Canad. Math. Bull. Vol. **64** (4), 2021, pp. 737–751
http://dx.doi.org/10.4153/S0008439520000788
© Canadian Mathematical Society 2020

# Optimal free export/import regions

Samer Dweik

*Abstract.* We consider the problem of finding two free export/import sets $E^+$ and $E^-$ that minimize the total cost of some export/import transportation problem (with export/import taxes $g^\pm$), between two densities $f^+$ and $f^-$, plus penalization terms on $E^+$ and $E^-$. First, we prove the existence of such optimal sets under some assumptions on $f^\pm$ and $g^\pm$. Then we study some properties of these sets such as convexity and regularity. In particular, we show that the optimal free export (resp. import) region $E^+$ (resp. $E^-$) has a boundary of class $C^2$ as soon as $f^+$ (resp. $f^-$) is continuous and $\partial E^+$ (resp. $\partial E^-$) is $C^{2,1}$ provided that $f^+$ (resp. $f^-$) is Lipschitz.

## 1 Introduction

In this paper, we study a shape optimization problem where the functional to be minimized is given by an export/import transportation problem with free export/import zones. Before entering the details of this problem, let us introduce the standard export/import transportation problem. Let $f^+$ and $f^-$ be two given masses in some bounded region $\Omega$ and assume that we want to transport $f^+$ to $f^-$ paying a transport cost $|x-y|$, for each unit of mass that moves from $x$ to $y$. But, as the total mass of $f^+$ can be different than the one for $f^-$, we are allowed to export or import masses from the boundary $\partial\Omega$ paying two additional costs on the boundary (called *export/import taxes*) $g^+(x)$ and $g^-(y)$, for each unit of mass that comes out/enters at some point of $\partial\Omega$. We note that this problem has already been considered in many papers [8, 3, 4, 5, 2]. In other words, we consider the problem

$$\min_{\gamma\in\Pi(f^+,f^-)}\left\{\int_{\Omega\times\Omega}|x-y|\,\mathrm{d}\gamma+\int_{\partial\Omega}g^+(y)\,\mathrm{d}(\Pi_y)_\#\gamma(y)+\int_{\partial\Omega}g^-(x)\,\mathrm{d}(\Pi_x)_\#\gamma(x)\right\},\tag{1.1}$$

where

$$\Pi(f^+,f^-):=\left\{\gamma\in\mathcal{M}^+(\Omega\times\Omega):\left[(\Pi_x)_\#\gamma\right]_{|\overline{\Omega}}^{\circ}=f^+,\ \left[(\Pi_y)_\#\gamma\right]_{|\overline{\Omega}}^{\circ}=f^-\right\}.$$

In [3, 8], the authors proved, using two different approaches, that problem (1.1) has a dual formulation, which is

$$\sup\left\{\int_\Omega u(f^+-f^-)\,\mathrm{d}x\ :\ u\in\mathrm{Lip}_1(\Omega),\ -g^-\le u\le g^+\text{ on }\partial\Omega\right\}.$$

Received by the editors May 27, 2020; revised September 5, 2020.
Published online on Cambridge Core September 17, 2020.
AMS subject classification: **49Q10, 49Q20, 90B20**.
Keywords: Shape optimization, optimal regions, import/export transport problem.

https://doi.org/10.4153/S0008439520000788 Published online by Cambridge University Press

Moreover, this problem has an equivalent minimal flow formulation (see [3, 4]):

$$\min_{\sigma\in\mathcal{M}^d(\Omega),\,\chi\in\mathcal{M}(\partial\Omega)}\Bigg\{\int_\Omega|\sigma|\,\mathrm{d}x+\int_{\partial\Omega}g^+\,\mathrm{d}\chi^+ + \int_{\partial\Omega}g^-\,\mathrm{d}\chi^- : \nabla\cdot\sigma = f+\chi \text{ in } \Omega\Bigg\}. \tag{1.2}$$

Now, assume that we have two regions $E^+$ and $E^-$ inside $\Omega$ where the export/import transport is free of charge; *i.e.,* there are no taxes on these special regions. Then problem (1.1) becomes

$$W(E^+,E^-)=\min_{\gamma\in\Pi(f^+,f^-)}\Bigg\{\int_{\Omega\times\Omega}|x-y|\,\mathrm{d}\gamma+\int_{(\Omega\setminus E^+)\times\partial\Omega}g^+(y)\,\mathrm{d}\gamma(x,y) + \int_{\partial\Omega\times(\Omega\setminus E^-)}g^-(x)\,\mathrm{d}\gamma(x,y)\Bigg\}. \tag{1.3}$$

Then our shape optimization problem consists of minimizing the total cost $W(E^+,E^-)$ of this export/import transport problem (1.3) plus some penalties on $E^\pm$ (such as paying a cost proportional to the perimeter of $E^\pm$) among all subsets $E^\pm\subset\Omega$, *i.e.*, we minimize

$$\min\big\{W(E^+,E^-)+\mathrm{Per}(E^+)+\mathrm{Per}(E^-):E^\pm\subset\Omega\big\}. \tag{1.4}$$

We note that the authors of [1] already considered a shape optimization problem slightly similar to problem (1.4); we now give a brief description of their problem. Let $f^+$ and $f^-$ be two densities (having the same total mass) in some region $\Omega$ and assume that the traffic congestion in $\Omega\setminus E$ is higher than the one in $E$, so their aim was to find a set $E$ that minimizes

$$\min\big\{\mathcal{J}(E)+\mathrm{Per}(E):E\subset\Omega\big\}, \tag{1.5}$$

where

$$\mathcal{J}(E):=\min\Bigg\{\int_E H_1(\sigma)\,\mathrm{d}x+\int_{\Omega\setminus E}H_2(\sigma)\,\mathrm{d}x:\nabla\cdot\sigma = f^+-f^- \text{ in } \Omega,\ \sigma\cdot n=0 \text{ on } \partial\Omega\Bigg\},$$

where $H_1$ and $H_2$ are two continuous superlinear convex functions such that $0\le H_1\le H_2$. They proved that there exists at least an optimal set $E$ for problem (1.5). On the contrary, an optimal set $E$ may fail to exist if we replace $\mathrm{Per}(E)$ with $|E|$; *i.e.,* the problem

$$\min\big\{\mathcal{J}(E)+|E|:E\subset\Omega\big\} \tag{1.6}$$

may have no solution. For this reason, they considered instead a relaxed formulation of problem (1.6) (*i.e.,* with a function $0\le\theta\le1$ instead of $E$). More precisely, they showed that the optimal choice for $\theta$ is to have $\theta=0$ on some region $E_0$ (which represents a high-congestion area), $\theta=1$ on another region $E_1$ (a low-congestion area) and $0<\theta<1$ on $\Omega\setminus(E_0\cup E_1)$ (an intermediate congestion area). That is why we

https://doi.org/10.4153/S0008439520000788 Published online by Cambridge University Press

consider a penalization with perimeter, since otherwise it is not clear how to prove existence, and a solution may not even exist.

