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The structure of first-order causality

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Game semantics describe the interactive behaviour of proofs by interpreting formulas as games on which proofs induce strategies. Such a semantics is introduced here for capturing dependencies induced by quantifications in first-order propositional logic. One of the main difficulties that has to be faced during the elaboration of this kind of semantics is to characterise definable strategies, that is, strategies that actually behave like a proof. This is usually done by restricting the model to strategies satisfying subtle combinatorial conditions, whose preservation under composition is often difficult to show. In this paper we present an original methodology to achieve this task, which requires a combination of advanced tools from game semantics, rewriting theory and categorical algebra. We introduce a diagrammatic presentation of the monoidal category of definable strategies of our model using generators and relations: these strategies can be generated from a finite set of atomic strategies, and the equality between strategies admits a finite axiomatisation, and this equational structure corresponds to a polarised variation of the bialgebra notion. The work described in this paper thus forms a bridge between algebra and denotational semantics in order to reveal the structure of dependencies induced by first-order quantifiers, and lays the foundations for a mechanised analysis of causality in programming languages.

1. Introduction

Denotational semantics were introduced to provide useful abstract invariants of proofs and programs modulo cut-elimination or reduction. In particular, various game semantics, which were introduced in the nineties, have been very successful in capturing precisely the interactive behaviour of programs. In these semantics, every type is interpreted as a *game* (that is as a set of *moves* that can be played during the game) together with the rules of the game (formalised by a partial order on the moves of the game indicating the dependencies between them). Every move is to be played by one of the two players, called the *Proponent* and the *Opponent*, who should be thought of as the program and its environment, respectively. The interactions between these two players are sequences of moves respecting the partial order of the game, called *plays*. In this setting, a program is characterised by the set of plays that it can exchange with its environment during an execution, and thus defines a *strategy* reflecting the interactive behaviour of the program inside the game specified by the type of the program.

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The notion of a *pointer game*, which was introduced by Hyland and Ong (Hyland and Ong 2000), gave one of the first fully abstract models of PCF (a simply typed λ-calculus extended with recursion, conditional branching and arithmetical constants). It has revealed that PCF programs generate strategies with partial memory, which are called *innocent* because they react to Opponent moves according to their own *view* of the play. In this setting, innocence is the main ingredient for characterising *definable* strategies, that is, strategies that are the interpretation of a PCF term, because it describes the behaviour of the purely functional core of the language (that is, λ-terms), which also corresponds to proofs in propositional logic. This seminal work has lead to an extremely successful series of semantics: by relaxing in various ways the innocence constraint on strategies, it became suddenly possible to generalise this characterisation to PCF programs extended with imperative features such as references, control, non-determinism, and so on.

Unfortunately, these constraints are quite specific to game semantics and remain difficult to link with other areas of computer science or algebra. They are, moreover, very subtle and combinatorial, and thus sometimes difficult to work with. The work presented in this paper is an attempt to find new ways to describe the behaviour of proofs.

1.1. *Generating instead of restricting*

In this paper, we introduce a game semantics capturing dependencies induced by quantifiers in first-order propositional logic, forming a strict monoidal category called **Games**. Instead of characterising definable strategies of the model by restricting to strategies satisfying particular conditions, we show here that we can equivalently use a kind of converse approach. We show how to *generate* definable strategies by giving a *presentation* of those strategies: a finite set of definable strategies can be used to generate all definable strategies by composition and tensoring, and the equality between strategies obtained in this way can be finitely axiomatised.

What we mean precisely by a presentation is a generalisation of the usual notion of a presentation of a monoid to monoidal categories. For example, consider the additive monoid $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. It admits the presentation $\langle\, p, q \mid qp = pq \,\rangle$, where p and q are two *generators* and $qp = pq$ is a relation between two elements of the free monoid M on $\{p, q\}$. This means that $\mathbb{N}^2$ is isomorphic to the free monoid M on the two generators, quotiented by the smallest congruence $\equiv$ (with respect to multiplication) such that $qp \equiv pq$. More generally, a (strict) monoidal category $\mathscr{C}$ (such as **Games**) can be presented by a *polygraph*, consisting of typed generators in dimension 1 and 2, and relations in dimension 3, such that the category $\mathscr{C}$ is monoidally equivalent to the free monoidal category on the generators, quotiented by the congruence generated by the relations.

1.2. *Reasoning locally*

The usefulness of our construction is both theoretic and practical. It reveals that the essential algebraic structure of dependencies induced by quantifiers is a polarised variation of the well-known structure of bialgebra, thus forming a bridge between game semantics and algebra. It also proves very useful from a technical point of view since this presentation

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allows us to reason locally about strategies. In particular, it enables us to deduce *a posteriori* that these strategies actually *compose*, which is not trivial, and it also enables us to deduce that the strategies of the category **Games** are *definable* (one only needs to check that generators are definable). Finally, the presentation gives a finite description of the category, which we can hope to manipulate with a computer, paving the way for a series of new tools to automate the study of the semantics of programming languages.

1.3. *A game semantics capturing first-order causality*

Game semantics has revealed that proofs in logic describe particular strategies for exploring formulas, or, more generally, sequents. Namely, a formula (or a sequent) is a syntactic tree expressing the order in which its connectives must be introduced in cut-free proofs. In this sense, it can be seen as the rules of a game whose moves correspond to connectives. For instance, consider a sequent of the form

$$\forall x.P \quad \vdash \quad \forall y.\exists z.Q \tag{1}$$

where P and Q are propositional formulas that may contain free variables. When searching for a proof of (1), the $\forall y$ quantification must be introduced before the $\exists z$ quantification, and the $\forall x$ quantification can be introduced independently. Here, introducing an existential quantification on the right of a sequent should be thought of as playing a Proponent move (the strategy gives a witness for which the formula holds) and introducing an universal quantification as playing an Opponent move (the strategy receives a term from its environment, for which it has to show that the formula holds); introducing a quantification on the left of a sequent is similar, but with polarities inverted since it is the same as introducing the dual quantification on the right of the sequent. So, the game associated with the formula (1) will be the partial order on the first-order quantifications appearing in the formula, depicted below (it should be read from the top to the bottom):

$$\begin{array}{ccc} \forall x & & \forall y \\ & & \vert \\ & & \exists z \end{array} \tag{2}$$

This partial order is sometimes called the *syntactic partial order* generated by the sequent. Possible proofs of sequent (1) in first-order propositional logic have one of the three following shapes:

$$\dfrac{\dfrac{\dfrac{\vdots}{P[t/x] \vdash Q[t'/z]}}{P[t/x] \vdash \exists z.Q}}{P[t/x] \vdash \forall y.\exists z.Q}{\forall x.P \vdash \forall y.\exists z.Q} \qquad \dfrac{\dfrac{\dfrac{\vdots}{P[t/x] \vdash Q[t'/z]}}{P[t/x] \vdash \exists z.Q}}{\forall x.P \vdash \exists z.Q}{\forall x.P \vdash \forall y.\exists z.Q} \qquad \dfrac{\dfrac{\dfrac{\vdots}{P[t/x] \vdash Q[t'/z]}}{\forall x.P \vdash Q[t'/z]}}{\forall x.P \vdash \exists z.Q}{\forall x.P \vdash \forall y.\exists z.Q}$$

where $P[t/x]$ denotes the formula P where every occurrence of the free variable x has been replaced by the term t. These proofs introduce the connectives in the following

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orders, respectively,

$$\begin{array}{ccc} \forall x & \forall y & \forall y \\ | & | & | \\ \forall y & \forall x & \exists z \\ | & | & | \\ \exists z & \exists z & \forall x \end{array}$$

and these are all total orders extending the partial order of the game (2): these correspond to the plays in the strategies interpreting the proofs in the game semantics. In this sense, they have more dependencies between moves: proofs add causal dependencies between connectives.

Some sequentialisations induced by proofs are not really relevant. For example, consider a proof of the form

$$\dfrac{\dfrac{\dfrac{\pi}{P \vdash Q}}{P \vdash \forall y.Q}}{\exists x.P \vdash \forall y.Q}$$

The order in which the introduction rules of the universal quantifications are introduced is not really significant here since this proof can always be reorganised into the proof

$$\dfrac{\dfrac{\dfrac{\pi}{P \vdash Q}}{\exists x.P \vdash Q}}{\exists x.P \vdash \forall y.Q}$$

by 'permuting' the introduction rules. Similarly, the following permutations of rules are always possible:

$$\dfrac{\dfrac{\dfrac{\pi}{P[t/x] \vdash Q[u/y]}}{P[t/x] \vdash \exists y.Q}}{\forall x.P \vdash \exists y.Q} \rightsquigarrow \dfrac{\dfrac{\dfrac{\pi}{P[t/x] \vdash Q[u/y]}}{\forall x.P \vdash Q[u/y]}}{\forall x.P \vdash \exists y.Q} \quad \text{and} \quad \dfrac{\dfrac{\dfrac{\pi}{P[t/x] \vdash Q}}{P[t/x] \vdash \forall y.Q}}{\forall x.P \vdash \forall y.Q} \rightsquigarrow \dfrac{\dfrac{\dfrac{\pi}{P[t/x] \vdash Q}}{\forall x.P \vdash Q}}{\forall x.P \vdash \forall y.Q}$$

Interestingly, the permutation

$$\dfrac{\dfrac{\dfrac{\pi}{P \vdash Q[t/y]}}{P \vdash \exists y.Q}}{\exists x.P \vdash \exists y.Q} \rightsquigarrow \dfrac{\dfrac{\dfrac{\pi}{P \vdash Q[t/y]}}{\exists x.P \vdash Q[t/y]}}{\exists x.P \vdash \exists y.Q}$$

is only possible if the term t used in the introduction rule of the $\exists y$ connective does not have x as free variable. If the variable x is free in t, the rule introducing $\exists y$ can only be used after the rule introducing the $\exists x$ connective. Now, the sequent $\exists x.P \vdash \exists y.Q$ will be

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interpreted by the following game

$$\exists x \qquad\qquad \exists y$$

Whenever the $\exists y$ connective depends on the $\exists x$ connective (that is, whenever x is free in the witness term t provided for y), the strategy corresponding to the proof will contain a causal dependency, which will be depicted by an oriented wire

$$\exists x \longrightarrow \exists y$$

and we sometimes say that the move $\exists x$ *justifies* the move $\exists y$. A simple further study of permutability of introduction rules of first-order quantifiers shows that this is the only kind of relevant dependencies. These permutations of rules were the motivation for the introduction of non-alternating asynchronous game semantics (Melliès and Mimram 2007), where plays are considered modulo certain permutations of consecutive moves. However, we will focus here on causality and define strategies by the dependencies they induce on moves (a precise description of the relation between these two points of view was investigated in Mimram (2008)). They are also very closely related to the motivations for the introduction of Hintikka's games and independence friendly logic (Hintikka and Sandu 1997).

We will thus build a strict monoidal category whose objects are games and whose morphisms are strategies, in which we can interpret formulas and proofs in the connective-free fragment of first-order propositional logic, and write **Games** for the subcategory of definable strategies. It will thus be useful to keep in mind the following correspondences while reading this paper:

category	logic	game semantics	combinatorial objects
object morphism	formula proof	game strategy	syntactic order justification order

This paper is devoted to the construction of a presentation for this category. We formally introduce the notion of the presentation of a monoidal category in Section 2 and recall some useful classical algebraic structures in Section 4. Then, we give a presentation of the category of relations in Section 5 and extend this presentation to the category **Games**, which we define formally in Section 6.

2. Presentations of monoidal categories

In this section we briefly recall some basic category theory definitions – for a more detailed presentation of these concepts, see the reference book Mac Lane (1971).

2.1. *Monoidal categories*

A *monoidal category* $(\mathscr{C}, \otimes, I)$ is a category $\mathscr{C}$ together with a functor

$$\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$$

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and natural isomorphisms

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C), \quad \lambda_A : I \otimes A \to A \quad \text{and} \quad \rho_A : A \otimes I \to A$$

satisfying coherence axioms (Mac Lane 1971). A symmetric monoidal category $\mathscr{C}$ is a monoidal category $\mathscr{C}$ together with a natural isomorphism

$$\gamma_{A,B} : A \otimes B \to B \otimes A$$

satisfying coherence axioms and such that $\gamma_{B,A} \circ \gamma_{A,B} = \mathrm{id}_{A \otimes B}$. A monoidal category $\mathscr{C}$ is *strictly* monoidal when the natural isomorphisms α, λ and ρ are identities. For simplicity, in the rest of this paper we only consider strict monoidal categories. It can be shown formally using Mac Lane's coherence theorem (Mac Lane 1971) that this is not restrictive, since it tells us that every monoidal category is monoidally equivalent to a strict one.

A (strict) *monoidal functor* $F : \mathscr{C} \to \mathscr{D}$ between two strict monoidal categories $\mathscr{C}$ and $\mathscr{D}$ is a functor F between the underlying categories such that $F(A \otimes B) = F(A) \otimes F(B)$ for all objects A and B of $\mathscr{C}$, and $F(I) = I$. A *monoidal natural transformation* $\theta : F \to G$ between two monoidal functors $F, G : \mathscr{C} \to \mathscr{D}$ is a natural transformation between the underlying functors F and G such that $\theta_{A \otimes B} = \theta_A \otimes \theta_B$ for all objects A and B of $\mathscr{C}$, and $\theta_I = \mathrm{id}_I$. Two monoidal categories $\mathscr{C}$ and $\mathscr{D}$ are *monoidally equivalent* when there exists a pair of monoidal functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ and two invertible monoidal natural transformations $\eta : \mathrm{Id}_{\mathscr{C}} \to GF$ and $\varepsilon : FG \to \mathrm{Id}_{\mathscr{D}}$.

3. Monoidal theories

A *monoidal theory* $\mathbb{T}$ is a strict monoidal category whose objects are the natural integers such that the tensor product on objects is the addition of integers. By an integer $\underline{n}$, we mean here the finite ordinal $\underline{n} = \{0, 1, \ldots, n-1\}$, and addition is given by $\underline{m} + \underline{n} = \underline{m+n}$ (we will simply write n instead of $\underline{n}$ in the following). An *algebra* F of a monoidal theory $\mathbb{T}$ in a strict monoidal category $\mathscr{C}$ is a strict monoidal functor from $\mathbb{T}$ to $\mathscr{C}$; we write $\mathbf{Alg}_{\mathbb{T}}^{\mathscr{C}}$ for the category of algebras from $\mathbb{T}$ to $\mathscr{C}$ and monoidal natural transformations between them. Monoidal theories are sometimes called PRO, this terminology was introduced in Mac Lane (1965) as an abbreviation for 'category with products'. They generalise equational theories – or Lawere theories (Lawvere 1963) – in the sense that operations are typed and can, moreover, have multiple outputs as well as multiple inputs, and are monoidal but not necessarily cartesian.

