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Wandering polygons and recurrent critical leaves

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Abstract. Let T be a finite subset of the complex unit circle $\mathbb{S}^1$ and define $f : \mathbb{S}^1 \mapsto \mathbb{S}^1$ by $f(z) = z^d$. Let $\mathrm{CH}(T)$ denote the convex hull of T. If $\mathrm{card}(T) = N \geq 3$, then $\mathrm{CH}(T)$ defines a polygon with N sides. The N-gon $\mathrm{CH}(T)$ is called a *wandering N-gon* if for every two non-negative integers $i \neq j$, $\mathrm{CH}(f^i(T))$ and $\mathrm{CH}(f^j(T))$ are disjoint N-gons. A non-degenerate chord of $\mathbb{S}^1$ is said to be *critical* if its two endpoints have the same image under f. Then for a critical chord, it is natural to define its (forward) orbit by the forward iterates of the endpoints. Similarly, we call a critical chord *recurrent* if one of its endpoints is recurrent under f. The main result of our study is that a wandering N-gon has at least $N - 1$ recurrent critical chords in its limit set (defined in a natural way) having pairwise disjoint, infinite orbits. Using this result, we are able to strengthen results of Blokh, Kiwi, Levin and Thurston about wandering polygons of laminations. We also discuss some applications to the dynamics of polynomials. In particular, our study implies that if v is a wandering non-precritical vertex of a locally connected polynomial Julia set, then there exists at least $\mathrm{ord}(v) - 1$ recurrent critical points with pairwise disjoint orbits, all having the same ω-limit set as v. Thus, we likewise strengthen results about wandering vertices of polynomial Julia sets.

1. *Introduction*

It is well known that the dynamical behavior of critical points is closely related to a number of interesting dynamical phenomena exhibited by rational functions [**16**]. In particular, the presence of *recurrent* critical points is necessary for some types of complicated dynamics to occur for rational maps [**13**]. In addition, critical points and their analogs have been shown to play a significant role in the combinatorial and topological aspects of rational dynamical systems [**2, 11, 17**]. The aim of this paper is to establish and specify their impact on the existence and behavior of wandering polygons in the complex unit circle, an important phenomenon in the combinatorial theory of polynomials.

In what follows we use standard terminology without a formal introduction, while defining the less standard notions. Always let $f : \mathbb{S}^1 \mapsto \mathbb{S}^1$ denote a map on the complex

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unit circle, $\mathbb{S}^1$, defined by $f(z) = z^d$ ($d \geq 2$). Thus d will always be the degree of f. Also, for a set $B \subset \mathbb{S}^1$, let $\mathrm{CH}(B)$ denote the convex hull of B, and define $B_i = f^i(B)$. Finally, two closed sets $A, B \subset \mathbb{S}^1$ are said to be *unlinked* if $\mathrm{CH}(A) \cap \mathrm{CH}(B) = \emptyset$.

Definition. (Wandering polygon) Let $T \subset \mathbb{S}^1$ be a finite set with $N = \mathrm{card}(T) \geq 3$. Then T *forms a wandering N-gon* if:
(1) *non-precritical*—for every i, $\mathrm{card}(T_i) = N$;
(2) *pairwise unlinked*—for every $i \neq j$, T_i and T_j are unlinked.

Remark. There is no cyclic order preserving hypothesis in the definition above since it will be proven to be irrelevant.

Thurston [**17**] proved that wandering N-gons do not exist under the map $z \mapsto z^2$. About this result, he writes, 'The question whether this generalizes to higher degree, and if not what is the nature of counterexamples, seems to me the key unresolved question about invariant laminations' [**17**]. After Thurston, Kiwi was the first to make significant progress towards understanding the necessary conditions for the presence of a wandering polygon. This was done in the language of polynomial Julia sets [**10**], although stated below is his result in terms of a wandering polygon (compare Theorem A.2 in [**11**]).

THEOREM 1.1. (Kiwi [**11**]) *If $T \subset \mathbb{S}^1$ forms a wandering polygon under f, then* $\mathrm{card}(T) \leq d$.

Other results in this direction have been obtained by Levin in [**12**] (who was the first to discover a more general setting where wandering polygons do not exist, even when $d \geq 3$), Blokh and Levin in [**2**] and [**3**] (where several co-existing polygons are considered), and Blokh in [**1**] (where the cubic case is considered in detail). We discuss some of these results in §1.2, where full definitions of necessary notions are also given (including that of a *lamination*). In addition to the results discussed in §1.2, Blokh and Oversteegen have recently shown the existence of a wandering triangle under the maps $f(z) = z^d$ of degree $d \geq 3$ [**5**]; as well as the existence of an uncountable family of polynomials with *dendritic* (i.e. tree-like locally connected) Julia sets that admit a wandering non-precritical branch point (equivalently, one can say that the corresponding laminations admit a wandering triangle). With this knowledge, one might expect the existence of wandering N-gons with any number of sides, perhaps even several co-existing in the same lamination, providing d is sufficiently large—this makes necessary a detailed study of the dynamical implications of the existence of wandering N-gons. To this end, we use the notion of a polygon's limit set (defined below). From now on, suppose $T \subset \mathbb{S}^1$ forms a wandering N-gon.

Definition. (Limit set) Define the *limit set* of $\mathrm{CH}(T)$, denoted by $\mathcal{L}_{\omega(T)}$, as the collection of closed sets $A \subset \overline{\mathbb{D}}$ such that $\lim_{n\to\infty} \mathrm{CH}(T_{i_n}) = A$ (in the Hausdorff metric for the space $2^{\overline{\mathbb{D}}} = \{B \subset \overline{\mathbb{D}} : B \text{ closed}\}$; see [**15**, Ch. 4]), where $\{i_n\}$ is a strictly increasing, infinite subsequence.

It is not too hard to see from the definition of a wandering polygon that the area of $\mathrm{CH}(T_i)$ must tend to zero, so that $\mathcal{L}_{\omega(T)}$ is a collection of chords (including degenerate, i.e. singleton points) of $\mathbb{S}^1$ that are pairwise disjoint in $\mathbb{D}$ (moreover, that their union is a compact subset of $\overline{\mathbb{D}}$); any element of $\mathcal{L}_{\omega(T)}$ is called a *leaf*.

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A non-degenerate chord of $\mathbb{S}^1$ is called *critical* if both of its endpoints have the same image under f. For a critical chord q, define its *critical value* as the unique image of the endpoints of q, denoted $f(q)$. Critical chords in the limit set $\mathcal{L}_{\omega(T)}$, called *critical leaves*, play a significant role in the dynamics of T. Some of the intimate relationship between critical chords and wandering polygons is revealed in [**2, 11, 17**]. We further specify this relationship by proving the existence of $N-1$ *recurrent* critical leaves, which are *recurrent* critical chords in the limit set $\mathcal{L}_{\omega(T)}$. We will now give the definition of a recurrent critical chord.

Definition. (Dynamics of a critical chord) Let ℓ be a critical chord with critical value $f(\ell) = v$. Define the ω-limit set of ℓ by $\omega(\ell) = \omega(v)$. We say that ℓ is *recurrent* if $\omega(\ell) \cap \ell \neq \emptyset$.

As a result of our study, we can state Theorem 1.2, below, which follows by putting together Theorem 3.6 and the Recurrence Theorem 3.8.

THEOREM 1.2. *If $T \subset \mathbb{S}^1$ forms a wandering N-gon, then there exist at least $N-1$ recurrent critical leaves with pairwise disjoint infinite orbits, all having the same ω-limit set $X \subset \mathbb{S}^1$. Moreover, $p \cap X \neq \emptyset$ for every $p \in \mathcal{L}_{\omega(T)}$.*

More generally, we also consider the possibility of several wandering polygons co-existing in the following way.

Definition. Let Γ be a collection of pairwise disjoint finite subsets of $\mathbb{S}^1$. Then Γ is said to *form a wandering collection of polygons* if:

(1) for every set $T \in \Gamma$, T forms a wandering polygon under f;
(2) for every two different sets $A, B \in \Gamma$ and any two integers $n, m \geq 0$, A_n and B_m are unlinked.

Given a wandering collection Γ of polygons, define $\mathcal{L}_{\omega(\Gamma)} = \bigcup_{T\in\Gamma} \mathcal{L}_{\omega(T)}$. Let R denote the maximum number of recurrent critical chords in $\mathcal{L}_{\omega(\Gamma)}$ with pairwise disjoint infinite orbits. Let Ω be the collection of distinct ω-limit sets of recurrent critical chords in $\mathcal{L}_{\omega(\Gamma)}$. Using Theorem 1.2 and a counting argument, we prove in §4 the following theorem.

THEOREM 1.3. *If Γ is a wandering collection of polygons, then*

$$\operatorname{card}(\Gamma) \leq \sum_{T\in\Gamma} (\operatorname{card}(T) - 2) \leq R - \operatorname{card}(\Omega) \leq d - 1 - \operatorname{card}(\Omega).$$

Our results allow us to improve some known results on wandering polygons in laminations [**2–4, 17, 19, 20, 27**]. Consequently, we also improve results on wandering vertices in locally connected polynomial Julia sets. The history of this subject and applications of our results are discussed in §1.2, in the framework of laminations. Let us first comment on the techniques of our study.

1.1. *Techniques.* Our study builds on the techniques of *orbit portraits*, which have been used in the past by Milnor–Goldberg [**9**], Thurston [**17**] and Kiwi [**10, 11**]. Another technique of approach uses *growing trees*, which is the approach taken by Blokh and Levin. Both have been successful techniques; it would be interesting to know whether our results

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may be accomplished with growing trees. Although our approach is quite different from *growing trees*, both of these approaches are similar in that they mostly use combinatorics and topology. Even more so than in Kiwi's study, for instance, which uses holomorphic conditions to find critical points in the limit set of a wandering vertex (whereas we use the Hausdorff metric for $2^{\overline{\mathbb{D}}}$ to find critical chords). Hence, philosophically, one would expect that it is also possible to achieve our results using growing trees. Indeed, a mathematical justification for this philosophical reasoning can be found in [**1**].