Coming back to our shape optimization problem, we can also consider a more general version of problem (1.4): assume that the export/import transport on $E^{\pm}$ is not free, but in order to export some mass from $E^+$, we pay a cost $g_0^+$, while outside of $E^+$ we pay a higher cost $g_1^+$, and to import some mass to $E^-$, we pay a cost $g_0^-$, while to $\Omega\backslash E^-$, we pay a higher cost $g_1^-$. In other words, we minimize

$$\min\left\{\widetilde{W}(E^+,E^-)+\mathrm{Per}(E^+)+\mathrm{Per}(E^-) : E^{\pm}\subset\Omega\right\},$$

where

$$\begin{aligned}\widetilde{W}(E^+,E^-)=\min_{\gamma\in\Pi(f^+,f^-)}\Bigg\{&\int_{\Omega\times\Omega}|x-y|\,\mathrm{d}\gamma+\int_{E^+\times\partial\Omega}g_0^+(y)\,\mathrm{d}\gamma(x,y)\\&+\int_{(\Omega\backslash E^+)\times\partial\Omega}g_1^+(y)\,\mathrm{d}\gamma(x,y)\\&+\int_{\partial\Omega\times E^-}g_0^-(x)\,\mathrm{d}\gamma(x,y)+\int_{\partial\Omega\times(\Omega\backslash E^-)}g_1^-(x)\,\mathrm{d}\gamma(x,y)\Bigg\}.\end{aligned}$$

For simplicity of exposition, we will assume that $g_0^{\pm}=0$ on $\partial\Omega$. In fact, all the results about the convexity of the optimal free export/import regions hold true in the general case under the assumption that $g_0^{\pm}\le g_1^{\pm}$ on $\partial\Omega$.

The aim of this paper is to show the existence of two optimal export/import sets $E^+_{\mathrm{opt}}$ and $E^-_{\mathrm{opt}}$ for problem (1.4) and then to study some properties of these optimal sets. More precisely, we prove that $E^+_{\mathrm{opt}}$ and $E^-_{\mathrm{opt}}$ are uniformly convex under some assumptions on $f^{\pm}$ and $g^{\pm}$. Moreover, we will study the regularity of $E^{\pm}_{\mathrm{opt}}$; in particular, we will show that $E^{\pm}_{\mathrm{opt}}$ is smooth (say $C^{2,1}$) inside $\mathrm{spt}(f^{\pm})$ as soon as $f^{\pm}$ is Lipschitz.

## 2 Export/Import Transport Problem with Free Export/Import Regions

Let $f^+$ and $f^-$ be two nonnegative Borel measures on a compact domain $\Omega\subset\mathbb{R}^d$ such that $\mathrm{spt}(f^{\pm})\subset\mathring{\Omega}$. Let $g^{\pm}:\partial\Omega\mapsto\mathbb{R}^+$ be two given functions. Let $E^{\pm}$ be two subsets of $\Omega$; then we consider the problem

$$\begin{aligned}\min_{\gamma\in\Pi(f^+,f^-)}\Bigg\{&\int_{\Omega\times\Omega}|x-y|\,\mathrm{d}\gamma+\int_{(\Omega\backslash E^+)\times\partial\Omega}g^+(y)\,\mathrm{d}\gamma(x,y)\\&+\int_{\partial\Omega\times(\Omega\backslash E^-)}g^-(x)\,\mathrm{d}\gamma(x,y)\Bigg\}.\end{aligned}\tag{2.1}$$

We have the following proposition.

***Proposition 2.1*** *Assume that $f^{\pm}\in L^1(\Omega)$ and $g^{\pm}\in C(\partial\Omega)$. Then problem (2.1) reaches a minimum.*

https://doi.org/10.4153/S0008439520000788 Published online by Cambridge University Press

**Proof** Let $(\gamma_n)_n \subset \Pi(f^+, f^-)$ be a minimizing sequence. Then it is clear that we can assume that

$$\gamma_n(\partial\Omega \times \partial\Omega) = 0.$$

In this case, we get

$$\begin{aligned}\gamma_n(\Omega \times \Omega) &\leq \gamma_n(\mathring{\Omega} \times \Omega) + \gamma_n(\Omega \times \mathring{\Omega}) \\ &= f^+(\Omega) + f^-(\Omega).\end{aligned}$$

Hence, up to a subsequence, $\gamma_n \rightharpoonup \gamma$ for some $\gamma \in \Pi(f^+, f^-)$. We define the three parts of $\gamma_n$ as follows

$$\gamma_n^{ii} := \gamma_n{}_{|\mathring{\Omega}\times\mathring{\Omega}}, \quad \gamma_n^{ib} := \gamma_n{}_{|\mathring{\Omega}\times\partial\Omega}, \quad \gamma_n^{bi} := \gamma_n{}_{|\partial\Omega\times\mathring{\Omega}}.$$

Yet, we see that $\gamma_n^{ii} \rightharpoonup \gamma_1$, $\gamma_n^{ib} \rightharpoonup \gamma_2$, and $\gamma_n^{bi} \rightharpoonup \gamma_3$ such that $\gamma = \gamma_1 + \gamma_2 + \gamma_3$. Moreover, we have $\mathrm{spt}(\gamma_1) \subset \mathring{\Omega} \times \mathring{\Omega}$, $\mathrm{spt}(\gamma_2) \subset \mathring{\Omega} \times \partial\Omega$ and $\mathrm{spt}(\gamma_3) \subset \partial\Omega \times \mathring{\Omega}$. This implies that $\gamma_1 = \gamma^{ii}$, $\gamma_2 = \gamma^{ib}$, and $\gamma_3 = \gamma^{bi}$. On the other hand, let us divide $\gamma_n^{ib}$ in two parts:

$$\gamma_n^{ib} = \gamma_n \cdot 1_{E^+\times\partial\Omega} + \gamma_n \cdot 1_{(\Omega\setminus E^+)\times\partial\Omega}.$$

We see that $\gamma_n \cdot 1_{E^+\times\partial\Omega} \rightharpoonup \gamma_4$ with $\mathrm{spt}(\gamma_4) \subset \overline{E^+} \times \partial\Omega$ and $\gamma_n \cdot 1_{(\Omega\setminus E^+)\times\partial\Omega} \rightharpoonup \gamma_5$ with $\mathrm{spt}(\gamma_5) \subset (\overline{\Omega\setminus E^+}) \times \partial\Omega$ such that

$$\gamma^{ib} = \gamma_4 + \gamma_5.$$

Yet, $f^+ \in L^1(\Omega)$ and $(\Pi_x)_\# \gamma^{ib} \leq (\Pi_x)_\#(\gamma \cdot 1_{\mathring{\Omega}\times\Omega}) = f^+$, then $\gamma^{ib}(\partial E^+ \times \partial\Omega) = 0$. Hence, we get that

$$\gamma_n \cdot 1_{(\Omega\setminus E^+)\times\partial\Omega} \rightharpoonup \gamma \cdot 1_{(\Omega\setminus E^+)\times\partial\Omega}.$$

Similarly, thanks to the fact that $f^- \in L^1(\Omega)$ and $(\Pi_y)_\# \gamma^{bi} \leq (\Pi_y)_\#(\gamma \cdot 1_{\Omega\times\mathring{\Omega}}) = f^-$, one can prove that

$$\gamma_n \cdot 1_{\partial\Omega\times(\Omega\setminus E^-)} \rightharpoonup \gamma \cdot 1_{\partial\Omega\times(\Omega\setminus E^-)}.$$

Consequently,

$$\begin{aligned}\liminf_n &\int_{\Omega\times\Omega} |x-y| \,\mathrm{d}\gamma_n + \int_{(\Omega\setminus E^+)\times\partial\Omega} g^+(y) \,\mathrm{d}\gamma_n(x,y) \\ &\quad + \int_{\partial\Omega\times(\Omega\setminus E^-)} g^-(x) \,\mathrm{d}\gamma_n(x,y) \\ &= \int_{\Omega\times\Omega} |x-y| \,\mathrm{d}\gamma + \int_{(\Omega\setminus E^+)\times\partial\Omega} g^+(y) \,\mathrm{d}\gamma(x,y) \\ &\quad + \int_{\partial\Omega\times(\Omega\setminus E^-)} g^-(x) \,\mathrm{d}\gamma(x,y).\end{aligned}$$