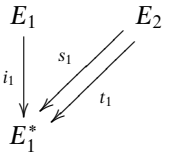
3.1. *Presentations of monoidal categories*

We now recall the notion of a *presentation* of a monoidal category by means of typed 1- and 2-dimensional generators and relations.

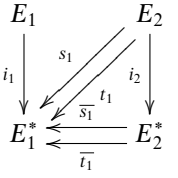
Suppose we are given a set E_1 whose elements are called *atomic types* or *generators for objects*. We write E_1^* for the free monoid on the set E_1 and $i_1 : E_1 \to E_1^*$ for the corresponding injection; the product of this monoid is written $\otimes$. The elements of E_1^* are called *types*. Suppose, moreover, that we are given a set E_2, whose elements are called *generators* (*for morphisms*), together with two functions $s_1, t_1 : E_2 \to E_1^*$, which to every

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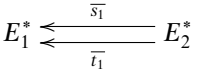
generator associate a type called its *source* and *target*, respectively. We call such a 4-uple (E_1, s_1, t_1, E_2) a *signature*:



All such signatures (E_1, s_1, t_1, E_2) generate a free strict monoidal category $\mathscr{E}$, whose objects are the elements of E_1^* and whose morphisms are formal composite and formal tensor products of elements of E_2, quotiented by suitable laws imposing associativity of composition and tensor and compatibility of composition with tensor (Burroni 1993). If we write E_2^* for the morphisms of this category and $i_2 : E_2 \to E_2^*$ for the injection of the generators into this category, we get a diagram



in **Set** together with a structure of a monoidal category $\mathscr{E}$ on the graph



where the morphisms $\overline{s_1}, \overline{t_1} : E_2^* \to E_1^*$ are the morphisms (which are unique by the universality of E_2^*) such that $s_1 = \overline{s_1} \circ i_2$ and $t_1 = \overline{t_1} \circ i_2$. The *size* $|f|$ of a morphism $f : A \to B$ in E_2^* is defined inductively by

$$|\mathrm{id}| = 0 \qquad |f| = 1 \quad \text{if } f \text{ is a generator}$$
$$|f_1 \otimes f_2| = |f_1| + |f_2| \qquad |f_2 \circ f_1| = |f_1| + |f_2|\,.$$

In particular, a morphism is of size 0 if and only if it is an identity.

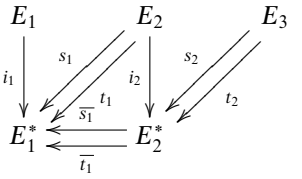
Our constructions are a 2-dimensional instance of Burroni's polygraphs (Burroni 1993), and Street's 2-computads (Street 1976), which made precise the sense in which the generated monoidal category is free on the signature. Namely, the following notion of equational theory is a specialisation of the definition of a 3-polygraph to the case where there is only one generator for 0-cells.

Definition 1. A monoidal equational theory is a 7-uple

$$\mathfrak{E} = (\mathfrak{E}_1, \mathfrak{s}_1, \mathfrak{t}_1, \mathfrak{E}_2, \mathfrak{s}_2, \mathfrak{t}_2, \mathfrak{E}_3)$$

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where (E_1, s_1, t_1, E_2) is a signature together with a set E_3 of *relations* and two morphisms $s_2, t_2 : E_3 \to E_2^*$, as shown in the diagram



such that $\overline{s_1} \circ s_2 = \overline{s_1} \circ t_2$ and $\overline{t_1} \circ s_2 = \overline{t_1} \circ t_2$.

Every equational theory defines a monoidal category $\mathbb{E} = \mathscr{E}/\!\equiv$ obtained from the monoidal category $\mathscr{E}$ generated by the signature (E_1, s_1, t_1, E_2) by quotienting the morphisms by the congruence $\equiv$ generated by the relations of the equational theory $\mathfrak{E}$: it is the smallest congruence (with respect to both composition and tensoring) such that $s_2(e) \equiv t_2(e)$ for every element e of E_3.

We say that a monoidal equational theory $\mathfrak{E}$ is a *presentation* of a strict monoidal category $\mathscr{M}$ when $\mathscr{M}$ is monoidally equivalent to the category $\mathbb{E}$ generated by $\mathfrak{E}$. Any monoidal category $\mathscr{M}$ admits a presentation (for example, the trivial presentation with E_1 the set of objects of $\mathscr{M}$, E_2 the set of morphisms of $\mathscr{M}$, and E_3 the set of all equalities between morphisms holding in $\mathscr{M}$), which is not unique in general. In such a presentation, the category $\mathscr{E}$ generated by the signature underlying $\mathfrak{E}$ should be thought as a category of 'terms' (which will be considered modulo the relations described by E_2), and is thus sometimes called the *syntactic category* of $\mathfrak{E}$.

We sometimes informally say that an equational theory has a *generator* $f : A \to B$ to mean that f is an element of E_2 such that $s_1(f) = A$ and $t_1(f) = B$. We also say that the equational theory has a *relation* $f = g$ when there exists an element e of E_3 such that $s_2(e) = f$ and $t_2(e) = g$.

We say that two equational theories are *equivalent* when they generate monoidally equivalent categories. A generator f in an equational theory $\mathfrak{E}$ is *superfluous* when the equational theory $\mathfrak{E}'$ obtained from $\mathfrak{E}$ by removing the generator f and all equations involving f is equivalent to $\mathfrak{E}$. Similarly, an equation e is *superfluous* when the equational theory $\mathfrak{E}'$ obtained from $\mathfrak{E}$ by removing the equation e is equivalent to $\mathfrak{E}$. An equational theory is *minimal* when it does not contain any superfluous generator or equation.

It is remarkable that every monoidal equational theory $(E_1, s_1, t_1, E_2, s_2, t_2, E_3)$ for which the set E_1 is reduced to only one object $\{1\}$ generates a monoidal category that is a monoidal theory ($\mathbb{N}$ is the free monoid on one object), thus giving a notion of presentation of those categories.

3.2. *Presented categories as models*

Suppose that a strict monoidal category $\mathscr{M}$ is presented by an equational theory $\mathfrak{E}$, generating a category $\mathbb{E} = \mathscr{E}/\!\equiv$. The proof that $\mathfrak{E}$ presents $\mathscr{M}$ can generally be decomposed in two parts:

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(1) $\mathscr{M}$ *is a model of the equational theory* $\mathfrak{E}$: there exists a functor $M : \mathbb{E} \to \mathscr{M}$.
This amounts to checking that there exists a functor $M' : \mathscr{E} \to \mathscr{M}$ such that for all morphisms $f, g : A \to B$ in $\mathscr{E}$, $f \equiv g$ implies $M'f = M'g$.

(2) $\mathscr{M}$ *is a fully-complete model of the equational theory* $\mathfrak{E}$: the functor M is full and faithful.

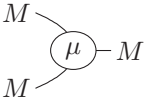
We say that a morphism $f : A \to B$ of $\mathbb{E}$ *represents* the morphism $Mf : MA \to MB$ of $\mathscr{M}$.

Usually, the first point is a straightforward verification, but proving that the functor M is full and faithful often requires more work. In this paper, we use the methodology introduced in Burroni (1993) and later refined in Lafont (2003). We first define *canonical forms*, which are canonical representatives of the equivalence classes of morphisms of $\mathscr{E}$ under the congruence $\equiv$ generated by the relations of $\mathfrak{E}$. Proving that every morphism is equal to a canonical form can be done by induction on the size of the morphisms. Then we show that the functor M is full and faithful by showing that the canonical forms are in bijection with the morphisms of $\mathscr{M}$.

It should be noted that this is not the only technique for proving that an equational theory presents a monoidal category. In particular, Joyal and Street have used topological methods (Joyal and Street 1991) by giving a geometrical construction of the category generated by a signature, in which morphisms are equivalence classes under continuous deformation of progressive plane diagrams (we will give more details about these diagrams, which are also called string diagrams, later on). Their work is, for example, extended in Baez and Langford (2003) to give a presentation of the 2-category of 2-tangles in 4 dimensions. The other general methodology we know of is given in Lack (2004), and involves constructing elaborate monoidal theories from simpler monoidal theories. Namely, a monoidal theory can be seen as a monad in a particular span bicategory, and monoidal theories can therefore be 'composed' given a distributive law between their corresponding monads. We chose not to use these methods because, even though they can be very helpful for building intuitions, they are difficult to formalise and even more difficult to mechanise: indeed, we believe that some of the tedious proofs given in this paper could be somewhat automated, and a first step in this direction was given in Mimram (2010) where we describe an algorithm for computing critical pairs in polygraphic rewriting systems of dimension 2.

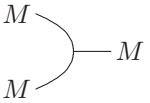
3.3. *String diagrams*

String diagrams provide a convenient way to represent and manipulate the morphisms in the category generated by a presentation. Given an object M in a strict monoidal category $\mathscr{C}$, a morphism $\mu : M \otimes M \to M$ can be drawn graphically as a device with two inputs and one output of type M as follows:



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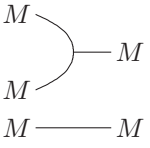
Or simply as



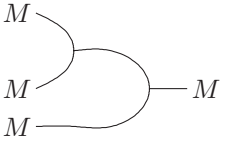
when it is clear from the context which morphism of type $M \otimes M \to M$ we are depicting (we will sometimes even omit the source and target of the morphisms). Similarly, the identity $\mathrm{id}_M : M \to M$ (which we sometimes simply write M) can be drawn as a wire



The tensor $f \otimes g$ of two morphisms $f : A \to B$ and $g : C \to D$ is obtained by putting the diagram corresponding to f above the diagram corresponding to g. So, for instance, the morphism $\mu \otimes M$ can be drawn diagrammatically as



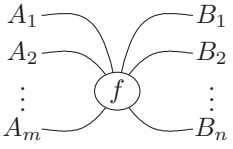
Finally, the composite $g \circ f : A \to C$ of two morphisms $f : A \to B$ and $g : B \to C$ can be drawn diagrammatically by putting the diagram corresponding to g on the right of the diagram corresponding to f and 'linking the wires'. The diagram corresponding to the morphism $\mu \circ (\mu \otimes M)$ is thus



Now suppose (E_1, s_1, t_1, E_2) is a signature. Every element f of E_2 such that

$$s_1(f) = A_1 \otimes \cdots \otimes A_m$$
$$t_1(f) = B_1 \otimes \cdots \otimes B_n,$$

where the A_i and B_i are elements of E_1 can be similarly represented by the diagram

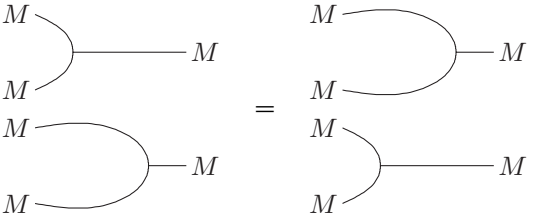


where wires correspond to generators for objects and circled points to generators for morphisms. Bigger diagrams can be constructed from these diagrams by composing and tensoring them, as explained above. Joyal and Street showed in detail (Joyal and Street 1991) that the category of these diagrams, modulo continuous deformations, is precisely the free category generated by a signature (which they call a 'tensor scheme'). For example, the equality

$$(M \otimes \mu) \circ (\mu \otimes M \otimes M) \quad = \quad (\mu \otimes M) \circ (M \otimes M \otimes \mu)$$

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in the category $\mathscr{C}$ of the above example, which holds because of the axioms satisfied in any monoidal category, can be shown by continuously deforming the diagram on the left-hand side below into the diagram on the right-hand side:



All the equalities satisfied in any monoidal category generated by a signature have a similar geometrical interpretation. And, conversely, any deformation of diagrams corresponds to an equality of morphisms in monoidal categories.

4. Algebraic structures

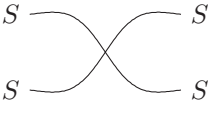
In this section, we recall the categorical formulation of some well-known algebraic structures, the most fundamental for the current work perhaps being the notion of a *bialgebra*. We give these definitions in the setting of a strict monoidal category, which is *not* required to be symmetric. We suppose that $(\mathscr{C}, \otimes, I)$ is a strict monoidal category, which is fixed throughout the section.

4.1. *Symmetric objects*

A *symmetric object* of $\mathscr{C}$ is an object S together with a morphism

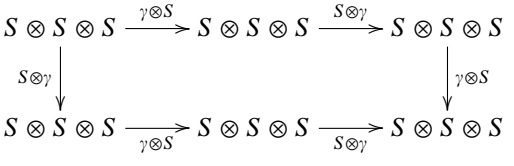
$$\gamma : S \otimes S \to S \otimes S$$

called *symmetry* and depicted by

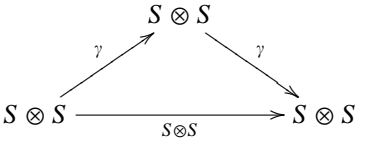


(3)

such that the diagrams

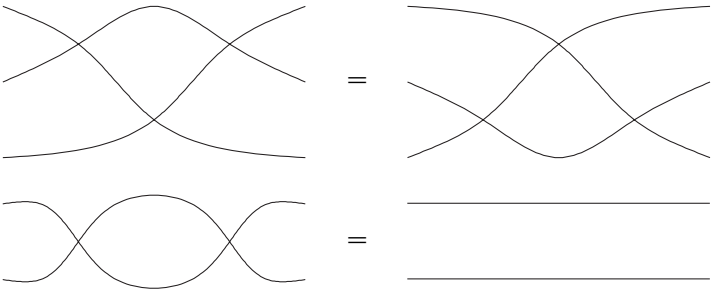


and



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commute. Graphically, we have



(4)

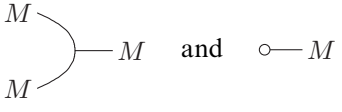
(the first equation is sometimes called the Yang–Baxter equation for braids). In particular, in a symmetric monoidal category, every object is canonically equipped with the structure of a symmetric object.