1.2. *Applications: invariant laminations and polynomial Julia sets.* As previously mentioned, the motivation for studying wandering polygons lies on their implications in polynomial dynamics. To explain this connection, one needs to introduce the notion of an *invariant lamination*. In our study, we will not define an invariant lamination as Thurston did in [**17**]. Instead, we will define laminations as certain equivalence relations of $\mathbb{S}^1$ (compare to [**6, 14**]). Recall that two closed sets $A, B \subset \mathbb{S}^1$ are said to be *unlinked* if $\mathrm{CH}(A) \cap \mathrm{CH}(B) = \emptyset$.

Definition. (Invariant lamination) An equivalence relation $\sim$ of $\mathbb{S}^1$ is called an *invariant lamination* if it satisfies the following four conditions.

(1) *Closed*—the graph of $\sim$ is closed in $\mathbb{S}^1 \times \mathbb{S}^1$.
(2) *Pairwise unlinked*—any two equivalence classes are unlinked.
(3) *Forward invariant*—if $B \subset \mathbb{S}^1$ is an equivalence class then $f(B)$ is also an equivalence class.
(4) *Gap invariant*—if $B \subset \mathbb{S}^1$ is an equivalence class with $\mathrm{card}(B) \geq 3$, then $f|_B$ is a covering map with positive orientation.

Given an invariant lamination $\sim$, define the quotient map $\pi : \mathbb{S}^1 \mapsto J_\sim = J$, where $J = \mathbb{S}^1/\sim$ denotes the quotient space defined by $\sim$. By the closed property of $\sim$, J is a locally connected continuum (by a *continuum* we mean a compact, connected metric space). Without loss of generality, since $\sim$ is unlinked, we can assume J is imbedded in the Riemann Sphere $\mathbb{C}_\infty$. The forward invariance of $\sim$ allows us to define the *induced* map $F : J \mapsto J$ by $F = \pi \circ f \circ \pi^{-1}$. The dynamics of F is quite similar to the dynamics of polynomials on their Julia sets. Thus, such a continuum J is a *topological Julia set* (of F). Fittingly, each component of $\mathbb{C}_\infty \setminus J$ is called a *Fatou domain*.

It can be shown that every locally connected polynomial Julia set is a topological Julia set. In fact, for the locally connected Julia set J_P of a polynomial P of degree d, we can naturally define its *corresponding invariant lamination* in the following way. Let $\mathbb{D} \subset \mathbb{C}_\infty$ be the open unit disk. Then infinity is a super-attracting fixed point of P. Let us denote its basin of attraction by A_∞, and define the Riemann map $\psi : \mathbb{D} \mapsto A_\infty$ such that $\psi^{-1} \circ P \circ \psi = z^d$. Note that $\partial A_\infty = J_P$. Since J_P is locally connected, ψ extends to a continuous surjection $\hat{\psi} : \overline{\mathbb{D}} \mapsto \overline{A}_\infty$. The *corresponding (to P) invariant lamination* $\sim$ is defined by the equivalence classes $\{\hat{\psi}^{-1}(y)\}_{y \in J}$ (it is easy to see that $\sim$ has properties (1)–(4)). We can think of $\pi = \hat{\psi}|_{\mathbb{S}^1}$ as the quotient map $\pi : \mathbb{S}^1 \mapsto J_P \approx J_\sim = \mathbb{S}^1/\sim$ such that, on the Julia set J, $P|_J = \pi \circ f \circ \pi^{-1}$ (even though π is not one-to-one, the formula on the right is well-defined).

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In the context of laminations that are not defined by polynomials, let $J_\sim = J \subset \mathbb{C}_\infty$ denote the topological Julia set of the induced map F. We will always use $\sim$ to denote an invariant lamination. Given J, F and the corresponding $\sim$, use π to denote the quotient map defined by $\sim$ such that $F = \pi \circ f \circ \pi^{-1}$. Let us give some basic definitions for points in a topological Julia set.

For a point $x \in J$, define the *order* of x by $\text{ord}(x) = \text{card}(\pi^{-1}(x))$. Note that $\text{ord}(x) = \text{card}(\ Q : Q \text{ is a component of } J \setminus \{x\}\ )$. A point $b \in J$ is called an *endpoint* (of J) if $\text{ord}(b) = 1$, a *cutpoint* (of J) if $\text{ord}(b) \geq 2$, and a *vertex* or *branch point* (of J) if $\text{ord}(b) \geq 3$.

Definition. (Wandering vertex) A point $b \in J$ is called a *wandering vertex* if b is a vertex of J such that for every non-negative integer i, $F^i(b)$ is neither critical nor periodic.

Remark. It follows that if b is a wandering vertex, then $\text{ord}(b) = \text{ord}(F^i(b))$ for every $i \geq 0$.

Clearly, the $\sim$-classes that form wandering polygons correspond to wandering vertices in J; both of which are, necessarily, non-precritical. Kiwi studied the dynamics of such vertices in [**10**]. The main result of his study, stated below, provides us with yet another example of the influence critical points have on the dynamics of polynomials. We need the following definition to state his result.

Definition. (Narrow) A point $x \in J$ is *narrow* if it does not lie in the boundary of any bounded Fatou domain and $\text{ord}(x) = 1$.

Remark. A point $x \in J$ is narrow if and only if for every arc $I \subset J$ with $x \in I$, x is an endpoint of I.

THEOREM 1.4. (Kiwi [**10**]) *Let J be the locally connected Julia set of a degree d polynomial, and suppose b is a wandering vertex of J. Then there exist at least* $\text{ord}(b) - 1$ *narrow critical values in the ω-limit set of b. Consequently,* $\text{ord}(b) \leq d$.

We previously mentioned that the presence of recurrent critical points is necessary for some specific types of complicated dynamics to occur in rational maps. It turns out that wandering vertices are included in this class of dynamics. Indeed, the ω-limit set of a wandering vertex in J coincides with the limit set of some recurrent critical point [**3**]. In fact, the ω-limit set of a wandering vertex coincides, moreover, with the ω-limit set of at least two recurrent critical points. This was shown by Blokh in [**1**] for cubic topological Julia sets. We show this for arbitrary degree. In addition, we are able to strengthen these results, while simultaneously improving the bounds given in Theorem 1.4.

THEOREM 1.5. *Let J be a topological Julia set, and suppose b is a wandering vertex of J. Then there exists, at least,* $\text{ord}(b) - 1$ *recurrent critical points from pairwise disjoint grand orbits in $\omega(b)$, each one having a narrow critical value and with their ω-limit set coinciding with $\omega(b)$.*

Theorem 1.2 is the fundamental step in proving Theorem 1.5; however, it also relies upon Proposition 2.10.

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It was Levin who introduced, in [**12**], the idea of studying the dynamics of wandering polygons that come from a lamination by means of studying the dynamics of the map on its topological Julia set (the main aim of [**12**] was to introduce and study an important notion of *backward stability*). Ideas of [**12**] were further developed in [**2**] and [**3**]. In particular, this was done in [**2**] to consider, for the first time, the possibility of several wandering polygons (with pairwise disjoint orbits) co-existing in an invariant lamination. To state the main result from [**2**] in this direction, we need the following definition.

Definition. (All-critical) Let $\sim$ be an invariant lamination. A non-degenerate equivalence class C of $\sim$ is said to be *all-critical* in $\sim$, denoted *all-c*, if all points in C have the same image under f.

All-critical classes correspond under the quotient map to critical points in the topological Julia set whose values (i.e. images) are endpoints. Given an invariant lamination $\sim$, let $k_\sim$ be the maximum number of all-c classes from pairwise disjoint infinite orbits.

THEOREM 1.6. (Blokh and Levin [**2**]) *Let Γ be a non-empty collection of equivalence classes of $\sim$ forming wandering polygons with pairwise disjoint orbits. Then*

$$\mathrm{card}(\Gamma) \le \sum_{T\in\Gamma}(\mathrm{card}(T)-2) \le k_\sim - 1 \le d-1.$$

Theorem 1.6 strengthens Kiwi's results. The approach taken by Blokh and Levin is very different from Kiwi's; it involves *growing trees*, which were introduced in [**12**] and [**2**], and inspired by the theory of Hubbard trees (see [**7, 8**]). This approach has proven fruitful in the further study of wandering polygons (see [**1, 3**]).

As a result of our study, we can replace $k_\sim$ with a finer characteristic of the lamination, which involves the number of recurrent all-c classes and the number of their limit sets. Thus, let us introduce these dynamical notions.

Definition. (Recurrent) Let C be an all-c class in the invariant lamination $\sim$. Define its ω-limit set by $\omega(C) = \omega(s)$ for any $s \in C$. Then, C is called *recurrent* if $\omega(C) \cap C \neq \emptyset$.

We will call an *infinite* subset $W \subset \mathbb{S}^1$ a *recurrent all-c limit set*, with respect to the invariant lamination $\sim$, if there is a recurrent all-c class C in $\sim$ with $\omega(C) = W$ (it follows that the orbit of C must be infinite). Denote by $R_\sim$ the maximal number of recurrent all-c classes in $\sim$ with pairwise disjoint infinite orbits; then $R_\sim \le k_\sim$. Now, let $\Omega_\sim$ be the collection of *all* the recurrent all-c limit sets. Theorem 1.3 implies Theorem 1.7, which strengthens Theorem 1.6.

THEOREM 1.7. *Let Γ be a collection of equivalence classes forming wandering polygons with pairwise disjoint orbits. Then*

$$\mathrm{card}(\Gamma) \le \sum_{T\in\Gamma}(\mathrm{card}(T)-2) \le R_\sim - \mathrm{card}(\Omega_\sim) \le d-1-\mathrm{card}(\Omega_\sim).$$

Likewise, we strengthen the bound from [**2**] on the number of wandering vertices in a topological Julia set J. To this end, we need some notation. Namely, let R_J be the

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maximum number of recurrent critical points in J with narrow values and pairwise disjoint infinite orbits. In addition, let Ω_J be the collection of distinct ω-limit sets of recurrent critical points with narrow values and infinite orbits. As a corollary of Theorem 1.7, we have the following.

