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Some local maximum principles along Ricci flows

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Abstract. In this work, we obtain a local maximum principle along the Ricci flow $g(t)$ under the condition that $\mathrm{Ric}(g(t)) \le \alpha t^{-1}$ for $t > 0$ for some constant $\alpha > 0$. As an application, we will prove that under this condition, various kinds of curvatures will still be nonnegative for $t > 0$, provided they are non-negative initially. These extend the corresponding known results for Ricci flows on compact manifolds or on complete noncompact manifolds with *bounded curvature*. By combining the above maximum principle with the Dirichlet heat kernel estimates, we also give a more direct proof of Hochard's [15] localized version of a maximum principle by Bamler et al. [1] on the lower bound of different kinds of curvatures along the Ricci flows for $t > 0$.

1 Introduction

Given a Riemannian manifold (M, g_0), the Ricci flow on M is a family of metrics $g(t)$ on M satisfying

$$\begin{cases} \partial_t g(t) &= -2\mathrm{Ric}(g(t)), \quad \text{on } M \times [0, T]; \\ g(0) &= g_0. \end{cases}$$

Here, we denote $g(x, t)$ simply by $g(t)$. In this work, we always assume that the family is smooth in space and time.

Ricci flow is a useful tool in the study of structures of manifolds. Ricci flow is useful because it tends to preserve certain geometric structures. In many cases, the behavior of a geometric structure is reflected by the behavior of a scalar function φ, which satisfies certain differential inequalities. One of the simplest ways to obtain useful information on φ, and hence on the corresponding geometric structure, for $t > 0$ is to apply maximum principles. In this work, we are interested in the following two frequently used differential inequalities along the Ricci flows:

$$\left(\partial_t - \Delta_{g(t)}\right)\varphi \le L\varphi, \tag{1.1}$$

for some continuous function $L(x, t)$ on $M \times [0, T]$ and

$$\left(\partial_t - \Delta_{g(t)}\right)\varphi \le \mathcal{R}\varphi + K\varphi^2 \tag{1.2}$$

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where K is a positive constant and $\mathcal{R}$ is the scalar curvature of $g(t)$. We will obtain two maximum principles for the two cases. The first one is the following:

Theorem 1.1 *Let $(M^n, g(t))$, $t \in [0, T]$ be a smooth solution to the Ricci flow which is possibly incomplete. Suppose*

$$\mathrm{Ric}(g(t)) \le \alpha t^{-1} \tag{1.3}$$

on $M \times (0, T]$ for some $\alpha > 0$. Let $\varphi(x, t)$ be a continuous function on $M \times [0, T]$ which satisfies $\varphi(x, t) \le \alpha t^{-1}$ on $M \times (0, T]$ and

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\varphi\Big|_{(x_0,t_0)} \le L(x_0, t_0)\varphi(x_0, t_0) \tag{1.4}$$

whenever $\varphi(x_0, t_0) > 0$ in the sense of barrier, for some continuous function $L(x, t)$ on $M \times [0, T]$ with $L(x, t) \le \alpha t^{-1}$. Suppose $p \in M$ such that $B_{g(0)}(p, 2) \Subset M$ and $\varphi(x, 0) \le 0$ on $B_{g(0)}(p, 2)$. Then for any $l > \alpha + 1$, there exists $\widehat{T}(n, \alpha, l) > 0$ such that for $t \in [0, T \wedge \widehat{T}]$,

$$\varphi(p, t) \le t^l.$$

Here and below, we denote

$$a \wedge b =: \min\{a, b\}.$$

For the definition of "*in the sense of barrier*," we refer readers to [6, Chapter 18].

Theorem 1.1 is known to be true if M is compact without boundary or M is noncompact and $g(t)$ is a complete solution with uniformly bounded curvature, see [11, 23] for example. Nevertheless, there are interesting results of the existence of the Ricci flows in which the initial metrics and the flows $g(t)$ may not be complete and may have unbounded curvatures, see [1, 2, 3, 7, 10, 12, 14, 15, 16, 24, 26, 28]. However, most of the Ricci flow solutions mentioned above satisfy the condition (1.3), which is invariant under parabolic rescaling. This motivates us to obtain the maximum principles, Theorem 1.1 and Theorem 1.2 below. As an immediate application, Theorem 1.1 will imply the preservation of non-negativity of most known curvature conditions under the assumption that $|\mathrm{Rm}(g(t))| \le \alpha t^{-1}$ in the complete noncompact case. See Theorem 3.1 for the full list of the curvature conditions. The theorem also implies the preservation of the Kähler condition, which is the first step in the use of the Kähler-Ricci flow to study the uniformization of complete noncompact Kähler manifolds with nonnegative bisectional curvature. See [23, 11] for more information.

In Theorem 1.1, the condition that $\varphi(0) \le 0$ is crucial and the analogous result is not true if φ is only assumed to be bounded from above initially. This can be seen by considering Euclidean space with the time function $\varphi_\varepsilon(t) = (\frac{t+\varepsilon}{\varepsilon})^\alpha$. The function satisfies $\varphi_\varepsilon(0) = 1$ and (1.4) with $L(x, t) = \alpha(t + \varepsilon)^{-1}$, but $\varphi_\varepsilon(t_0) \to +\infty$ as $\varepsilon \to 0$ for any fixed $t_0 > 0$. Hence, if the geometric quantity $\varphi(0)$ is only assumed to be bounded from above, one cannot expect the analogous conclusion of Theorem 1.1 holds. However, if φ satisfies (1.2), we have the following local upper estimates of φ for a short time. This was first proved by Hochard in [13, Proposition I.2.1].

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Theorem 1.2 *Let $g(t)$ be a smooth Ricci flow on $M^n \times [0,T]$ which is possibly incomplete. Suppose $g(t)$ satisfies the following:*

$$\begin{cases} |\mathrm{Rm}(g(x,t))| \le \alpha t^{-1}, & \text{for all } (x,t) \in M \times (0,T]; \\ \mathrm{inj}_{g(t)}(x) \ge \sqrt{\alpha^{-1}t}, & \text{for all } (x,t) \in M \times (0,T] \text{ with} \\ & B_{g(t)}(x, \sqrt{\alpha^{-1}t}) \Subset M; \\ \mathcal{R}(g(0)) \ge -\sigma T^{-1} & \text{on } M; \\ V_0(x,r) \le r^n \exp(v_0 r T^{-\frac{1}{2}}) & \text{for all } r > 0, \text{and } x \in M \text{ with } B_0(x,r) \Subset M, \end{cases}$$

for some $\alpha, v_0 > 0, \sigma \ge 0$ where $\mathcal{R}(g(0))$ is the scalar curvature of $g(0)$. Let φ be a nonnegative continuous function on $M \times [0,T]$ satisfying (1.2) in the sense of barrier. Assume that

$$\begin{cases} \varphi(0) \le \delta, & \text{on } M \text{ for some } \delta > 0; \\ \varphi(t) \le \alpha t^{-1}, & \text{on } M \times (0,T]. \end{cases}$$

Suppose $p \in M$ is a point such that $B_{g_0}(p, 3RT^{\frac{1}{2}}) \Subset M$ for some $R > 0$. Then

$$\varphi(p,t) \le C((RT^{\frac{1}{2}})^{-2} + \delta)$$

for $t \in [0,T]$ for some constant $C > 0$ depending only on $n, K, \alpha, v_0, \sigma$.

Remark 1.1 By volume comparison, if $\mathrm{Ric}(g(0)) \ge -\frac{1}{n}\sigma T^{-1}$, then the conditions on $\mathcal{R}(g(0))$ and $V_0(x,r)$ in the theorem will be satisfied for σ and for some $v_0 > 0$.

Maximum principle for the evolution equation (1.2) along the Ricci flow was first considered by Bamler et al. in [1]. In particular, they showed that if φ is the negative part of the smallest eigenvalue of $\mathrm{Rm}(x,t)$ with respect to certain curvature cones, then φ satisfies (1.2) in the barrier sense. They proved that for the Ricci flow $g(t)$ on a compact manifold or on a complete noncompact manifold with bounded curvature, if $g(t)$ and $\varphi(t)$ satisfy the conditions in Theorem 1.2 and $\varphi(0) \le \delta$, then $\varphi(t) \le C\delta$ within a short time-interval $[0, T_0]$ for some constant $C > 0$ both depending only on n, α, σ, and v_0. Theorem 1.2 is a localized version of this result. In [13], Hochard proved Theorem 1.2 by obtaining estimates of the heat kernels together with their gradients for the backward heat equation on a nested sequences of domains. In this work, we will give a more direct proof by combining the Dirichlet heat kernel estimates on a fixed $g(0)$-geodesic ball with Theorem 1.1. The proof is in the spirit of work [1].

