INVARIANT GROUPS ASSOCIATED WITH AN ALGEBRAIC SURFACE

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Alexander (1, 2) has introduced certain topological invariants of a manifold which arise from the intersections of cycles of non-complementary dimensions, and he points out that they are not derivable from the Betti and torsion numbers, nor from the fundamental group. In the present paper we consider some topological invariants of this type on an algebraic surface, and, although we cannot define them completely, we show that they are intimately connected with the multiplications of the period matrix of the simple integrals of the first kind. We are then led to a concept which we call the "intersection group" of the surface, which is, by its definition, topologically invariant, and we show that it is also invariant under birational transformations. The proofs are based on Lefschetz's theory of cycles for an algebraic surface (4) and some simple properties of the period matrix of an algebraic curve. The results obtained here have a number of applications to the theory of ∞^3 correspondences between algebraic surfaces, as we propose to show in a later paper.

The notation required is somewhat involved, and in parts unconventional. Matrices are denoted throughout by symbols in Clarendon type, and the elements of a matrix $\mathbf{a}$ are denoted by a_{ij}. The symbol $\mathbf{1}$, or $\mathbf{1}_r$ if we wish to emphasize its order, denotes the unit matrix, and δ_{ij} denotes its elements. $\mathbf{a}'$ denotes the transposed matrix of $\mathbf{a}$ and $\hat{\mathbf{a}}$ its complementary matrix; i.e. the matrix such that $\hat{\mathbf{a}}\mathbf{a}' = \mathbf{1}$. The conjugate imaginary of a number z is denoted by $\bar{z}$, with a similar notation for conjugate imaginary matrices. We also use bars as superscripts in a quite different sense, in a context in which there should be no possibility of confusion.

We are concerned almost entirely with homology with division, and, unless we specify strict homology, the word is always to be understood in this sense, and statements about dependence or independence of cycles must be interpreted accordingly. We also find it convenient to deal mostly with fractional cycles; i.e. some of the "cycles" which occur are not cycles in the ordinary sense of the word but fractional multiples of such cycles. If we wish to stress the fact that a cycle is not fractional we call it a real cycle. All our results are expressible in terms of real cycles, but a free use of fractional cycles justifies itself by a considerable simplification of the arguments. By a *base* for p-cycles of a manifold M

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we mean any set of R_p independent p-cycles, where R_p is the p-dimensional Betti number of M. By *fundamental base* we mean R_p independent real cycles such that any other real cycle is homologous (in the strict sense) to a sum of integral multiples of our base-cycles and, possibly, a zero-divisor. We use the usual notation for topological products.

The summation convention which we adopt is that every repeated literal *lower* suffix in any expression is to be summed over all its admissible values. Such suffixes may be attached either to numbers (such as the elements of matrices) or to cycles: e.g. expressions such as $a_{ij}b_{jk}$, $a_{ij}\gamma_j$, $\gamma_i \times \gamma_i$, where $\mathbf{a}$ and $\mathbf{b}$ are matrices and the γ_i are cycles, are all to be taken as sums.

1. Let us take an algebraic surface and construct a Riemannian manifold F from some suitable non-singular model of it, which will also be denoted by F. Let us choose on F a base $\gamma_1, \ldots, \gamma_{2q}$ for the 1-cycles of F, and a set $u_1, \ldots, u_q$ of independent simple integrals of the first kind on F, the period matrix of these integrals over the chosen cycles being $\boldsymbol{\omega}$ $\left(\text{i.e. } \omega_{ij} = \int_{\gamma_j} du_i\right)$. Then, given $\gamma_1, \ldots, \gamma_{2q}$, we can find a base $\Gamma_1, \ldots, \Gamma_{2q}$ for the 3-cycles of F such that

$$(\gamma_i . \Gamma_j) = -(\Gamma_j . \gamma_i) = \delta_{ij}. \tag{1·1}$$

We suppose that our γ_i are all real. The Γ's need not be real unless the γ's are a fundamental base, which we do not necessarily suppose to be the case.

Let $P_1, \ldots, P_\rho$ be a base for the algebraic 2-cycles, where each P_i is a generic curve of a linear system of positive grade and positive freedom r, such that the system does not contain ∞^{r-1} reducible curves. A curve of this type will henceforth be described as "proper", and we can apply to proper curves the theory of cycles developed by Lefschetz ((4), Chaps. II and III).