This implies that the transport plan $\gamma \in \Pi(f^+, f^-)$ is in fact a minimizer for problem (2.1). ∎

https://doi.org/10.4153/S0008439520000788 Published online by Cambridge University Press

Let $\gamma_{\text{opt}}$ be an optimal transport plan for problem (2.1). We divide $\gamma_{\text{opt}}$ into three parts $\gamma^{ii}_{\text{opt}}$, $\gamma^{ib}_{\text{opt}}$ and $\gamma^{bi}_{\text{opt}}$, where

$$\gamma^{ii}_{\text{opt}} = \gamma_{\text{opt}} \cdot 1_{\overset{\circ}{\Omega}\times\overset{\circ}{\Omega}}, \quad \gamma^{ib}_{\text{opt}} = \gamma_{\text{opt}} \cdot 1_{\overset{\circ}{\Omega}\times\partial\Omega}, \quad \text{and} \quad \gamma^{bi}_{\text{opt}} = \gamma_{\text{opt}} \cdot 1_{\partial\Omega\times\overset{\circ}{\Omega}}.$$

Set

$$\nu^{+}_{\text{opt}} := (\Pi_x)_{\#}(\gamma^{ib}_{\text{opt}}) \quad \text{and} \quad \nu^{-}_{\text{opt}} := (\Pi_y)_{\#}(\gamma^{bi}_{\text{opt}}).$$

We consider the three following problems:

$$\text{(P1)} \qquad \min\left\{\int_{\Omega\times\Omega} |x-y|\,\mathrm{d}\gamma : \gamma \in \Pi(f^+ - \nu^+_{\text{opt}}, f^- - \nu^-_{\text{opt}})\right\},$$

$$\text{(P2)} \qquad \min\left\{\int_{\Omega\times\Omega} |x-y|\,\mathrm{d}\gamma + \int_{(\Omega\setminus E^+)\times\partial\Omega} g^+(y)\,\mathrm{d}\gamma : (\Pi_x)_{\#}\gamma = \nu^+_{\text{opt}},\ \text{spt}((\Pi_y)_{\#}\gamma) \subset \partial\Omega\right\},$$

$$\text{(P3)} \qquad \min\left\{\int_{\Omega\times\Omega} |x-y|\,\mathrm{d}\gamma + \int_{\partial\Omega\times(\Omega\setminus E^-)} g^-(x)\,\mathrm{d}\gamma : (\Pi_y)_{\#}\gamma = \nu^-_{\text{opt}},\ \text{spt}((\Pi_x)_{\#}\gamma) \subset \partial\Omega\right\}.$$

Then we have the following proposition.

***Proposition 2.2*** *The plans $\gamma^{ii}_{\text{opt}}$, $\gamma^{ib}_{\text{opt}}$ and $\gamma^{bi}_{\text{opt}}$ solve* (P1), (P2), *and* (P3), *respectively. In addition, $\gamma^{ib}_{\text{opt}} = (Id, T_{E^+})_{\#}\nu^+_{\text{opt}}$ where $T_{E^+}$ is the Borel selector function of the multivalued map*

$$T_{E^+}(x) = \begin{cases} P(x) := \text{argmin}\{|x-y| : y \in \partial\Omega\} & \text{if } x \in E^+, \\ T^+(x) := \text{argmin}\{|x-y| + g^+(y) : y \in \partial\Omega\} & \text{if } x \in \Omega\setminus E^+. \end{cases}$$

*On the other hand, $\gamma^{bi}_{\text{opt}} = (T_{E^-}, Id)_{\#}\nu^-_{\text{opt}}$ where $T_{E^-}$ is the following Borel selector function:*

$$T_{E^-}(y) = \begin{cases} P(y) := \text{argmin}\{|x-y| : x \in \partial\Omega\} & \text{if } y \in E^-, \\ T^-(y) := \text{argmin}\{|x-y| + g^-(x) : x \in \partial\Omega\} & \text{if } y \in \Omega\setminus E^-. \end{cases}$$

**Proof** This follows immediately from the fact that $\gamma^{ii}_{\text{opt}}$, $\gamma^{ib}_{\text{opt}}$, and $\gamma^{bi}_{\text{opt}}$ are admissible in (P1), (P2), and (P3), respectively. Moreover, if $\gamma_1$, $\gamma_2$, and $\gamma_3$ minimize (P1), (P2), and (P3), respectively, then $\gamma_1+\gamma_2+\gamma_3$ minimizes problem (2.1), and so, $\gamma^{ii}_{\text{opt}}$, $\gamma^{ib}_{\text{opt}}$ and $\gamma^{bi}_{\text{opt}}$ minimize (P1), (P2), and (P3), respectively. On the other hand, if $\gamma \neq (Id, T_{E^+})_{\#}\nu^+_{\text{opt}}$ is admissible in (P2), then we have

$$\int_{\Omega\times\partial\Omega} |x-y|\,\mathrm{d}\gamma + \int_{(\Omega\setminus E^+)\times\partial\Omega} g^+(y)\,\mathrm{d}\gamma = \int_{E^+\times\partial\Omega} |x-y|\,\mathrm{d}\gamma + \int_{(\Omega\setminus E^+)\times\partial\Omega} [|x-y| + g^+(y)]\,\mathrm{d}\gamma$$

https://doi.org/10.4153/S0008439520000788 Published online by Cambridge University Press

$$> \int_{E^+\times\partial\Omega} |x - T_{E^+}(x)|\,\mathrm{d}\gamma + \int_{(\Omega\setminus E^+)\times\partial\Omega} \left[|x - T_{E^+}(x)| + g^+(T_{E^+}(x))\right]\mathrm{d}\gamma.$$

This shows that $\gamma^{ib}_{\mathrm{opt}} = (Id, T_{E^+})_{\#}v^+_{\mathrm{opt}}$. Similarly, we also get that $\gamma^{bi}_{\mathrm{opt}} = (T_{E^-}, Id)_{\#}v^-_{\mathrm{opt}}$. ∎

We conclude this section by the following proposition.

***Proposition 2.3*** *Assume $f^{\pm} \in L^1(\Omega)$. Let $\gamma_{\mathrm{opt}}$ be an optimal transport plan for (2.1) and consider the transport plans $\gamma^{ii}_{\mathrm{opt}}$, $\gamma^{ib}_{\mathrm{opt}}$ and $\gamma^{bi}_{\mathrm{opt}}$. Let $v^+_{\mathrm{opt}} := (\Pi_x)_{\#}\gamma^{ib}_{\mathrm{opt}}$ and $v^-_{\mathrm{opt}} := (\Pi_y)_{\#}\gamma^{bi}_{opt}$. Then there are two sets $E^{\pm} \subset \mathrm{spt}(f^{\pm})$ such that $v^{\pm}_{\mathrm{opt}} = f^{\pm}\cdot 1_{E^{\pm}}$.*

**Proof** Assume that this is not the case for $v^+_{\mathrm{opt}}$ (of course, the proof will be the same for $v^-_{\mathrm{opt}}$), *i.e.,* there is some set $A$ such that $0 < v^+_{\mathrm{opt}} < f^+$ on $A$. This means that on $A$, we split the mass $f^+$ in two parts: one is going to $f^- - v^-_{\mathrm{opt}}$, and the second one is exported to $\partial\Omega$. But this is a contradiction, since $|A| > 0$, while it is well known that the set of double points (that is, the set of points that belong to different transport rays) is negligible for the Lebesgue measure (see [9]). ∎