The localized maximum principle Theorem 1.2 is particularly useful when we consider the partial Ricci flow. Combining the maximum principle with the partial Ricci flow method [12, 26], Lai [14] constructed a complete Ricci flow solution starting from a complete noncollapsed metric which is of almost weakly $\mathbf{PIC}_1$, and remains almost weakly $\mathbf{PIC}_1$ for a short time. In [21], McLeod and Topping combined Theorem 1.2, Lai's Ricci flow solutions [14] and the techniques developed in their earlier work [20] to obtain a smooth structure on the noncollapsed $\mathbf{IC}_1$-limit space. In [16], the authors used Theorem 1.2 to construct a local Kähler-Ricci flow starting from a noncollapsed Kähler manifold with almost nonnegative curvature and improve a result of Liu [18] on the complex structure of the corresponding Gromov–Hausdorff limit of this class of Kähler manifolds. See the recent work by Lott [19] for further development.

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The paper is organized as follows: In Section 2, we will collect some useful lemmas which allow us to compare $g(0)$-geodesic balls and $g(t)$-geodesic balls. In Section 3, we will give a proof of Theorem 1.1 and a unified proof for preservation of non-negativity of some curvature conditions. In Section 4, we will obtain Dirichlet heat kernel estimates for the backward heat equation and give a proof of Theorem 1.2.

2 Shrinking and expanding balls Lemmas

Let $(M^n, g(t))$ be a Ricci flow defined on $M \times [0, T]$. Since $g(t)$ may not be complete, we use the following convention: Let (M, g) be a Riemannian manifold without boundary which may be incomplete. Let $x \in M$, $r > 0$. If $\exp_{g,x}$ is defined on the ball $B(r)$ of radius r in the tangent space $T_x(M)$ with center at the origin, then we denote $B_g(x, r) = \text{Image}(\exp_{g,x}(B(r))$. We say that $B_g(x, r) \Subset M$ if it is compactly contained in M. We say that the injectivity radius $\text{inj}_g(x)$ of x satisfies $\text{inj}_g(x) \geq \iota_0$, if $B_g(x, \iota_0) \Subset M$ and $\exp_{g,x}$ is a diffeomorphism from the ball of radius ι_0 onto its image $B_g(x, \iota_0)$. Observe that if $B_g(x, r) \Subset M$, then any point in $B_g(x, r)$ can be joined to x by a minimizing geodesic in M. If $B_g(x, 2r) \Subset M$, then any two points in $B_g(x, r)$ can be joined by a minimizing geodesic lying inside $B_g(x, 2r)$. In this case, the distance function is well-defined on $B_g(x, r)$. We will omit the subscript g when the content is clear. In the rest of the work, we denote the ball of radius r with respect to $g(t)$ by $B_t(x, r)$ and its volume $\text{Vol}_{g(t)}(B_t(x, r))$ by $V_t(x, r)$. Moreover, the distance function with respect to $g(t)$ is denoted by d_t.

Since $g(t)$ is not necessarily complete, it is important to compare balls with respect to $g(t)$ at different time. Some basic results on this will be used later. The first one is the following shrinking balls Lemma by Simon-Topping [25, Corollary 3.3]:

Lemma 2.1 *There exists a constant $\beta = \beta(n) \geq 1$ depending only on n such that the following is true. Suppose $(M^n, g(t))$ is a Ricci flow for $t \in [0, T]$ and $x_0 \in M$ with $B_0(x_0, r) \Subset M$ for some $r > 0$. Suppose $g(t)$ satisfies $\text{Ric}(g(t)) \leq (n-1)a/t$ on $B_0(x_0, r)$ for some $a > 0$ for all $t \in (0, T]$. Then*

$$B_t\left(x_0, r - \beta\sqrt{at}\right) \subset B_0(x_0, r),$$

and in general for $0 < s < t < T$,

$$B_t\left(x_0, r - \beta\sqrt{at}\right) \subset B_s\left(x_0, r - \beta\sqrt{as}\right).$$

In particular,

$$d_t(y, x_0) \geq d_s(y, x_0) - \beta\sqrt{a}(t^{\frac{1}{2}} - s^{\frac{1}{2}})$$

for all $y \in B_t\left(x_0, r - \beta\sqrt{at}\right)$.

We also need the following expanding balls Lemma by He [10].

Lemma 2.2 *For any positive integer $n \in \mathbb{N}$ and for any $v_0, \alpha, \sigma > 0$, there exists $\mu(n, v_0, \alpha, \sigma) > 1$ and $R_0 = R_0(n, v_0, \alpha, \sigma) > 0$ such that the following is true: Let $(M^n, g(t))$ be a Ricci flow for $t \in [0, T]$ with $T \leq 1$. Suppose $p \in M$ with $B_0(p, R) \Subset M$ for some $R \geq R_0$ such that:*

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(a) $|\mathrm{Rm}(x,t)| \le \alpha t^{-1}$ *for all* $x \in B_0(p,R)$ *and* $t \in (0,T]$;
(b) $V_t(x,\sqrt{t}) \ge \alpha^{-1}t^{n/2}$ *for all* $t \in (0,T]$ *and for all* x *with* $B_t(x,\sqrt{t}) \subset B_0(p,R)$;
(c) $V_0(x,r) \le v_0 r^n$, *for all* $0 < r \le 1$ *and* $x \in B_0(p,R)$ *with* $B_0(x,r) \subset B_0(p,R)$;
(d) $\mathcal{R}_0 \ge -\sigma$, *in* $B_0(p,R)$, *where* $\mathcal{R}_t$ *is the scalar curvature of* $g(t)$.

Then for all $t \in [0,T]$, *we have*

$$B_0(p,\mu^{-1}R) \subset B_t(p,\frac{1}{2}R).$$

Proof By Lemma 2.1, we have $B_t(p,\frac{3}{4}R) \Subset M$ for all $t \in [0,T]$ provided

(c1): $R \ge C_1$.

Here and below, C_l will denote a positive constant depending only on n, v_0, α, σ.

By [25, Lemma 8.1], there is $T_1 = T_1(n,\alpha,\sigma) > 0$, such that $\mathcal{R}_t \ge -2\sigma$ on $B_t(p,\frac{2}{3}R)$ for $t \in [0, T \wedge T_1]$, if C_1 is large enough.

Let $0 < \tau \le T \wedge T_1 \le 1$ and let $\beta = \beta(n)$ be the constant from Lemma 2.1. Let $R_1 = \varepsilon R$ where $0 < \varepsilon < 1/2$ is a constant to be chosen later. Define

$$r_0 = \max\{r \in [0,R_1] : B_0(p,r) \Subset B_\tau(p,R_1/2)\}.$$

By Lemma 2.1 again, $r_0 \le \frac{1}{2}R_1 + \beta\sqrt{\alpha} \le \frac{1}{2}R$, provided C_1 in **(c1)** is large enough. By definition, there exists $y \in M$ such that $d_0(p,y) = r_0$ and $d_\tau(p,y) = R_1/2$. Let $\gamma : [0,r_0]$ be a minimizing $g(0)$-geodesic from p to y. Let N be the positive integer such that

$$r_0 + 2\beta\tau^{\frac{1}{2}} \ge 2\beta\tau^{\frac{1}{2}}N \ge r_0. \tag{2.1}$$

Then we can find $\{x_i\}_{i=1}^N$ on γ so that $B_0(x_i,\beta\sqrt{\tau})$ are all disjoint and γ is covered by $\bigcup_{i=1}^N B_0(x_i,2\beta\sqrt{\tau})$ which is a subset of $B_0(p,R)$ provided C_1 is large. Choose C_1 large enough so that for each i, and for each $z \in B_0(x_i,2\beta\sqrt{\tau})$, we have $B_\tau(z,2\sqrt{\tau}) \subset B_0(p,R)$. For each i, let $\{z_j^{(i)}\}_{j=1}^{k_i}$ be the maximal set of points in $B_0(x_i,2\beta\sqrt{\tau})$ such that $B_\tau(z_j^{(i)},\sqrt{\tau})$ are mutually disjoint and

$$\bigcup_{j=1}^{k_i} B_\tau(z_j^{(i)},\sqrt{\tau}) \subset B_0(x_i,2\beta\sqrt{\tau}) \subset \bigcup_{j=1}^{k_i} B_\tau(z_j^{(i)},2\sqrt{\tau})$$

Then γ will be covered by $\cup_{i=1}^N \cup_{j=1}^{k_i} B_\tau(z_j^{(i)},2\sqrt{\tau})$. Hence by (2.1), we have

$$\frac{1}{2}R_1 = d_\tau(p,y) \le 2\sqrt{\tau}\textstyle\sum_{i=1}^N k_i. \tag{2.2}$$

We want to estimate k_i from above.