THEOREM 1.8. *Let J be a topological Julia set. Let Γ be a non-empty collection of wandering vertices in J with pairwise disjoint orbits. Then*

$$\text{card}(\Gamma) \leq \sum_{b\in\Gamma}(\text{ord}(b)-2) \leq R_J - \text{card}(\Omega_J) \leq d-1-\text{card}(\Omega_J).$$

2. *Holes in a finite set*

In this section, we introduce some notation and make some introductory observations. We take the measure of the full angle in $\mathbb{S}^1$ to be equal to 1 and define the length of an arc $\gamma \subset \mathbb{S}^1$ as its angle measure, denoted $\text{len}(\gamma)$. For any two points $v, w \in \mathbb{S}^1$, use $[v, w]$ to denote the arc in $\mathbb{S}^1$ running counterclockwise from v to w.

For a closed set $B \subset \mathbb{S}^1$, call each component of $\mathbb{S}^1 \setminus B$ a *hole* in B. Thus if two sets Q and B are unlinked, then Q is contained one of the holes in B. The following proposition makes this observation more precise.

PROPOSITION 2.1. *Let $Q, B \subset \mathbb{S}^1$ be unlinked sets such that* $\text{card}(B), \text{card}(Q) \geq 3$. *Suppose there exists a $\tau > 0$ such that both Q and B have at least two holes of length $\geq \tau$. Then Q is contained in a hole in B whose length is $> \tau$.*

If $B \subset S^1$ then we say that a map $g|_B$ is *orientation preserving* if it is injective and preserves the cyclic order on any three points of B. To continue, we introduce the following notation, in §2.1 and 2.2, for a finite set $B \subset \mathbb{S}^1$ with $\text{card}(B) = M \geq 2$.

2.1. *Ordering holes by size.*

Denote the holes in B by $\{H_k(B)\}_{k=1}^M$ such that $\text{len}(H_k(B)) \geq \text{len}(H_{k-1}(B))$ for every $k > 1$, and define $s_k(B) = \text{len}(H_k(B))$. Even though the above numbering of the H_k terms is not always well-defined, the numbers s_k are; still, we make a choice for the H_k terms and fix it for the given set B. Note that $\bigcup_{k=1}^M \overline{H_k(B)} = \mathbb{S}^1$, so $\sum_{k=1}^M s_k(B) = 1$. Also, note that each hole $H_k(B)$ is an open arc in the circle whose closure intersects B at exactly two points. These two points define a chord which is one of the edges of the polygon $\text{CH}(B)$. Let us denote this chord by $e_k(B)$. So the chord $e_k(B)$ denotes the edge of the polygon $\text{CH}(B)$ that corresponds to the kth-smallest hole in B.

2.2. *Image-holes and remainders.*

For every $k \in \{1, \ldots, M\}$, define $\tilde{s}_k(B) = \min\{s_k(B) - (j/d) \mid (j/d) \leq s_k(B) \text{ for } j = 0, 1, \ldots, d-1\}$. We call $\tilde{s}_k(B)$ the *remainder of* $s_k(B)$. Remainders are useful for the analysis of the lengths of holes in $f(B)$. Next, for a hole $H = (u, w)$ in B, the arc $(f(u), f(w))$ is called the *image-hole* of H. This terminology is justified in Lemma 2.3, below, where we show that the remainders, $\{\tilde{s}_k(B)\}_{k=1}^M$, of the lengths of the holes in B determine the lengths of the holes in $f(B)$. To do this, first observe the following theorem [**4**, Theorem 2.1].

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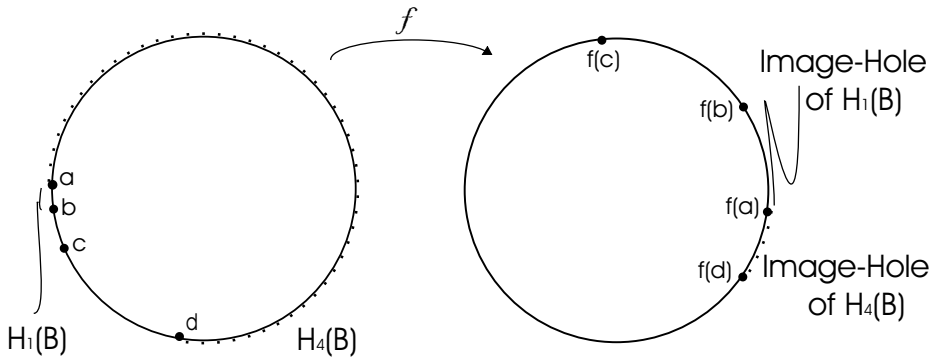


FIGURE 1. A subset $B = \{a, b, c, d\}$ of the circle and its image under $f(z) = z^4$.

THEOREM 2.2. *If $B \subset S^1$ is a closed set and $f|_B$ is injective then $f|_B$ preserves orientation if and only if $S^1 \setminus B$ contains $d-1$ pairwise disjoint open intervals of length $1/d$.*

Now we are ready to prove Lemma 2.3.

LEMMA 2.3. *Let $B \subset \mathbb{S}^1$ be a finite set with* $\text{card}(B) = M \geq 2$, *such that $f|_B$ is injective. If $f|_B$ is orientation preserving, then for every $k \in \{1, \ldots, M\}$, the* image-hole *of $H_k(B) = (u, w)$ is a well-defined hole $(f(u), f(w))$ in $f(B)$ of length $ds_k(B) \pmod 1 = d\tilde{s}_k(B)$. Moreover, $f|_B$ is orientation preserving if and only if $\sum_{k=1}^{M} \tilde{s}_k(B) = 1/d$.*

Proof. The first claim in the lemma is left to the reader. Let us show that this claim implies the rest of the lemma. If $f|_B$ is orientation preserving then, by the claim, $\sum_{k=1}^{M} \tilde{s}_k(B) = 1/d$ because there is a one-to-one correspondence between the holes in B and their image-holes, which are the holes in $f(B)$ (note that the lengths of the holes in $f(B)$ sum up to 1).

Now, suppose that $\sum_{k=1}^{M} \tilde{s}_k(B) = 1/d$. Observe that $d(s_k(B) - \tilde{s}_k(B))$ is the maximum number of pairwise disjoint open arcs of length $1/d$ that are contained in $\bigcup_{k=1}^{M} H_k(B) = \mathbb{S}^1 \setminus B$. So the number of pairwise disjoint open arcs with length $1/d$ in the complement of B is $d\left[\sum_{k=1}^{M}(s_k(B) - \tilde{s}_k(B))\right] = d\left[1 - \sum_{k=1}^{M} \tilde{s}_k(B)\right] = d - 1$. By Theorem 2.2, this implies that $f|_B$ is orientation preserving. □

The image-hole of H does not necessarily coincide with the image of H under f. However, $f(H)$ is equal to the image-hole of H when $\text{len}(H) < 1/d$. In particular, as an awkward consequence of the first claim in Lemma 2.3, the image-hole of the kth-smallest hole in B is not necessarily the kth-smallest hole in $f(B)$ (see Figure 1).

However, Lemma 2.3 does give a one-to-one correspondence between the holes in B and their image-holes, the holes in $f(B)$. Using this, one can say that $f : B \mapsto f(B)$ *induces the permutation* $k \mapsto j_k$, defined so that the (j_k)th-smallest hole in $f(B)$ is the image-hole of the kth-smallest hole in B. Our study relies heavily upon the observation that for large i, the only possible permutations induced by $T_i \mapsto f(T_i) = T_{i+1}$ are r-cycles of the form $(k, k+1, \ldots, k+r)$ *with* $k + r \in \{N-1, N\}$. For example, pictured below in Figure 2 are all the possible permutations induced by a wandering 4-gon.

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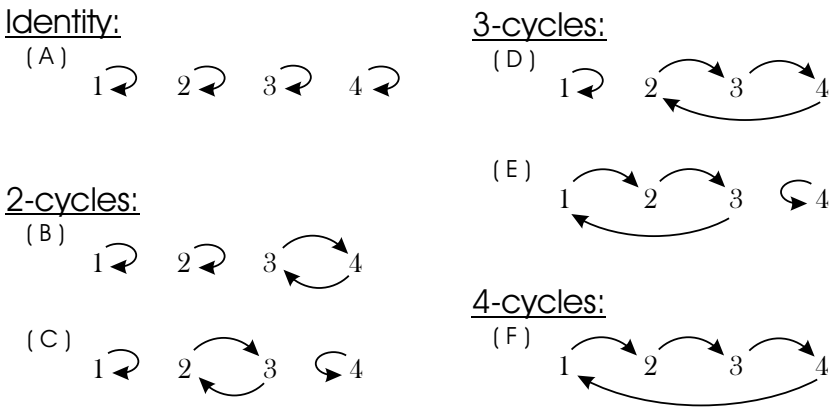


FIGURE 2. The complete list of possible permutations that can be induced by $f(T_i) = T_{i+1}$, for large i, provided that T forms a wandering 4-gon under $f(z) = z^d$ (for any degree $d \geq 3$). The 4-cycle, labeled (F), is the permutation induced by the map $f : B \mapsto f(B)$ in Figure 1.

2.3. *Holes in a wandering N-gon.* The motivation for understanding the permutations that can be induced by a wandering N-gon is that these permutations dictate the dynamics of *jumping critical leaves*, defined in §3.2; these are critical leaves that satisfy a specific relationship with $\mathrm{CH}(T_i)$ at the moment i that $f : T_i \mapsto T_{i+1}$ induces certain non-identity permutations. Before describing these permutations and their relationship with critical leaves, we want to establish that each map $f : T_i \mapsto T_{i+1}$ induces a well-defined permutation (as described). In particular, we need that the map $f : T_i \mapsto T_{i+1}$ is order-preserving (Proposition 2.4) and that the holes in T_i all have distinct sizes (Proposition 2.7)—so that our numbering of the holes is unique. Let us begin by showing the first of these two propositions; it's proof relies upon Theorem 2.2.