4.2. *Monoids*

A *monoid* (M, μ, η) in $\mathscr{C}$ is an object M together with two morphisms

$$\mu : M \otimes M \to M \quad \text{and} \quad \eta : I \to M$$

called *multiplication* and *unit*, respectively, and depicted by

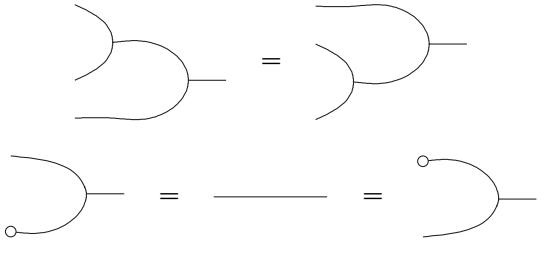


(5)

respectively, such that the diagrams

$$\begin{array}{ccc} M \otimes M \otimes M & \xrightarrow{\mu \otimes M} & M \otimes M \\ {\scriptstyle M \otimes \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ M \otimes M & \xrightarrow[\mu]{} & M \end{array} \quad \text{and} \quad \begin{array}{ccccc} I \otimes M & \xrightarrow{\eta \otimes M} & M \otimes M & \xleftarrow{M \otimes \eta} & M \otimes I \\ & {\scriptstyle M}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle M} & \\ & & M & & \end{array}$$

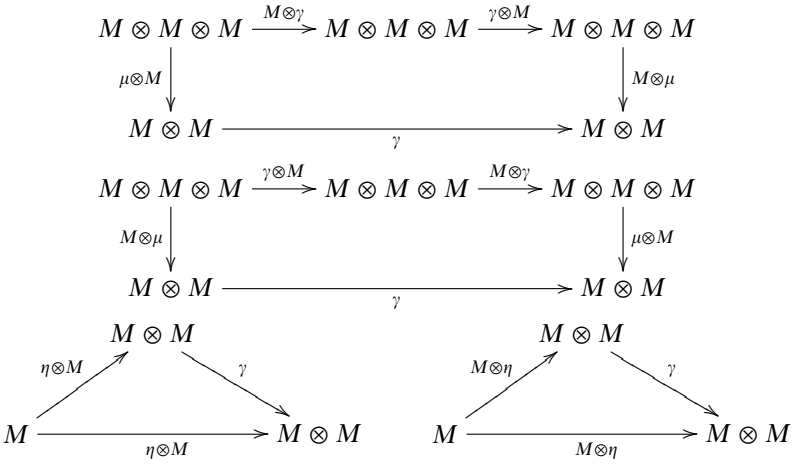
commute. Graphically, we have



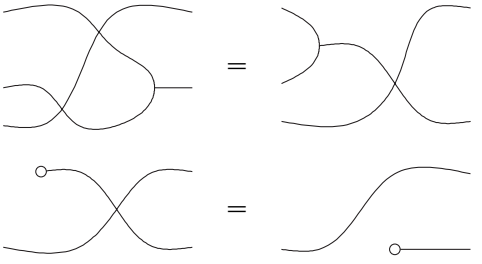
(6)

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A *symmetric monoid* is a monoid that admits a symmetry $\gamma : M \otimes M \to M \otimes M$ that is compatible with the operations of the monoid in the sense that it makes the diagrams

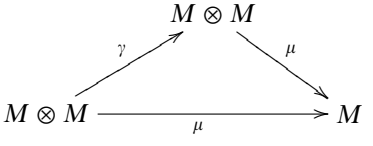


commute. Graphically, we have

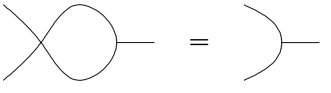


(7)

are satisfied, as well as the equations obtained by turning the diagrams upside down. A *commutative monoid* is a symmetric monoid such that the diagram



commutes. Graphically, we have



(8)

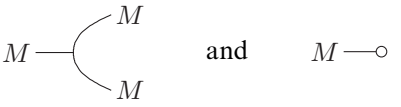
In particular, a commutative monoid in a symmetric monoidal category is a commutative monoid whose symmetry corresponds to the symmetry of the category: $\gamma = \gamma_{M,M}$. In this case, the equations (7) can always be deduced from the naturality of the symmetry of the monoidal category.

A *comonoid* (M, δ, ε) in $\mathscr{C}$ is an object M together with two morphisms

$$\delta : M \to M \otimes M \quad \text{and} \quad \varepsilon : M \to I$$

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depicted by



(9)

respectively, satisfying dual coherence diagrams. Similarly, the notions of a symmetric comonoid and a cocommutative comonoid can be defined by duality.

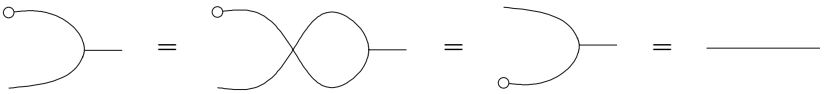
The definition of a monoid can be reformulated internally in the language of equational theories as follows.

Definition 2. The *equational theory of monoids* $\mathfrak{M}$ has one generator for objects 1 and two generators for morphisms $\mu : 2 \to 1$ and $\eta : 0 \to 1$ subject to the three relations

$$\begin{array}{ccccc} \mu \circ (\mu \otimes \mathrm{id}_1) & = & \mu \circ (\mathrm{id}_1 \otimes \mu) \\ \mu \circ (\eta \otimes \mathrm{id}_1) & = & \mathrm{id}_1 & = & \mu \circ (\mathrm{id}_1 \otimes \eta) \end{array} \tag{10}$$

The equations (10) correspond precisely to the equations for a monoid object (6). If we write $\mathbb{M}$ for the monoidal category generated by the equational theory $\mathfrak{M}$, the algebras of $\mathbb{M}$ in a strict monoidal category $\mathscr{C}$ are precisely its monoids: the category $\mathbf{Alg}^{\mathscr{C}}_{\mathbb{M}}$ of algebras of the monoidal theory $\mathbb{M}$ in $\mathscr{C}$ is monoidally equivalent to the category of monoids in $\mathscr{C}$. Similarly, all the algebraic structures introduced in this section can be defined using algebraic theories.

Remark 3. The presentations given here are not necessarily minimal. For example, in the theory of commutative monoids, one equation for units of monoids (5) is derivable from the equation (8), one of the equations (7) and one of the equations for units of monoids (5):



A minimal presentation of this equational theory with three generators and seven equations is given in Massol (1997). However, not all the equational theories introduced in this paper have a known presentation that has been proved to be minimal.

4.3. *Bialgebras*

A *bialgebra* $(B, \mu, \eta, \delta, \varepsilon, \gamma)$ in $\mathscr{C}$ is an object B together with five morphisms

$$\begin{aligned} \mu &: B \otimes B \to B \\ \delta &: B \to B \otimes B \\ \eta &: I \to B \\ \varepsilon &: B \to I \\ \gamma &: B \otimes B \to B \otimes B \end{aligned}$$

such that:

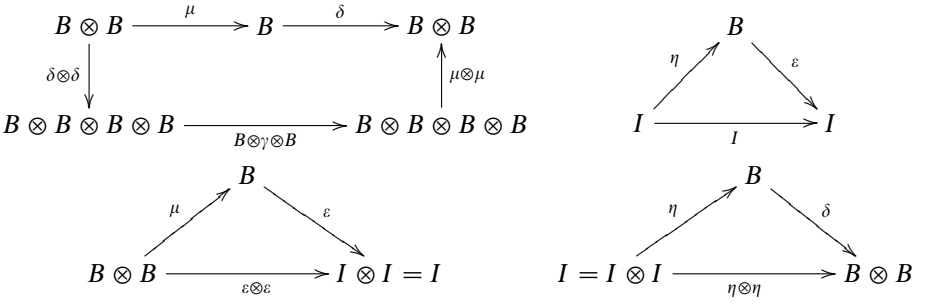
— $\gamma : B \otimes B \to B \otimes B$ is a symmetry for B;

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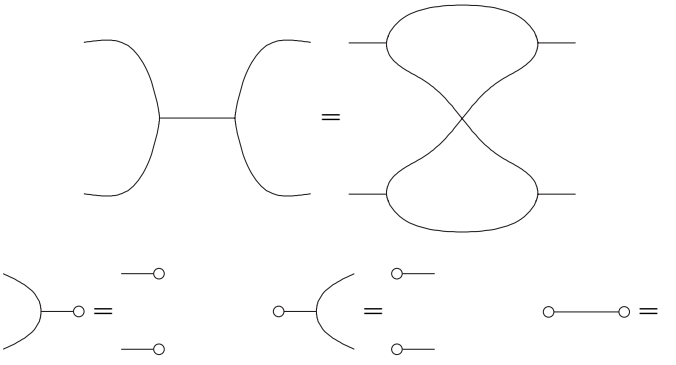
— (B, μ, η, γ) is a symmetric monoid; and

— $(B, \delta, \varepsilon, \gamma)$ is a symmetric comonoid.

The morphism γ is depicted as in (3), μ and η as in (5), and δ and ε as in (9). These two structures should be coherent, in the sense that the diagrams



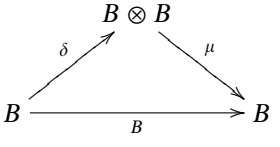
should commute. Graphically, we have



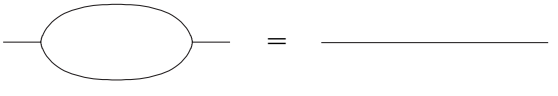
(11)

should be satisfied.

A bialgebra is *commutative* (respectively, *cocommutative*) when the induced symmetric monoid (B, μ, η, γ) (respectively, symmetric comonoid $(B, \delta, \varepsilon, \gamma)$) is commutative (respectively, cocommutative), and it is *bicommutative* when it is both commutative and cocommutative. A bialgebra is *qualitative* when the diagram



commutes. Graphically, we have



(12)

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Definition 4. We write $\mathfrak{B}$ for the *equational theory of bicommutative bialgebras*. It has one generator for objects 1, five generators for morphisms

$$\begin{array}{ll} \mu : 2 \to 1 & \eta : 0 \to 1 \\ \delta : 1 \to 2 & \varepsilon : 1 \to 0 \end{array} \quad \text{and} \quad \gamma : 2 \to 2$$

and twenty-two relations: the two relations of symmetry (4), the eight relations of commutative monoids (6), (7) and (8), the eight relations of cocommutative comonoids that are the duals of (6), (7) and (8), and the four compatibility relations for bialgebras (11).

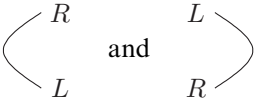
We also write $\mathfrak{R}$ for the *equational theory of qualitative bicommutative bialgebras*, which is defined as $\mathfrak{B}$, with the relation (12) added.

4.4. *Dual objects*

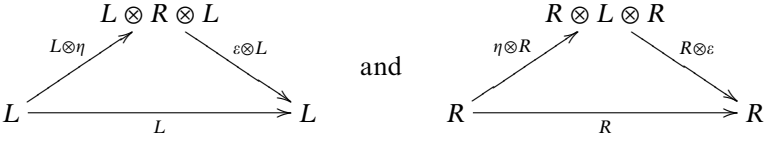
An object L of $\mathscr{C}$ is said to be *left dual* to an object R when there exist two morphisms

$$\eta : I \to R \otimes L \quad \text{and} \quad \varepsilon : L \otimes R \to I,$$

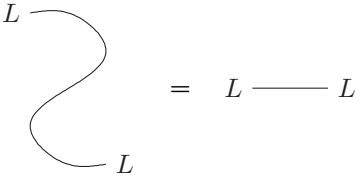
which are called the *unit* and *counit*, respectively, of the duality and are depicted by



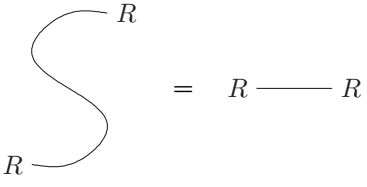
respectively, making the diagrams



commute. Graphically, we have



and



We write $\mathfrak{D}$ for the equational theory associated with dual objects.

Remark 5. If $\mathscr{C}$ is a category, two dual objects in the monoidal category End($\mathscr{C}$) of endofunctors of $\mathscr{C}$, with tensor product given on objects by composition of functors, are

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adjoint endofunctors of $\mathscr{C}$. The theory of adjoint functors in a 2-category was described precisely in Schanuel and Street (1986), the definition of $\mathfrak{D}$ is a specialisation of this construction to the case where there is only one 0-cell.

5. Presenting the category of relations

In this section we introduce a presentation of the category **Rel** of finite ordinals and relations by refining presentations of simpler categories. This result was mentioned in Examples 6 and 7 of Hyland and Power (2000), and is proved in three different ways in Lafont (1995), Pirashvili (2002) and Lack (2004). The methodology adopted here to build this presentation has the advantage of being simple to check (although very repetitive) and can be extended to give the presentation of the category of games and strategies described in Section 6.5.

5.1. *The simplicial category*

The simplicial category Δ is the monoidal theory whose morphisms $f : m \to n$ are the monotone functions from m to n. It has been known for a long time that this category is closely related to the notion of a monoid – see Mac Lane (1971) or Lafont (2003), for example. This result can be formulated as follows.

Property 6. The monoidal category Δ is presented by the equational theory of monoids $\mathfrak{M}$.

In this sense, the simplicial category Δ impersonates the notion of a monoid. We will extend this result to more complex categories.

5.2. *Multirelations*

A *multirelation* R between two finite sets A and B is a function $R : A \times B \to \mathbb{N}$. It can be equivalently be seen as a multiset whose elements are in $A \times B$ or as a matrix over $\mathbb{N}$, or as a span

$$\begin{array}{ccc} & R & \\ {\scriptstyle s}\swarrow & & \searrow{\scriptstyle t} \\ A & & B \end{array}$$

in the category **FinSet** of finite sets. In the final case, the multiset representation can be recovered from the span using

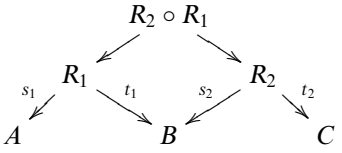
$$R(a,b) \quad = \quad |\{\, e \in R \quad | \quad s(e) = a \text{ and } t(e) = b \,\}|$$

for every element $(a,b) \in A \times B$. If $R_1 : A \to B$ and $R_2 : B \to C$ are two multirelations, their composition is defined by

$$R_2 \circ R_1(a,c) \quad = \quad \sum_{b \in B} R_1(a,b) \times R_2(b,c).$$

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This corresponds to the usual composition of matrices if we view R_1 and R_2 as matrices over $\mathbb{N}$, and as the span obtained by computing the pullback



if we view R_1 and R_2 as spans in **Set**. The cardinal $|R|$ of a multirelation $R : A \to B$ is the sum

$$|R| = \sum_{(a,b) \in A \times B} R(a,b)$$

of its coefficients. We write **MRel** for the monoidal theory of multirelations: its objects are finite ordinals and morphisms are multirelations between them. It is a strict symmetric monoidal category with the tensor product $\otimes$ defined on objects and morphisms by disjoint union, and thus a monoidal theory. In this category, the object 1 can be equipped with the obvious structure of bicommutative bialgebra

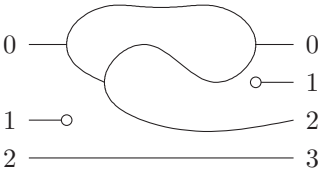
$$(1, R^\mu, R^\eta, R^\delta, R^\varepsilon). \tag{13}$$

In this structure, $R^\mu : 2 \to 1$ is the multirelation defined by $R^\mu(i,0) = 1$ for $i = 0$ or $i = 1$, $R^\delta : 1 \to 2$ is the multirelation dual to R^μ, and $R^\eta : 0 \to 1$ and $R^\varepsilon : 1 \to 0$ are uniquely defined by the fact that the object 0 is both initial and terminal in **MRel**. We now show that the category of multirelations is presented by the equational theory $\mathfrak{B}$ of bicommutative bialgebras. We write $\mathscr{B}$ for the syntactic category of $\mathfrak{B}$ (that is, the monoidal category generated by the underlying signature of $\mathfrak{B}$), so $\mathscr{B}/\equiv$ is the monoidal category generated by $\mathfrak{B}$, where $\equiv$ is the congruence generated by the relations of $\mathfrak{B}$. The bicommutative bialgebra structure (13) induces an 'interpretation functor' $I : \mathscr{B} \to \mathbf{MRel}$ such that $I(1) = 1$, $I(\mu) = R^\mu$, $I(\eta) = R^\eta$, $I(\delta) = R^\delta$ and $I(\varepsilon) = R^\varepsilon$. Since the morphisms (13) satisfy the equations of bicommutative bialgebra, the interpretations of two morphisms of $\mathscr{B}$ related by $\equiv$ will be equal. The interpretation functor thus extends to a functor $I/\!\equiv : \mathscr{B}/\!\equiv \to \mathbf{MRel}$.