Let $$(P_i . P_j) = y_{ij} \quad \text{and} \quad \mathbf{z} = \mathbf{y}^{-1}. \tag{1·2}$$

Let $P^* \approx \lambda_l^* P_l$ $(l = 1, \ldots, \rho)$ be a proper curve of F of genus p^*. We can choose on P^* (Zariski (10)) a set $\delta_1^*, \ldots, \delta_{2p^*}^*$ of 1-cycles of P^*, the first $2q$ of them being invariant and the remainder vanishing cycles, as well as a set $u_1^*, \ldots, u_q^*$ of independent simple integrals of the first kind on F, and a set $u_{q+1}^*, \ldots, u_{p^*}^*$ of similar integrals of P^*, such that the period matrix of the integrals $u_1^*, \ldots, u_{p^*}^*$ over the cycles $\delta_1^*, \ldots, \delta_{2p^*}^*$ is of the form

$$\begin{pmatrix} \boldsymbol{\tau}^* & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\omega}^* \end{pmatrix}, \tag{1·3}$$

where $\boldsymbol{\tau}^*$ is a $(q, 2q)$ matrix.

But, since $\delta_1^*, \ldots, \delta_{2q}^*$ are independent invariant cycles of P^*, any cycle of F, and in particular γ_i, is dependent on them (Lefschetz (4), p. 34). We thus have $2q$ relations of the form

$$t^i \gamma_i \approx d_{ij} \delta_j^* \text{ on } F \quad (i, j = 1, \ldots, 2q: i \text{ not summed}). \tag{1·4}$$

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Let us take a new base $\gamma_1^*, \ldots, \gamma_{2q}^*$ of (fractional) 1-cycles of P^* defined by

$$\gamma_i^* \approx (t^i)^{-1} d_{ij} \delta_j^* \text{ on } P^* \quad (i,j = 1, \ldots, 2q: i \text{ not summed}) \tag{1·5}$$

and by $$\gamma_i^* \approx \delta_i^* \quad (i = 2q+1, \ldots, 2p^*).$$

Then clearly, by (1·4) and (1·5), we have

$$\gamma_i^* \approx \gamma_i \text{ on } F \quad (i = 1, \ldots, 2q). \tag{1·6}$$

Let us also take as a new base for the integrals of P^* the set

$$u_1, \ldots, u_q, \quad u_{q+1}^*, \ldots, u_{p^*}^*.$$

These changes of base affect only the sub-matrix $\boldsymbol{\tau}^*$ in (1·3), and thus change (1·3) into the matrix

$$\boldsymbol{\Omega}^* = \begin{pmatrix} \boldsymbol{\omega} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\omega}^* \end{pmatrix}, \tag{1·7}$$

since, by (1·6), $\int_{\gamma_j^*} du_i = \int_{\gamma_j} du_i = \omega_{ij}$. Then, since the invariant cycles have zero-intersection number with the vanishing cycles (Lefschetz (4), p. 24), the intersection matrix of the cycles $\gamma_1^*, \ldots, \gamma_{2p^*}^*$ is of the form

$$\begin{pmatrix} \mathbf{b}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^* \end{pmatrix}, \tag{1·8}$$

where $\mathbf{b}^*$ is a $(2q, 2q)$ matrix.

The topological intersection cycle $\Gamma_i . P^*$ belongs to a well-defined class of homologous 1-cycles of F. Let us write

$$\Gamma_i . P^* \approx P^* . \Gamma_i \approx a_{ij}^* \gamma_j \text{ on } F \quad (i,j = 1, \ldots, 2q).$$

Thus, when the base for the 1-cycles, and hence for the 3-cycles, is given, $\mathbf{a}^*$ is completely defined by P^*. The cycles γ_i^* $(i = 1, \ldots, 2q)$ are, as we have seen, defined to within homologies on P^*, and this suffices to define $\mathbf{b}^*$. Hence, if we let P^* coincide with a curve P_l of the base, we define the corresponding matrices $\mathbf{a}^l$ and $\mathbf{b}^l$, as well as cycles γ_i^l $(i = 1, \ldots, 2q)$ of P_l. Then, since $\Gamma . (P + P') \approx \Gamma . P + \Gamma . P'$, it follows that $\mathbf{a}^* = \sum_{l=1}^{\rho} \lambda_l^* \mathbf{a}^l$, and for any (not necessarily proper) curve $C \approx \lambda_l P_l$ it follows that