PROPOSITION 2.4. *Let $\mathcal{A}$ be a collection of pairwise unlinked closed sets in $\mathbb{S}^1$ such that f is injective on each set. Then there are no more than $d-1$ sets $A \in \mathcal{A}$ for which $f|_A$ is not orientation preserving.*

Proof. For any collection $\mathcal{A}$ of pairwise unlinked closed subsets of $\mathbb{S}^1$, let $c(\mathcal{A})$ denote the cardinality of the collection of distinct sets $A \in \mathcal{A}$ for which $f|_A$ is not orientation preserving. We want to show that for such every collection $\mathcal{A}$ such that f is injective on each set, $c(\mathcal{A}) \leq d-1$. We prove this bound by means of induction on $d \geq 1$.

The base case is $d = 1$. If $f(z) = z$, then, clearly, for every collection $\mathcal{A}$ of pairwise unlinked sets, $c(\mathcal{A}) = 0$. So next, we assume by induction that for all $k < d$, if $f(z) = z^k$, then every collection $\mathcal{A}$ of pairwise unlinked sets such that f is injective on each set satisfies $c(\mathcal{A}) \leq k-1$. Set $f(z) = z^d$ and let $\mathcal{A}$ be such a collection for f. We want to prove that $c(\mathcal{A}) \leq d-1$.

Without loss of generality, we suppose there exists a set $A \in \mathcal{A}$ such that $f|_A$ is not orientation preserving. Note that by assumption, $f|_A$ is injective. Let $\mathcal{H}$ be the collection of distinct holes in A. For every $H \in \mathcal{H}$, define the collection of pairwise unlinked sets $\mathcal{A}_H = \{B \in \mathcal{A} \mid B \subset H\}$. Recall that, since all sets in $\mathcal{A}$ are pairwise unlinked, for every set $B \in \mathcal{A}$ with $B \neq A$, there is a unique $H \in \mathcal{H}$ such that $B \subset H$. Consequently, $c(\mathcal{A}) - 1 = \sum_{H \in \mathcal{H}} c(\mathcal{A}_H)$. We will show that $\sum_{H \in \mathcal{H}} c(\mathcal{A}_H) \leq d-2$.

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To do this, note that for every $H \in \mathcal{H}$, there is a unique integer j_H such that $j_H/d \leq \text{len}(H) < (j_H+1)/d$. Since $f|_A$ is injective, while not orientation preserving, we have that $\sum_{H\in\mathcal{H}} j_H \leq d-2$ (Theorem 2.2). Thus it suffices to show that $c(\mathcal{A}_H) \leq j_H$ for every $H \in \mathcal{H}$. Pick $H \in \mathcal{H}$ and note that (by definition of j_H) there must be a closed arc of length $(j_H+1)/d$ that contains $\overline{H}$. Given such an arc γ, we can define a continuous orientation preserving map $\pi : \gamma \mapsto \mathbb{S}^1$ that identifies only the endpoints of γ (in particular, so that $\pi|_H$ is injective) and defined in such a way that $g \circ \pi = f|_\gamma$, where $g(z) = z^{j_H+1}$. Since $\pi|_H$ is injective and orientation preserving, the collection of sets $\mathcal{V} = \{\pi(B) \mid B \in \mathcal{A}_H\}$ is pairwise unlinked and in a one-to-one correspondence (induced by π) to the collection $\mathcal{A}_H$. Moreover, for every set $B \in \mathcal{A}_H$, $\pi|_B$ is an orientation preserving bijection. Thus we are done, by induction, because j_H is an upper bound for the number of distinct sets $V \in \mathcal{V}$ with $g|_V$ not being orientation preserving. □

We note that by Lemma 2.3 and the proposition above it follows that for large i there is a well-defined bijection between the holes in T_i and their image-holes, the holes in T_{i+1}. The next proposition, which is a consequence of the unit disk having finite area, will help to show that the holes in T_i all have distinct lengths. It's proof is left to the reader.

PROPOSITION 2.5. *If $T \subset \mathbb{S}^1$ form a wandering N-gon then $s_k(T_i) \to 0$ as $i \to \infty$, for $1 \leq k \leq N-2$.*

Before showing that the holes in T_i have distinct sizes, we note that by Propositions 2.5 and 2.4 we can make the following assumption without loss of generality.

STANDING ASSUMPTION 2.6. *If a set $T \subset \mathbb{S}^1$ forms a wandering N-gon then we may assume that $f|_{T_i}$ preserves the orientation and for every $k \in \{1,\ldots,N-2\}$ that $1/dN > s_k(T_i)$, so that the image-hole of $H_k(T_i)$ is of length ds_k $(i = 1, 2, \ldots)$.*

From now on, let us use T to denote a finite set which forms a wandering N-gon and satisfies Standing Assumption 2.6. Then the image-holes of any hole in T are all well-defined and not degenerate for all powers of f. Moreover, the lengths of the image-holes of a hole H depend only on the length of H. Thus, using Standing Assumption 2.6, we can show that the lengths of any two distinct holes in T_i are different.

PROPOSITION 2.7. *Let T be as in Standing Assumption 2.6. For every integer $i \geq 0$, the lengths of the holes in T_i are pairwise distinct; moreover, their remainders are also pairwise distinct. In particular,*

$$s_N(T_i) > s_{N-1}(T_i) > \cdots > s_1(T_i).$$

Proof. By Lemma 2.3, a hole in T has infinitely many image-holes of length greater than $1/d$. By Standing Assumption 2.6, any hole in T_i whose length is greater than $1/d$ is one of the two longest holes in T_i. Hence, if two holes in T have the same length, then there exist two non-negative integers $q \neq p$ with $s_{N-1}(T_p) = s_N(T_p) \geq s_{N-1}(T_q) = s_N(T_q) > 1/d$, which contradicts the fact that T_p and T_q are unlinked (being unlinked implies that $H_{N-1}(T_p)$ or $H_N(T_p)$ is contained in a hole in T_q, while each hole in T_q has length at most equal to $s_N(T_p) = s_{N-1}(T_p)$). The claim about the remainders follows from Lemma 2.3. □

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In conclusion, by Standing Assumption 2.6 and Proposition 2.7, we can apply Lemma 2.3 to each map $f : T_i \mapsto T_{i+1}$ in order to define the induced permutation $k \mapsto j_k$ as we described earlier, so that the j_kth-smallest hole in T_{i+1} is the image-hole of the kth-smallest hole in T_i.

Next, we would like to establish the technical results that will be used (in §3.2) to describe the permutations that can be induced by $f : T_i \mapsto T_{i+1}$ and their relationship with critical leaves. We begin with the following definitions.

2.3.1. *Critical hole.* By Proposition 2.7, there is a unique integer denoted by $cr(i) = cr \in \{1, \ldots, N\}$ such that $\tilde{s}_{\mathrm{cr}}(T_i)$ is minimal among all the remainders of holes whose lengths are greater than $1/d$. By Standing Assumption 2.6, we see that $cr \in \{N-1, N\}$. We call $H_{\mathrm{cr}}(T_i)$ the *critical hole* in T_i. For certain values of T_i, the image-hole of the critical hole will contain the critical value of the critical leaf 'closest' to the polygon $\mathrm{CH}(T_i)$; later on, this will give us information on the dynamics of this critical leaf.

Next, we introduce a notion of distance between two chords that are disjoint in the unit open disk $\mathbb{D}$. This notion may be used to define a metric on any collection of chords that are pairwise disjoint in $\mathbb{D}$, which is equivalent to the Hausdorff metric.

2.3.2. *The ρ-metric.* For two chords p, q that are disjoint in $\mathbb{D}$, there exist a unique component of $\overline{\mathbb{D}} \setminus (p \cup q)$ whose closure contains $p \cup q$. Denote the closure of this component by $C_{p,q}$. Note that $C_{p,q} \cap \mathbb{S}^1$ forms two arcs, or possibly an arc union a point. Let $\rho(p, q)$ denote the sum of the lengths of these two arcs. We will call $\rho(p, q)$ the *amount of arc between p and q*. Let us observe some immediate properties of ρ.

LEMMA 2.8. *Let p be a chord and $\{q_i\}_{i=1}^{\infty}$ a sequence of chords disjoint from p.*

(1) *If l is a chord such that l separates p from q_1 in $\mathbb{D}$ then $\rho(p, l) + \rho(q_1, l) = \rho(p, q_1)$.*
(2) *If q_1 and p are both critical chords then $\rho(p, q_1) \geq 1/d$.*
(3) *$\lim_{i\to\infty} q_i = p$ (in the Hausdorff metric on $2^{\mathbb{D}}$) if and only if $\rho(p, q_i)$ converges to zero.*

By Lemma 2.8, we can use ρ to determine if the chords from a subsequence of $\mathrm{CH}(T_i)$ converge onto a given critical chord. To this end, we need Lemma 2.9, which describes the relationship between the remainder of a specific hole in T_i and the amount of arc between its corresponding edge and certain critical chords. The proof of this lemma is left to the reader; recall that $e_k(T_i)$ denotes the edge of the polygon corresponding to the hole $H_k(T_i)$. See Figure 3 for an illustration of the following lemma.