Example 7. Consider the morphism

$$((\mu \otimes \eta \otimes 1) \circ (1 \otimes \delta) \circ (\delta \otimes \varepsilon)) \otimes 1 \quad : \quad 3 \to 4$$

of $\mathscr{B}$ whose graphical representation is



(14)

Its interpretation is the multirelation

$$((R^\mu \otimes R^\eta \otimes 1) \circ (1 \otimes R^\delta) \circ (R^\delta \otimes R^\varepsilon)) \otimes 1. \tag{15}$$

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This multirelation is a function $3 \times 4 \to \mathbb{N}$ (where 3 and 4 are the sets $\{0,1,2\}$ and $\{0,1,2,3\}$ respectively), and can thus be represented as the following $\mathbb{N}$-valued matrix of size 3×4:

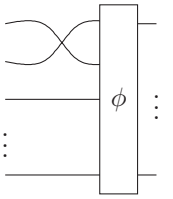
$$\begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This matrix is computed by evaluating the formula (15), but has, in fact, a very natural interpretation if we consider the string diagrammatic representation (14) of the morphism: an entry (i, j) of the matrix is precisely the number of different paths in wires linking the object i on the input to the object j on the output (for example, from 0 there are two paths to 0 and one to 2, which gives the first line of the matrix).

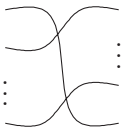
For every morphism $\phi : m+1 \to n$ in $\mathscr{B}$, where $m > 0$, we define a morphism written $S^{m \to n}\phi : m+1 \to n$ by

$$S^{m \to n}\phi \;=\; \phi \circ (\gamma \otimes \mathrm{id}_{m-1}). \tag{16}$$

Graphically, we have

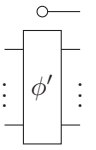
$S^{m \to n}\phi \quad =$ 

The *stairs* morphisms are defined inductively as either id_1 or $S\phi'$ where ϕ' is a stair, and are represented graphically by

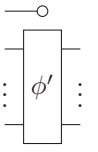


The *length* of a stairs morphism is defined as 0 if it is an identity id_1, or as the length of the stairs ϕ' plus one if it is of the form $S\phi'$. The stairs of length $n+1$ is written $\gamma_n : n \to n$.

Morphisms ϕ that are *precanonical forms* are defined inductively: ϕ is either empty or

$H^{m \to n}\phi' =$ 

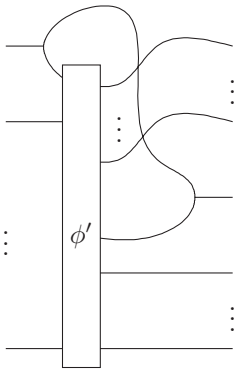
or

$E^{m \to n}\phi' =$ 

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or

$W_i^{m\to n}\phi' \quad =$



where $\phi : m \to n$ is a precanonical form. In this case, we write ϕ as, respectively:

— $Z : 0 \to 0$ (the identity morphism id_0);
— $H^{m\to n}\phi' : m \to n+1$;
— $E^{m\to n}\phi' : m+1 \to n$; or
— $W_i^{m\to n}\phi' : m \to n$ (where i is the length of the stairs in the morphism).

Algebraically:

$$\begin{aligned} Z &= \mathrm{id}_0 \\ E^{m\to n}\phi' &= \varepsilon \otimes \phi' \\ H^{m\to n}\phi' &= \eta \otimes \phi' \\ W_i^{m\to n}\phi' &= (i \otimes \mu \otimes (n-1-i)) \circ (\gamma_i \otimes (n-i)) \circ (1 \otimes \phi') \circ (\delta \otimes (m-1)). \end{aligned}$$

Precanonical forms ϕ are thus the well-formed morphisms (where compositions respect types) generated by the following grammar:

$$\phi \quad ::= \quad Z \quad | \quad H^{m\to n}\phi \quad | \quad E^{m\to n}\phi \quad | \quad W_i^{m\to n}\phi. \tag{17}$$

In order to simplify the notation, we will remove the superscripts in the following and simply write $W_i\phi$ instead of $W_i^{m\to n}\phi$.

It is easy to see that every non-identity morphism ϕ of a category generated by a monoidal equational theory (such as $\mathfrak{B}$) can be written as $\phi = (m \otimes \pi \otimes n) \circ \phi'$, where π is a generator, thus allowing us to reason inductively about morphisms by case analysis on the integer m and on the generator π. Using this technique, we can prove the following lemma.

Lemma 8. Every morphism $\phi : m \to n$ of $\mathscr{B}$ is equivalent (with respect to the relation $\equiv$) to a precanonical form.

Proof. We use induction on the size $|\phi|$ of ϕ:

— If $|\phi| = 0$, then $m = n$ and $\phi = \mathrm{id}_n$. If $n = 0$, then $\phi = Z$. Otherwise, we have $\phi = \mathrm{id}_{n+1} = 1 \otimes \mathrm{id}_n = W_0 E H \mathrm{id}_n$ and id_n is equivalent to a canonical form by induction on n.
— Otherwise, the morphism ϕ is of the form $\phi = \xi \circ \psi$ with $|\xi| = 1$ and $|\xi| + |\psi| = |\phi|$. By the induction hypothesis, the morphism ψ is equivalent to a canonical form. Moreover,

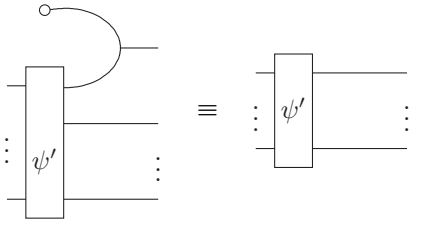
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the morphism ξ is of the form $m_1 \otimes \pi \otimes m_2$ where π is μ, η, δ, ε or γ. We show the result by distinguishing these five cases for π, and for each case by distinguishing whether the precanonical form of ψ is of the form Z, $H\psi'$, $E\psi'$ or $W_i\psi'$:

(1) Case $\pi = \mu$:

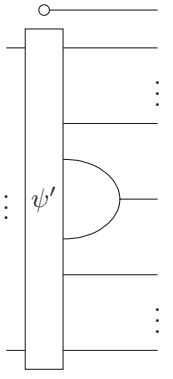
(a) If $\psi = H\psi'$, we distinguish two cases:

– If $m_1 = 0$, we have the equivalence



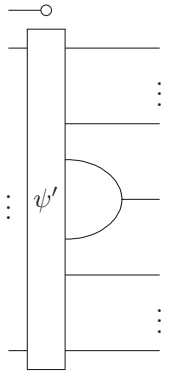
where ψ' is equivalent to a precanonical form by the induction hypothesis.

– Otherwise, the morphism ϕ can be represented by



and is of the form $H(((m_1 - 1) \otimes \mu \otimes m_2) \circ \psi')$, where the morphism $((m_1 - 1) \otimes \mu \otimes m_2) \circ \psi'$ is equivalent to a precanonical form by the induction hypothesis.

(b) If $\psi = E\psi'$, the morphism ϕ can be represented by

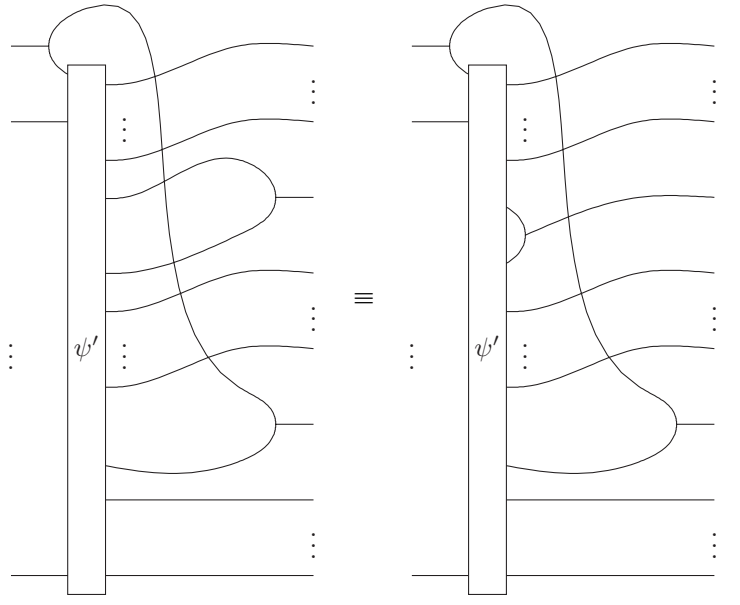


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and is of the form $E(\xi \circ \psi')$ where the morphism $\xi \circ \psi'$ is equivalent to a precanonical form by the induction hypothesis.

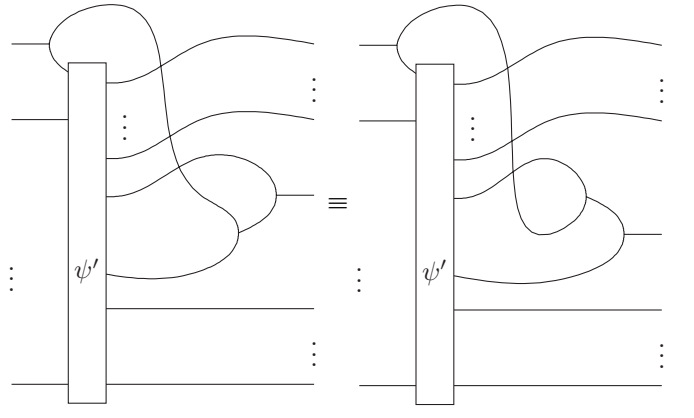
(c) If $\psi = W_i'\psi'$, we distinguish four cases:

– If $m_1 < i-1$, we have the equivalence

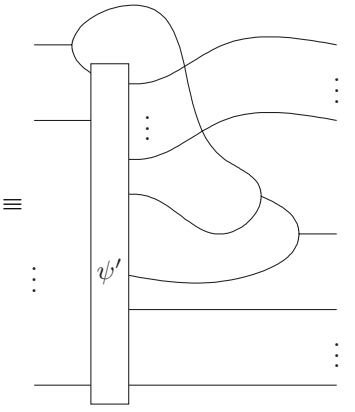


and ϕ is of the form $W_{i-1}(((m_1 - 1 \otimes \mu \otimes m_2)) \circ \psi')$ where the morphism $((m_1 - 1 \otimes \mu \otimes m_2)) \circ \psi'$ is equivalent to a precanonical form by the induction hypothesis.

– If $m_1 = i-1$, we have the equivalences

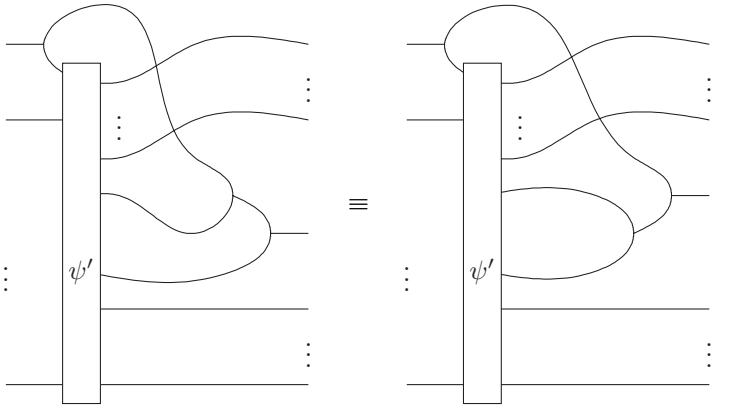


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and we are actually in the case that is handled next.

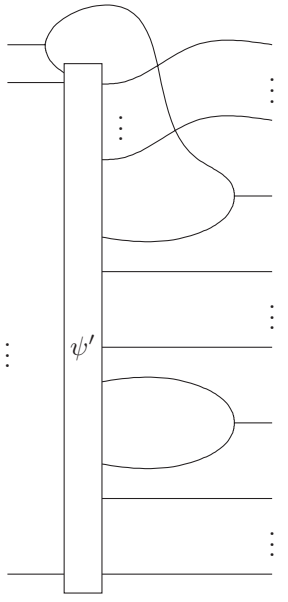
– If $m_1 = i$, we have the equivalence



and ϕ is of the form $W_i(\xi \circ \psi')$ where the morphism $\xi \circ \psi'$ is equivalent to a precanonical form by the induction hypothesis.

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– If $m_1 > i$, then ϕ can be represented by



and is of the form $W_i(\xi \circ \psi')$ where the morphism $\xi \circ \psi'$ is equivalent to a precanonical form by the induction hypothesis.

(2) Case $\pi = \eta$:

(a) If $\psi = Z$, then $\phi = HZ$, which is a precanonical form.

(b) If $\psi = H\psi'$, we distinguish two cases:

– If $m_1 = 0$, then $\phi = HH\psi'$, which is a precanonical form.

– Otherwise, $\phi = H(((m_1 - 1) \otimes \eta \otimes m_2) \circ \psi')$ where $(m_1 - 1) \otimes \eta \otimes m_2$ is equivalent to a precanonical form by the induction hypothesis.