$$\Gamma_i . C \approx C . \Gamma_i \approx a_{ij} \gamma_j \text{ on } F, \tag{1·9}$$

where

$$\mathbf{a} = \sum_{l=1}^{\rho} \lambda_l \mathbf{a}^l. \tag{1·10}$$

2. Our problem now is to discover the form of the matrix $\mathbf{a}$ of (1·9), which in future we call the a-matrix of C (over the cycles $\gamma_1, \ldots, \gamma_{2q}$). But first we notice that the a-matrices are connected also with the cycle of intersection of two 3-cycles. For consider the 2-cycle of F defined by $\Gamma_i . \Gamma_j$. Let us write

$$\Gamma_i . \Gamma_j \approx g_{ijl} P_l + c_{ij}, \tag{2·1}$$

where the g's are to be determined and c_{ij} is a transcendental 2-cycle of F, i.e. one having zero-intersection number with every algebraic cycle (Lefschetz (4), p. 130).

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Then $$(P_m.\Gamma_i.\Gamma_j) = g_{ijl}(P_m.P_l)+(P_m.c_{ij});$$

hence $$a^m_{ik}(\gamma_k.\Gamma_j) = g_{ijl}y_{ml},$$

that is, $$a^m_{ij} = g_{ijl}y_{lm}, \tag{2·2}$$

using (1·1) and the fact that $\mathbf{y}$ is a symmetric matrix.

Hence, multiplying (2·2) by z_{mn} and summing over $m = 1, \ldots, \rho$, we have

$$\sum_{m=1}^{\rho} a^m_{ij}z_{mn} = g_{ijl}y_{lm}z_{mn} = g_{ijn}, \tag{2·3}$$

since $\mathbf{yz} = \mathbf{1}$.

This is very similar to a result obtained by Alexander (2).

We notice also, from the left side of (2·2), that $(P_m.\Gamma_i.\Gamma_j) = a^m_{ij}$, and hence for any curve C it follows that

$$(C.\Gamma_i.\Gamma_j) = a_{ij}, \tag{2·4}$$

where $\mathbf{a}$ is the $\mathbf{a}$-matrix of C.

Consider now the algebraic fourfold $F \times F'$, where F' is a copy of F. The diagonal surface Δ on $F \times F'$, which represents the identical correspondence on F, is given (Lefschetz (6), 266) by

$$\Delta \approx x \times F' + F \times x' + \gamma_i \times \Gamma'_i - \Gamma_i \times \gamma'_i + z_{lm}P_m \times P'_l + w_{rs}c_r \times c'_s \text{ on } F \times F'$$
$$(i = 1, \ldots, 2q;\ l, m = 1, \ldots, \rho;\ r, s = \rho+1, \ldots, R_2),$$

where $c_{\rho+1}, \ldots, c_{R_2}$ form a base for the transcendental 2-cycles, $\hat{\mathbf{w}}$ is their intersection matrix, and x is a generic point of F.

Consider $\Delta.P^* \times F'$, where $P^* \approx \lambda^*_n P_n$ is a proper curve of F. Then, by the usual rules for intersections on a product complex (Lefschetz (6), 239),

$$\Delta.P^* \times F' \approx P^* \times x' - (P^*.\Gamma_i) \times \gamma'_i + z_{lm}(P^*.P_m)x \times P'_l$$
$$\approx P^* \times x' + z_{lm}\lambda_n y_{nm} x \times P'_l - a^*_{ij}\gamma_j \times \gamma'_i,$$

that is, $$\Delta.P^* \times F' \approx P^* \times x' + x \times P^{*\prime} - a^*_{ij}\gamma_j \times \gamma'_i \text{ on } F \times F'. \tag{2·5}$$