## 3 Properties of the Optimal Free Export/Import Regions

In this section, we prove existence of optimal free export/import sets $E^+_{\mathrm{opt}}$ and $E^-_{\mathrm{opt}}$ for problem (1.4). Then we introduce some regularity results on $E^{\pm}_{\mathrm{opt}}$. Fix $\lambda^{\pm} > 0$; then we consider the following shape optimization problem

$$\min\left\{W(E^+, E^-) + \lambda^+\,\mathrm{Per}(E^+) + \lambda^-\,\mathrm{Per}(E^-) \,:\, E^{\pm} \subset \Omega\right\}, \tag{3.1}$$

where

$$W(E^+, E^-) = \min_{\gamma\in\Pi(f^+,f^-)} \left\{\int_{\Omega\times\Omega} |x-y|\,\mathrm{d}\gamma + \int_{(\Omega\setminus E^+)\times\partial\Omega} g^+(y)\,\mathrm{d}\gamma(x,y) + \int_{\partial\Omega\times(\Omega\setminus E^-)} g^-(x)\,\mathrm{d}\gamma(x,y)\right\}.$$

First, we have the following proposition.

***Proposition 3.1*** *Assume that $f^{\pm} \in L^p(\Omega)$, for some $p > 1$, and $g^{\pm} \in C(\partial\Omega)$. Then there exist two optimal free export/import regions $E^+_{\mathrm{opt}}$ and $E^-_{\mathrm{opt}}$ for problem (3.1).*

**Proof** Let $(E^+_k, E^-_k)_k$ be a minimizing sequence in problem (3.1). For each $k$, let $\gamma_k$ be an optimal transport plan for $W(E^+_k, E^-_k)$ (see Proposition 2.1). Recall that $\gamma_k \in \Pi(f^+, f^-)$, and one can assume that $\gamma_k$ gives zero mass to $\partial\Omega\times\partial\Omega$, so there is some constant $C$ such that $\gamma_k(\Omega\times\Omega) \le C$, for every $k$. As $g^{\pm} \in C(\partial\Omega)$, we have $W(E^+_k, E^-_k) \ge -C\|g\|_{\infty}$, for all $k$. Then we get that

$$\mathrm{Per}(E^{\pm}_k) \le C \quad \text{for every } k.$$

https://doi.org/10.4153/S0008439520000788 Published online by Cambridge University Press

Yet, $E_k^{\pm} \subset \Omega$ for every $k$, and $\Omega$ is bounded. Hence, up to a subsequence, $(E_k^+, E_k^-) \to (E^+, E^-)$ in $L^1$, where $E^+$ and $E^-$ are two subsets of $\Omega$. In particular, we have

$$\mathrm{Per}(E^{\pm}) \le \liminf_k \mathrm{Per}(E_k^{\pm}).$$

Moreover, we have

$$W(E_k^+, E_k^-) = \int_{\Omega\times\Omega} |x-y|\,\mathrm{d}\gamma_k + \int_{(\Omega\setminus E_k^+)\times\partial\Omega} g^+(y)\,\mathrm{d}\gamma_k(x,y) + \int_{\partial\Omega\times(\Omega\setminus E_k^-)} g^-(x)\,\mathrm{d}\gamma_k(x,y).$$

Set

$$\nu_k^+ := (\Pi_x)_{\#}(\gamma_k \cdot 1_{\Omega\times\partial\Omega}) \quad \text{and} \quad \nu_k^- := (\Pi_y)_{\#}(\gamma_k \cdot 1_{\partial\Omega\times\Omega}).$$

From Proposition 2.2, we know that

$$\langle \gamma_k \cdot 1_{\Omega\times\partial\Omega}, \phi\rangle = \int_{E_k^+} \phi\big(x, P(x)\big)\,\mathrm{d}\nu_k^+ + \int_{\Omega\setminus E_k^+} \phi\big(x, T^+(x)\big)\,\mathrm{d}\nu_k^+,$$

for all $\phi \in C(\Omega\times\Omega)$, and

$$\langle \gamma_k \cdot 1_{\partial\Omega\times\Omega}, \psi\rangle = \int_{E_k^-} \psi(P(y), y)\,\mathrm{d}\nu_k^- + \int_{\Omega\setminus E_k^-} \psi(T^-(y), y)\,\mathrm{d}\nu_k^-,$$

for all $\psi \in C(\Omega\times\Omega)$. Yet, up to a subsequence, we see that $\gamma_k \rightharpoonup \gamma$ (in the sense of measures) for some $\gamma \in \Pi(f^+, f^-)$ and $\nu_k^{\pm} \rightharpoonup \nu^{\pm}$ in $L^p$, where

$$\langle \gamma \cdot 1_{\Omega\times\partial\Omega}, \phi\rangle = \int_{E^+} \phi(x, P(x))\,\mathrm{d}\nu^+ + \int_{\Omega\setminus E^+} \phi(x, T^+(x))\,\mathrm{d}\nu^+,$$

for all $\phi \in C(\Omega\times\Omega)$ and

$$\langle \gamma \cdot 1_{\partial\Omega\times\Omega}, \psi\rangle = \int_{E^-} \psi(P(y), y)\,\mathrm{d}\nu^- + \int_{\Omega\setminus E^-} \psi(T^-(y), y)\,\mathrm{d}\nu^-,$$

for all $\psi \in C(\Omega\times\Omega)$. Consequently, we have

$$\begin{aligned} W(E^+, E^-) &\le \int_{\Omega\times\Omega} |x-y|\,\mathrm{d}\gamma + \int_{(\Omega\setminus E^+)\times\partial\Omega} g^+(y)\,\mathrm{d}\gamma(x,y) \\ &\quad + \int_{\partial\Omega\times(\Omega\setminus E^-)} g^-(x)\,\mathrm{d}\gamma(x,y) \\ &= \liminf_k W(E_k^+, E_k^-). \end{aligned}$$

Finally, we infer that

$$W(E^+, E^-) + \lambda^+ \mathrm{Per}(E^+) + \lambda^- \mathrm{Per}(E^-) \le \liminf_k \big[W(E_k^+, E_k^-) + \lambda^+ \mathrm{Per}(E_k^+) + \lambda^- \mathrm{Per}(E_k^-)\big],$$

which means that the pair $(E^+, E^-)$ minimizes problem (3.1). ∎

On the other hand, we have the following monotonicity property.

https://doi.org/10.4153/S0008439520000788 Published online by Cambridge University Press