(c) If $\psi = E\psi'$, then $\phi = E(\xi \circ \psi')$ where the morphism $\xi \circ \psi'$ is equivalent to a precanonical form by the induction hypothesis.

(d) If $\psi = W_i\psi'$, we distinguish two cases:

– If $m_1 \leqslant i$, then $\phi \equiv W_{i+1}(\xi \circ \psi')$ where the morphism $\xi \circ \psi'$ is equivalent to a precanonical form by the induction hypothesis.

– Otherwise, $\phi = W_i(\xi \circ \psi')$ where the morphism $\xi \circ \psi'$ is equivalent to a precanonical form by the induction hypothesis.

(3) Case $\pi = \delta$:

(a) If $\psi = H\psi'$, we distinguish two cases:

– If $m_1 = 0$, then $\phi \equiv HH\psi'$ where ψ' is a precanonical form.

– Otherwise, $\phi \equiv H(((m_1 - 1) \otimes \delta \otimes m_2) \circ \psi')$ where $((m_1 - 1) \otimes \delta \otimes m_2) \circ \psi'$ is equivalent to a precanonical form by the induction hypothesis.

(b) If $\psi = E\psi'$, then $\phi = E(\xi \circ \psi')$ where the morphism $\xi \circ \psi'$ is equivalent to a precanonical form by the induction hypothesis

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(c) If $\psi = W_i\psi'$, we distinguish three cases:

- If $m_1 < i$, then $\phi \equiv W_{i+1}(\xi \circ \psi')$ where the morphism $\xi \circ \psi'$ is equivalent to a precanonical form by the induction hypothesis
- If $m_1 = i$, then $\phi \equiv W_i W_{i+1}(\xi \circ \psi')$ where the morphism $\xi \circ \psi'$ is equivalent to a precanonical form by the induction hypothesis.
- Otherwise, $\phi = W_i(\xi \circ \psi')$ where the morphism $\xi \circ \psi'$ is equivalent to a precanonical form by the induction hypothesis.

(4) Case $\pi = \varepsilon$.

(a) If $\psi = H\psi'$, we distinguish two cases:

- If $m_1 = 0$, then $\phi \equiv \psi'$ where the morphism ψ' is a precanonical form.
- Otherwise, $\psi = H(((m_1 - 1) \otimes \varepsilon \otimes m_2) \circ \psi')$ where $((m_1 - 1) \otimes \varepsilon \otimes m_2) \circ \psi'$ is equivalent to a precanonical form by the induction hypothesis.

(b) If $\psi = E\psi'$, then $\phi = E(\xi \circ \psi')$ where the morphism $\xi \circ \psi'$ is equivalent to a precanonical form by the induction hypothesis.

(c) If $\psi = W_i\psi'$, we distinguish three cases:

- If $m_1 < i$, then $\phi \equiv W_{i-1}(\xi \circ \psi')$ where the morphism $\xi \circ \psi'$ is equivalent to a precanonical form by the induction hypothesis.
- If $m_1 = i$, then $\phi \equiv E(\xi \circ \psi')$ where the morphism $\xi \circ \psi'$ is equivalent to a precanonical form by the induction hypothesis.
- Otherwise, $\phi = W_i(\xi \circ \psi')$ where the morphism $\xi \circ \psi'$ is equivalent to a precanonical form by the induction hypothesis.

(5) Case $\pi = \gamma$:

(a) If $\psi = H\psi'$, we distinguish two cases:

- If $m_1 = 0$, then $\phi \equiv ((1 \otimes \eta \otimes m_2) \circ \psi')$ where the morphism $(1 \otimes \eta \otimes m_2) \circ \psi'$ is equivalent to a precanonical form by the induction hypothesis.
- Otherwise, $\phi = H(\xi \circ \psi')$ where the morphism $\xi \circ \psi'$ is equivalent to a precanonical form by the induction hypothesis.

(b) If $\psi = E\psi'$, then $\phi = E(\xi \circ \psi')$ where the morphism $\xi \circ \psi'$ is equivalent to a precanonical form by the induction hypothesis.

(c) If $\psi = W_i\psi'$, we distinguish four cases:

- If $m_1 < i - 1$, then $\phi \equiv W_i(\xi \circ \psi')$ where the morphism $\xi \circ \psi'$ is equivalent to a precanonical form by the induction hypothesis.
- If $m_1 = i - 1$, then $\phi \equiv W_{i-1}(((m_1 + 1) \otimes \gamma \otimes m_2) \circ \psi')$ where the morphism $((m_1 + 1) \otimes \gamma \otimes m_2) \circ \psi'$ is equivalent to a precanonical form by the induction hypothesis.
- If $m_1 = i$, then $\phi \equiv W_{i+1}(((m_1 + 1) \otimes \gamma \otimes m_2) \circ \psi')$ where the morphism $((m_1 + 1) \otimes \gamma \otimes m_2) \circ \psi'$ is equivalent to a precanonical form by the induction hypothesis.

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– Otherwise, $\phi = W_i(\xi \circ \psi')$ where the morphism $\xi \circ \psi'$ is equivalent to a precanonical form by the induction hypothesis. □

The *canonical forms* are precanonical forms that are normal with respect to the following rewriting system:

$$\begin{array}{rcll} HW_i & \Longrightarrow & W_{i+1}H & \\ HE & \Longrightarrow & EH & \\ W_iW_j & \Longrightarrow & W_jW_i & \text{when } i < j \end{array} \tag{18}$$

when considered as words generated by the grammar (17). It is a routine verification to show that two precanonical forms ϕ and ψ such that ϕ rewrites to ψ are equivalent. This rewriting system thus provides us with a notion of a canonical form for precanonical forms.

Lemma 9. The rewriting system (18) is normalising.

Proof. We first show that the rewriting system is terminating by defining an interpretation of precanonical forms into $\mathbb{N} \times \mathbb{N}$, ordered lexicographically. This interpretation $[\![-]\!]$ is defined on generators by

$$[\![Z]\!] = (0,0) \qquad [\![H]\!] = (0,0) \qquad [\![E]\!] = (1,0) \qquad [\![W_i]\!] = (1,i),$$

and on composition and identities by

$$[\![G \circ F]\!] = ([\![G]\!]_1 + 2 \times [\![F_1]\!], [\![G]\!]_2 + 2 \times [\![F]\!]_2) \quad \text{and} \quad [\![\mathrm{id}]\!] = (0,0)$$

where F and G are such that

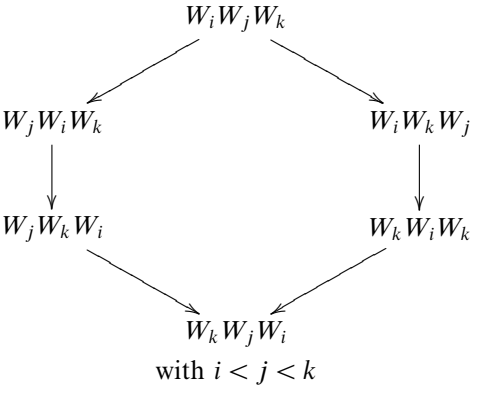
$$[\![F]\!] = ([\![F]\!]_1, [\![F]\!]_2)$$
$$[\![G]\!] = ([\![G]\!]_1, [\![G]\!]_2).$$

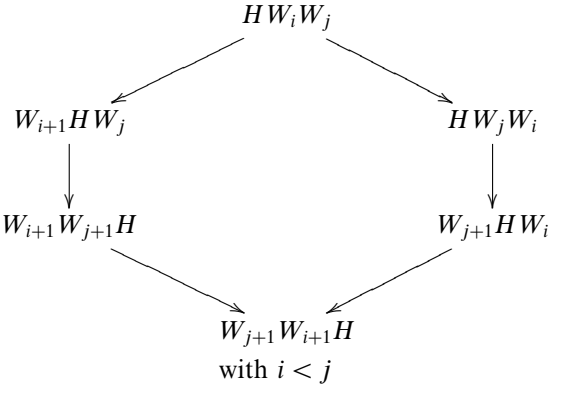
Note that the rules are strictly decreasing with respect to this interpretation:

$$[\![HW_i]\!] = (2,2i) > (1,i) = [\![W_iH]\!]$$
$$[\![HE]\!] = (2,0) > (1,0) = [\![EH]\!]$$
$$[\![W_iW_j]\!] = (3,i+2j) > (3,j+2i) = [\![W_jW_i]\!]$$

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The rewriting system is therefore terminating. Moreover, it is locally confluent, since the two critical pairs are joinable:





The rewriting system being terminating, it is thus confluent. □

Remark 10. Canonical forms are the precanonical forms of the form

$$W_{i^n_{k_n}} \cdots W_{i^n_1} E \cdots\cdots W_{i^1_{k_1}} \cdots W_{i^1_1} EH \cdots HZ \tag{19}$$

with $i^p_1 \geqslant \ldots \geqslant i^p_{k_p}$ for every p such that $1 \leqslant p \leqslant n$.

From Lemmas 8 and 9, we can finally deduce that every morphism of the category $\mathscr{B}$ is equivalent to a unique canonical form.

Lemma 11. The interpretation functor $I/\equiv : \mathscr{B}/\equiv \to$ **MRel** is full.

Proof. We show the result by showing that the functor $I : \mathscr{B} \to$ **MRel** is full, that is, that every multirelation $R : m \to n$ is the image of a precanonical form $\phi : m \to n$ in $\mathscr{B}$. We use induction on m and on the cardinal $|R|$ of R:

(1) If $m = 0$, then R is the interpretation of the precanonical form $H \ldots HZ$, with n occurrences of H.

(2) If $m > 0$ and for every $j < n$ we have $R(0, j) = 0$, then R is of the form $R = R^{\varepsilon} \otimes R'$, where $R' : m - 1 \to n$ is the multirelation such that $R'(i, j) = R(i + 1, j)$. By the

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induction hypothesis, R' is the interpretation of a precanonical form ϕ' and R is therefore the interpretation of the precanonical form $E\phi'$.

(3) Otherwise, we necessarily have $n \neq 0$, and that there exists an index k' such that $R(0,k) \neq 0$. We write k for the greatest such index. The multirelation R is of the form

$$R = (k \otimes R^{\mu} \otimes n-1-k) \circ (R^{\gamma_k} \otimes n-k) \circ (1 \otimes R') \circ (R^{\delta} \otimes m-1),$$

where $R' : m \to n$ is the multirelation defined by $R'(0,k) = R(0,k) - 1$ and $R'(i,j) = R(i,j)$ for every $(i,j) \neq (0,k)$. The multirelation R' is thus of cardinal $|R'| = |R| - 1$ and is the interpretation of a precanonical form $\phi' : m \to n$ by the induction hypothesis. Finally, we have R is the interpretation of the precanonical form $W_k\phi'$. □

The proof of the previous lemma provides us with an algorithm that, given a multirelation R, builds a precanonical form ϕ whose interpretation is R. The execution of this algorithm consists of enumerating the coefficients of the multirelation column by column. We suppose we are given a multirelation $R : m \to n$. In pseudo-code, the algorithm can be written as follows:

```
for i = 0 to m − 1 do
    for j = n − 1 downto 0 do
        for k = 0 to R(i, j) do
            print 'W_j'
        done
        print 'H'
    done
done
for j = 0 to n − 1 do
    print 'E'
done
print 'Z'
```

The word printed by the algorithm will be a precanonical form whose interpretation is R.

Knowing the general form (19) of canonical forms, it is easy to show that the precanonical forms produced by the algorithm are actually canonical forms. Conversely, every canonical form (19) can be read as an 'enumeration' of the coefficients of a multirelation in a way similar to the previous algorithm. This shows that, in fact, multirelations $R : m \to n$ are in bijection with the canonical forms $\phi : m \to n$. A morphism of $\mathscr{B}$ being equivalent to an unique canonical form, we finally deduce the following theorem.

Theorem 12. The categories $\mathscr{B}/\equiv$ and **MRel** are isomorphic, that is, the category **MRel** of natural numbers and multirelations is presented by the theory $\mathfrak{B}$ of bicommutative bialgebras.

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5.3. *Relations*

The monoidal category **Rel** has finite ordinals as objects and relations as morphisms. This category can be obtained from **MRel** by quotienting the morphisms by the equivalence relation $\sim$ on multirelations such that two multirelations $R_1, R_2 : m \to n$ are equivalent when they have the same null coefficients. We can therefore easily adapt the previous presentation to show that

Theorem 13. The category **Rel** of relations is presented by the equational theory $\mathfrak{R}$ of *qualitative* bicommutative bialgebras.

In particular, precanonical forms are the same, and canonical forms are defined by adding the rule

$$W_i W_i \Longrightarrow W_i \tag{20}$$

to the rewriting system (18), which remains normalising.

6. A game semantics for first-order causality

Suppose we are given a fixed first-order language $\mathscr{L}$, that is:

— a set of proposition symbols $P, Q, \ldots$ with given arities;
— a set of function symbols $f, g, \ldots$ with given arities;
— and a set of first-order variables $x, y, \ldots$.

Terms t and *formulas* A are generated by the following grammars:

$$\begin{array}{rcl} t & ::= & x \mid f(t, \ldots, t) \\ A & ::= & P(t, \ldots, t) \mid \forall x.A \mid \exists x.A \end{array}$$

(we only consider formulas without connectives here). We suppose that application of propositions and functions always respect arities. Formulas are considered modulo renaming of bound variables, and substitution $A[t/x]$ of a free variable x by a term t in a formula A is defined as usual, avoiding capture of variables. In the following, we sometimes omit the arguments of propositions when they are clear from the context. We also suppose we are given a set Ax of *axioms* (that is, pairs of propositions) that is reflexive, transitive and closed under substitution (so that the resulting logic has the cut-elimination property). The logic associated with these formulas has the following inference rules:

$$\frac{A[t/x] \vdash B}{\forall x.A \vdash B}(\forall\text{-L}) \qquad \frac{A \vdash B}{A \vdash \forall x.B}(\forall\text{-R}) \quad (\text{with } x \text{ not free in } A)$$

$$\frac{A \vdash B}{\exists x.A \vdash B}(\exists\text{-L}) \quad (\text{with } x \text{ not free in } B) \qquad \frac{A \vdash B[t/x]}{A \vdash \exists x.B}(\exists\text{-R})$$

$$\frac{(P, Q) \in Ax}{P \vdash Q}(\text{Ax}) \qquad \frac{A \vdash B \qquad B \vdash C}{A \vdash C}(\text{Cut})$$

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6.1. *Games and strategies*

Games are defined as follows.