Let $\tau = \min\{T, T_1, (2\beta)^{-2}\}$. By (b), we have

$$\begin{aligned} k_i\alpha^{-1}\tau^{n/2} &\le \textstyle\sum_{j=1}^{k_i} V_\tau(z_j^{(i)},\sqrt{\tau}) \\ &\le V_\tau(B_0(x_i,2\beta\sqrt{\tau})) \\ &\le C_2 V_0(B_0(x_i,2\beta\sqrt{\tau})) \\ &\le C_2 v_0 \tau^{\frac{n}{2}}. \end{aligned}$$

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The third inequality follows from $\partial_t d\mu_t = -\mathcal{R}_t d\mu_t$ and the lower bound on $\mathcal{R}_t$. Hence $k_i \le C_3$. By (2.2) and (2.1), we have:

$$\frac{1}{2}R_1 \le 2C_3 N\tau^{\frac{1}{2}} \le 2C_3\tau^{\frac{1}{2}} \cdot \frac{r_0 + 2\beta\tau^{\frac{1}{2}}}{2\beta\tau^{\frac{1}{2}}}$$

Therefore $r_0 \ge C_4^{-1}R_1 - 2\beta\tau^{\frac{1}{2}} \ge C_4^{-1}R_1 - 1$ and hence

$$B_0(p, \varepsilon C_4^{-1}R - 1) \subset B_\tau(p, \frac{\varepsilon}{2}R). \tag{2.3}$$

Suppose $T \le T_1 \wedge (2\beta)^{-2}$, then $\tau = T$. For all $t \le T = \tau$, by Lemma 2.1,

$$B_\tau(p, \frac{\varepsilon}{2}R) \subset B_t(p, \frac{\varepsilon}{2}R + \beta\tau^{\frac{1}{2}}) \subset B_t(p, \frac{1}{2}R)$$

provided C_1 in **(c1)** is large enough and $\varepsilon < \frac{1}{4}$. If $T > T_1 \wedge (2\beta)^{-2}$, then for $t \le \tau$, the above inequality is still true. For $T \ge t \ge \tau$, by condition (a), we have

$$B_\tau(p, \frac{\varepsilon}{2}R) \subset B_t(p, \varepsilon C_5 R) \subset B_t(p, \frac{1}{4}R).$$

provided that we choose $\varepsilon = \frac{1}{4(C_5+1)}$. By (2.3), one can see that if C_1 is large enough, then the Lemma is true. ∎

3 Local maximum principle Theorem 1.1

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1 Let $g(t)$ be a Ricci flow on $M^n \times [0,T]$ so that $\mathrm{Ric}(g(t)) \le \alpha t^{-1}$ for some $\alpha > 0$. Let φ be a continuous function defined on $M \times [0,T]$ which satisfies (1.1) in the sense of barrier at those points where $\varphi > 0$. Assume $\varphi \le 0$ at $t = 0$. Let $p \in M$ with $B_0(p,2) \Subset M$. We want to prove that $\varphi(p,t) \le t^l$ for large l provided $t \le T \wedge \widehat{T}(n,\alpha,l)$. We will first show that $\varphi \le t^{\frac{1}{2}}$ near $t = 0$, and then we improve the estimate to higher powers of t.

By replacing L by its positive part if necessary, we may assume that $L \ge 0$. By Lemma 2.1, there is $T_1 = T_1(n,\alpha) > 0$ such that $B_t(p, \frac{3}{2}) \Subset M$ if $t \le T \wedge T_1$. Let $d_t(x)$ to be the distance function from p with respect to $g(t)$. Then $d_t(x)$ is defined on $B_t(p,1)$ and is realized by a minimizing geodesic from p. By [22, Lemma 8.3], there exists $c_1(n) > 0$ such that

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)(d_t(x) + c_1\alpha\sqrt{t}) \ge 0 \tag{3.1}$$

for $x \in B_t(p,1)\backslash B_t(p,\sqrt{t})$ in the sense of barrier.

Let $\phi : [0,\infty) \to [0,1]$ be a smooth function such that

$$\phi(s) = \begin{cases} 1 & \text{for } 0 \le s \le \frac{1}{2}; \\ \exp(-\frac{1}{(1-s)}) & \text{for } \frac{3}{4} \le s \le 1; \\ 0, & \text{for } s \ge 1, \end{cases} \tag{3.2}$$

and such that $\phi' \le 0$, $\phi'' \ge -c\phi$ for some absolute constant $c > 0$.

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For $r \in [\frac{1}{2^4}, 1]$, let

$$\Phi_r(x,t) = \exp(-cr^{-2}t) \cdot \phi\left(\frac{d_t(x) + c_1\alpha\sqrt{t}}{r}\right).$$

Then $\operatorname{supp}(\Phi_r(\cdot,t)) \subset B_t(p,r)$. Note that

$$\frac{r}{2} \le d_t(x) + c_1\alpha\sqrt{t} \le r$$

if and only if $r - c_1\alpha\sqrt{t} \ge d_t(x) \ge \frac{r}{2} - c_1\alpha\sqrt{t}$. Hence if $t \le T_1 = \frac{1}{2^5}(c_1\alpha+1)^{-2}$, and $0 < \phi(x,t) < 1$, then $1 \ge d_t(x) \ge \sqrt{t}$. Therefore, by (3.1), we have

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\Phi_r \le 0 \tag{3.3}$$

in the sense of barrier on $B_0(p,2) \times [0, T_1 \wedge T]$. We may also choose T_1 small enough so that $e^{-cr^{-2}T_1} \ge \frac{1}{2}$. Let $\eta(t) \ge 0$ be a smooth function in t such that $\eta(t) > 0$ for $t > 0$. We consider the function

$$F = -\Phi_r^m\varphi + \eta.$$

Here, m is a positive integer to be determined later. Assume η is chosen so that $F > 0$ near $t = 0$. In this case, if $F(x,t) < 0$ for some $(x,t) \in B_0(p,2) \times [0, T_1]$ then there is $(x_0,t_0) \in B_0(p,2) \times (0, T_1]$ with $0 < t_0 \le T_1$ such that $F(x_0,t_0) = 0$, and $F(x,t) \ge 0$ on $B_0(p,2) \times [0,t_0]$. At (x_0,t_0), $\Phi_r > 0$ and $\varphi > 0$.

By (3.3) and (1.1), for any $\varepsilon > 0$, there exists C^2 functions $\sigma(x)$, $\zeta(x)$ near x_0 such that $\sigma(x) \le \Phi_r(x,t_0)$, $\sigma(x_0) = \Phi_r(x_0,t_0)$, $\zeta(x) \le \varphi(x,t_0)$, and $\zeta(x_0) = \varphi(x_0,t_0)$. Moreover, the following are true:

$$\frac{\partial_-}{\partial t}\Phi_r(x_0,t_0) - \Delta_{g(t)}\sigma(x) \le \varepsilon,$$

and

$$\frac{\partial_-}{\partial t}\varphi(x_0,t_0) - \Delta_{g(t)}\zeta(x_0) - L(x_0,t_0)\zeta(x_0) \le \varepsilon.$$

Here, for a function $f(x,t)$,

$$\frac{\partial_-}{\partial t}f(x_0,t_0) = \liminf_{h\to 0^+}\frac{f(x_0,t_0) - f(x_0,t_0-h)}{h}.$$