But the geometric intersection of Δ and $P^* \times F'$ is the diagonal curve on $P^* \times P^{*\prime}$, which we denote by $S(P^*)$. And since for two algebraic varieties which intersect in an irreducible variety of the appropriate dimensions the topological intersection is homologous (strictly) to a multiple of the geometric one (van der Waerden (8)), it follows, by using the theorem on the intersection number of three varieties with only a finite number of discrete simple intersections (Lefschetz (5)), that $S(P^*)$ actually belongs to the class of homologous cycles defined by $\Delta.P^* \times F'$. We therefore have $S(P^*) \approx \Delta.P^* \times F'$ on $F \times F'$. But, by using the formula for the diagonal cycle on $P^* \times P^{*\prime}$ (Lefschetz (6), p. 266), we have

$$S(P^*) \approx P^* \times x' + x \times P^{*\prime} + e^*_{ij}\gamma^*_i \times \gamma^*_j \text{ on } P^* \times P^{*\prime} \quad (i, j = 1, \ldots, 2p^*), \tag{2·6}$$

where $$\hat{\mathbf{e}}^* = \begin{pmatrix} \mathbf{b}^* & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^* \end{pmatrix}.$$

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Now from (2·6) we obtain

$$S(P^*) \approx P^* \times x' + x \times P^{*\prime} + e^*_{ij} \gamma_i \times \gamma_j \text{ on } F \times F',$$

where now i, j vary from 1 up to $2q$ only. Hence, comparing the above with (2·5), we have $e^*_{ij} = -a^*_{ji}$ $(i, j = 1, \ldots, 2q)$, that is $\hat{\mathbf{b}}^* = -\mathbf{a}^{*\prime}$, and hence, since $\mathbf{b}^*$ is skew-symmetric, so is $\mathbf{a}^*$, and we have

$$\mathbf{a}^* = \hat{\mathbf{b}}^*. \tag{2·7}$$

It is from this result and (2·4) that the rest of this paper follows.

Since the matrix (1·8) is the intersection matrix of the 1-cycles of P^* it follows (Lefschetz (7)) that its complementary matrix $\begin{pmatrix} \hat{\mathbf{b}}^* & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{B}}^* \end{pmatrix}$ is a principal† matrix of $\mathbf{\Omega}^* = \begin{pmatrix} \boldsymbol{\omega} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\omega}^* \end{pmatrix}$. From which it follows that $\mathbf{a}^*$, which we have just proved equal to $\hat{\mathbf{b}}^*$, is a principal matrix of $\boldsymbol{\omega}$. As special cases of this result we have the theorem that every $\mathbf{a}^t$ is a principal matrix of $\boldsymbol{\omega}$. Hence, if $C \approx \lambda_i P_i$ is any curve of F, not necessarily proper, it follows that its $\mathbf{a}$-matrix is an alternate matrix of $\boldsymbol{\omega}$, since by (1·10) and the result just proved it is the sum of a finite number of such matrices. But it is not implied that $\mathbf{a}$ must be principal or even non-singular. We shall produce examples in which it is neither.

An immediate corollary of this is that a *necessary* condition for a curve C to have some multiple which is proper is that $i\boldsymbol{\omega}\mathbf{a}\bar{\boldsymbol{\omega}}'$ is a definite Hermitian matrix.

3. The set of the $\mathbf{a}$-matrices of all the effective and virtual curves of F form an additive group. This group is a linear set over the integers, since any member of it is expressible as a sum of integral multiples of the $\mathbf{a}$-matrices of a fundamental base, which thus form a set of generating elements of the group. The group is a subset of the set of all alternate matrices of $\boldsymbol{\omega}$, and every member of it is expressible as a sum of multiples of principal matrices. We denote this group by $\mathfrak{J}_A$, and call it the algebraic intersection group of F over the cycles $\gamma_1, \ldots, \gamma_{2q}$. Since $\mathfrak{J}_A$ has a finite number of generators, and no elements of finite order, it follows from the principal theorem on Abelian groups (van der Waerden (9)) that it has a finite basis $\mathbf{a}^1, \ldots, \mathbf{a}^t$ $(t \leqslant \rho)$ such that every element of $\mathfrak{J}_A$ is expressible uniquely as a sum of integral multiples of $\mathbf{a}^1, \ldots, \mathbf{a}^t$. Let $C_1, \ldots, C_t$ be curves having these as their respective $\mathbf{a}$-matrices.

† A principal matrix of a Riemann matrix $\boldsymbol{\omega}$ is a rational skew-symmetric matrix $\mathbf{c}$ such that

(1) $\boldsymbol{\omega}\mathbf{c}\boldsymbol{\omega}' = 0$,

(2) $i\boldsymbol{\omega}\mathbf{c}\bar{\boldsymbol{\omega}}'$ is a positive definite Hermitian matrix.