Definition 14. A *game* $A = (M_A, \lambda_A, \leqslant_A)$ consists of a set M_A whose elements are called *moves*, a function λ_A from M_A to $\{-1, +1\}$, which to every move m associates its *polarity*, and a partial order $\leqslant_A$ on moves, called the *causality* or *justification*, which should be well-founded, that is, be such that every move $m \in M_A$ defines a finite downward closed set

$$m{\downarrow} \quad = \quad \{\, n \in M_A \quad | \quad n \leqslant_A m \,\}.$$

A move m is said to be a *Proponent move* when $\lambda_A(m) = +1$, and an *Opponent move* otherwise.

The size $|A|$ of a game A is the cardinal of its set of moves M_A.

Remark 15. More generally, games should be defined as event structures (Winskel 1987) in order to be able to model additive connectives. We will not give the details here since we only consider formulas without connectives.

If A and B are two games, their tensor product $A \otimes B$ is defined by disjoint union on moves, polarities and causality:

$$M_{A\otimes B} = M_A \uplus M_B, \quad \lambda_{A\otimes B} = \lambda_A + \lambda_B \quad \text{and} \quad \leqslant_{A\otimes B} = \leqslant_A \cup \leqslant_B .$$

The opposite game A^* of the game A is obtained from A by inverting the polarities of moves:

$$A^* = (M_A, -\lambda_A, \leqslant_A).$$

Finally, the arrow game $A \multimap B$ is defined by

$$A \multimap B \quad = \quad A^* \otimes B.$$

A game A is *filiform* when the associated partial order is total (we will be mostly interested in such games in the following).

Definition 16. A *strategy* σ on a game A is a partial order $\leqslant_\sigma$ on the moves of A that satisfies the following two properties:

(1) *Polarity*: for every pair of moves $m, n \in M_A$,

$$m <_\sigma n \qquad \text{implies} \qquad \lambda_A(m) = -1 \quad \text{and} \quad \lambda_A(n) = +1.$$

(2) *Acyclicity*: the partial order $\leqslant_\sigma$ is compatible with the partial order of the game, in the sense that the transitive closure of their union is still a partial order (that is, is acyclic).

The *size* $|A|$ of a game A is the cardinal of M_A and the *size* $|\sigma|$ of a strategy $\sigma : A$ is the cardinal of the relation $\leqslant_\sigma$.

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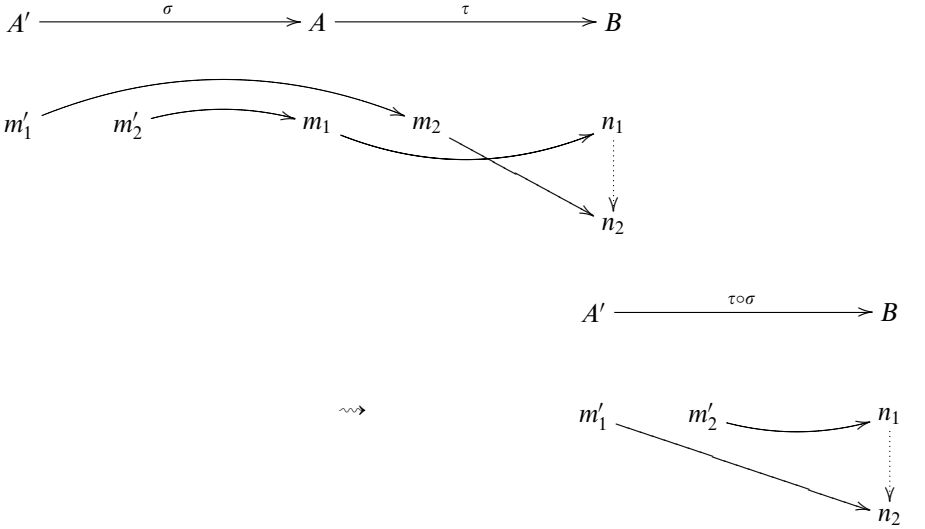
6.2. *A category of games*

At this point it would be very tempting to build a category whose:

— objects are games;
— morphisms $\sigma : A \to B$ are strategies on the game $A \multimap B$.

The identity strategy $\mathrm{id}_A : A' \to A$ (the apostrophe sign is only used here to identify the two copies of A unambiguously) would be the strategy such that for every move m in A and m' in A' that are instances of the same move m, we have $m' \leqslant_{\mathrm{id}_A} m$ whenever $\lambda_A(m) = +1$ and $m \leqslant_{\mathrm{id}_A} m'$ whenever $\lambda_A(m) = -1$ (it is easy to check that this definition satisfies the axioms for strategies). Now consider two strategies $\sigma : A \to B$ and $\tau : B \to C$. The partial order $\leqslant_\sigma$ on the set $M_A \uplus M_B$ is a relation on $M_A \uplus M_B$, that is, a subset of $(M_A \uplus M_B)^2$, and similarly for τ. The partial order $\leqslant_{\tau\circ\sigma}$ corresponding to the composite $\tau \circ \sigma : A \to C$ of the two strategies σ and τ would be defined as the transitive closure of the relation $\leqslant_\sigma \cup \leqslant_\tau$ on $M_A \uplus M_B \uplus M_C$ restricted to the set $M_A \uplus M_C$. It is easy to check that identities act as neutral elements for composition. Similar ideas for composing strategies were, in particular, developed in the appendix of Hyland and Schalk (2002).

For example, consider the game A with two Proponent moves m_1 and m_2 and the empty causality relation, the game B with two Proponent moves n_1 and n_2 and the causality relation $n_1 \leqslant_B n_2$, the strategy $\sigma : A' \to A$ such that $m'_1 \leqslant_\sigma m_2$ and $m'_2 \leqslant_\sigma m_1$ and the strategy $\tau : A \to B$ such that $m_1 \leqslant_\tau n_1$ and $m_2 \leqslant_\tau n_2$. Their composite is the strategy $\tau \circ \sigma : A' \to B$ such that $m'_2 \leqslant_{\tau\circ\sigma} n_1$ and $m'_1 \leqslant_{\tau\circ\sigma} n_2$. This can be viewed graphically as follows:

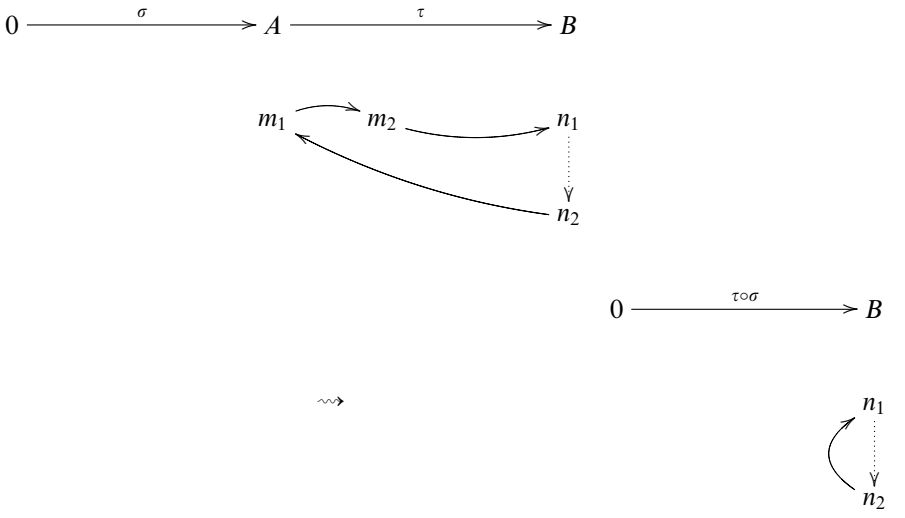


In the diagram above, the dotted arrows represent the causal dependencies in the games and solid arrows the dependencies in the strategies.

However, the composite of two strategies is not necessarily a strategy! For example, consider the game A defined as before except that m_1 is now an Opponent move, the game B defined as before except that n_2 is now an Opponent move, the strategy $\sigma : 0 \to A$ (where 0 denotes the empty game) such that $m_1 \leqslant_\sigma m_2$ and the strategy $\tau : A \to B$ such

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that $n_2 \leqslant_\tau m_1$ and $m_2 \leqslant_\tau n_1$. Their 'composite' is *not* a strategy because it does not satisfy the acyclicity property:



This is a typical example of the fact that compositionality of strategies in game semantics is often a subtle property that needs to be checked very carefully.

Remark 17. A more conceptual explanation of this compositionality problem can be given as follows. If we write P for the game with only one Proponent move, the game A should correspond, in a model of linear logic, to either $P \otimes P^*$ or $P \parr P^*$. However, we have not included in our strategies conditions that are necessary to distinguish between the interpretation of tensor ($\otimes$) and par ($\parr$). This explains why we are not able to recover the compositionality of the acyclicity property, which is deeply linked with the correctness criterion of linear logic. We leave a precise investigation of this for future work, in which we plan to extend our model to first-order linear logic.

Fortunately, if we restrict the previous attempt to construct a category by only allowing *finite filiform games* as objects, then we actually do construct a category (that is, the composite of two morphisms is a morphism), which we denote by **Games**. Moreover, we will show that the connective-free fragment of first-order propositional logic can be interpreted in this category and that the conditions imposed on strategies characterise exactly the strategies interpreting proofs (Theorem 26).

We could give a direct proof of the fact that **Games** is actually a category. However, a direct proof of the fact that the composite of two acyclic strategies is acyclic is combinatorial, lengthy and requires global reasoning about strategies. This proof would show, by *reductio ad absurdum*, that if the composite of two strategies contains a cycle (together with the causality of the game), then one of the strategies already contains a cycle. So, it would, moreover, not be very satisfactory in the sense that it would not be constructive. Instead of proceeding in this way, we define the category **Games** in an abstract fashion, construct a presentation of this category, and conclude *a posteriori* that,

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in fact, its only morphisms are strategies, which implies, in particular (Theorem 25), that strategies do actually compose!

We first define a weaker notion of a strategy.

Definition 18. A *cyclic strategy* σ on a game A is a relation on the moves of A, that is, a subset of $M_A \times M_A$, such that:

(1) The relation σ is reflexive and transitive.
(2) *Polarity*: for every pair of moves $m, n \in M_A$,

$$m \,\sigma\, n \quad \text{and} \quad m \neq n \qquad \text{implies} \qquad \lambda_A(m) = -1 \quad \text{and} \quad \lambda_A(n) = +1.$$

In particular, every strategy is a cyclic strategy. From this definition it is very easy to build a category **CGames** whose:

— objects are games;
— morphisms $\sigma : A \to B$ are strategies on the game $A \multimap B$;
— identities and composition are defined as above.

Since the definition of a cyclic strategy is much weaker than the notion of a strategy, it is routine to check that the category is well defined. We now define the category **Games** as the category generated in **CGames** by finite filiform games and strategies, that is, the smallest category whose:

— objects are finite filiform games;
— for all objects A and B, and every strategy $\sigma : A \multimap B$ (in the sense of the definition (6)), we have that σ is a morphism in $\mathrm{Hom}(A, B)$;
— for all objects A, B and C, if σ is a morphism in $\mathrm{Hom}(A, B)$ and τ is a morphism in $\mathrm{Hom}(B, C)$, then their composite $\tau \circ \sigma$ (in the category **CGames**) is a morphism in $\mathrm{Hom}(A, C)$.

As mentioned above, we will show in Theorem 25 that the only morphisms of this category are actually strategies.

6.3. *A monoidal structure on* **Games**

If A and B are two games, the game $A \Subset B$ (to be read A *before* B) is the game defined as $A \otimes B$ on moves and polarities, and $\leqslant_{A \Subset B}$ is the transitive closure of the relation

$$\leqslant_{A \otimes B} \cup \{\, (a, b) \quad | \quad a \in M_A \text{ and } b \in M_B \,\}.$$

This operation is extended as a bifunctor on strategies as follows. If $\sigma : A \to B$ and $\tau : C \to D$ are two strategies, the strategy $\sigma \Subset \tau : A \Subset C \to B \Subset D$ is defined as the relation $\leqslant_{\sigma \Subset \tau} = \leqslant_\sigma \uplus \leqslant_\tau$. This bifunctor induces a monoidal structure $(\mathbf{Games}, \Subset, I)$ on the category **Games**, where I denotes the empty game.

We write O for a game with only one Opponent move and P for a game with only one Proponent move. It is easy to see that finite filiform games A are generated by the grammar

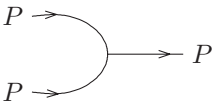
$$A \qquad ::= \qquad I \quad | \quad O \Subset A \quad | \quad P \Subset A.$$

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A game $X_1 \parr \cdots \parr X_n \parr I$ where the X_i are either O or P is represented graphically by

$$\begin{array}{c} X_1 \\ \vdots \\ X_n \end{array}$$

and a strategy $\sigma : A \to B$ is represented graphically by drawing a line from a move m to a move n whenever $m \leqslant_\sigma n$. For example, the strategy $\mu^P : P \parr P \to P$



is the strategy on the game $(O \parr O) \otimes P$ in which both Opponent moves of the left-hand game justify the Proponent move of the right-hand game. When a move does not justify (or is not justified by) any other move, we draw a line with a small circle at its end. For example, the strategy $\varepsilon^P : P \to I$, drawn as



is the unique strategy from P to the terminal object I. With these conventions, we can now introduce notation for some morphisms, which are shown in Figure 1.