LEMMA 2.9. *Suppose for some $k \in \{1, \ldots, N\}$ there is a $j \in \{1, \ldots, d-1\}$ such that $(j+1)/d > s_k(T_i) > j/d$. Let $[c, d]$ be an arc of length j/d contained in $H_k(T_i)$ and define the critical chord $q = \overline{cd}$. Then $\rho(e_k(T_i), q) = \tilde{s}_k(T_i)$, and $f(q)$ lies in the image-hole of $H_k(T_i)$.*

2.4. *Critical leaves exist.* Recall that a critical chord is called a *critical leaf* if it appears as the limit of chords from a subsequence of the polygons $\mathrm{CH}(T_i)$. In this section,

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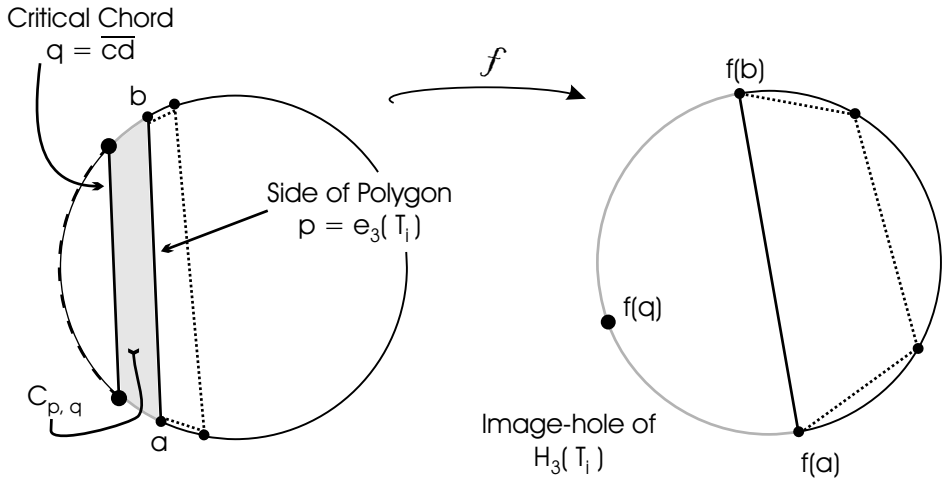


FIGURE 3. An illustration of Lemma 2.9 with $f(z) = z^4$ and $\frac{1}{2} > s_3(T_i) > \frac{1}{4}$. Observe that $\text{len}(C_{p,q} \cap \mathbb{S}^1) = \rho(e_3(T_i), q) = \tilde{s}_3(T_i)$.

we will prove that critical leaves exist. Also, we include a detailed account of how the critical leaves appear. We believe this is the foundation for understanding the relationship between the wandering polygon and the dynamics of its critical leaves.

Let us begin with the following definition (see Figure 4 for an illustration).

Definition. A point $x \in S^1$ is called *T-narrow* if there exists a subsequence T_{i_n} such that $\lim_{n\to\infty} \text{CH}(T_{i_n}) = x$ and $x \in \bigcup_{j=1}^{N-1} H_j(T_{i_n})$ for every n.

Proposition 2.10 immediately follows from Lemma 2.8, the properties of ρ and the definitions; the proof is left to the reader.

PROPOSITION 2.10. *Every critical leaf has T-narrow image. Consequently, for every critical leaf $p \in \mathcal{L}_{\omega(T)}$, $\text{CH}(T_i) \cap p = \emptyset$ for every i.*

To conclude this section, we describe the relationship between critical remainders and critical leaves.

THEOREM 2.11. *If T forms a wandering N-gon then critical leaves exist. In particular, there exists a positive $\varepsilon_T < 1/dN$ such that for every non-negative integer i, if $\tilde{s}_{\text{cr}}(T_i) < \varepsilon_T$ then there is a unique critical leaf q such that*

$$\rho(e_{\text{cr}}(T_i), q) = \tilde{s}_{\text{cr}}(T_i).$$

Consequently, in this case, the critical value $f(q)$ lies in the image-hole of the critical hole $H_{\text{cr}}(T_i)$.

Proof. By Proposition 2.5, for at least $N-2$ holes H in T the lower limit of the sequence of lengths of their image-holes is 0. Fix one such hole H. Denote by H^i the ith image-hole of H (so that $H^0 = H$), and by e^i the chord connecting the endpoints of H^i. Then there exists a sequence k_i such that $\text{len}(H^{k_i}) = \min\{\text{len}(H^j)\ 0 \le j \le k_i\}$, so $\text{len}(H^{k_i}) \searrow 0$.

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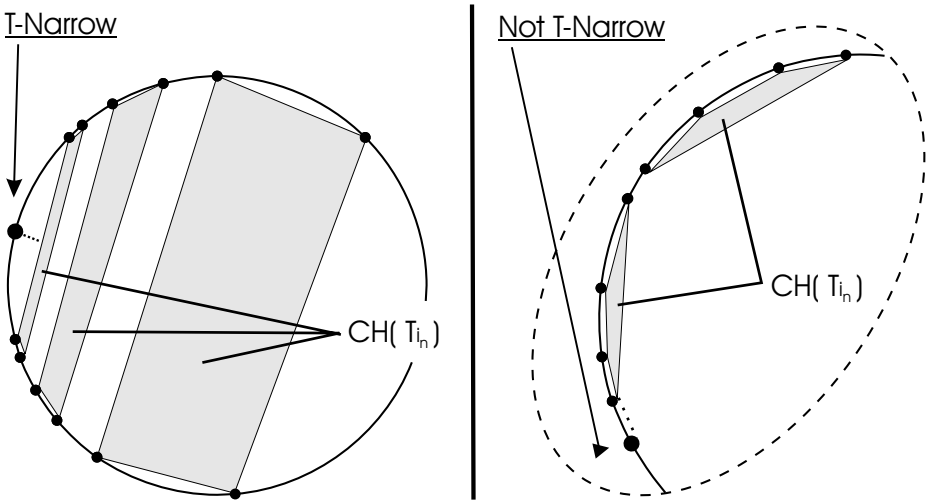


FIGURE 4. (a) A T-narrow point. (b) A point in $\mathcal{L}_{\omega(T)}$ that is not T-narrow.

By Lemma 2.3, the remainders of the image-holes H^{k_i-1} converge monotonically to 0, while the lengths of the image-holes H^{k_i-1} are greater than $1/d$. By refining this sequence and by using Lemma 2.8 and the properties of ρ (Lemma 2.8), we may assume that the chords e^{l_i} converge to a critical leaf l. Hence, critical leaves exist. Also observe that since T is wandering, any two critical leaves are disjoint in $\mathbb{D}$, so there are only finitely many of them.

By way of contradiction, suppose that the number ε_T does not exist. Since $\liminf \tilde{s}_{\mathrm{cr}}(T_i) = 0$, this means that there exists a sequence i_n such that $\tilde{s}_{\mathrm{cr}}(T_{i_n}) < 1/n$, while a unique critical leaf q with $\rho(e_{\mathrm{cr}}(T_{i_n}), q) = \tilde{s}_{\mathrm{cr}}(T_{i_n})$ does not exist. By the properties of ρ and because the ρ-distance between distinct critical leaves is at least $1/d$; we see that if n is sufficiently large, and there is a critical leaf q with $\rho(e_{\mathrm{cr}}(T_{i_n}), q) = \tilde{s}_{\mathrm{cr}}(T_{i_n}) < 1/n$, then this leaf is unique. Thus for each n, there is no critical leaf q with $\rho(e_{\mathrm{cr}}(T_{i_n}), q) = \tilde{s}_{\mathrm{cr}}(T_{i_n}) < 1/n$. However, as in the first paragraph of the proof, we can now choose a subsequence along which chords $e_{\mathrm{cr}}(T_{i_n})$ will converge to a chord q, which then will have to be a critical leaf. Clearly, this contradiction implies that the desired number ε_T exists; we fix ε_T and use it in what follows. □

3. *Jumping critical leaves*

We begin with extending Standing Assumption 2.6, which further requires that $s_{N-2}(T_i) < \varepsilon_T$ and can be done without loss of generality, by Proposition 2.5; from now on we assume that Standing Assumption 3.1 holds for T.

STANDING ASSUMPTION 3.1. *If a set $T \subset \mathbb{S}^1$ forms a wandering N-gon then we may assume that $f|_{T_i}$ preserves orientation and $1/3dN > \varepsilon_T > s_{N-2}(T_i)$, so that the image-hole of $H_k(T_i)$, $1 \le k \le N-2$, is of length $ds_k(T_i)$ $(i = 1, 2, \dots)$.*

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The main objective of this section is to understand the dynamics of *jumping critical leaves*. Jumping critical leaves are defined later; at this point we introduce and study a few related notions.

3.1. *Set of jumps.* The iterates i for which $ds_{N-2}(T_i) > s_{N-2}(T_{i+1})$ are called the *jumps*; the set of jumps is denoted by $\mathrm{JUMP}(T) = \{\mathrm{JM}_1(T), \mathrm{JM}_2(T), \dots\}$. In fact, jumps are exactly the times i when Theorem 2.11 is satisfied (i.e. $\tilde{s}_{\mathrm{cr}}(T_i) < \varepsilon_T$) *and* the image-hole of $H_{\mathrm{cr}}(T_i)$ is one of the $N-2$ smallest holes in T_{i+1}. To see this, we need to better understand the relationship between the holes in T_i and their image-holes in T_{i+1}. The following lemma gives a sufficient condition on the length $s_{N-1}(T_i)$ for when $f(H_k(T_i)) = H_k(T_{i+1})$ for every $k \in \{1, \dots, N-1\}$. It also gives a useful property on the sparseness of the set of jumps.

LEMMA 3.2. *Let T be as in Standing Assumption 3.1.*

(1) *If $1/d^{m+1}N > s_{N-1}(T_i)$ then for all $k \in \{1, \dots, N-1\}$ and all $j \le m$ we have $f^j(H_k(T_i)) = H_k(T_{i+j})$.*

(2) $\lim_{n\to\infty} s_{N-1}(T_{i_n+1}) = 0$, *where $i_n = \mathrm{JM}_n$—and thus*

$$\lim_{n\to\infty} \mathrm{JM}_{n+1} - \mathrm{JM}_n = \infty.$$

Proof. (1) First we show that if $1/dN > s_{N-1}(T_i)$ then for all $k \in \{1, \dots, N-1\}$ we have $f(H_k(T_i)) = H_k(T_{i+1})$ and $s_k(T_{i+1}) = ds_k(T_i)$. Clearly, if $s_{N-1}(T_i) < 1/dN$ then $H_N(T_i)$ is the critical hole. More importantly, $\sum_{k=1}^{N-1} \tilde{s}_k(T_i) < (N-1)/dN$ and hence $\tilde{s}_N(T_i) = (1/d) - \sum_{k=1}^{N-1} \tilde{s}_k(T_i) > 1/dN$. So $ds_N(T_i) \pmod 1 = d\tilde{s}_N(T_i) > ds_{N-1}(T_i)$, which implies that the $N-1$ smallest holes in T_{i+1} have lengths $ds_{N-1}(T_i) > ds_{N-2}(T_i) > \cdots > ds_1(T_i)$. Since by Proposition 2.7, the holes in T_{i+1} have pairwise distinct lengths; thus Lemma 2.3 completes the argument. Claim (1) of the lemma now easily follows from the above argument.