The function $G(x,t) = -\sigma^m(x)\zeta(x) + \eta(t)$ is C^2 in space and time so that $G(x_0,t_0) = F(x_0,t_0) = 0$. For x near x_0, since $\varphi(x,t_0) > 0$ near x_0 and φ is continuous, for x sufficiently close to x_0,

$$G(x,t_0) \ge -\Phi_r^m(x,t_0)\varphi(x,t_0) + \eta(t_0) = F(x,t_0) \ge 0.$$

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Hence at (x_0, t_0), we have

$$\begin{cases} \zeta = \dfrac{\eta}{\Phi_r^m}; \\ \nabla\zeta = -\dfrac{m\zeta\nabla\sigma}{\sigma}; \\ |\nabla\sigma| \le |\nabla\Phi_r|, \end{cases}$$

and

$$\begin{aligned} 0 &\le \Delta_{g(t)} G \\ &= -\sigma^m \Delta_{g(t)}\zeta - \zeta\Delta_{g(t)}\sigma^m - 2\langle\nabla\sigma^m, \nabla\zeta\rangle \\ &= -\sigma^m \Delta_{g(t)}\zeta - m\zeta\sigma^{m-1}\Delta_{g(t)}\sigma - m(m-1)\zeta\sigma^{m-2}|\nabla\sigma|^2 - 2m\sigma^{m-1}\langle\nabla\sigma, \nabla\zeta\rangle \\ &\le \Phi_r^m\left(-\frac{\partial_-}{\partial t}\varphi + L\varphi + \varepsilon\right) + m\Phi_r^{m-1}\varphi\left(-\frac{\partial_-}{\partial t}\Phi_r(x_0, t_0) + \varepsilon\right) - 2m\Phi_r^{m-1}\langle\nabla\sigma, \nabla\zeta\rangle \\ &= \frac{\partial_-}{\partial t}F - \eta' + \Phi_r^m\left(L\varphi + \varepsilon\right) + \varepsilon n\Phi_r^{m-1}\varphi + 2m^2\sigma^{m-2}\zeta|\nabla\sigma|^2 \\ &\le -\eta' + L\eta + \varepsilon\Phi_r^m + \varepsilon m\Phi_r^{m-1}\varphi + 2m^2\eta\frac{|\nabla\Phi_r|^2}{\Phi^2}, \end{aligned}$$

because

$$\frac{\partial_-}{\partial t}F\Big|_{(x_0,t_0)} \le 0.$$

By letting $\varepsilon \to 0$, we conclude that at (x_0, t_0),

$$\begin{aligned} \eta'(t_0) &\le \eta(t_0)\left(L(x_0, t_0) + 2m^2\frac{|\nabla\Phi_r|^2}{\Phi_r^2(t_0)}\right) \\ &\le \begin{cases} \eta(t_0)\left(L_0 + C_1^2 m^2\left(\dfrac{a_0}{\eta(t_0)}\right)^{\frac{2}{m}}\right); \text{ or} \\ \eta(t_0)\left(\dfrac{\alpha}{t_0} + C_1^2 m^2\left(\dfrac{\alpha}{t_0\eta(t_0)}\right)^{\frac{2}{m}}\right), \end{cases} \end{aligned} \tag{3.4}$$

where $L_0 = \max_{B_0(p,2)\times[0,T]} L$, $a_0 = \max_{B_0(p,2)\times[0,T]} |\varphi|$. In the above inequalities, we have used the fact that at (x_0, t_0),

$$\frac{1}{\Phi_r^m} = \frac{\varphi}{\eta} \le \min\left\{\frac{\alpha}{t_0\eta(t_0)}, \frac{a_0}{\eta(t_0)}\right\}.$$

Here and below, C_k will denote a positive constant depending only on n, α.

First, we show that $\varphi(t) = O(t^{1/2})$. For any $1 > \delta > 0$, let $\eta(t) = t^{\frac{1}{2}} + \delta$. Then $F > 0$ near $t = 0$. By the first inequality on the second line of (3.4), we have

$$\frac{1}{2}t_0^{-\frac{1}{2}} \le (t_0^{\frac{1}{2}} + \delta)\left(L_0 + \frac{C_1^2 m^2 a_0^{\frac{2}{m}}}{(t_0^{\frac{1}{2}} + \delta)^{\frac{2}{m}}}\right).$$

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Choose $m = 2$ and $r = 1$, we see that there is $\tau > 0$, small enough but independent of δ so that $t_0 \geq \tau$. Hence by letting $\delta \to 0$, we conclude that $\varphi \leq 2t^{\frac{1}{2}}$ on $B_t(p, \frac{1}{2} - c_1\alpha\sqrt{t})$ near $t = 0$.

Next, we improve the estimate of φ as $t \to 0^+$. Given an integer $k \geq 1$ and $\delta > 0$, let $\eta = \delta t^{\frac{1}{4}} + t^k$ and $r = \frac{1}{2}$. By the first inequality on the second line of (3.4), we have

$$\frac{1}{4}\delta t_0^{-\frac{1}{4}} + k t_0^{k-1} \leq (\delta t_0^{\frac{1}{4}} + t_0^k)\left(L_0 + \frac{C_1^2 m^2 a_0^{\frac{2}{m}}}{(\delta t_0^{\frac{1}{4}} + t_0^k)^{\frac{2}{m}}}\right).$$

Choose m large enough so that $2k/m < 1$, then we can find $\tau_1 > 0$ such that $t_0 > \tau_1$. Therefore, we may conclude that $\varphi(x, t) \leq 2t^k$ near $t = 0$ on $B_0(p, \frac{1}{4} - c_1\alpha\sqrt{t})$.

Now, we will show that under (1.3), for each $l \geq \alpha + 1$, the above τ_1 can be chosen so that it is bounded from below away from zero depending only on n, α, l. Let $\eta = \frac{1}{2}t^l$, $r = \frac{1}{4}$. By the above upper estimate of φ near $t = 0$, we see that $F > 0$ near $t = 0$. Therefore, we can use the second inequality on the second line of (3.4) to show that

$$l t_0^{l-1} \leq t_0^l \left(\frac{\alpha}{t_0} + C_1^2 m^2 \left(\frac{\alpha}{t_0^{l+1}}\right)^{\frac{2}{m}}\right).$$

This implies:

$$t_0^{l-1} \leq C_1^2 m^2 \alpha^{\frac{2}{m}} t_0^{l - \frac{2}{m}(l+1)}.$$

Choose m sufficiently large so that $\frac{2}{m}(l+1) < \frac{1}{2}$, we conclude that $t_0 \geq T_2(n, \alpha, l)$ and hence

$$\varphi(p, t) \leq t^l$$

if $t \leq T_2 \wedge T_1 \wedge T$. This completes the proof. ∎

Theorem 1.1 is invariant under parabolic rescaling in the following sense: Let $(M, g(t))$, φ, L be as in the theorem. For any $\lambda > 0$, we define $g_1(x, t) = \lambda g(x, \lambda^{-1}t)$, $\varphi_1(x, t) = \lambda^{-1}\varphi(x, \lambda^{-1}t)$, and $L_1(x, t) = \lambda^{-1}L(x, \lambda^{-1}t)$. Then $g_1(t)$ satisfies the curvature condition (1.3) with the same α and

$$\varphi_1(x, t) = \lambda^{-1}\varphi(x, \lambda^{-1}t) \leq \lambda^{-1}\alpha(\lambda^{-1}t)^{-1} = \alpha t^{-1}.$$

Similarly, $L_1(x, t) \leq \alpha t^{-1}$. Moreover, let $s = \lambda^{-1}t$

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_{g_1(t)}\right)\varphi_1(x, t) &= \lambda^{-2}\left(\frac{\partial}{\partial s} - \Delta_{g(s)}\right)\varphi(x, s) \\ &\leq \lambda^{-2}L(x, s)\varphi(x, s) \\ &= L_1(x, t)\varphi_1(x, t)\end{aligned}$$

in the sense of barrier whenever $\varphi_1(x, t) = \lambda^{-1}\varphi(x, \lambda^{-1}t) > 0$. Hence, we have the following rescaled version of Theorem 1.1.

Corollary 3.1 *Let $(M, g(t))$, $t \in [0, T]$, φ, L be as in the* Theorem 1.1. *Let $p \in M$, $r > 0$ with $B_0(p, r) \Subset M$. Then for any positive integer $l \geq \alpha + 1$, there is $T_1(n, \alpha, l) > 0$*

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depending only on n, α, l such that

$$\varphi(p,t) \le r^{-2(l+1)} t^l$$

for all $t \le [0, T \wedge r^2 T_1]$.