6.4. *A game semantics*

A formula A is interpreted as a filiform game $[\![A]\!]$ by

$$[\![P]\!] = I \qquad [\![\forall x.A]\!] = O \parr [\![A]\!] \qquad [\![\exists x.A]\!] = P \parr [\![A]\!].$$

A cut-free proof $\pi : A \vdash B$ is interpreted as a strategy $\sigma : [\![A]\!] \multimap [\![B]\!]$ whose causality partial order $\leqslant_\sigma$ is defined as follows. For every Proponent move P interpreting a quantifier introduced by a rule that is either

$$\frac{A[t/x] \vdash B}{\forall x.A \vdash B}(\forall\text{-L}) \qquad \text{or} \qquad \frac{A \vdash B[t/x]}{A \vdash \exists x.B}(\exists\text{-R})$$

we have that every Opponent move O interpreting a universal quantification $\forall x$ on the right-hand side of a sequent or an existential quantification $\exists x$ on the left-hand side of a sequent is such that $O \leqslant_\sigma P$ whenever the variable x is free in the term t. For example, a proof

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash Q[t/z]}(\text{Ax})}{P \vdash \exists z.Q}(\exists\text{-R})}{\exists y.P \vdash \exists z.Q}(\exists\text{-L})}{\exists x.\exists y.P \vdash \exists z.Q}(\exists\text{-L})$$

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$$\begin{array}{llll} \mu^O & : & O \otimes O \to O & \mu^P : P \otimes P \to P \\ \eta^O & : & I \to O & \eta^P : I \to P \\ \delta^O & : & O \to O \otimes O & \delta^P : P \to P \otimes P \\ \varepsilon^O & : & O \to I & \varepsilon^P : P \to I \\ \gamma^O & : & O \otimes O \to O \otimes O & \gamma^P : P \otimes P \to P \otimes P \\ \eta^{OP} & : & I \to O \otimes P & \varepsilon^{OP} : P \otimes O \to I \\ & & \gamma^{OP} : P \otimes O \to O \otimes P & \end{array}$$

which are depicted, respectively, by

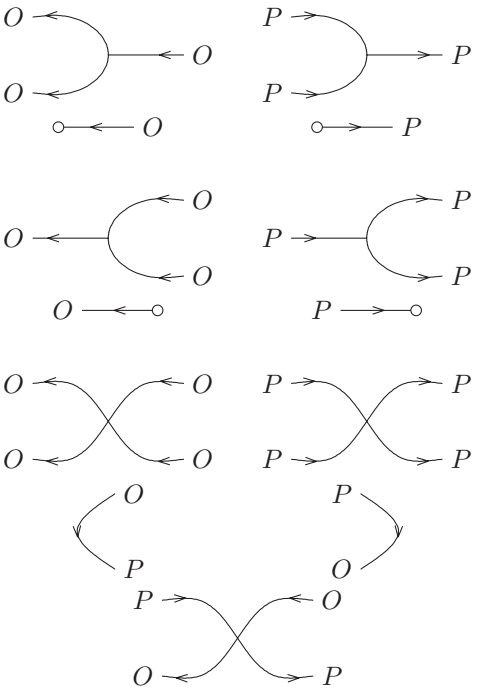
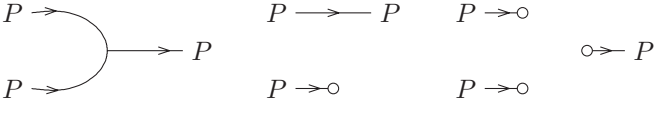


Figure 1. Generators of the strategies.

is interpreted by the strategies



(21)

respectively, where the free variables of t are $\{x, y\}$, $\{x\}$ or $\varnothing$.

Remark 19. This interpretation could be generalised to proofs with cuts using the composition of the category **Games**, and one could show that the interpretation is invariant under cut-elimination. However, we do not give the details here since it is best expressed using connectives, and thus we leave it for future work.

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6.5. *An equational theory of strategies*

We can now introduce the equational theory that will be shown to present the category **Games**.

Definition 20. The *equational theory of strategies* is the equational theory $\mathfrak{G}$ with two atomic types O and P, and the thirteen generators shown in Figure 1 such that:

— The Opponent structure
$$(O, \mu^O, \eta^O, \delta^O, \varepsilon^O, \gamma^O) \tag{22}$$
is a bicommutative qualitative bialgebra.
— The object P is left dual to the object O with η^{OP} as unit and ε^{OP} as counit.
— The Proponent structure $(P, \mu^P, \eta^P, \delta^P, \varepsilon^P, \gamma^P)$, as well as the morphism γ^{OP}, are deduced from the Opponent structure (22) by composition with the duality morphisms η^{OP} and ε^{OP} in the sense that the equations of Figure 2 hold.

We write $\mathscr{G}/\!\!\equiv$ for the monoidal category generated by $\mathfrak{G}$. Note that the generators μ^P, η^P, δ^P, ε^P, γ^P and γ^{OP} are superfluous in this presentation (since they can be deduced from the Opponent structure and duality). However, removing them would seriously complicate the proofs.

Remark 21. If we adopt the point of view of logic, the relations of Figure 2 (as well as, in fact, all the relations of our presentation) can be understood as rules for cut-elimination. For example, suppose for clarity that function symbols include a nullary symbol 0, that proposition symbols include a nullary symbol $\top$ and a binary symbol $=$, and that the set Ax of axioms contains the reasonable axioms for equality, for example, $(\top, x = x) \in Ax$, and so on. In the third equation of Figure 2, the left and right members are the interpretation of the proofs

$$\dfrac{\dfrac{}{\top \vdash 0 = 0}(\text{Ax})}{\top \vdash \exists x.x = 0}(\exists\text{-R})$$

and

$$\dfrac{\dfrac{\dfrac{}{\top \vdash y = y}(\text{Ax})}{\top \vdash \exists z.y = z}(\exists\text{-R})}{\top \vdash \forall y.\exists z.y = z}(\forall\text{-R}) \qquad \dfrac{\dfrac{\dfrac{}{0 = z \vdash z = 0}(\text{Ax})}{0 = z \vdash \exists x.x = 0}(\exists\text{-R})}{\exists z.0 = z \vdash \exists x.x = 0}(\exists\text{-L})}{\forall y.\exists z.y = z \vdash \exists x.x = 0}(\exists\text{-L})$$
$$\top \vdash \exists x.x = 0 \qquad (\text{Cut})$$

respectively, and the second proof reduces to the first by cut-elimination.

Lemma 22. With the notations of Definition 20, we have:

— $(P, \mu^P, \eta^P, \delta^P, \varepsilon^P, \gamma^P)$ is a qualitative bicommutative bialgebra.

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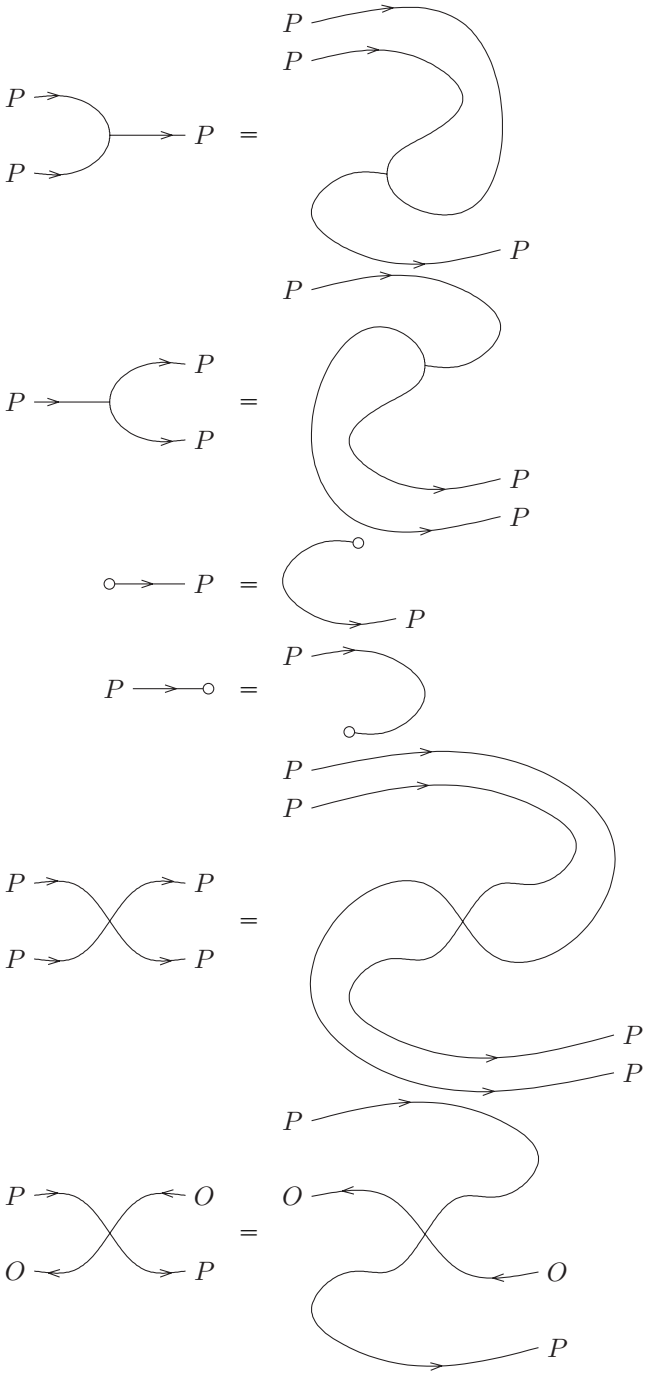
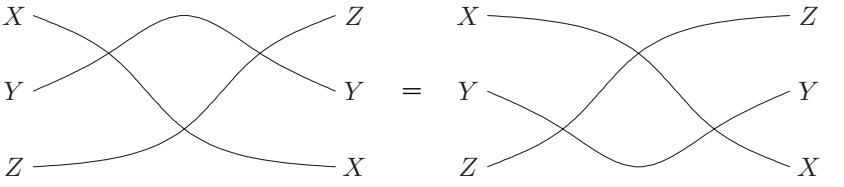


Figure 2. Proponent is left dual to Opponent.

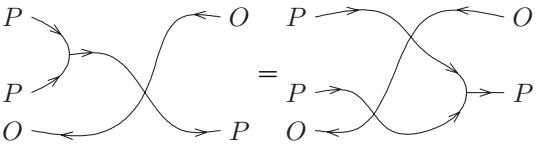
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— The Yang–Baxter equalities

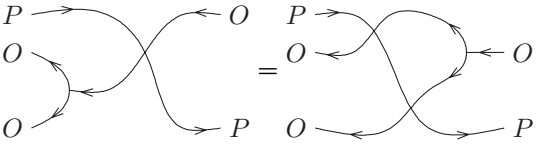


hold whenever (X, Y, Z) is either (O, O, O), (P, O, O), (P, P, O) or (P, P, P).

— The equalities

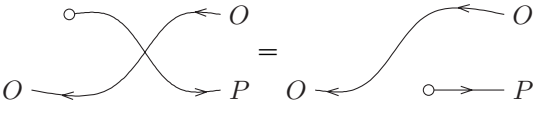


and

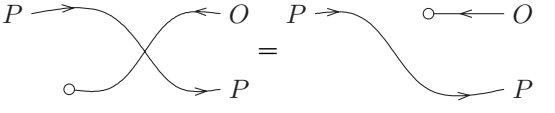


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— The equalities

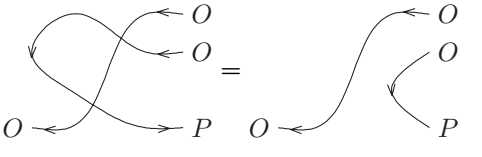


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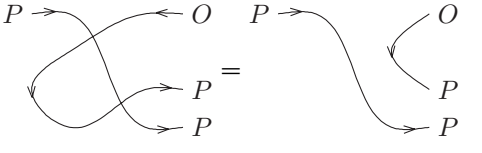


hold (and dually for counits).

— The equalities



and



hold (and dually for the counit of duality).

We can now proceed as in Section 5 to show that the theory $\mathfrak{G}$ introduced in Definition 20 presents the category **Games**. First, in the category **Games** with the monoidal

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structure induced by $\Subset$, the objects O and P can be canonically equipped with thirteen morphisms as shown in Figure 1 in order to form a model of the theory $\mathfrak{G}$.

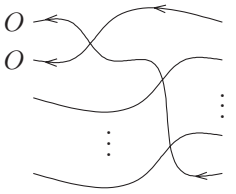
Conversely, we need to introduce a notion of canonical form for the morphisms of $\mathscr{G}$. Stairs are defined in a similar way to earlier, but are now constructed from the three kinds of polarised crossings γ^O, γ^P and γ^{OP} instead of simply γ in (16): a *stair* is:

$$\mathrm{id}_O$$

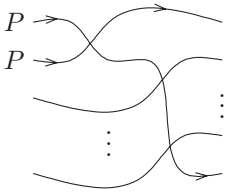
or

$$\mathrm{id}_P$$

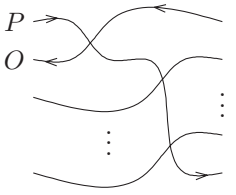
or



or



or



The notion of a *precanonical form* ϕ is now defined inductively as shown in Figure 3, where the object X is either O or P and ϕ' is a precanonical form. These cases correspond, respectively, to the productions of the grammar

$$\phi \quad ::= \quad Z \quad | \quad A_i\phi \quad | \quad B_i\phi \quad | \quad W_i\phi \quad | \quad E^X\phi \quad | \quad H^X\phi.$$

It can be shown by induction on the size of morphisms that every morphism of $\mathscr{G}$ is equivalent to a precanonical form, and a notion of canonical form can be defined by adapting the rewriting system (18) into a rewriting system for precanonical forms by

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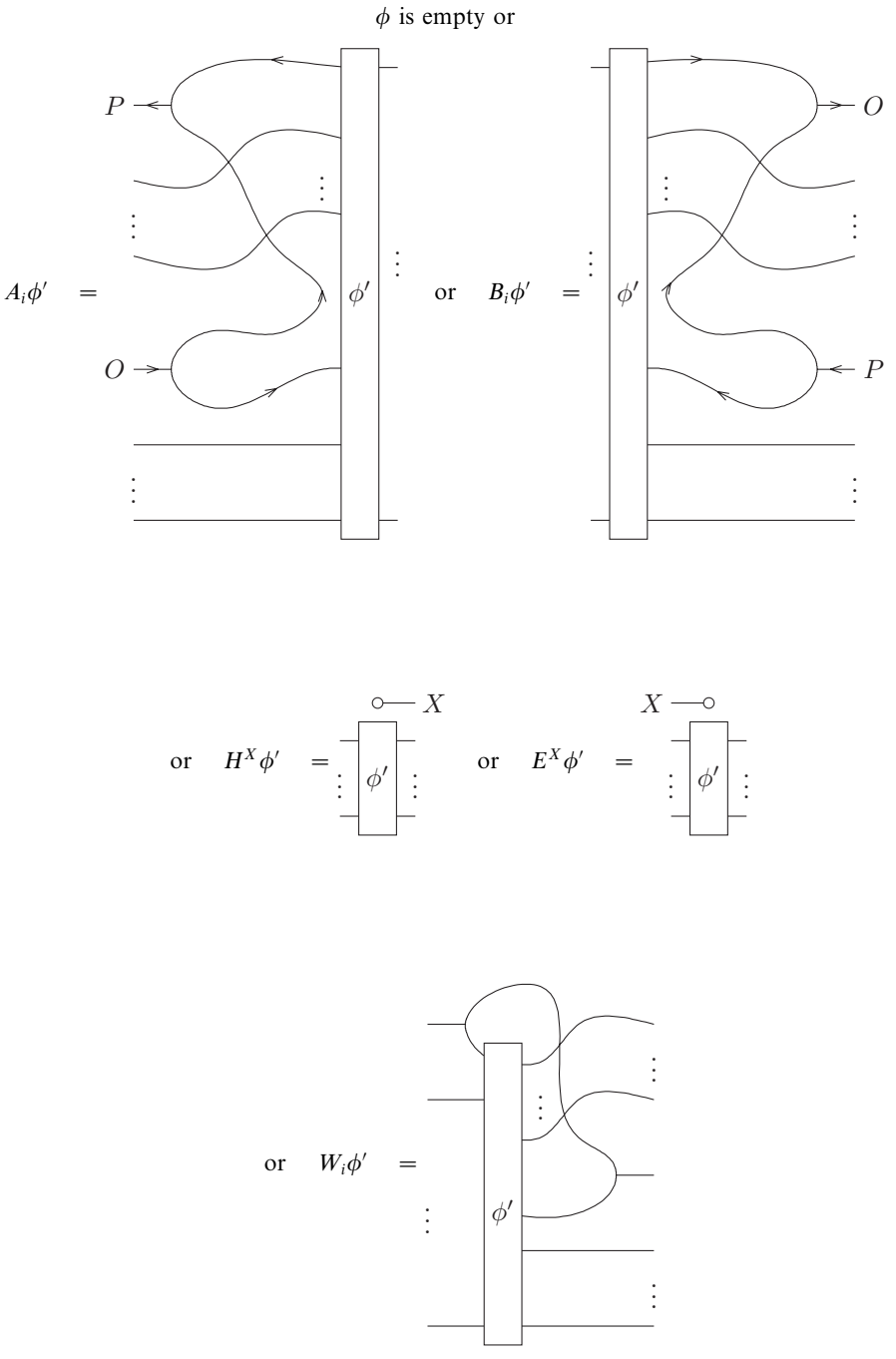


Figure 3. Precanonical forms for strategies.