If $\mathbf{c}$ satisfies the first but not necessarily the second condition it is called an alternate matrix of $\boldsymbol{\omega}$.

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Then any curve C has **a**-matrix $\mathbf{a} = \sum_{n=1}^{t} \lambda_n \mathbf{a}^n$. It follows that

$$C \sim \lambda_n C_n + O \quad (n = 1, \ldots, t), \tag{3·1}$$

where the λ's are uniquely defined by C and O is a curve whose (strict) homology class is defined by C and whose **a**-matrix is zero. But the (strict) homology classes of curves such as O form a subgroup $\mathfrak{O}$ of the homology group. It follows, by another application of the principal theorem on Abelian groups, that there is a finite set of independent elements of $\mathfrak{O}$ such that any element of $\mathfrak{O}$ is expressible uniquely as a sum of integral multiples of these elements and an element of finite order, which is simply the homology class of a zero-divisor. If the base elements of $\mathfrak{O}$ are the homology classes of curves $C_{t+1}, \ldots, C_{t+u}$, the curve O in (3·1) is given uniquely in the form

$$O \sim \lambda_{t+k} C_{t+k} + D \quad (k = 1, \ldots, u),$$

where D is a zero-divisor.

Thus C is expressible by the homology $C \sim \lambda_m C_m + D$ $(m = 1, \ldots, t+u)$, where the λ's are all uniquely determined. Since C is an arbitrary curve, it follows that $C_1, \ldots, C_{t+u}$ form a fundamental base and that $t+u = \rho$. Thus if t is the order† of $\mathfrak{J}_A$ we can find a fundamental base $C_1, \ldots, C_t, C_{t+1}, \ldots, C_\rho$, whose **a**-matrices are respectively $\mathbf{a}^1, \ldots, \mathbf{a}^t, \mathbf{0}, \ldots, \mathbf{0}$. It will appear presently that the set of curves, other than zero-divisors, with zero **a**-matrices is not necessarily vacuous.

If we change the base for the 1-cycles of F we change the intersection group. Consider the change in the **a**-matrix of a curve C when we change the base for the 1-cycles to $g_1, \ldots, g_{2q}$, where $g_i \approx p_{ij}\gamma_j$. Then the corresponding 3-cycles are given by $G_i \approx q_{ij}\Gamma_j$, where $\mathbf{q} = \hat{\mathbf{p}}$. Then if $\boldsymbol{\alpha}$ is the new **a**-matrix of C it is easily seen that $\boldsymbol{\alpha} = \mathbf{q}\mathbf{a}\mathbf{q}'$, and, since $\mathbf{p}$ is an arbitrary non-singular matrix, we may take $\mathbf{q}$ as any non-singular matrix. Thus it follows that the change of base gives an isomorphism of $\mathfrak{J}_A$ into $\mathbf{q}\mathfrak{J}_A\mathbf{q}'$, where to any element $\mathbf{a}$ of the first group corresponds $\mathbf{q}\mathbf{a}\mathbf{q}'$ of the second. The class of such isomorphic intersection groups is a topological invariant of F.

If the γ's are a fundamental base, the Γ's are all real cycles and the relations (1·9) correspond to real intersection cycles. The **a**-matrices are thus matrices of integers. An intersection group over a fundamental base we call real, otherwise it is called virtual. All possible real groups are obtained from one of them by means of the transformations $\mathfrak{J}_A \to \mathbf{q}\mathfrak{J}_A\mathbf{q}'$, where $\mathbf{q}$ is a unimodular matrix of integers. The knowledge of a virtual group is clearly sufficient to define the real groups; the former are introduced chiefly for convenience in examples (cf. the second example in § 5).

† By the order of $\mathfrak{J}_A$ we mean its order when regarded as a linear set.

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4. Starting with an algebraic surface we have constructed a Riemannian F from some suitable non-singular model. Had we taken another non-singular model we might have found a Riemannian not topologically equivalent to F, since the two non-singular models might be connected by a transformation involving fundamental curves. We now prove that all possible non-singular models of the surface give the same groups, i.e. that the class of isomorphic intersection groups is not merely a topological, but an absolute invariant of the surface.