Observe that by the definition of a jump and Claim (1), if $1/d^{m+1}N > s_{N-1}(T_i)$ then none of the moments $i, i+1, \dots, i+m$ is a jump.

(2) Suppose that n is very large, then $i_n = JM_n(T)$ is very large. So by Proposition 2.5, the holes $H_1(T_{i_n}), \dots, H_{N-2}(T_{i_n})$ are very small. Hence by Lemma 2.3, the order among these $N-2$ holes and their image-holes is the same. Also, the size, $ds_{N-2}(T_{i_n})$, of the image-hole of $H_{N-2}(T_{i_n})$ is still very small. Since i_n is a jump then $s_{N-2}(T_{i_n+1}) < ds_{N-2}(T_{i_n})$, which implies that in addition to the $N-3$ image-holes of $H_1(T_{i_n}), \dots, H_{N-3}(T_{i_n})$ (which are all shorter than $ds_{N-2}(T_{i_n})$) there is *at least* one more hole in T_{i_n+1} which is shorter than $ds_{N-2}(T_{i_n})$. However, by Lemma 2.3, the sum of remainders of all the holes is $1/d$. Hence, there must exist *exactly* one more hole in T_{i_n+1} which is shorter than $ds_{N-2}(T_{i_n})$ and so $s_{N-1}(T_{i_n+1}) = ds_{N-2}(T_{i_n})$, which is very small; in particular $\lim_{n\to\infty} s_{N-1}(T_{i_n+1}) = d \lim_{N-2}(T_{i_n}) = 0$ by Proposition 2.5. Now note that using this along with (1) (see the observation directly below the proof of (1)), as n goes to ∞, the stretch of numbers $i_n + 1, \dots, i_n + m$ contains no jump and m goes to ∞. Thus $\lim_{n\to\infty}(\mathrm{JM}_{n+1}(T) - \mathrm{JM}_n(T)) = \infty$, as desired. □

Using a similar argument, we describe in Proposition 3.3 the possibilities for the image-hole of each one of the $N-2$ smallest holes in T_i.

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PROPOSITION 3.3. *For every $k \in \{1, \ldots, N-2\}$, exactly one of the following must occur.*
(1) $s_k(T_i) > \tilde{s}_{\mathrm{cr}}(T_i)$ *and* $f(H_k(T_i)) = H_{k+1}(T_{i+1})$.
(2) $\tilde{s}_{\mathrm{cr}}(T_i) > s_k(T_i)$ *and* $f(H_k(T_i)) = H_k(T_{i+1})$.

Furthermore, if $k \in \{1, \ldots, N-2\}$ is such that (1) holds, then the image-hole of the critical hole is one of the k smallest holes in T_{i+1} and $\tilde{s}_{\mathrm{cr}}(T_i) < \varepsilon_T$.

Proof. Let us introduce the following notation: (a) if $s_{N-1}(T_i) < 1/d$ then set $H' = H_{N-1}$ and $s' = s_{N-1}$; (b) if $s_{N-1}(T_i) > 1/d$ then set H' to be the *non-critical* hole of length greater than $1/d$ and s' to be equal the remainder of this length. Let us show then, in either case, that the length of the image-hole of H' is $ds' > s_{N-2}(T_{i+1})$. Indeed, it is enough to consider the case when the length of H' is greater than $1/d$. In this case

$$s' + \tilde{s}_{\mathrm{cr}}(T_i) = 1 - \sum_{k=1}^{N-2} \tilde{s}_k(T_i) > 1 - \frac{N-2}{3dN} > \frac{1}{Nd},$$

which implies (by the choice of the critical hole) that $s' > 1/2Nd$, and so $s' > \tilde{s}_k(T_i)$, $(1 \le k \le N-2)$. In fact, in the case (b) (i.e. when H' is the non-critical hole of length greater than $1/d$), we always have that $ds' = s_N(T_{i+1})$. However, in general (i.e. when either (a) or (b) can take place), we have that only if (1) holds for i is $ds' = s_N(T_{i+1})$ (by (1) the image-hole of the critical hole of T_i is shorter than ds').

Hence, the only way the image-holes of the $N-2$ smallest holes in T_i are not the $N-2$ smallest holes in T_{i+1} is when (1) holds; in which case, the image-hole of the critical hole is one of the k smallest holes in T_{i+1}, and so $\tilde{s}_{\mathrm{cr}}(T_i) < d\tilde{s}_{\mathrm{cr}}(T_i) < \varepsilon_T$. □

By Proposition 3.3, the inequality $ds_{N-2}(T_i) > s_{N-2}(T_{i+1})$ is necessary and sufficient for the image-holes of the $N-2$ smallest holes in T_i not to be the $N-2$ smallest holes in T_{i+1}. Also, these times i are *exactly* the times when the image-hole of the critical hole is one of the $N-2$ smallest holes in T_{i+1}. Thus, it follows from this proposition that the set of jumps is equal to the set of times i when the permutation induced by $f : T_i \mapsto T_{i+1}$ is a non-identity r-cycle of the form, $(k, k+1, k+2, \ldots, k+r)$ with $k \in \{1, \ldots, N-2\}$ such that $k + r \in \{N-1, N-2\}$ (when T forms a wandering 4-gon, these are the permutations (C)–(F) illustrated in Figure 2). Moreover, if $i \notin \mathrm{JUMP}(T)$ then the only possible non-identity permutation that can be induced by $f : T_i \mapsto T_{i+1}$ is the transposition $(N-1, N)$, the unique 2-cycle that fixes $\{1, \ldots, N-2\}$ (e.g., see (B) in Figure 2). Also note that by Proposition 3.3 and Standing Assumption 3.1, $\tilde{s}_{\mathrm{cr}} < \varepsilon_T$ for any $i \in \mathrm{JUMP}(T)$, so we can apply Theorem 2.11 to any integer $i \in \mathrm{JUMP}(T)$. This implies the following theorem, whose proof we leave to the reader.

THEOREM 3.4. *Let T be as in Standing Assumption 3.1.*
(1) *If $i \in \mathrm{JUMP}(T)$ then there exists a unique critical leaf q_i such that for the chord $e_{\mathrm{cr}}(T_i)$ we have $\rho(e_{\mathrm{cr}}(T_i), q_i) = \tilde{s}_{\mathrm{cr}}(T_i) < \varepsilon_T$. It follows that the critical value $f(q_i)$ lies in the image-hole of the critical hole, which is one of the $N-2$ smallest holes in T_{i+1}.*
(2) *If $i \notin \mathrm{JUMP}(T)$ then for every $k \in \{1, \ldots, N-2\}$, $f(H_k(T_i)) = H_k(T_{i+1})$.*

Below, we use the notation q_i as in Theorem 3.4. At the time i of a jump, the polygon $\mathrm{CH}(T_i)$ is 'leaf-like' and very close to a critical leaf q_i, while $\mathrm{CH}(T_{i+1})$ is 'point-like'

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and $f(q_i)$ is 'trapped' in one of the $N-2$ smallest holes in T_{i+1}. We will see the dynamical importance of these moments later in this section. First, we state a corollary of Theorem 3.4 (the proof is left to the reader).

COROLLARY 3.5. *Let $j \in \mathrm{JUMP}(T)$ and suppose the integer $m > j$ is chosen so that $i \notin \mathrm{JUMP}(T)$ for every integer i with $j < i < m$. Then $f^{m-j}(q_j)$ lies in one of the $N-2$ smallest holes in T_m.*

3.2. *Jumping critical leaf.* We are interested in the critical leaves l such that $q_i = l$ for infinitely many $i \in \mathrm{JUMP}(T)$, which we call the *jumping critical leaves*. It should be noted that while every critical leaf has its image coinciding with a narrow critical value, a critical leaf may not be one of the jumping critical leaves. However, the critical leaves with narrow critical values that Kiwi found in [**10**] correspond to the images of jumping critical leaves. This follows from our proof of Theorem 3.6, which is inspired by Kiwi's arguments from [**10**]. Theorem 3.6 says that there are at least $N-1$ jumping critical leaves with pairwise disjoint orbits. Note that there might be more than $N-1$ jumping critical leaves (e.g. consider the case when T is strictly contained in some finite set P, where P also forms a wandering polygon). We show in §3.3 that every jumping critical leaf is recurrent.

THEOREM 3.6. *If T forms a wandering N-gon, then there exist at least $N-1$ jumping critical leaves with pairwise disjoint orbits.*

Proof. By Proposition 2.5, for every positive integer n there exists M_n such that $1/d^{2n}N > s_{N-2}(T_i)$ for all $i \geq M_n$. Also by Proposition 2.5, we can find a strictly increasing sequence $\{i_n\}_{n=1}^{\infty}$ of integers such that for every n, $ds_1(T_{i_n}) > s_1(T_{i_n+1})$. We assume without loss of generality that $i_n \geq M_n$ for every n.

Since there are finitely many critical leaves, by choosing subsequences of i_n, we may also assume that the leaf q_{i_n} (defined in Theorem 3.4) is the same critical leaf l_0 for every n. Note that by Proposition 3.3, all of this implies that $f(l_0) \in H_1(T_{i_n+1})$ and $s_{N-1}(T_{i_n+1}) = ds_{N-2}(T_{i_n}) < 1/d^{2n-1}N$. Then by applying Lemma 3.2 to T_{i_n+1}, we get that $f^n(l_0) \in H_1(T_{i_n+n})$.

Set $\tau_n = s_1(T_{i_n+n})$. By our assumptions and Lemma 3.2 we have $\tau_n < 1/d^nN$. The number τ_n serves as a threshold below which the lengths $s_1(r), \ldots, s_{N-2}(r)$ of holes in T_r must stay for some time on by Proposition 2.5. Choose $j_n(k)$ as the moment when the size of the hole $s_k(j_n(k))$ drops below τ_n for the last time (so that for $r > j_n(k)$ we have $s_k(r) < \tau_n$). We now show that the finite sequence $j_n(1), \ldots, j_n(N-2)$ is strictly increasing.