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adding the rules

$$\begin{array}{rcll} H^X W_i & \Longrightarrow & W_{i+1} H^X & \\ H^X E^Y & \Longrightarrow & E^Y H^X & \\ W_i W_j & \Longrightarrow & W_j W_i & \text{when } i < j \\ W_i W_i & \Longrightarrow & W_i & \\ H^X A_i & \Longrightarrow & A_i H^X & \\ A_i W_j & \Longrightarrow & W_j A_i & \\ A_i A_j & \Longrightarrow & A_j A_i & \text{when } i < j \\ A_i A_i & \Longrightarrow & A_i & \\ E^X B_i & \Longrightarrow & E^X & \\ B_i W_j & \Longrightarrow & W_j B_i & \\ B_i B_j & \Longrightarrow & B_j B_i & \text{when } i < j \\ B_i B_i & \Longrightarrow & B_i & \\ B_i A_j & \Longrightarrow & A_j B_i & \end{array}$$

to the rewriting system containing the rules (18) and (20). It is simple to extend the proof of Lemma 9 in order to show that this rewriting system is normalising. The general form for canonical forms is

$$\begin{aligned} W_{i^n_{k_n}} \cdots W_{i^n_1} A_{j^n_{l_n}} \cdots A_{j^n_1} E \cdots\cdots W_{i^1_{k_1}} \cdots W_{i^1_1} A_{j^1_{l_1}} \cdots A_{j^1_1} E \\ \cdots B_{h^p_{m_p}} \cdots B_{h^p_1} H \cdots B_{h^1_{m_1}} \cdots B_{h^1_1} HZ \end{aligned} \tag{23}$$

with

— $i^p_{k_p} > \ldots > i^p_1$ for every integer r such that $1 \leqslant r \leqslant k_n$;
— $j^p_{l_p} > \ldots > j^p_1$ for every integer r such that $1 \leqslant r \leqslant l_n$;
— $h^p_{l_p} > \ldots > h^p_1$ for every integer r such that $1 \leqslant r \leqslant m_n$.

Lemma 23. Every strategy $\sigma : A \to B$ is the interpretation of a unique canonical form.

Proof. We will show that every strategy $\sigma : A \to B$ is the interpretation of a precanonical form $\phi : A \to B$ by induction on the triple $(|A|, |\sigma|, |B|)$, ordered lexicographically:

(1) If $A = B = I$, then σ is the interpretation of the precanonical form Z.
(2) If $A = I$ and $B = X \otimes B'$, where X is either P or O, we distinguish two cases:
   — If no move depends on X in the strategy, this strategy is the image of a precanonical form $H_X \phi'$, where ϕ' is a precanonical form, obtained by the induction hypothesis whose interpretation is the strategy $\sigma' : I \to B'$ obtained by restricting σ to the codomain B (the size of σ' is $|\sigma'| = |\sigma|$).
   — Otherwise, we write i for the index in B of the move of minimal index that depends on X in the strategy. The strategy is the image of a precanonical form $B_i \phi'$, where ϕ' is precanonical form, obtained by the induction hypothesis, whose interpretation is the strategy $\sigma' : I \to B$ obtained from σ by removing the dependency of the ith move of B on the first move of B (its size is such that $|\sigma'| < |\sigma|$).
(3) If $A = X \otimes A'$, where X is either P or O, we distinguish three cases:
   — If no move depends on X in the strategy, this strategy is the image of a precanonical form $E^X \phi'$, where ϕ' is a precanonical form, obtained by the induction hypothesis,

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whose interpretation is the strategy $\sigma' : A' \to B$ obtained by restricting σ to the domain A'.

— If there exists a move of X that depends on X, we write i for the index in A of such a move of minimal index. The strategy is the interpretation of a precanonical form $A_i\phi'$, where ϕ' is a precanonical form, obtained by the induction hypothesis, whose interpretation is the strategy $\sigma' : A \to B$ obtained from σ by removing the dependency of the ith move of A on the first move of A (its size is such that $|\sigma'| < |\sigma|$).

— Otherwise, there exists a move in B that depends on the move X. We write i for the index in B of such a move of minimal index. The strategy is the interpretation of a precanonical form $W_i\phi'$, where ϕ' is a precanonical form, obtained by the induction hypothesis, whose interpretation is the strategy $\sigma' : A \to B$, obtained from σ by removing the dependency of the ith move of B on the first move of A (its size is such that $|\sigma'| < |\sigma|$).

Knowing the general form (23) of canonical forms, it is easy to show that the precanonical forms thus constructed are actually canonical and that canonical forms $\phi : A \to B$ are in bijection with strategies $\sigma : A \to B$, as in the proof of Theorem 12. □

We can now, therefore, deduce the main theorem of this paper.

Theorem 24. The monoidal category **Games** (with the $\circledcirc$ tensor product) is presented by the equational theory $\mathfrak{G}$.

As a direct consequence of this Theorem, we can deduce the following two properties, which show the technical benefits of our construction.

Theorem 25. The composite of two strategies, in the sense of Definition 16, is itself a strategy (in particular, the acyclicity property is preserved by composition).

Proof. Two strategies $\sigma : A \to B$ and $\tau : B \to C$ can be viewed as morphisms $\tilde{\sigma}$ and $\tilde{\tau}$ of the category $\mathscr{G}/\!\equiv$, and the image of their composite is $\widetilde{\tau \circ \sigma} = \tilde{\tau} \circ \tilde{\sigma}$, which corresponds to the image of a unique acyclic strategy. □

Theorem 26. The strategies of **Games** are definable (when the set Ax of axioms is reasonably large enough): it is enough to check that generators are definable – for example, the first case of (21) shows that μ^P is definable.

Proof. Suppose there is a countable number of variable symbols. Suppose, moreover, that there exists a unary propositional symbol I, which enables us to see every term t as a proposition $I(t)$, for which we will simply write t by abuse of notation. We also assume that the set of propositions contains two nullary propositions $\top$ and $\bot$ and is closed under formal conjunctions and disjunctions: if $P(x_1, \ldots, x_n)$ and $Q(y_1, \ldots, y_m)$ are propositions, then $P(x_1, \ldots, x_n) \wedge Q(y_1, \ldots, y_m)$ and $P(x_1, \ldots, x_n) \vee Q(y_1, \ldots, y_m)$ are also propositions. We then define a set Ax of axioms as the smallest set of pairs of propositions that is reflexive and transitive and such that:

— for every proposition P:

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- $(P, \top) \in Ax$;
- $(\bot, P) \in Ax$;

— for all propositions P, P_1 and P_2:

- if $(P, P_1) \in Ax$ and $(P, P_2) \in Ax$, then $(P, P_1 \wedge P_2) \in Ax$;
- if $(P, P_1) \in Ax$ or $(P, P_1) \in Ax$, then $(P, P_1 \vee P_2) \in Ax$;
- if $(P_1, P) \in Ax$ or $(P_2, P) \in Ax$, then $(P_1 \wedge P_2, P) \in Ax$;
- if $(P_1, P) \in Ax$ and $(P_2, P) \in Ax$, then $(P_1 \vee P_2, P) \in Ax$;

where, for brevity, we have omitted the arguments of the propositions. By Theorem 24, every strategy can be expressed as a tensor and composite of the generating strategies shown in Figure 1. It is therefore enough to show that those strategies are definable:

— The strategies μ^P and η^P are, respectively, the interpretations of the proofs

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x \wedge y \vdash x \wedge y}(\text{Ax})}{x \wedge y \vdash \exists z.z}(\exists\text{-R})}{\exists y.x \wedge y \vdash \exists z.z}(\exists\text{-L})}{\exists x.\exists y.x \wedge y \vdash \exists z.z}(\exists\text{-L})$$

and

$$\dfrac{\dfrac{}{\top \vdash \top}(\text{Ax})}{\top \vdash \exists x.x}(\exists\text{-R})$$

— the strategies δ^P and ε^P are, respectively, the interpretations of the proofs

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x \vdash x \wedge x}(\text{Ax})}{x \vdash \exists z.x \wedge z}(\exists\text{-R})}{x \vdash \exists y.\exists z.y \wedge z}(\exists\text{-R})}{\exists x.x \vdash \exists y.\exists z.y \wedge z}(\exists\text{-L})$$

and

$$\dfrac{\dfrac{}{x \vdash \top}(\text{Ax})}{\exists x.x \vdash \top}(\exists\text{-L})$$

— the strategies η^{OP} and ε^{OP} are, respectively, the interpretations of the proofs

$$\dfrac{\dfrac{\dfrac{}{\top \vdash x \vee (x \vee \top)}(\text{Ax})}{\top \vdash \exists y.x \vee y}(\exists\text{-R})}{\top \vdash \forall x.\exists y.x \vee y}(\forall\text{-R})$$

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and

$$\dfrac{\dfrac{\dfrac{}{x \wedge (x \wedge \bot) \vdash \bot}(\text{Ax})}{\forall y.x \wedge y \vdash \bot}(\forall\text{-L})}{\exists x.\forall y.x \wedge y \vdash \bot}(\exists\text{-L})$$

— the strategies γ^P and γ^{OP} are, respectively, the interpretations of the proofs

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x \wedge y \vdash x \wedge y}(\text{Ax})}{x \wedge y \vdash \exists t.t \wedge y}(\exists\text{-R})}{x \wedge y \vdash \exists z.\exists t.t \wedge z}(\exists\text{-R})}{\exists y.x \wedge y \vdash \exists z.\exists t.t \wedge z}(\exists\text{-L})}{\exists x.\exists y.x \wedge y \vdash \exists z.\exists t.t \wedge z}(\exists\text{-L})$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x \wedge z \vdash x \wedge z}(\text{Ax})}{x \wedge z \vdash \exists t.t \wedge z}(\exists\text{-R})}{\forall y.x \wedge y \vdash \exists t.t \wedge z}(\forall\text{-L})}{\exists x.\forall y.x \wedge y \vdash \exists t.t \wedge z}(\exists\text{-L})}{\exists x.\forall y.x \wedge y \vdash \forall z.\exists t.t \wedge z}(\forall\text{-R})$$

— and so on. □

A given strategy is not necessarily the interpretation of a unique proof. In particular, as explained in the introduction, two proofs that only differ in the order of introduction of some successive connectives are identified in the semantics.

In the preceding proof, we could, of course, have taken the set of all pairs of propositions as the set Ax of axioms. However, the set we have used shows that our definability result can be obtained with a reasonable set of axioms: in particular, it is *coherent*, which means that there exists a sequent that cannot be proved (the sequent $\top \vdash \bot$ for example), which would not have been the case with the trivial set of axioms.

7. Conclusion

We have constructed a game semantics for first-order propositional logic and given a presentation of the category **Games** of games and definable strategies. This has revealed the essential structure of causality induced by quantifiers as well as provided technical tools for showing definability and composition of strategies.

We consider this work to be more a starting point for forming a bridge between semantics and algebra than a final result. The methodology presented here seems to be very general and many tracks remain to be explored.

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First, we would like to extend the presentation to a game semantics for richer logic systems containing connectives (such as conjunction or disjunction). Whilst we do not expect there to be any essential technical complications, this case is much more difficult to grasp and manipulate since a presentation of such a semantics would have generators up to dimension 3: games would be modelled as trees of connectives and strategies as 'surface diagrams' between these trees. It would be particularly interesting to do this for the multiplicative fragment of linear logic (MLL) with first-order quantifiers since it would provide us with a local reformulation of the Danos–Regnier criterion for MLL extended with the MIX rule (this is hinted at in Remark 17).

Some of the proofs (such as the proof of Lemma 8) are very repetitive, which we think is a good point since we believe that they could be mechanically checked or automated. It turns out that it is quite difficult to find a good representation of morphisms in monoidal categories, which would be suitable for a computer to manipulate without having to handle complex congruences such as the exchange law. We have proposed such a representation as well as a unification algorithm for monoidal rewriting systems (Mimram 2010), but many properties and generalisations of these techniques remain to be investigated in order to have really useful tools. Formulated in categorical terms, this amounts to generalising term rewriting techniques from Lawvere theories (which are categories with products, and thus monoidal categories, and thus 2-categories with one object) to the general setting of 2-categories. In particular, it would also be interesting to know whether it is possible to orient the equalities in the presentations in order to obtain strongly normalising rewriting systems for the algebraic structures described in this paper. Such rewriting systems are given in Lafont (2003) for monoids and commutative monoids, and so on, but, for example, finding a strongly normalising rewriting system presenting the theory of bialgebras is a difficult problem (Mimram 2008), not to mention a strongly normalising presentation of our category of games. Such a presentation would have a very large number of critical pairs, which makes us regard the development of automated tools to compute them a necessary preliminary step.

Finally, there is a striking analogy between the string diagrams we have used and the wires in electronic circuits. This was actually one of the starting points of current work by Ghica (as well as game semantics), who is currently elaborating a compiler from a high-level language into integrated circuits (Ghica 2007). The categorical string-diagrammatic axioms are revealed to be crucial in this setting for establishing design principles for the circuits. Following this point of view, we believe that a deep understanding of the algebraic structure of the categories of semantics of programming languages will prove very useful for designing and optimising circuits implementing programs in these languages.

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