Let $\bar{F}$ be the Riemannian of a non-singular model of F without exceptional curves (it being supposed that F is not transformable into a ruled surface). Let there be on F a number of exceptional curves $E_1, \ldots, E_v$ arising from points $\bar{e}_1, \ldots, \bar{e}_v$ of $\bar{F}$. Let $\bar{G} = \bar{F} - \Sigma \bar{e}_i$ and $G = F - \Sigma E_i$. Then G and $\bar{G}$ are homeomorphic open manifolds. Let us take bases $\bar{\gamma}_1, \ldots, \bar{\gamma}_{2q}$; $\bar{\Gamma}_1, \ldots, \bar{\Gamma}_{2q}$ for the 1- and 3-cycles of $\bar{G}$ such that $(\bar{\gamma}_i . \bar{\Gamma}_j) = \delta_{ij}$. Then these are also bases for $\bar{F}$. Let $\gamma_1, \ldots, \gamma_{2q}$; $\Gamma_1, \ldots, \Gamma_{2q}$, be the corresponding bases for G. Then $(\gamma_i . \Gamma_j) = (\bar{\gamma}_i . \bar{\Gamma}_j) = \delta_{ij}$. Hence these bases will also be bases for F, on account of the invariance of q.

Now Γ_i, Γ_j are 3-cycles of G, and therefore their intersection, calculated by means of the usual approximating chains, is a 2-cycle of G. Hence it does not meet ΣE_i, and thus it does not meet any particular component, E say, of an exceptional curve. Hence $(E . \Gamma_i . \Gamma_j) = 0$. But $(E . \Gamma_i . \Gamma_j) = a_{ij}$ (by 2·4), where $\mathbf{a}$ is the $\mathbf{a}$-matrix of E. Thus the $\mathbf{a}$-matrix of any component of an exceptional curve is the zero-matrix.

Now a base for the curves of F can be taken as the exceptional curves and a number $\bar{\rho}$ of curves of G, where $\bar{\rho}$ is the base-number for $\bar{F}$. But each such curve has the same $\mathbf{a}$-matrix as the corresponding curve of $\bar{F}$ (because of the homeomorphism between G and $\bar{G}$). And since the exceptional curves, as we have seen, add nothing to the intersection group, it follows that the intersection group of F over $\gamma_1, \ldots, \gamma_{2q}$ is the same as that of $\bar{F}$ over $\bar{\gamma}_1, \ldots, \bar{\gamma}_{2q}$. This proves the absolute invariance of the intersection groups.

5. In this paragraph we consider some special cases. Suppose first that $\boldsymbol{\omega}$ is pure and has essentially only one principal matrix (i.e. all principal matrices are positive multiples of the same matrix). Then for our base $P_1, \ldots, P_\rho$ the $\mathbf{a}$-matrices must be multiples of the unique principal matrix $\boldsymbol{\pi}$, say. Hence for any curve C of F we must have $\mathbf{a}^{\cdot} = \lambda \boldsymbol{\pi}$, where λ is a scalar depending only on C.

If we transform the 1-cycles of F by the relation $g_i \approx \pi_{ij} \gamma_j$ $(i, j = 1, \ldots, 2q)$, and take the g_i as a new base, leaving the 3-cycles unchanged, we have

$$\Gamma_i . C \approx \lambda g_i \text{ on } F, \tag{5·1}$$

where λ is a constant independent of i and depending only on C.

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Next suppose that $\boldsymbol{\omega}$ is reducible to the form

$$\boldsymbol{\omega} = \begin{pmatrix} \boldsymbol{\omega}^1 & & & \\ & \boldsymbol{\omega}^2 & & \\ & & \ddots & \\ & & & \boldsymbol{\omega}^s \end{pmatrix}, \tag{5·2}$$

where $\boldsymbol{\omega}^i$ is a pure $(q_i, 2q_i)$ Riemann matrix for each i, no two $\boldsymbol{\omega}^i$ are isomorphic, and each has essentially only one principal matrix $\boldsymbol{\pi}^i$. (It is not necessarily possible to ensure at the same time that the γ's form a fundamental base.) Then for any of the curves P_l we must have

$$\mathbf{a}^l = \begin{pmatrix} \lambda_1^l \boldsymbol{\pi}^1 & & & \\ & \lambda_2^l \boldsymbol{\pi}^2 & & \\ & & \ddots & \\ & & & \lambda_s^l \boldsymbol{\pi}^s \end{pmatrix}, \tag{5·3}$$

where the λ's are all positive. A similar result then holds for any curve with the restriction on the λ's removed.

