ldots, 0).$$

Taking $u = (D\varphi(a))^{-1}(\varphi(a))$, we get

$$D(\psi \circ v)(a)(u) = \psi(v(a)). \qquad \square$$

We define here $w_U(x) = D\chi(e)(\mathrm{Ad}_{\bar{h}(v_U(x))}(v_U(x)))$, where we recall that $\bar{h}$ is defined in (6.14). We emphasize that changing h to $\bar{h}$ does not affect the role of w_U, in particular, Theorem 6.4 applies verbatim. We finally get that w_U is regular in the following sense.

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

LEMMA 6.11. *Suppose that v_U is a $\mathcal{C}^1$ subimmersion with a rank greater than 1 μ-almost everywhere. We get that w_U is $\mathcal{C}^1$ on a full measure open set such that*

$$\mu\Big(\bigcup_{\xi\in\mathbb{Z}^d\setminus\{0\}}\{x\in U : d\langle\xi|w_U(x)\rangle = 0\}\setminus\{x\in U : \langle\xi|w_U(x)\rangle = 0\}\Big) = 0.$$

Proof. By hypothesis, v_U is a subimmersion with a rank greater than 1 μ-almost everywhere. Thus, by Lemma 6.10, for μ-almost all $x\in U$, there exists a neighborhood W of x, V an open neighborhood of $v_U(x)$, $\psi : V\to V'\subset \mathrm{Lie}(G)$ a $\mathcal{C}^1$-diffeomorphism such that for all $a\in W$,

$$\psi(v_U(a))\in \mathrm{Im}(D(\psi\circ v_U)(a)).$$

Let $x\in U$ such that $\langle\xi|w_U(x)\rangle\neq 0$. Therefore,

$$\langle\xi|D\chi(e)(\mathrm{Ad}_{\bar{h}(v_U(x))}(\psi^{-1}(\psi(v_U(x)))))\rangle\neq 0.$$

We have

$$d\langle\xi|w_U(x)\rangle(u) = D(S_\xi\circ\psi^{-1})(\psi\circ v_U(x))(D(\psi\circ v_U)(x)(u)).$$

By Lemma 6.10, we can get u such that

$$d\langle\xi|w_U(x)\rangle(u) = D(S_\xi\circ\psi^{-1})(\psi\circ v_U(x))(v_U(x)) = \langle\xi|w_U(x)\rangle\neq 0.$$

Thus,

$$\mu\Big(\bigcup_{\xi\in\mathbb{Z}^d\setminus\{0\}}\{x\in U : d\langle\xi|w_U(x)\rangle = 0\}\setminus\{x\in U : \langle\xi|w_U(x)\rangle = 0\}\Big) = 0. \qquad \square$$

Proof of Corollary 6.6. By Lemma 6.11, we get that w_{U_j} is $\mathcal{C}^1$ and

$$\mu\Big(\bigcup_{\xi\in\mathbb{Z}^d\setminus\{0\}}\{x\in U_j : d\langle\xi|w_{U_j}(x)\rangle = 0\}\setminus\{x\in U_j : \langle\xi|w_{U_j}(x)\rangle = 0\}\Big) = 0.$$

Applying Theorem 6.4, we get the result. $\square$

6.4. *Measurable selection from the orbital group to the maximal torus.* We detail here how to build a measurable function h which fulfills all the requirements of (6.1). First, we ensure the existence of a measurable selection of the element g, which conjugates an element of G to another one in the fixed maximal torus T. To select this g, we will use the Kuratowski theorem proved in [**Par**].

THEOREM 6.12. (Kuratowski) *Let (X,d) be a Polish space and $(\Omega,\mathcal{F})$ a measurable space. Let F a multifunction from Ω to X such that for all $\omega\in\Omega$ and for all open sets U of X,*

$$F(\omega) = \overline{F(\omega)} \quad \text{and} \quad F^{-1}(U)\in\mathcal{F}.$$

There exists an $\mathcal{F}$-$\mathcal{B}(X)$ measurable function $f:\Omega\to X$ such that for all $\omega\in\Omega$,

$$f(\omega)\in F(\omega).$$

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

Now, we build the multifunction F, which allows us to identify a measurable function f such that for all $z \in G$, $f(z)z(f(z))^{-1} \in T$. Let

$$F : z \in G \mapsto \{g \in G : z \in gTg^{-1}\}.$$

Now, we need to have $F(z)$ closed and non-empty.