Proof Let $\lambda = r^{-2}4$. Define g_1, φ_1, L_1 as above. Then $B_{g_1(0)}(p,2) \Subset M$. By Theorem 1.1, for any $l \ge \alpha + 1$

$$\varphi_1(p,t) \le t^l$$

for $t \in [0, T_1' \wedge \lambda T]$ for some $T_1' > 0$, depending only on n, α, l. Hence

$$\begin{aligned}\varphi(p,t) &= \lambda \varphi_1(p, \lambda t)\\ &\le \lambda^{l+1} t^l\\ &= r^{-2(l+1)} 4^{-(l+1)} t^{l+1}\\ &\le r^{-2(l+1)} t^l\end{aligned}$$

for $t \in [0, (r^2 T_1) \wedge T]$ where $T_1 = 4T_1'$. From this, the result follows. ∎

When $g(t)$ is a complete solution to the Ricci flow, then the corollary implies that $\varphi \le 0$ for $t > 0$ by letting $r \to \infty$. In fact, in this case by using the trick of Chen [5], we do not need the assumption that $\mathrm{Ric}(g(t)) \le \alpha t^{-1}$. Namely, we have following corollary of our method:

Corollary 3.2 *Let $(M^n, g(t))$ be a complete solution of the Ricci flow on $M \times [0, T]$. Let φ, L be as in* Theorem 1.1. *Then $\varphi \le 0$ for $t > 0$.*

Proof For any compact set Ω, we have $\mathrm{Ric}(g(t)) \le t^{-1}$ on Ω provided t is small enough. Hence by Corollary 3.1, for any $l \ge 1$ and compact set Ω, we have $\varphi \le t^l$ on Ω provided t is small depending only on n, l, and Ω.

Let $p \in M$ and let b be a positive number such that $\mathrm{Ric}(g(t)) \le b^2$ on $B_t(p,1)$ for all $t \in [0, T]$. Then as before

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)(d_t(x) + c_1 b t) \ge 0$$

in the sense of barrier on $M \backslash B_t(p, \frac{1}{b})$ for some $c_1 = c_1(n)$. Let ϕ be as in (3.2). Then for $A > 1$ sufficiently large,

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\phi\left(\frac{d_t(x) + c_1 b t}{A}\right) \le \frac{c}{A^2}\phi\left(\frac{d_t + c_1 b t}{A}\right)$$

in the sense of barrier. Define $\Phi(x,t) = \exp(-\frac{c}{A^2}t)\phi$ so that

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\Phi \le 0.$$

in the sense of barrier. For any integer $m \ge 2$, $l > 1 + \alpha$, and $\varepsilon > 0$, we define $F = -\Phi^m \varphi + \varepsilon t^l$. The argument in the beginning of the proof of Theorem 1.1 shows that $F(x,t) > 0$ on $M \times (0, t')$ for some $t' > 0$. If $F(x,t) < 0$ somewhere, then there is $x_0 \in M$, $T \ge t_0 > 0$ so that $F(x_0, t_0) = 0$ and $F(x,t) \ge 0$ on $M \times [0, t_0]$. As in (3.4),

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we have

$$\varepsilon l t_0^{l-1} \le \varepsilon t_0^l \left(\frac{\alpha}{t_0} + \frac{C_1^2 m^2}{A^2} \left(\frac{\alpha}{\varepsilon t_0^{l+1}} \right)^{\frac{2}{m}} \right).$$

And hence,

$$\varepsilon^{\frac{2}{m}} A^2 t_0^{l-1} \le t_0^{l-\frac{2}{m}(l+1)} \alpha^{\frac{2}{m}} C_1^2 m^2,$$

where we have used the fact that $\ell > \alpha + 1$.

For a fixed l, we choose m sufficiently large such that $\frac{2}{m}(l+1) < \frac{1}{2}$, then we have

$$\varepsilon^{\frac{2}{m}} A^2 C_2 \le t_0^{1-\frac{2}{m}(l+1)}.$$

Therefore, if we choose A large enough so that $\varepsilon^{\frac{2}{m}} A^2 C_2 > T^{1-\frac{2}{m}(l+1)}$. We conclude that $t_0 > T$ which is impossible. By letting $A \to +\infty$ and followed by $\varepsilon \to 0$, we conclude that $\varphi(p,t) \le 0$. Since p is an arbitrary point on M, the result follows. ∎

Corollary 3.3 *Let (M^n, g) be a complete Riemannian manifold with* $\mathrm{Ric}(g) \ge -1$. *Let φ, L be as in Theorem 1.1 with respect to $\partial_t - \Delta_g$ instead. Then $\varphi \le 0$ for $t > 0$.*

Proof By Laplacian comparison, we have $(\partial_t - \Delta_g)(d_g(x,p) + Ct) \ge 0$ in the sense of barrier for some fixed $p \in M$ and $C > 0$ whenever $d_g(x,p) > 1$. Hence, the proof of Corollary 3.2 can be carried over. ∎

Using the idea in [23], we may use Corollary 3.2 to prove that complete Ricci flows satisfying curvature condition $|\mathrm{Rm}(g(t))| \le \alpha t^{-1}$ preserve the Kähler condition. This recovers results in [11, 23]. Another application of Corollary 3.2 is on the preservation of non-negativity of various curvatures along the Ricci flows which may not be complete or may have unbounded curvature. We will follow the set-up in [1]. See [27] for a unified approach in compact case and the case that $g(t)$ is complete noncompact with bounded curvature. Information about previous contributions by others can also be found in [27].

Theorem 3.1 *Let $(M^n, g(t))$ be a smooth solution to the Ricci flow on $M \times [0,T]$, where $g(t)$ may not be complete. Assume the scalar curvature $\mathcal{R}$ satisfies $\mathcal{R}(g(t)) \le \alpha t^{-1}$ for some $\alpha > 0$ on $M \times (0,T]$. Consider one of the following curvature conditions $\mathcal{C}$:*

(1) *non-negative curvature operator;*
(2) *2-non-negative curvature operator, (i.e. the sum of the lowest two eigenvalues is non-negative);*
(3) *weakly* PIC_2 *(i.e. taking the Cartesian product with $\mathbb{R}^2$ produces a non-negative isotropic curvature operator);*
(4) *weakly* PIC_1 *(i.e. taking the Cartesian product with $\mathbb{R}$ produces a non-negative isotropic curvature operator);*
(5) *non-negative bisectional curvature, in the case in which $(M, g(t))$ is Kähler;*
(6) *non-negative orthogonal bisectional curvature, in the case in which $(M, g(t))$ is Kähler .*

Let $p \in M$ and $r > 0$ be such that $B_0(p,r) \Subset M$, $\mathrm{Rm}(g(x,0)) \in \mathcal{C}$, *and* $\mathrm{Rm}(g(t)) + \alpha t^{-1} \mathrm{I}$ *are in the same $\mathcal{C}$ for $(x,t) \in B_0(p,r) \times (0,T]$. Then for all $l > \alpha + 1$, there is*

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$\widehat{T}(n,\alpha,l) > 0$ *such that for all* $t \in [0, T \wedge \widehat{T}r^2]$, $\mathrm{Rm}(g(p,t)) + r^{-2(l+1)}t^l \mathrm{I}$ *is in the same* $\mathcal{C}$. *In particular, if* $g(t)$ *is a complete solution and the assumption holds for all* $r > 0$, *then* $\mathrm{Rm}(g(t)) \in \mathcal{C}$ *for all* $t > 0$.

Proof Fix a curvature condition $\mathcal{C}$. Let

$$\ell(x,t) = \inf\{\varepsilon > 0 | \mathrm{Rm}(g(x,t)) + \varepsilon I \in \mathcal{C}\}.$$

Then by [1] for (1)–(5) and by [17] for (6), we have

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\ell \le \mathcal{R}\ell + c(n)\ell^2$$

in the sense of barrier for some constant $c(n)$ depending only on n. By the assumption on $\mathcal{R}$ and the assumption that $\mathrm{Rm}(g(t)) + \alpha t^{-1} I \in \mathcal{C}$, we conclude that

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\ell \le at^{-1}\ell$$

for some $a > 0$. Since $\ell = 0$ at $t = 0$ as $\mathrm{Rm}(g(0)) \in \mathcal{C}$, the conclusion at p follows from Corollary 3.1. When $g(t)$ is a complete solution, we can let $r \to +\infty$ to conclude that $\mathrm{Rm}(g(p,t)) \in \mathcal{C}$. Since p is arbitrary, the result follows. ∎

4 Local maximum principle Theorem 1.2

In this section, we will use Theorem 1.1 to prove Theorem 1.2. Let $g(t)$ be a Ricci flow on $M \times [0,T]$ satisfying:

$$(4.1)\quad \begin{cases} |\mathrm{Rm}(g(x,t))| \le \alpha t^{-1}, & \text{for all } (x,t) \in M \times (0,T] \\ \mathrm{inj}_{g(t)}(x) \ge \sqrt{\alpha^{-1}t}, & \text{for all } (x,t) \in M \times (0,T] \text{ with } B_t(x, \sqrt{\alpha^{-1}t}) \Subset M, \end{cases}$$

for some $\alpha > 1$. We will consider the continuous function $\varphi(x,t) \ge 0$ on $M \times [0,T]$ which satisfies:

$$(4.2)\qquad \left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\varphi \le \mathcal{R}\varphi + K\varphi^2$$

in the sense of barrier, where $\mathcal{R}(g(t))$ is the scalar curvature of $g(t)$ and $K \ge 0$ is a constant.