If we put

$$\boldsymbol{\pi} = \begin{pmatrix} \boldsymbol{\pi}^1 & & & \\ & \boldsymbol{\pi}^2 & & \\ & & \ddots & \\ & & & \boldsymbol{\pi}^s \end{pmatrix},$$

then the transformation $g_i \approx \pi_{ij} \gamma_j$ (the Γ's being left unaltered) gives us

$$\Gamma_i . C \approx \lambda_k g_i \quad [i = 2(q_1 + q_2 + \ldots + q_{k-1}) + 1, \ldots, 2(q_1 + q_2 + \ldots + q_k), k = 1, \ldots, s],$$

where the λ_k depend only on C and not on the values of i within the ranges given.

Examples of this theorem are easily verified in the cases $s = 2$ or $s = 1$ by taking a product of two curves with pure non-isomorphic Riemann matrices, each having essentially only one principal matrix, or a ruled surface which is the product of one such curve and a line.

It is obvious that the order t of the intersection group is not greater than ρ, and it follows from (5·3) that if $\boldsymbol{\omega}$ is of the form (5·2) it is also not greater than s. That this last result is not necessarily true, if the condition of (5·2) that no two of the $\boldsymbol{\omega}^i$ are isomorphic is not fulfilled, is easily seen from the following example in which s is *two* and t at least *three*.

Consider the product $C \times C'$ where C is a curve whose Riemann matrix is pure and has essentially only one principal matrix, and C' is a copy of C. Take as a base for the 1-cycles of C the retrosections $\delta_1, \ldots, \delta_{2p}$ whose intersection matrix is $\begin{pmatrix} \mathbf{0} & \mathbf{1}_p \\ -\mathbf{1}_p & \mathbf{0} \end{pmatrix}$. On $C \times C'$ take the bases $\gamma_1, \ldots, \gamma_{4p}$; $\Gamma_1, \ldots, \Gamma_{4p}$ for the 1- and 3-cycles as follows:

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$$\gamma_i \approx \delta_i \times x' \quad (i=1,\ldots,2p), \qquad \gamma_i \approx x \times \delta'_{i-2p} \quad (i=2p+1,\ldots,4p),$$

$$\Gamma_i \approx \delta_{p+i} \times C' \quad (i=1,\ldots,2p), \qquad \Gamma_i \approx C \times \delta'_{i-p} \quad (i=2p+1,\ldots,4p),$$

with the convention that $\delta_{2p+i} = -\delta_i$. Then it follows that $(\gamma_i . \Gamma_j) = \delta_{ij}$.

The base for the algebraic 2-cycles of $C \times C'$ consists of $C_1 \approx C \times x'$, $C_2 \approx x \times C'$, C_3 the diagonal cycle, and possibly other cycles corresponding to complex multiplications of ω. But we know (Lefschetz (6), 266) that

$$C_3 \approx C \times x' + x \times C' + \delta_i \times \delta'_{p+i} \quad (i=1,\ldots,2p).$$

A little examination of the intersections then gives

$$\mathbf{a}^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{a}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

and

$$\mathbf{a}^3 = \mathbf{a}^1 + \mathbf{a}^2 + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix},$$

all the blocks being of p rows and columns.

Thus the intersection group of $C \times C'$ has at least *three* independent elements, although s is only equal to *two*.

6. We could, just as we defined $\mathfrak{J}_A$, the algebraic intersection group, define also a transcendental group $\mathfrak{J}_T$ by using the transcendental 2-cycles instead of the algebraic ones. The matrices of this group are less easily characterized than those of the algebraic group, but some results for them will be given in a subsequent note. The transcendental groups are clearly absolutely invariant. We can also define the full intersection group $\mathfrak{J}$ as the join of $\mathfrak{J}_A$ and $\mathfrak{J}_T$. The groups $\mathfrak{J}$ are also absolutely invariant.

In conclusion I wish to express my very warmest appreciation of the encouragement and helpful criticism I have at all times received from Prof. Hodge, and to say how much this work owes to him. In particular the ideas of his paper(3) have been found especially suggestive. I have also to acknowledge my obligation to Magdalene College, Cambridge, for the award of a research studentship which enabled me to do the greater part of this work.

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