LEMMA 6.13. *For all $z \in G$, $F(z) \neq \emptyset$ is closed.*

Proof. Let $z \in G$. By Proposition 6.1(ii), we get that $F(z) \neq \emptyset$. Let $g \in \overline{F(z)}$. There exists $(g_n)_{n\in\mathbb{N}} \in (F(z))^{\mathbb{N}}$ such that

$$g_n \xrightarrow[n\to+\infty]{} g.$$

Thus, for all n, $g_n^{-1} z g_n \in T$. We know that T is closed and G is a Lie group, thus,

$$g^{-1} z g \in T. \qquad \square$$

Now, we have to prove that for all open sets U of G, $F^{-1}(U)$ is a measurable set.

LEMMA 6.14. *For all open sets U, $F^{-1}(U)$ is a measurable set.*

Proof. Let

$$P : (g, t) \in G \times T \mapsto gtg^{-1}.$$

Here, P is continuous because G is a Lie group. Let U be an open set of G. We see immediately that

$$F^{-1}(U) = P(U \times T).$$

We know that G is a compact connected Lie group; thus, G is metrizable and U is a countable union of compact sets. By compactness of T and continuity of P, $P(U \times T)$ is a countable union of compact set. Then, $F^{-1}(U)$ is a measurable set. $\square$

Now, by Lemmas 6.13, 6.14, and Theorem 6.12, there exists a measurable function $f : G \to G$ such that for all $z \in G$,

$$f(z)z(f(z))^{-1} \in T. \tag{6.17}$$

With the flow orbit, we can make a semi-fibration on the Lie group to use the torus property as in the previous theorem in the tori bundle case. We note that for all $v \in \mathrm{Lie}(G)$, $v_{\mathrm{inj}} := (\delta_{\mathrm{inj}}/(\|v\|_2 + 1))v \in B(0, \delta_{\mathrm{inj}})$. Let $h(v) = f(\exp(v_{\mathrm{inj}}))$. Writing $T' = h(v)^{-1} T h(v)$, by injectivity of exp on $B(0, \delta_{\mathrm{inj}})$ and the definition of f, $v_{\mathrm{inj}} \in \mathrm{Lie}(T')$. Thus, for all $t \in \mathbb{R}$, $tv \in \mathrm{Lie}(T')$. Taking the exponential shows that $H_v \subset T'$, since T' is closed. Thus, (6.1) is satisfied by h.

6.5. *Main examples of compact Lie groups where the results can be applied.*

Example 6.15. (Torus) The simplest example of a compact Lie group is a torus $\mathbb{T}^d$, which is abelian. This is the framework of the previous sections. In particular, with $\mathbb{T}^2$,

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

a measurable velocity vector $v(x)$, and for a single fiber to torus, the flow is

$$g_t : (x, y) \in \mathbb{T}^2 \mapsto (x, y + tv(x)).$$

Example 6.16. (Spinorial groups) The spin group for $n \geq 2$, $\mathrm{Spin}(n)$, is a compact and connected Lie group, which allows us to use it as an example.

Example 6.17. (Orthogonal groups) The orthogonal groups for $n \geq 2$, $\mathrm{SO}_n(\mathbb{R})$, are connected compact Lie groups, non-abelian if $n \geq 3$. We can consider the flow $g_t : M \in \mathrm{SO}_3(\mathbb{R}) \mapsto \exp(tA)M$ with A an antisymmetric matrix.

Example 6.18. (Special unitary group) A special unitary group for $n \geq 2$, $\mathrm{SU}_n(\mathbb{R})$, is also a connected compact Lie group and so we can use the previous theorems. Moreover, these Lie groups are even simply connected. They are always non-abelian groups. The simplest example is $\mathrm{SU}_2(\mathbb{R})$, which is isomorphic to the hypersphere $\mathbb{S}^3$ of $\mathbb{R}^4$. The flow

$$g_t : M \in \mathrm{SU}_2(\mathbb{R}) \mapsto \exp(tA)M$$

is well defined for $A \in \mathcal{M}_2(\mathbb{R})$ such that ${}^tA = -A^*$.