Before we prove Theorem 1.2, we first give the following application of the theorem.

Corollary 4.1 *Let* $(M^n, g(t))$ *be as in Theorem 1.2. Suppose*

$$\mathrm{Rm}(g_0) + \delta I \in \mathcal{C}$$

for some $\delta > 0$ *where* $\mathcal{C}$ *is one of the curvature cones (1)–(6) in Theorem 3.1. Let* p, R *be as in Theorem 1.2. Then there is a constant* $C_0 = C_0(n,\alpha,v_0,\sigma) > 0$ *such that*

$$\mathrm{Rm}(g(p,t)) + \delta^* I \in \mathcal{C}$$

for $t \in [0,T]$ *where* $\delta^* = C_0((RT^{\frac{1}{2}})^{-2} + \delta)$. *In particular, if* $g(t)$ *is a complete solution, then* $\mathrm{Rm}(g(t)) + C_0\delta \in \mathcal{C}$.

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Proof As in the proof of Theorem 3.1, let

$$\ell(x,t) = \inf\{\varepsilon > 0 | \operatorname{Rm}(g(x,t)) + \varepsilon I \in \mathcal{C}\}.$$

Then

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\ell \le \mathcal{R}\ell + c(n)\ell^2$$

in the sense of barrier for some constant $c(n)$ depending only on n. By (4.1), $\ell(x,t) \le \alpha' t^{-1}$ for some $\alpha' > 0$ depending only on α, n. The first result follows from the Theorem 1.2. The second result follows by letting $R \to \infty$. ∎

We may reduce the proof of Theorem 1.2 to the case that $T = 1$. More precisely, let $g_1(t) = T^{-1}g(Tt)$, $\varphi_1(x,t) = T\varphi(x,Tt)$. Then $g_1(t)$ satisfies (4.1) and $\varphi_1(x,t)$ satisfies (4.2) with $\mathcal{R}(g(t))$ replaced by $\mathcal{R}(g_1(t))$. Moreover, $g_1(0)$ and φ_1 satisfy

$$\text{(4.3)} \quad \begin{cases} \mathcal{R}(g_1(0)) \ge -\sigma & \text{on } M; \\ V_{g_1(0)}(x,r) \le r^n \exp(v_0 r) & \text{for all } r > 0, \text{ and } x \in M \text{ with } B_0(x,r) \Subset M, \end{cases}$$

and

$$\begin{cases} \varphi_1(0) \le T\delta, & \text{on } M; \\ \varphi_1(t) \le \alpha t^{-1}, & \text{on } M \times (0,T]. \end{cases}$$

If we can prove Theorem 1.2 with $T = 1$, then the upper bound for φ_1 will imply the required upper bound for φ.

Theorem 1.2 has been obtained earlier by Hochard [13, Propositions I.2.1 and II.2.6]. The approach here is a localized version of the original method in [1]. The main ingredients are the local maximum principle Theorem 1.1 and an upper bound of the Dirichlet heat kernel for a fixed domain.

4.1 Upper estimates of the Dirichlet heat kernel

Let (M^n, g_0) be a Riemannian manifold and let $g(t)$ be a Ricci flow on $M \times [0,T]$ with $g(0) = g_0$. Here $g(t)$ may be incomplete. Let $\Omega \Subset M$ be an open set with smooth boundary. We let $G_\Omega(x,t;y,s)$, $t > s$ be the Dirichlet heat kernel for the backward heat equation coupled with the Ricci flow $g(t)$:

$$\begin{cases} (\partial_s + \Delta_{y,s})\, G_\Omega(x,t;y,s) = 0, & \text{in int } (\Omega) \times \text{int } (\Omega) \times [0,t); \\ \lim_{s \to t^-} G_\Omega(x,t;y,s) = \delta_x(y), & \text{for } x \in \text{int } (\Omega); \\ G_\Omega(x,t;y,s) = 0, & \text{for } y \in \partial\Omega, x \in \text{int } (\Omega), \end{cases}$$

where $\Delta_{y,g(s)}$ is denoted by $\Delta_{y,s}$. Then

$$\begin{cases} (\partial_t - \Delta_{x,t} - \mathcal{R}_t)\, G_\Omega(x,t;y,s) = 0, & \text{in int } (\Omega) \times \text{int } (\Omega) \times (s,T]; \\ \lim_{t \to s^+} G_\Omega(x,t;y,s) = \delta_y(x), & \text{for } y \in \text{int } (\Omega); \\ G_\Omega(x,t;y,s) = 0, & \text{for } x \in \partial\Omega, y \in \text{int } (\Omega), \end{cases}$$

where $\mathcal{R}_t$ is the scalar curvature of $g(t)$. Such G_Ω exists and is positive in the interior of Ω, see [8].

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We want to estimate the upper bound of $G_\Omega(x,t;y,s)$ with respect to y and $g(s)$ under the conditions (4.1). The following Dirichlet heat kernel estimate was implicitly proved in [4, Theorem 5.1].

Lemma 4.1 *Let (M^n, g_0) be a Riemannian manifold and $p \in M$. Suppose $g(t)$ is a solution to the Ricci flow on $M \times [0,1]$ with $g(0) = g_0$ such that $B_0(p, 2(r+1)) \Subset M$ for some $r \geq 1$ and $|\mathrm{Rm}(x,t)| \leq A$ on $M \times [0,1]$. If Ω is an open set in M with smooth boundary such that $\Omega \Subset B_{g_0}(p,r)$ and $G_\Omega(x,t;y,s)$ is the Dirichlet heat with respect to the backward heat equation on $\Omega \times \Omega \times [0,1]$. Then there is $C(n,A) > 0$ such that for all $0 \leq s < t \leq 1$, $x, y \in \Omega$,*

$$G_\Omega(x,t;y,s) \leq \frac{C}{V_0^{\frac{1}{2}}(x,\sqrt{t-s})V_0^{\frac{1}{2}}(y,\sqrt{t-s})} \times \exp\left(-\frac{d_0^2(x,y)}{C(t-s)}\right).$$

Proof Let $\widetilde{\Omega}$ be a bounded open domain with smooth boundary so that $B_0(p, r+\frac{1}{2}) \Subset \widetilde{\Omega} \Subset B_0(p, r+1)$. In particular, any two points in $\widetilde{\Omega}$ can be joined by a minimizing geodesic in M. Let $G_{\widetilde{\Omega}}$ be the heat kernel on $\widetilde{\Omega} \times \widetilde{\Omega} \times [0,1]$. By the maximum principle, we have $G_\Omega \leq G_{\widetilde{\Omega}}$ on $\Omega \times \Omega \times [0,1]$. In the following, C_i will denote positive constants depending only on n, A.

Step 1 : Denote $G_{\widetilde{\Omega}}$ by G. For $0 < s < t \leq 1$,

$$\frac{\partial}{\partial t}\left(\int_{\widetilde{\Omega}} G(x,t;y,s)d\mu_{x,t}\right) = \int_{\widetilde{\Omega}} \Delta_{x,t} G \, d\mu_{x,t} = \int_{\partial\widetilde{\Omega}} \frac{\partial G}{\partial \nu} \leq 0,$$

because $G > 0$ on $\mathrm{int}(\tilde{\Omega})$ and $G = 0$ on $\partial\tilde{\Omega}$. Since $\lim_{t\to s^+} \int_{\widetilde{\Omega}} G(x,t;y,s)d\mu_{x,t} = 1$, we have for all $y \in \widetilde{\Omega}$.

$$\int_{\widetilde{\Omega}} G_{\widetilde{\Omega}}(x,t;y,s)d\mu_{x,t} \leq 1.$$

Let $f \in C^\infty(\tilde{\Omega})$ with $0 \leq f \leq 1$ on $\tilde{\Omega}$ and $f = 0$ on $\partial\tilde{\Omega}$. Let

$$u(x,t) = \int_{\widetilde{\Omega}} G_{\widetilde{\Omega}}(x,t;y,s) f(y) \, d\mu_{y,s}.$$