***Proposition 3.2*** *For $\lambda_1^{\pm}, \lambda_2^{\pm} > 0$, let $(E^+_{\lambda_1^+}, E^-_{\lambda_1^-})$ and $(E^+_{\lambda_2^+}, E^-_{\lambda_2^-})$ be the corresponding optimal free export/import regions for problem (1.4) with $\lambda^{\pm} = \lambda_1^{\pm}$ and $\lambda^{\pm} = \lambda_2^{\pm}$, respectively. Let $\gamma_{\lambda_1}$ and $\gamma_{\lambda_2}$ be two optimal transport plans for $W(E^+_{\lambda_1^+}, E^-_{\lambda_1^-})$ and $W(E^+_{\lambda_2^+}, E^-_{\lambda_2^-})$, respectively. Set $\nu^+_{\lambda_1} = (\Pi_x)_{\#}(\gamma_{\lambda_1} \cdot 1_{\Omega\times\partial\Omega})$, $\nu^+_{\lambda_2} = (\Pi_x)_{\#}(\gamma_{\lambda_2} \cdot 1_{\Omega\times\partial\Omega})$, $\nu^-_{\lambda_1} = (\Pi_y)_{\#}(\gamma_{\lambda_1} \cdot 1_{\partial\Omega\times\Omega})$, and $\nu^-_{\lambda_2} = (\Pi_y)_{\#}(\gamma_{\lambda_2} \cdot 1_{\partial\Omega\times\Omega})$. Assume that $\frac{\lambda_2^{\pm}}{\lambda_1^{\pm}} < \frac{\min g^{\pm}}{\max g^{\pm}}$ and $\mathrm{spt}(\nu^{\pm}_{\lambda_1}) \subset \mathrm{spt}(\nu^{\pm}_{\lambda_2})$; then we have $E^{\pm}_{\lambda_1^{\pm}} \subset E^{\pm}_{\lambda_2^{\pm}}$.*

**Proof** From the optimality of $(E^+_{\lambda_1^+}, E^-_{\lambda_1^-})$ and $(E^+_{\lambda_2^+}, E^-_{\lambda_2^-})$ in problem (1.4) with $\lambda^{\pm} = \lambda_1^{\pm}$ and $\lambda^{\pm} = \lambda_2^{\pm}$, respectively, we have

$$\begin{aligned} &W(E^+_{\lambda_1^+}, E^-_{\lambda_1^-}) + \lambda_1^+ \operatorname{Per}(E^+_{\lambda_1^+}) + \lambda_1^- \operatorname{Per}(E^-_{\lambda_1^-}) \\ &\quad \le W(E^+_{\lambda_1^+} \cap E^+_{\lambda_2^+}, E^-_{\lambda_1^-}) + \lambda_1^+ \operatorname{Per}(E^+_{\lambda_1^+} \cap E^+_{\lambda_2^+}) + \lambda_1^- \operatorname{Per}(E^-_{\lambda_1^-}) \end{aligned}$$

and

$$\begin{aligned} &W(E^+_{\lambda_2^+}, E^-_{\lambda_2^-}) + \lambda_2^+ \operatorname{Per}(E^+_{\lambda_2^+}) + \lambda_2^- \operatorname{Per}(E^-_{\lambda_2^-}) \\ &\quad \le W(E^+_{\lambda_1^+} \cup E^+_{\lambda_2^+}, E^-_{\lambda_2^-}) + \lambda_2^+ \operatorname{Per}(E^+_{\lambda_1^+} \cup E^+_{\lambda_2^+}) + \lambda_2^- \operatorname{Per}(E^-_{\lambda_2^-}). \end{aligned}$$

But,

$$\operatorname{Per}(E \cup E') + \operatorname{Per}(E \cap E') \le \operatorname{Per}(E) + \operatorname{Per}(E').$$

Then we get

$$\begin{aligned} &\frac{1}{\lambda_1^+}\big(W(E^+_{\lambda_1^+}, E^-_{\lambda_1^-}) - W(E^+_{\lambda_1^+} \cap E^+_{\lambda_2^+}, E^-_{\lambda_1^-})\big) \\ &\quad \le \frac{1}{\lambda_2^+}\big(W(E^+_{\lambda_1^+} \cup E^+_{\lambda_2^+}, E^-_{\lambda_2^-}) - W(E^+_{\lambda_2^+}, E^-_{\lambda_2^-})\big). \end{aligned}$$

Yet we have

$$W(E^+_{\lambda_1^+}, E^-_{\lambda_1^-}) - W(E^+_{\lambda_1^+} \cap E^+_{\lambda_2^+}, E^-_{\lambda_1^-}) \ge -\int_{E^+_{\lambda_1^+}\setminus E^+_{\lambda_2^+}} g^+(P(x))\,\mathrm{d}\nu_{\lambda_1^+}(x)$$

and

$$W(E^+_{\lambda_1^+} \cup E^+_{\lambda_2^+}, E^-_{\lambda_2^-}) - W(E^+_{\lambda_2^+}, E^-_{\lambda_2^-}) \le -\int_{E^+_{\lambda_1^+}\setminus E^+_{\lambda_2^+}} g^+(T^+(x))\,\mathrm{d}\nu_{\lambda_2^+}(x).$$

We infer that

$$\int_{E^+_{\lambda_1^+}\setminus E^+_{\lambda_2^+}} \left(\frac{g^+(T^+(x))\,\nu_{\lambda_2^+}(x)}{\lambda_2^+} - \frac{g^+(P(x))\,\nu_{\lambda_1^+}(x)}{\lambda_1^+}\right)\mathrm{d}x \le 0.$$

Similarly, we prove that

$$\int_{E^-_{\lambda_1^-}\setminus E^-_{\lambda_2^-}} \left(\frac{g^-(T^-(x))\,\nu_{\lambda_2^-}(x)}{\lambda_2^-} - \frac{g^-(P(x))\,\nu_{\lambda_1^-}(x)}{\lambda_1^-}\right)\mathrm{d}x \le 0.$$

This concludes the proof. ∎

https://doi.org/10.4153/S0008439520000788 Published online by Cambridge University Press

***Remark 3.1*** Assume that $f^- = 0$. For $\lambda_1,\ \lambda_2 > 0$, let $E_{\lambda_1}$ and $E_{\lambda_2}$ be the corresponding optimal free export regions for problem (1.4) with $\lambda^+ = \lambda_1$ and $\lambda^+ = \lambda_2$, respectively. Assume that $\frac{\lambda_2}{\lambda_1} < \frac{\min g^+}{\max g^+}$. Then we have $E_{\lambda_1} \subset E_{\lambda_2}$.

The aim now is to study some properties of the optimal free export/import regions $E^+_{\text{opt}}$ and $E^-_{\text{opt}}$. First, we start by introducing the following definition.

***Definition 3.1*** For a subset $E \subset \Omega$, we define the $\Omega$-convex hull of $E$ as $\bigcup_{k\in\mathbb{N}} P_k$, where $P_k \subset \Omega$ is a polygone with $k$-vertices on $\partial E$.

***Definition 3.2*** Let $E$ be a subset of $\Omega$. We say that $E$ is $\Omega$-convex if the $\Omega$-convex hull of $E$ is the set $E$ itself.

***Remark 3.2*** We note that the $\Omega$-convex hull of a subset $E \subset \Omega$ is not necessarily convex, unless $\Omega$ is convex or the convex hull of $E$ is contained in $\Omega$.