In other words, we show that for every n there exists a strictly increasing finite sequence of integers $\{j_n(k)\}_{k=1}^{N-2}$, with $j_n(1) \geq i_n + n$, such that for each k:

(a) $s_k(T_{j_n(k)}) \geq \tau_n$; and

(b) for every $i > j_n(k)$, $\tau_n > s_k(T_i)$.

To this end, we fix n and use finite induction on $k \in \{1, \ldots, N-2\}$. The base case is when $k = 1$. Take $j_n(1)$ to be the maximal integer $i \geq i_n + n$ such that $s_1(T_i) \geq \tau_n$,

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and note that $j_n(1)$ exists by Proposition 2.5. By our choice of $j_n(1)$, we have that $\tau_n > s_1(T_i)$ for all $i > j_n(1)$. Moreover, we also have that $s_1(T_{j_n(1)}) \geq \tau_n > s_1(T_{j_n(1)+1})$, thus concluding the base case.

Now we assume that for some $k \in \{1, \ldots, N-3\}$, we have found the integers $j_n(k)$ satisfying (a) and (b) above. Then by Proposition 3.3 (1), we have that $s_{k+1}(T_{j_n(k)+1}) = ds_k(T_{j_n(k)})$. Since $s_k(T_{j_n(k)}) \geq \tau_n$, this implies $s_{k+1}(T_{j_n(k)+1}) > \tau_n$. The rest of the argument is analogous to the proof in the base case.

We can do this for every n. Note that by Theorem 3.4(2), $j_n(k) \in \mathrm{JUMP}(T)$ for every n and k. Since the number of critical leaves is finite, there is a collection of critical leaves $\{l_1, \ldots, l_{N-2}\}$ and an infinite set $V \subset \mathbb{N}$ such that if $n \in V$ then for every k, $q_{j_n(k)} = l_k$. Recall that l_0 is the critical leaf corresponding to the sequence i_n. We want to show that the jumping critical leaves (given above) $l_0, \ldots, l_{N-2}$ have pairwise disjoint orbits. To see this, assume by way of contradiction that there exists two non-negative integers b, c such that $f^b(l_{k_0}) = f^c(l_{k_1})$, where $k_0 \in \{0, \ldots, N-2\}$, $k_1 \in \{1, \ldots, N-2\}$ and $k_0 \neq k_1$.

Since V is infinite, we can fix $n \in V$ with $n > \max\{b, c\}$. Then for each $k \in \{1, \ldots, N-2\}$, $q_{j_n(k)} = l_k$. Also, Theorem 3.4(1) implies that $f(l_k)$ lies in the image-hole of the critical hole $H_{\mathrm{cr}}(T_{j_n(k)})$. By the choice of our sequences and Proposition 3.3, we get the inequality $s_{N-1}(T_{j_n(k)+1}) = ds_{N-2}(T_{j_n(k)}) > \tau_n > s_k(T_{j_n(k)+1}) \geq d\tilde{s}_{\mathrm{cr}}(T_{j_n(k)})$. Moreover since $j_n(k) > i_n \geq M_n$, we have that $1/d^{2n-1}N > s_{N-1}(T_{j_n(k)+1})$. By these inequalities, Lemma 3.2 and property (b) imply the following.

For every $k \in \{1, \ldots, N-2\}$ and every positive integer $r < 2n$, there are at least two holes in $T_{j_n(k)+r}$ of length $\geq \tau_n$, while:

(1) $f^r(l_k)$ lies in a hole in $T_{j_n(k)+r}$ that has length no greater than τ_n; and

(2) if $k < N-2$ then $j_n(k+1) > j_n(k) + 2n - 1$.

Define

$$\Gamma = \{T_{i_n+n}, T_{j_n(1)+1}, \ldots, T_{j_n(1)+2n-1}, \ldots, T_{j_n(N-2)+1}, \ldots, T_{j_n(N-2)+2n-1}\}.$$

Note that the iterates in Γ are pairwise distinct by property (3) and because $j_n(1) \geq i_n + n$. In particular, since T is a wandering N-gon, Γ is a collection of pairwise unlinked distinct sets. Thus since $\tau_n = s_1(T_{i_n+n})$, and since $T_{j_n(k)+r}$ satisfies properties (1) and (3) for every $k \in \{1, \ldots, N-2\}$ and $r < 2n$, any two different sets $Q, B \in \Gamma$ satisfies the conditions of Proposition 2.1 with $\tau = \tau_n$. Hence, for any such two sets, Q is contained in a hole in B whose size is larger than τ_n.

Recall that $n > \max\{b, c\}$. It follows that $n \geq n - b > 0$, $0 < r := n - b + c < 2n$ and $f^n(l_{k_0}) = f^{n-b} \circ f^b(l_{k_0}) = f^{n-b} \circ f^c(l_{k_1}) = f^r(l_{k_1})$. Since $k_0 \in \{0, \ldots, N-2\}$, $B := T_{j_n(k_0)+n} \in \Gamma$ (where $j_n(0) = i_n$). Also, since $0 < r < 2n$ and $k_1 \in \{1, \ldots, N-2\}$, we have that $Q := T_{j_n(k_1)+r} \in \Gamma$. Observe that by property (3), $Q \neq B$ because $k_0 \neq k_1$; so that Q is contained in a hole in B whose size is larger than τ. By the symmetry of this argument, we also have that B is contained in a hole in Q whose size is larger than τ_n. However this contradicts property (2) of Q (or B), since $f^n(l_{k_0}) = f^r(l_{k_1})$. □

3.3. *Jumping critical leaves are recurrent.* Recall that the ω-limit set of a critical leaf l is defined by $\omega(l) = \omega(v)$, where $v = f(l) \in \mathbb{S}^1$ is the critical value. We call a critical leaf *recurrent* if $\omega(l) \cap l \neq \emptyset$. Lemma 3.7 shows the dynamical significance of critical values

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being 'trapped' in one of the $N-2$ smallest holes. The proof of Lemma 3.7 is left to the reader.

LEMMA 3.7. *Let T be a wandering N-gon. Suppose $\{i_n\}_{n=0}^{\infty}$ is an increasing sequence of non-negative integers such that $\mathrm{CH}(T_{i_n})$ converges onto a leaf $p \in \mathbb{S}^1$. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence of points from $\mathbb{S}^1$ such that for every n, x_n is in one of the $N-2$ smallest holes in T_{i_n}. Then the limit points of $\{x_n\}$ form a non-empty subset of $p \cap \mathbb{S}^1$.*

Now we are ready to prove the Recurrence Theorem.

THEOREM 3.8. (Recurrence Theorem) *All jumping critical leaves of a wandering N-gon are recurrent with the same ω-limit set.*

Proof. By Theorem 3.6 there are at least two jumping critical leaves. Let $\mathcal{C}_{\mathrm{J}}$ denote the collection of all jumping critical leaves. Let $q \in \mathcal{C}_{\mathrm{J}}$ and define the collection $\mathcal{C}_{\omega(q)} = \{l \in \mathcal{C}_{\mathrm{J}} : l = q \text{ or } l \cap \omega(q) \neq \emptyset\}$. Clearly, $\mathcal{C}_{\mathrm{J}}$ is non-empty, and $\omega(l) \subset \omega(q)$ for every $l \in \mathcal{C}_{\omega(q)}$. We show that $\mathcal{C}_{\omega(q)} = \mathcal{C}_{\mathrm{J}}$, which concludes the argument since there are at least two jumping critical leaves and since $q \in \mathcal{C}_{\mathrm{J}}$ was arbitrary.

By definition $\mathcal{C}_{\omega(q)} \subset \mathcal{C}_{\mathrm{J}}$. Assume by way of contradiction that $\mathcal{C}_{\mathrm{J}} \setminus \mathcal{C}_{\omega(q)} \neq \emptyset$. Then there exist infinitely many integers $i \in \mathrm{JUMP}(T)$ with $q_i \notin \mathcal{C}_{\omega(q)}$. Since $\mathcal{C}_{\omega(q)} \neq \emptyset$, we also know that there are infinitely many integers $i \in \mathrm{JUMP}(T)$ with $q_i \in \mathcal{C}_{\omega(q)}$. Hence we can find a strictly increasing sequence $\{j_n\}_{n=1}^{\infty}$ of jumps such that for every positive integer n, $q_{j_{2n-1}} \in \mathcal{C}_{\omega(q)}$ while $q_{j_{2n}} \in \mathcal{C}_{\mathrm{J}} \setminus \mathcal{C}_{\omega(q)}$ and $i \notin \mathrm{JUMP}(T)$ for all i with $j_{2n-1} < i < j_{2n}$. Since there are only finitely many critical leaves we may assume that for every n, $q_{j_{2n-1}}$ is the same critical leaf $l \in \mathcal{C}_{\omega(q)}$ and $q_{j_{2n}}$ is the same critical leaf $r \in \mathcal{C}_{\mathrm{J}} \setminus \mathcal{C}_{\omega(q)}$.

For each n, Theorem 3.4(1) implies that $s_{N-2}(T_{j_{2n}}) > \tilde{s}_{\mathrm{cr}}(T_{j_{2n}}) = \rho(e_{\mathrm{cr}}(T_{j_{2n}}), r)$ (recall that $e_{\mathrm{cr}}(T_i)$ is the edge of $\mathrm{CH}(T_i)$ corresponding to the critical hole in T_i). Thus, by Proposition 2.5 and Lemma 2.8 (3), this implies that $\mathrm{CH}(T_{j_{2n}})$ converges onto r as $n \to \infty$. Next, define $b_n = j_{2n} - j_{2n-1}$ for every n. By Corollary 3.5, $f^{b_n}(l)$ lies in one of the $N-2$ smallest holes in $T_{j_{2n}}$. In addition, by Lemma 3.2(2) we know that $\lim_{n\to\infty} b_n = \infty$. These facts together with Lemma 3.7 imply $\omega(l) \cap r \neq \emptyset$. Since $\omega(l) \subset \omega(q)$, this contradicts $r \in \mathcal{C}_{\mathrm{J}} \setminus C_{\omega(q)}$. □

By the Recurrence Theorem 3.8 there is a unique ω-limit set for the jumping critical leaves of T. Using the Recurrence Theorem, we give a specific relationship between the limit set $\mathcal{L}_{\omega(T)}$ and the limit set of the jumping critical leaves. Note that $\mathcal{L}_{\omega(T)}$ contains several chords, while $\omega(q) \subset \mathbb{S}^1$ for a jumping critical leaf q. Thus $\mathcal{L}_{\omega(T)} \neq \omega(q)$. However, we know that $\omega(q) \subset \mathcal{L}_{\omega(T)}$. The following corollary better describes the relationship between these two limit sets.