Then u satisfies $(\partial_t - \Delta_{g(t)} - \mathcal{R}_t)u = 0$ with zero boundary data and with initial data f. By the maximum principle, we have $u(x,t) \leq C_1$ for $t \geq s$. Letting $f \to 1$, we conclude that

$$\int_{\widetilde{\Omega}} G_{\widetilde{\Omega}}(x,t;y,s) \, d\mu_{y,s} \leq C_1.$$

Step 2 : Apply the argument of [4, Lemma 5.3] and Step 1, using the mean value inequality [4, Lemma 3.1], volume comparison and the fact that $B_{g_0}(x, \frac{1}{2}) \Subset \widetilde{\Omega}$ for all $x \in \Omega$, we have pointwise estimate:

$$G_\Omega(x,t;y,s) \leq G_{\widetilde{\Omega}}(x,t;y,s) \leq \min\left\{\frac{C_2}{V_0(x,\sqrt{t-s})}, \frac{C_2}{V_0(y,\sqrt{t-s})}\right\}.$$

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Combining this with the integral estimates in Step 1, we conclude that for $y \in \Omega$ and $s < t$,

$$\begin{cases} \int_\Omega G_\Omega^2(x,t;y,s)\, d\mu_{x,t} \le \dfrac{C_3}{V_0(y,\sqrt{t-s})}; \\ \int_\Omega G_\Omega^2(x,t;y,s)\, d\mu_{y,s} \le \dfrac{C_3}{V_0(x,\sqrt{t-s})}. \end{cases}$$

Step 3 : Apply the method of proof in [9, Theorem 2.1], (see also [4, Lemma 2.2]), we have

$$\begin{cases} \int_\Omega G_\Omega^2(x,t;y,s) e^{\frac{d_0^2(x,y)}{C_4(t-s)}}\, d\mu_{x,t} & \le \dfrac{C_4}{V_0(y,\sqrt{t-s})} \quad \text{for all } y \in \Omega; \text{ and} \\ \int_\Omega G_\Omega^2(x,t;y,s) e^{\frac{d_0^2(x,y)}{C_4(t-s)}}\, d\mu_{y,s} & \le \dfrac{C_4}{V_0(x,\sqrt{t-s})} \quad \text{for all } x \in \Omega. \end{cases}$$

Step 4: By the semigroup property of Dirichlet heat kernel (see [6, Lemma 26.12] for example), and by using arguments in the proof of [4, Theorem 5.5], we have

$$G_\Omega(x,t;y,s) \le \frac{C_5}{V_0^{\frac{1}{2}}(x,\sqrt{t-s})\, V_0^{\frac{1}{2}}(y,\sqrt{t-s})} \times \exp\left(-\frac{d_0^2(x,y)}{C_5(t-s)}\right).$$

■

Using Lemma 4.1, we can now proceed as in [1, Proposition 3.1] to obtain the following heat kernel estimate.

Proposition 4.1 *For any $n, \alpha > 0$, there exists $C(n,\alpha) > 0$ such that the following is true: Suppose $(M^n, g(t))$ is a solution to the Ricci flow on $M \times [0,1]$ with initial metric g_0 satisfying the conditions (4.1). Let $p \in M$ be a fixed point so that $B_t(p,4r) \Subset M$ for some $r \ge 1$ for all $t \in [0,1]$. Let Ω be a domain with smooth boundary so that $\Omega \Subset B_t(p,r)$ for all $t \in [0,1]$. Then the Dirichlet heat kernel $G(x,t;y,s)$ with respect to the backward heat equation on $\Omega \times \Omega \times [0,1]$ satisfies:*

$$G(x,t;y,s) \le \frac{C}{(t-s)^{\frac{n}{2}}} \exp\left(-\frac{d_s^2(x,y)}{C(t-s)}\right).$$

for all $0 \le s < t \le 1$ and $x, y \in \Omega$, where d_s is the distance function with respect to $g(s)$.

Remark 4.1 The condition that $\Omega \Subset B_t(p,r) \subset B_t(p,4r) \Subset M$ is assumed so that for all $x, y \in \Omega$, $d_t(x,y)$ is well-defined and is realized by a minimizing $g(t)$-geodesic lying in $B_t(p,4r)$.

By Lemma 2.2, we have the following:

Corollary 4.2 *There exist $R_0(n,\alpha,\sigma,v_0) > 1, \mu(n,\alpha,\sigma,v_0) > 1$ such that the following is true: Suppose (M^n, g_0) is a Riemannian manifold and $g(t)$ is a solution to the Ricci flow on $M \times [0,T]$ with $g(0) = g_0$ satisfying conditions (4.1) and*

$$\begin{cases} \mathcal{R}(g_0) \ge -\sigma T^{-1} & \text{on } M; \\ V_0(x,r) \le v_0 r^n & \text{for } 0 < r \le T^{\frac{1}{2}}. \end{cases}$$

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Then we can find $C(n,\alpha)>0$ so that if $p\in M$ with $B_0(p,R)\Subset M$ and $R\geq T^{\frac{1}{2}}R_0$, then the heat kernel $G(x,t;y,s)$ on $B_{g_0}(p,\mu^{-1}R)\times[0,T]$ satisfies

$$G(x,t;y,s)\leq \frac{C}{(t-s)^{\frac{n}{2}}}\exp\left(-\frac{d_s^2(x,y)}{C(t-s)}\right).$$

Proof Let $g_1(t)=T^{-1}g(Tt)$. Then $g_1(t)$ is a Ricci flow defined on $M\times[0,1]$ satisfying (4.1) and (4.3). By Lemma 2.2 and Proposition 4.1, the results follows by parabolic rescaling. ∎

4.2 Proof of Theorem 1.2

The following Lemma reduces the upper bound of φ to the integral bound of the heat kernel.

Lemma 4.2 *Suppose $(M^n,g(t))$, $t\in[0,T]$ is a smooth solution to the Ricci flow with initial metric g_0 which may not be complete. Suppose $g(t)$ satisfies:*

$$\mathrm{Ric}(x,t)\leq\frac{\alpha}{t}$$

for $(x,t)\in M\times(0,T]$ for some $\alpha>0$. Let φ be a nonnegative continuous function on $M\times[0,T]$ such that $\varphi(0)\leq\delta$ and $\varphi(t)\leq\alpha t^{-1}$ for some $\delta>0$. Assume φ satisfies

$$\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)\varphi\leq\mathcal{R}\varphi+K\varphi^2$$

in the sense of barrier, where $K>0$ is a constant and $\mathcal{R}=\mathcal{R}(g(t))$ is the scalar curvature of $g(t)$. Let $p\in M$ such that $B_{g_0}(p,r)\Subset M$. Then there exists constants $C(n,\alpha,K),T_1(n,\alpha,K)>0$ so that

$$\varphi(p,t)\leq C\left(r^{-2}+\delta\mathcal{S}\right)$$

for all $t\in[0,T\wedge r^2T_1]$, where

$$\mathcal{S}=\sup_{B_0(p,r)\times[0,r^2T_1\wedge T]}\int_{B_{g_0}(p,r)}G(x,t;y,0)d\mu_{y,0},$$

and $G(x,t;y,s)$ is the heat kernel for the backward heat equation on $B_{g_0}(p,r)$.

Proof Let $g_1(x,\tau)=r^{-2}g(x,r^2\tau)$ and $\varphi_1(x,\tau)=r^2\varphi(x,r^2\tau)$ which are defined on $M\times[0,r^{-2}T]$. Then the rescaled Ricci flow and the rescaled function satisfy $\mathrm{Ric}(g_1(\tau))\leq\alpha\tau^{-1},\varphi_1(\tau)\leq\alpha\tau^{-1},\varphi_1(0)\leq r^2\delta,\mathcal{R}_1(\tau)=\mathcal{R}(g_1(\tau))=r^2\mathcal{R}(g(t))$, where $t=r^2\tau$. Moreover, φ_1 satisfies:

$$\left(\frac{\partial}{\partial\tau}-\Delta_{g_1(\tau)}\right)\varphi_1\leq\mathcal{R}_1\varphi_1+K\varphi_1^2$$

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in the sense of barrier on $M \times [0, r^{-2}T]$. Let $G_1(x, \tau; y, u)$ be the heat kernel with respect to $g_1(\tau)$ and let $G(x, t; y, s)$ be the heat kernel with respect to $g(t)$. Then

$$r^{\frac{n}{2}} G(x, t; y, s) = G_1(x, \tau; y, u)$$

where $t = r^2\tau, s = r^2 u$. So

$$\int_{B_{g_1(0)}(p,1)} G_1(x, \tau; y, 0)(d\mu_1)_{y,0} = \int_{B_{g_0}(p,r)} G(x, t; y, 0) d\mu_{y,0}$$

where $(d\mu)_1$ is the volume element of g_1. Therefore, it is sufficient to prove the case of $r = 1$.