Acknowledgements. We would like to thank the anonymous referee, as well as Sébastien Gouezel and Thierry De La Rue for their time and valuable inputs to our manuscript. Their constructive suggestions and valuable comments were critical to improve the overall quality of the manuscript.

References

[AGZV] V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko. *Singularities of Differential Maps, Volume 2: Monodromy and Asymptotics of Integrals*. Birkhäuser, Boston, MA, 2012. Transl. from the Russian by H. Porteous and revised by the authors and J. Montaldi. Reprint of the 1988 hardback edition.

[Ath] J. Athreya. Logarithm laws and shrinking target properties. *Proc. Indian Acad. Sci. Math. Sci.* **119**(4) (2009), 541–557.

[Blu] C. Bluhm and L. Numbers. Rajchman measures, and small Cantor sets. *Proc. Amer. Math. Soc.* **128**(9) (2000), 2637–2640.

[Bou] N. Bourbaki. *Groupe et algèbre de Lie (Chapitre 2 et 3)*. Springer, Berlin, 2006.

[Bre] J. Brémont. Self-similar measures and the Rajchman property. *Ann. H. Lebesgue* **4** (2021), 973–1004.

[CK] J. Chaika and D. Constantine. Quantitative shrinking target properties for rotations and interval exchanges. *Israel J. Math.* **230** (2019), 275–334.

[CK2] N. Chernov and D. Kleinbock. Dynamical Borel-Cantelli lemmas for Gibbs measures. *Israel J. Math.* **122** (2001), 1–27.

[Del] A. Delzant. Groupes de Lie compacts et tores maximaux. *Sém. Henri Cartan* **12**(1) (1959–1960), 1–14.

[DF] D. Dolgopyat and B. Fayad. Deviation of ergodic sums for toral translation II. boxes. *Publ. Math. Inst. Hautes Études Sci.* **132** (2020), 293–352.

[Kah] J.-P. Kahane. Sur la distribution de certaines séries aléatoires, Colloque de théorie des nombres (Bordeaux, 1969). *Mém. Soc. Math. France* **25** (1971), 119–122.

[Kau] R. Kaufman. On the theorem of Jarnik and Besicovitch. *Acta Arith.* **39** (1981), 265–267.

[Kes] H. Kesten. Uniform distribution mod 1. *Ann. of Math. (2)* **71** (1960), 445–471.

[Kur] J. Kurzweil. On the metric theory of inhomogeneous diophantine approximations. *Stud. Math.* **15**(1) (1955), 84–112.

[Par] T. Parthasarathy. *Selection Theorems and their Applications (Lecture Notes in Mathematics, 263)*. Springer, Berlin, 1972.

[Lyo] R. Lyons. Seventy years of Rajchman measures. *Journal of Fourier Analysis and Applications Special Issue*, 1st edn. Ed. J. J. Benedetto. CRC Press, Boca Raton, FL, 1995, pp. 363–378.

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press

[Sol] B. Solomyak. Fourier decay for self-similar measures. *Proc. Amer. Math. Soc.* **149**(08) (2021), 1.

[Spr] V. G. Sprindzuk, *Metric Theory of Diophantine Approximations*. Trans. and Ed. R. A. Silverman and V. H. Winston. Wiley, Washington, DC, New York, 1979.

[Tho] D. Thomine. Keplerian shear in ergodic theory. *Ann. H. Lebesgue* **3** (2020), 649–676.

[Tse] J. Tseng. On circle rotations and the shrinking target properties. *Discrete Contin. Dyn. Syst.* **20**(4) (2008), 1111–1122.

[Xin] V. Xing. Dynamical Borel–Cantelli lemmas and applications. *Master's Thesis*, Mathematics, Faculty of Engineering, Lund University, 2020, LUTFMA-3402-2020.

[Zwo] M. Zworski. *Semiclassical Analysis (Graduate Studies in Mathematics, 138)*. American Mathematical Society, Providence, RI, 2012.

https://doi.org/10.1017/etds.2026.10360 Published online by Cambridge University Press