***Proposition 3.3*** *Assume $d = 2$. Let $E^+_{\text{opt}}$ (resp. $E^-_{\text{opt}}$) be the optimal free export (resp. import) region. Then, $E^+_{\text{opt}}$ (resp. $E^-_{\text{opt}}$) is $\Omega$-convex.*

**Proof** Let us prove that the optimal free export region $E^+_{\text{opt}}$ is $\Omega-$convex. Suppose that this is not the case and let $E^{++}_{\text{opt}}$ be the $\Omega-$convex hull of $E^+_{\text{opt}}$. It is not difficult to see that

$$\text{Per}(E^{++}_{\text{opt}}) < \text{Per}(E^+_{\text{opt}}).$$

Moreover, let $\gamma_{\text{opt}}$ be an optimal transport plan for $W(E^+_{\text{opt}}, E^-_{\text{opt}})$. Then the following holds:

$$\begin{aligned} &W(E^{++}_{\text{opt}}, E^-_{\text{opt}}) \\ &\quad\le \int_{\Omega\times\Omega} |x-y|\,\mathrm{d}\gamma_{\text{opt}} + \int_{(\Omega\setminus E^{++}_{\text{opt}})\times\partial\Omega} g^+(y)\,\mathrm{d}\gamma_{\text{opt}} + \int_{\partial\Omega\times(\Omega\setminus E^-_{\text{opt}})} g^-(x)\,\mathrm{d}\gamma_{\text{opt}} \\ &\quad\le \int_{\Omega\times\Omega} |x-y|\,\mathrm{d}\gamma_{\text{opt}} + \int_{(\Omega\setminus E^{+}_{\text{opt}})\times\partial\Omega} g^+(y)\,\mathrm{d}\gamma_{\text{opt}} + \int_{\partial\Omega\times(\Omega\setminus E^-_{\text{opt}})} g^-(x)\,\mathrm{d}\gamma_{\text{opt}} \\ &\quad= W(E^+_{\text{opt}}, E^-_{\text{opt}}). \end{aligned}$$

Hence, we get

$$\begin{aligned} W(E^{++}_{\text{opt}}, E^-_{\text{opt}}) + \lambda^+\,\text{Per}(E^{++}_{\text{opt}}) + \lambda^-\,\text{Per}(E^-_{\text{opt}}) < W(E^+_{\text{opt}}, E^-_{\text{opt}}) \\ + \lambda^+\,\text{Per}(E^+_{\text{opt}}) + \lambda^-\,\text{Per}(E^-_{\text{opt}}), \end{aligned}$$

which is a contradiction, as $(E^+_{\text{opt}}, E^-_{\text{opt}})$ minimizes problem (3.1). The proof of $\Omega$-convexity for $E^-_{\text{opt}}$ is eventually the same. ∎

More precisely, we have the following proposition.

***Proposition 3.4*** *Assume $d = 2$. Let $E^\pm_{\text{opt}}$ be the optimal free export/import region and let $v^\pm_{\text{opt}}$ the mass to be exported/imported. Then, $E^\pm_{\text{opt}}$ is convex in the interior of*

https://doi.org/10.4153/S0008439520000788 Published online by Cambridge University Press

$\mathrm{spt}(v^{\pm}_{\mathrm{opt}})$. *Moreover, the part of* $E^{\pm}_{\mathrm{opt}}$ *that is outside* $\mathrm{spt}(v^{\pm}_{\mathrm{opt}})$ *is a part of the* $\Omega$*-convex hull of* $\mathrm{spt}(v^{\pm}_{\mathrm{opt}})$.

**Proof** The convexity of $E^{\pm}_{\mathrm{opt}}$ in the interior of $\mathrm{spt}(v^{\pm}_{\mathrm{opt}})$ follows immediately from Proposition 3.3. Let $\mathcal{C}$ be a part of $\partial E^{+}_{\mathrm{opt}} \setminus \mathrm{spt}(v^{+}_{\mathrm{opt}})$ and $x$, $y$ be the endpoints of $\mathcal{C}$ such that $[x, y] \subset \Omega \setminus \mathrm{spt}(v^{+}_{\mathrm{opt}})$. Then it is clear that the segment $[x, y]$ is better than $\mathcal{C}$ in $E^{+}_{\mathrm{opt}}$. ∎

***Corollary 3.5*** *Assume that* $d = 2$ *and* $\mathrm{spt}(v^{\pm}_{\mathrm{opt}})$ *is* $\Omega$*-convex. Then* $E^{\pm}_{\mathrm{opt}}$ *is contained in* $\mathrm{spt}(v^{\pm}_{\mathrm{opt}})$.

We also have the following corollary.

***Corollary 3.6*** *Assume that* $d = 2$ *and* $\mathrm{spt}(f^{\pm})$ *is* $\Omega$*–convex. Then the optimal free export/import region* $E^{\pm}_{\mathrm{opt}}$ *is contained in* $\mathrm{spt}(f^{\pm})$.

Moreover, we have the following proposition.

***Proposition 3.7*** *Assume that* $d = 2$ *and the convex hull of* $\mathrm{spt}(v^{\pm}_{\mathrm{opt}})$ *is contained in* $\Omega$. *Then the optimal free export/import region* $E^{\pm}_{\mathrm{opt}}$ *is convex.*

**Proof** This follows immediately from Proposition 3.4. ∎

***Remark 3.3*** Notice that it is not obvious when $d > 2$ if the optimal free export/import regions $E^{+}_{\mathrm{opt}}$ and $E^{-}_{\mathrm{opt}}$ are $\Omega$-convex or not, since it is not true in dimensions 3 or greater that the perimeter of the $\Omega$-convex hull of a set is less than the perimeter of the set itself. However, it is possible to prove convexity of $E^{\pm}_{\mathrm{opt}}$ under the assumption that $E^{\pm}_{\mathrm{opt}}$ are smooth. In fact, from the optimality conditions on $E^{+}_{\mathrm{opt}}$ in the problem

$$\min_{E} \left\{ \lambda^{+} \mathrm{Per}(E) - \int_{E} \left[ |x - T^{+}(x)| + g^{+}(T^{+}(x)) - |x - P(x)| \right] \mathrm{d}v^{+}_{\mathrm{opt}}(x) \right\},$$

one can prove (see, for instance, [7]) that for any smooth vector field $V$ such that $V \cdot n = 0$ on $\partial\Omega$, one has

$$\int_{\partial E^{+}_{opt}} \left[ \lambda^{+} \mathcal{K}(x) - \left[ |x - T^{+}(x)| + g^{+}(T^{+}(x)) - |x - P(x)| \right] v^{+}_{opt}(x) \right] V \cdot n = 0,$$

where $\mathcal{K}$ denotes the mean curvature of $\partial E^{+}_{\mathrm{opt}}$ and $n$ is the exterior normal vector to $\partial E^{+}_{\mathrm{opt}}$. Since $V$ is arbitrary, we get

$$\mathcal{K} = \frac{1}{\lambda^{+}} \left[ |x - T^{+}(x)| + g^{+}(T^{+}(x)) - |x - P(x)| \right] v^{+}_{\mathrm{opt}}(x) \geq 0 \text{ on } \partial E^{+}_{\mathrm{opt}}.$$

***Proposition 3.8*** *Assume that* $\Omega$ *is convex. Then the optimal free export/import set* $E^{\pm}_{opt}$ *intersects the boundary of* $\mathrm{spt}(v^{\pm}_{\mathrm{opt}})$, *unless* $g^{\pm} = c$ *on some arc of* $\partial\Omega$.

https://doi.org/10.4153/S0008439520000788 Published online by Cambridge University Press

**Proof** Assume that $E^+_{opt} \cap \partial[spt(v^+_{opt})] = \emptyset$. Let $R$ be the union of all the transport rays between $v^+_{\text{opt}} \cdot 1_{E^+_{\text{opt}}}$ and its projection on the boundary $P_{\#}[v^+_{\text{opt}} \cdot 1_{E^+_{\text{opt}}}]$. Set $E := (R \cap \text{spt}(v^+_{\text{opt}}))\backslash E^+_{\text{opt}}$ and $\mathcal{C} := P(R)$. As the transport rays cannot intersect at their interiors, we then see that this set $E$ is also exported (with a tax $g^+$) to the same arc $\mathcal{C}$ with $T^+(x) = P(x) \in \mathcal{C}$, for all $x \in E$. Yet this implies that there is some constant $c$ such that $g^+ = c$ on $\mathcal{C}$. ∎

Now, we will study the regularity of $E^{\pm}_{\text{opt}}$.