Since $\varphi(t) \le \alpha t^{-1} \le \alpha$ for $t \ge 1$, we may assume that $T \le 1$. Let

$$\rho(x) = \sup\{r |\, B_0(x, r) \subset B_0(p, 1)\},$$

and set

$$f(x, t) = \delta \int_{B_0(p,1)} G(x, t; y, 0) d\mu_{y,0}.$$

Then $f(x, 0) = \delta$ for $x \in B_0(p, 1)$ and $\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) f = \mathcal{R} f$. Let $A > \delta$ be a constant. Then $A\rho^{-2} - \varphi > 0$ at $t = 0$ and near $\partial B_0(p, 1)$. If $A\rho^{-2} - \varphi < 0$ somewhere on $B_0(p, 1) \times [0, T]$, then there is $x_0 \in B_0(p, 1)$, $t_0 \le T$ such that

$$A\rho_0^{-2} = \varphi(x_0, t_0)$$

and $A\rho^{-2}(x) \ge \varphi(x, t)$ for all $x \in B_0(p, 1) \times [0, t_0]$. Here $\rho_0 =: \rho(x_0)$. Therefore for $x \in B_0(x_0, \frac{1}{2}\rho_0)$ and $t \in [0, t_0]$,

$$\varphi(x, t) \le A\rho^{-2}(x) \le 4A\rho_0^{-2}.$$

By the assumption on φ, we have

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) \varphi \le \mathcal{R}\varphi + 4AK\rho_0^{-2}\varphi.$$

in the sense of barrier on $B_0(x_0, \frac{1}{2}\rho_0) \times [0, t_0]$. Let $b = 4AK\rho_0^{-2}$ and

$$u = e^{-bt}\varphi - f.$$

Then u satisfies:

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) u \le \mathcal{R} u.$$

in the sense of barrier on $B_0(x_0, \frac{1}{2}\rho_0) \times [0, t_0]$ and $u(0) \le 0$. By Corollary 3.1, for any integer $l > \max\{\alpha, K\} + 1$, there is $T_1(l, n, \alpha, K) \in (0, 1]$ such that

$$u(x_0, t) \le \left(\frac{1}{2}\rho_0\right)^{-2(l+1)} t^l$$

for all $t \in [0, t_0 \wedge (\frac{1}{4}\rho_0^2 T_1)]$. On the other hand,

$$A\rho_0^{-2} = \varphi(x_0, t_0) \le \alpha t_0^{-1}. \tag{4.4}$$

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Suppose $A \geq 4\alpha/T_1$, then

$$t_0 \leq \frac{\alpha}{A}\rho_0^2 \leq \frac{1}{4}\rho_0^2 T_1 \leq \frac{1}{4}\rho_0^2.$$

Hence, we have

$$\begin{aligned} e^{-2bt_0}A\rho_0^{-2} - f(x_0,t_0) &= u(x_0,t_0) \\ &\leq 4\rho_0^{-2}\cdot\left((4\rho_0^{-2})t_0\right)^l \\ &\leq 4\rho_0^{-2}. \end{aligned}$$

This implies that

$$A \leq \rho_0^2 e^{2bt_0}\left(f(x_0,t_0) + 4\rho_0^{-2}\right) \leq C_1(\delta\mathcal{S}+1)$$

for some $C_1 > 1$ depending only on n, α, K, because $\rho_0 \leq 1$ and by (4.4)

$$bt_0 = 4AK\rho_0^{-2}t_0 \leq 4K\alpha.$$

By choosing a larger $C_1 = C_1(n,\alpha,T_1)$ and $A = C_1(\delta(\mathcal{S}+\varepsilon)+1)$ with $\varepsilon > 0$, we can conclude that

$$\varphi(x,t) \leq A\rho^{-2}(x)$$

for all $(x,t) \in B_0(p,1)\times[0, T_1 \wedge T]$. By letting $\varepsilon \to 0$ and taking $x = p$, we have

$$\varphi(p,t) \leq C_1(1+\delta\mathcal{S})\rho^{-2}(p) = C_1(1+\delta\mathcal{S}).$$

The result follows. ∎

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2 As mentioned before, we may assume $T = 1$ by parabolic rescaling. Let R_0, μ be as in Corollary 4.2 with v_0 replaced by e^{v_0}. And let T_1 be the constant obtained from Lemma 4.2. First consider the case $R \geq R_0$. Then the heat kernel $G(x,t;y,s)$ on $B_0(p,\mu^{-1}R)\times B_0(p,\mu^{-1}R)\times[0,T]$ satisfies:

$$G(x,t;y,0) \leq C_1 t^{-\frac{n}{2}}\exp\left(-\frac{d_0^2(x,y)}{C_1 t}\right) \tag{4.5}$$

for $x, y \in B_0(p,\mu^{-1}R)$ and $t \in [0,T]$. By Lemma 4.2, we have

$$\varphi(p,t) \leq C_2(\delta\mathcal{S} + R^{-2})$$

for some constant $C_2 = C_2(\alpha,n,K)$ and $t \in [0, T\wedge T_2R^2]$, where $T_2 = \mu^{-2}T_1$ depends only on n, α, K, and

$$\mathcal{S} = \sup_{(x,t)\in B_0(p,\mu^{-1}R)\times[0,T\wedge T_2R^2]}\int_{B_0(p,\mu^{-1}R)} G(x,t;y,0)\,d\mu_{y,0}.$$

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By (4.5), for $(x,t) \in B_0(p,\mu^{-1}R) \times [0, T \wedge T_2R^2]$,

$$\begin{aligned}
&\int_{B_0(p,\mu^{-1}R)} G(x,t;y,0)d\mu_{y,0} \\
&\leq C_1 t^{-\frac{n}{2}} \int_{B_0(p,\mu^{-1}R)} \exp\left(-\frac{d_0^2(x,y)}{C_1 t}\right) d\mu_{y,0} \\
&\leq C_1 t^{-\frac{n}{2}} \int_{B_0(x,2\mu^{-1}R)} \exp\left(-\frac{d_0^2(x,y)}{C_1 t}\right) d\mu_{y,0} \\
&\leq C_1 \int_0^{2\mu^{-1}R} \exp\left(-\frac{r^2}{C_1 t}\right) A(r)dr \\
&\leq C_1 \left(t^{-\frac{n}{2}} V_0(x, 2\mu^{-1}R) \exp\left(-\frac{4\mu^{-2}R^2}{C_1 t}\right) + 2C_1^{-1} t^{\frac{n}{2}+1} \int_0^{2\mu^{-1}R} r \exp\left(-\frac{r^2}{C_1 t}\right) V(r)dr \right) \\
&\leq C_3
\end{aligned}$$

for some $C_3 = C_3(n,\alpha,K,v_0)$. Here, we have used the fact that $V(r) = V_0(x,r) \leq r^n \exp(v_0 r)$ for $r > 0$. Here $A(r)$ is the area of $\partial B_0(x,r)$ with respect to g_0. To summarize, we have

$$\varphi(p,t) \leq C_4(R^{-2} + \delta)$$

for $t \in [0, T \wedge T_2R^2]$ for some $C_4(n,\alpha,K,v_0) > 0$. If $T > T_2R^2$ and $R^2T_2 \leq t \leq T$, then

$$\varphi(p,t) \leq \alpha t^{-1} \leq \alpha T_2^{-1} R^{-2}.$$

This completes the proof of the theorem in the case of $R \geq R_0$.

When $R < R_0$, let $T_3 = T_2 \wedge R_0^{-2}$. By Corollary 4.2, the heat kernel $G(x,t;y,s)$ on $B_0(p,\mu^{-1}R) \times B_0(p,\mu^{-1}R) \times [0, T_3R^2 \wedge T]$ satisfies the same bound as in (4.5). Now the same argument above shows that for all $t \in [0, T_3R^2 \wedge T]$,

$$\varphi(p,t) \leq C_5(R^{-2} + \delta).$$

For $t \in [T_3R^2, T]$, $\varphi(p,t) \leq C_6R^{-2}$ as $\varphi \leq \alpha t^{-1}$ for some $C_6(n,\alpha,K,v_0,\sigma)$. We complete the proof by combining two cases. ∎

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