COROLLARY 3.9. *If q is a jumping critical leaf then for every $p \in \mathcal{L}_{\omega(T)}$, $\omega(q) \cap p \neq \emptyset$.*

Proof. Let $\mathrm{CH}(T_{i_n})$ be a subsequence converging to a leaf $p \in \mathcal{L}_{\omega(T)}$. By Theorem 3.8, we need to show that there exists a jumping critical leaf q with $\omega(q) \cap p \neq \emptyset$. To do this, define the subsequence $\{m_n\}$ of jumps by $m_n = \max\{i < i_n : i \in \mathrm{JUMP}(T)\}$.

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We may assume that $m_{n+1} > m_n$ for every n and that the critical leaf q_{m_n} (defined in Theorem 3.4(1)) is the same critical leaf q for every jump m_n.

Define $b_n = i_n - m_n$. By Corollary 3.5, $f^{b_n}(q)$ lies in one of the $N-2$ smallest holes in T_{i_n} for every n. By Proposition 3.7, the (non-empty) set A of limit points of the sequence $f^{b_n}(q)$ is contained in $p \cap \mathbb{S}^1$. Since q is recurrent, $\overline{\mathrm{orb}(f(q))} = \omega(f(q))$, and so $A \subset \omega(f(q))$, which completes the proof. □

4. *Recurrent critical leaves from the limit set of co-existing wandering polygons*

Consider now several co-existing wandering polygons. Recall that for a finite set $A \subset \mathbb{S}^1$, $A_i = f^i(A)$.

Definition. Let Γ be a collection of pairwise disjoint finite subsets. Then Γ is said to *form a wandering collection of polygons* if the following holds:

(1) each finite set $T \in \Gamma$ forms a wandering polygon under f;

(2) for every two different finite sets $A, B \in \Gamma$ and any two integers $n, m \geq 0$, A_n and B_m are unlinked.

For a wandering collection of polygons Γ, define $\mathcal{L}_{\omega(\Gamma)} = \bigcup_{T\in\Gamma} \mathcal{L}_{\omega(T)}$. So, we call any chord (including degenerate chords, i.e. points in $\mathbb{S}^1$) in $\mathcal{L}_{\omega(\Gamma)}$ a *leaf*. Let R be the maximum number of recurrent critical leaves with pairwise disjoint grand orbits. Let Ω be the collection of distinct ω-limit sets of the recurrent critical leaves. Now we can prove the following theorem.

THEOREM 4.1. *Let $f(z) = z^d$ and suppose Γ forms a wandering collection of polygons. Then*

$$\mathrm{card}(\Gamma) \leq \sum_{T\in\Gamma} (\mathrm{card}(T) - 2) \leq R - \mathrm{card}(\Omega) \leq d - 1 - \mathrm{card}(\Omega).$$

Proof. First, let us consider a wandering collection of polygons Γ' with $\mathrm{card}(\Gamma') < \infty$. Then fix $T \in \Gamma'$ and set $N_T = \mathrm{card}(T)$. We can construct the sequences i_n, $j_n(1), \ldots, j_n(N_T - 2)$ of jumps as in the proof of Theorem 3.6 applied to this set $T \in \Gamma'$. Recall that $\tau_n = s_1(T_{i_n+n})$. As in the proof of Theorem 3.6, we can find a collection of critical leaves $\mathcal{J}_T = \{l_1, \ldots, l_{N_T-2}\} \subset \mathcal{L}_{\omega(\Gamma')}$ and an infinite subset $V \subset \mathbb{N}$ such that for every $n \in V$, $s_k(T_{j_n(k)}) \geq \tau_n$, $q_{j_n(k)} = l_k$, and $\tau_n > s_k(T_i)$ for all $i > j_n(k)$. Recall that $l_0 = q_{i_n}$ for every n. As in Theorem 3.6, $l_0, \ldots, l_{N_T-2}$ are jumping critical leaves of T with pairwise disjoint orbits.

Let $A \in \Gamma', A \neq T$, and set $N_A = \mathrm{card}(A)$. Since $\lim_{n\to\infty} \tau_n = 0$ and V is infinite, we can find an $n \in V$ with $s_1(A) > \tau_n$. Then by the same methods used to construct the collection $\mathcal{J}_T$, we can find a collection of jumping critical leaves of A, $\mathcal{J}_A = \{l'_1, \ldots, l'_{N_A-2}\} \subset \mathcal{L}_{\omega(\Gamma')}$, and an infinite subset $V' \subset V$ such that for all $n \in V'$ there is a subset $\{j'_n(1), j'_n(2), \ldots, j'_n(N_A - 2)\} \subset \mathrm{JUMP}(A)$ such that $s_k(A_{j'_n(k)}) \geq \tau_n$, $q_{j'_n(k)} = l'_k$, and $\tau_n > s_k(A_i)$ for all $i > j'_n(k)$. Since $V' \subset V$, this implies (as in the proof of Theorem 3.6) the critical leaves of the collection $\{l_0\} \cup \mathcal{J}_T \cup \mathcal{J}_A \subset \mathcal{L}_{\omega(\Gamma')}$ have pairwise disjoint orbits. Note that $\mathrm{card}(\mathcal{J}_A) = \mathrm{card}(A) - 2$.

This process of extending our collection of critical leaves can be continued as long as we have a supply of wandering polygons with pairwise disjoint orbits. Thus, for every finite wandering collection of polygons Γ' there exists a collection of critical

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leaves $\mathcal{J}_{\Gamma'} = \{l_0\} \cup \bigcup_{Q\in\Gamma'} mcJ_Q \subset \mathcal{L}_{\omega(\Gamma')}$ with pairwise disjoint orbits such that $\text{card}(\mathcal{J}_{\Gamma'}) = \sum_{Q\in\Gamma'}(\text{card}(Q) - 2) + 1$, and every $l \in \mathcal{J}_{\Gamma'}$ is a jumping critical leaf of some $A \in \Gamma'$. For any finite wandering collection of polygons Γ', let us denote by $\mathcal{J}_{\Gamma'}$ the just-constructed collection of critical leaves; let us emphasize here that all critical leaves from this collection have pairwise disjoint forward orbits.

Now let Γ be the wandering collection of polygons. We proof the inequality for Γ. First, since the number of critical leaves is finite, and since $\text{card}(Q) - 2 \geq 1$ for any wandering N-gon, by applying the above paragraph to finite subcollections of Γ, we have that $\text{card}(\Gamma) < \infty$. Now, let us begin.

By the Recurrence Theorem 3.8, all of the jumping critical leaves of a polygon are recurrent and have the same ω-limit set. For every $A \in \Gamma$, let us denote by W_A this limit set. Furthermore, for every $X \in \Omega$, define the subcollection $\Gamma_X = \{A \in \Gamma : W_A = X\}$. Set $\Omega_1 = \{X \in \Omega : \Gamma_X \neq \emptyset\}$ and $\Omega_0 = \{X \in \Omega : \Gamma_X = \emptyset\}$. Notice that these two sets are a partition of Ω, so $\text{card}(\Omega_0) + \text{card}(\Omega_1) = \text{card}(\Omega)$.

By Theorem 3.6 every wandering N-gon has at least $N - 1 > 0$ jumping critical leaves, and so the collection $\{\Gamma_X\}_{X\in\Omega_1}$ is a partition of Γ into non-empty (by the choice of Ω_1) sets. For every $X \in \Omega_1$, we have $\text{card}(\mathcal{J}_{\Gamma_X}) = 1 + \sum_{Q\in\Gamma_X}(\text{card}(T) - 2)$. Summing up these formulas over all $X \in \Omega_1$, we see that $\sum_{X\in\Omega_1} \text{card}(\mathcal{J}_{\Gamma_X}) = \text{card}(\Omega_1) + \sum_{T\in\Gamma}(\text{card}(T) - 2)$.

Now, for every $X \in \Omega_0$ there exists at least one recurrent critical leaf whose limit set is X. So we can define a collection $\mathcal{R}_0$ of recurrent critical leaves such that for every $X \in \Omega_0$ there is a unique $l \in \mathcal{R}_0$ with $\omega(l) = X$. Note then that $\text{card}(\mathcal{R}_0) = \text{card}(\Omega_0)$.

Finally, let us show that critical leaves in the collection $\bigcup_{X\in\Omega_1} \mathcal{J}_{\Gamma_X} \cup \mathcal{R}_0$ have pairwise disjoint forward orbits. Indeed, if the leaves in question come from the same collection $\mathcal{J}_{\Gamma_X}$ for some X, then it follows from the properties of such collections. On the other hand, if they come from different collections Γ_X, Γ_Y with $X \neq Y$, or $\Gamma_X, \mathcal{R}_0$ for some X, or if they both come from $\mathcal{R}_0$, then, in either one of these three cases, they have different limit sets which again yields that their forward orbits are disjoint. Thus, $\sum_{X\in\Omega_1} \text{card}(\mathcal{J}_{\Gamma_X}) + \text{card}(\mathcal{R}_0) \leq R$, which implies that $\sum_{T\in\Gamma}(\text{card}(T) - 2) \leq R - (\text{card}(\Omega_1) + \text{card}(\Omega_0)) \leq R - \text{card}(\Omega)$, as desired. □

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