***Proposition 3.9*** *Assume $d = 2$ and $f^{\pm} \in L^p(\Omega)$ with $p > 2$. Then the optimal free export/import region $E^{\pm}_{\text{opt}}$ is $C^1$ in the interior of $\Omega$. Moreover, $E^{\pm}_{\text{opt}}$ is $C^1$ provided that $\partial\Omega$ is $C^1$.*

**Proof** Assume that this is not the case at some point $x \in \partial E^+_{\text{opt}}$. After a rotation and translation of axes, we can assume that $x = (0, 0)$ and the $x_1$-axis is below the two tangent lines to $\partial E^+_{\text{opt}}$ at $x$. Let $\alpha_1$ and $\alpha_2$ be the parameterizations of the two parts of $\partial E^+_{\text{opt}}$ around $x$. Take $\varepsilon > 0$ small enough and let $\delta > 0$ be such that $\alpha_1(\varepsilon) = \alpha_2(-\delta)$. Now, let $\mathcal{C}$ be the part of $\partial E^+_{\text{opt}}$ between $(\varepsilon, \alpha_1(\varepsilon))$ and $(-\delta, \alpha_2(-\delta))$ and let $\widehat{\mathcal{C}}$ be the segment joining these two points. Let $E^{++}_{\text{opt}}$ be such that $\partial E^{++}_{\text{opt}} = (\partial E^+_{\text{opt}}\backslash\mathcal{C}) \cup \widehat{\mathcal{C}}$. Then we have

$$\text{Per}(E^{++}_{\text{opt}}) - \text{Per}(E^+_{\text{opt}}) = \varepsilon + \delta - \int_0^{\varepsilon} \sqrt{1 + \alpha_1'(s)^2}\, \mathrm{d}s - \int_{-\delta}^{0} \sqrt{1 + \alpha_2'(s)^2}\, \mathrm{d}s.$$

Thanks to the convexity of $E^+_{\text{opt}}$ (see Proposition 3.4), we have that $|\alpha_1'(s)|, |\alpha_2'(s)| \geq c > 0$, for $s$ small enough. Hence, we get that

$$\text{Per}(E^{++}_{\text{opt}}) - \text{Per}(E^+_{\text{opt}}) \leq (1 - \sqrt{1 + c^2})(\varepsilon + \delta).$$

On the other hand, let $\gamma_{\text{opt}}$ be an optimal transport plan for $W(E^+_{\text{opt}}, E^-_{\text{opt}})$. Then we have

$$\begin{aligned} W(E^{++}_{\text{opt}}, E^-_{\text{opt}}) - W(E^+_{\text{opt}}, E^-_{\text{opt}}) &\leq \int_{(E^+_{\text{opt}}\backslash E^{++}_{\text{opt}}) \times \partial\Omega} g^+(y)\, \mathrm{d}\gamma_{\text{opt}} \\ &= \int_{(E^+_{\text{opt}}\backslash E^{++}_{\text{opt}})} g^+(P(x))\, \mathrm{d}v^+_{\text{opt}}(x) \\ &\leq \|g^+\|_{L^\infty} \|v^+_{\text{opt}}\|_{L^p} |E^+_{opt}\backslash E^{++}_{\text{opt}}|^{\frac{1}{q}}. \end{aligned}$$

Yet,

$$|E^+_{\text{opt}}\backslash E^{++}_{\text{opt}}| = (\varepsilon + \delta)\alpha_1(\varepsilon) - \int_0^{\varepsilon} \alpha_1(s)\, \mathrm{d}s - \int_{-\delta}^{0} \alpha_2(s)\, \mathrm{d}s \leq C(\varepsilon + \delta)^2.$$

Consequently, we get

$$\begin{aligned} W(E^{++}_{\text{opt}}, E^-_{\text{opt}}) + \lambda^+ \text{Per}(E^{++}_{\text{opt}}) - W(E^+_{\text{opt}}, E^-_{\text{opt}}) - \lambda^+ \text{Per}(E^+_{\text{opt}}) \leq \\ \left[\lambda^+(1 - \sqrt{1 + c^2}) + \|g^+\|_{L^\infty} \|v^+_{\text{opt}}\|_{L^p} C(\varepsilon + \delta)^{\frac{2}{q}-1}\right](\varepsilon + \delta), \end{aligned}$$

which is a contradiction for $\varepsilon$, $\delta$ small enough, as $(E^+_{\text{opt}}, E^-_{\text{opt}})$ is a minimizer for problem (2.1) and $p > 2$. ∎

https://doi.org/10.4153/S0008439520000788 Published online by Cambridge University Press

***Proposition 3.10*** *Assume $d = 2$, $g^{\pm} \geq c > 0$, and $f^{\pm}$ is continuous on* $\mathrm{spt}(f^{\pm})$. *Let $v^{\pm}_{\mathrm{opt}}$ the mass to be exported/imported and let $E^{\pm}_{\mathrm{opt}}$ be the optimal free export/import region. Then $E^{\pm}_{\mathrm{opt}}$ is strictly convex in $\{v^{\pm}_{\mathrm{opt}} > 0\}$. In particular, if $f^{\pm}$ is bounded from below, then $E^{\pm}_{\mathrm{opt}}$ is uniformly convex inside* $\mathrm{spt}(v^{\pm}_{\mathrm{opt}})$.

**Proof** Let $\gamma_{\mathrm{opt}}$ be an optimal transport plan for $W(E^{+}_{\mathrm{opt}}, E^{-}_{\mathrm{opt}})$ such that $v^{+}_{\mathrm{opt}} = (\Pi_x)_{\#}(\gamma_{\mathrm{opt}} \cdot 1_{\Omega\times\partial\Omega})$. It is easy to see that the set $E^{+}_{\mathrm{opt}}$ minimizes

$$\min_{E}\left\{\min\left\{\int_{\Omega\times\Omega}|x-y|\mathrm{d}\gamma+\int_{(\Omega\setminus E)\times\partial\Omega}g^{+}(y)\mathrm{d}\gamma:(\Pi_x)_{\#}\gamma = v^{+}_{\mathrm{opt}},\ (\Pi_y)_{\#}\gamma\subset\partial\Omega\right\}+\lambda^{+}\,\mathrm{Per}(E)\right\}.$$

On the other hand, from Proposition 2.2, the transport plan $\gamma^{ib}_{\mathrm{opt}} := (Id, T_E)_{\#}v^{+}_{opt}$, where the map $T_E$ is defined as follows

$$T_E(x) = \begin{cases} P(x) := \mathrm{argmin}\{|x-y| : y\in\partial\Omega\}, & \text{if } x\in E,\\ T^{+}(x) := \mathrm{argmin}\{|x-y| + g^{+}(y) : y\in\partial\Omega\}, & \text{if } x\in\Omega\setminus E,\end{cases}$$

is a minimizer for the problem

$$\min\left\{\int_{\Omega\times\Omega}|x-y|\,\mathrm{d}\gamma+\int_{(\Omega\setminus E)\times\partial\Omega}g^{+}(y)\,\mathrm{d}\gamma:(\Pi_x)_{\#}\gamma = v^{+}_{\mathrm{opt}},\ \mathrm{spt}((\Pi_y)_{\#}\gamma)\subset\partial\Omega\right\}.$$

So, we infer that $E^{+}_{\mathrm{opt}}$ minimizes

$$\min_{E}\left\{\int_{\Omega}|x-T_E(x)|\,\mathrm{d}v^{+}_{\mathrm{opt}}(x)+\int_{\Omega\setminus E}g^{+}(T_E(x))\,\mathrm{d}v^{+}_{\mathrm{opt}}(x)+\lambda^{+}\,\mathrm{Per}(E)\right\}.$$

Fix $x_0\in\partial E^{+}_{opt}$. Let $\mathcal{C}$ be a part of $\partial E^{+}_{\mathrm{opt}}$ around $x_0$ and let $x_1 := (s_1, t_1)$, $x_2 := (s_2, t_2)$ be the endpoints of $\mathcal{C}$. Assume that $\mathcal{C}$ is the graph of a function $\alpha_{\mathrm{opt}}$. Then we see that $\alpha_{\mathrm{opt}}$ minimizes the following problem:

$$\min\left\{\int_{s_1}^{s_2}\int_{0}^{\alpha(s)}u(s,t)\,v^{+}_{opt}(s,t)\,\mathrm{d}t\,\mathrm{d}s+\lambda^{+}\int_{s_1}^{s_2}\sqrt{1+\alpha'(s)^2}\,\mathrm{d}s:\alpha(s_1) = t_1,\ \alpha(s_2) = t_2\right\},$$

where

$$u(s,t) := |(s,t) - T^{+}(s,t)| + g^{+}(T^{+}(s,t)) - |(s,t) - P(s,t)|.$$

https://doi.org/10.4153/S0008439520000788 Published online by Cambridge University Press

From the optimality conditions on $\alpha_{\text{opt}}$ and thanks to the continuity of $v^+_{\text{opt}}$ on $\text{spt}(v^+_{\text{opt}})$ (see Proposition 2.3), we get that

$$\left[\frac{\alpha'_{opt}(s)}{\sqrt{1+\alpha'_{opt}(s)^2}}\right]' = \frac{1}{\lambda^+}\,u\big(s,\alpha_{\text{opt}}(s)\big)\,v^+_{\text{opt}}\big(s,\alpha_{\text{opt}}(s)\big).$$

This implies that

$$\kappa(s_0) = \frac{1}{\lambda^+}u\big(s_0,\alpha_{opt}(s_0)\big)\,v^+_{\text{opt}}\big(s_0,\alpha_{\text{opt}}(s_0)\big),$$

where $\kappa(s_0)$ denotes the curvature of $\partial E^+_{\text{opt}}$ at $x_0 := (s_0,\alpha_{\text{opt}}(s_0))$. Yet, we have $u(s,t) \geq g^+(T(s,t)) \geq c > 0$. This concludes the proof. ∎

***Corollary 3.11*** *Assume $d = 2$ and $f^\pm$ is continuous on $\text{spt}(f^\pm)$. Then, the optimal free export/import region $E^\pm_{opt}$ is $C^2$ in the interior of $\text{spt}(v^\pm_{opt})$ and $C^{1,1}$ in the interior of $\Omega$. Moreover, $E^\pm_{opt}$ is $C^{1,1}$ provided that $\partial\Omega$ is $C^{1,1}$.*

**Proof** This follows from Propositions 3.4 and 3.9, the continuity of $v^\pm_{\text{opt}}$ (see Proposition 2.3) and $u$, and the fact that the curvature $\kappa$ of $\partial E^+_{\text{opt}}$ in the interior of $\Omega$ satisfies (see Proposition 3.10)

$$\kappa(s) = \frac{1}{\lambda^+}u\big(s,\alpha_{\text{opt}}(s)\big)\,v^+_{\text{opt}}\big(s,\alpha_{\text{opt}}(s)\big).$$

Now, assume that $x_0 \in \partial E^+_{\text{opt}} \cap \partial\Omega$. Similarly to Proposition 3.10, we denote by $\mathcal{C}$ the arc of $\partial E^+_{\text{opt}}$ around $x_0$ and by $x_1 := (s_1,t_1)$, $x_2 := (s_2,t_2)$ the endpoints of $\mathcal{C}$. Assume that $\mathcal{C}$ (resp. $\partial\Omega$) is the graph of a function $\alpha_{\text{opt}}$ (resp. $\psi$). Then we have that $\alpha_{\text{opt}}$ solves

$$\min\left\{\int_{s_1}^{s_2}\int_0^{\alpha(s)} u(s,t)\,v^+_{\text{opt}}(s,t)\,\mathrm{d}t\mathrm{d}s + \lambda^+\int_{s_1}^{s_2}\sqrt{1+\alpha'(s)^2}\mathrm{d}s : \alpha \geq \psi,\ \alpha(s_1) = t_1,\ \alpha(s_2) = t_2\right\}.$$

From the optimality conditions on $\alpha_{\text{opt}}$, we get

$$\kappa(s) \leq \frac{1}{\lambda^+}u(s,\alpha_{\text{opt}}(s))\,v^+_{\text{opt}}(s,\alpha_{\text{opt}}(s)).$$

As $\partial\Omega$ is $C^{1,1}$, it is then clear that $\kappa$ is bounded from below as well. This concludes the proof. ∎

***Corollary 3.12*** *Assume that $d = 2$ and $f^\pm$ is $C^{0,\alpha}$ on $\text{spt}(f^\pm)$, with $\alpha \in (0,1)$. Then the optimal free export/import region $E^\pm_{\text{opt}}$ is $C^{2,\alpha}$ in the interior of $\text{spt}(v^\pm_{\text{opt}})$. In particular, $E^\pm_{\text{opt}}$ is $C^{2,1}$ inside $\text{spt}(v^\pm_{\text{opt}})$ as soon as $f^\pm$ is Lipschitz on $\text{spt}(f^\pm)$.*

***Remark 3.4*** We see that if $f^\pm$ is $C^{k,\alpha}$ on $\text{spt}(f^\pm)$ and $g^\pm = c^\pm > 0$, then the optimal free export/import set $E^\pm_{\text{opt}}$ is $C^{k+2,\alpha}$ in the interior of $\text{spt}(v^\pm_{\text{opt}})$.

https://doi.org/10.4153/S0008439520000788 Published online by Cambridge University Press

***Remark 3.5*** It is not clear how to adapt Proposition 3.9 to the case $d > 2$. On the other hand, we know that any part $E \subset \partial E^+_{\text{opt}}$ is a solution of the variational problem

$$\min\left\{\int_{B(\varepsilon)}\int_0^{\alpha(\bar{y})} u(\bar{y}, y_d)v^+_{opt}(\bar{y}, y_d)\mathrm{d}y_d\,\mathrm{d}\bar{y} + \lambda^+\int_{B(\varepsilon)}\sqrt{1+|\nabla\alpha(\bar{y})|^2}\mathrm{d}\bar{y} : Id\times\alpha_{|\partial B(\varepsilon)} = \partial E\right\}.$$

From the optimality conditions on $\alpha_{\text{opt}}$, we get that $\alpha_{\text{opt}}$ satisfies the following equation:

$$\nabla\cdot\left[\frac{\nabla\alpha_{opt}(\bar{y})}{\sqrt{1+|\nabla\alpha_{opt}(\bar{y})|^2}}\right] = \frac{1}{\lambda^+}\,u\big(\bar{y},\alpha_{\text{opt}}(\bar{y})\big)\,v^+_{\text{opt}}\big(\bar{y},\alpha_{\text{opt}}(\bar{y})\big)$$

or, equivalently,

$$\sum_{i,j} a_{i,j}\;\partial^2_{x_ix_j}\alpha_{\text{opt}} = \frac{1}{\lambda^+}\,u(\bar{y},\alpha_{\text{opt}}(\bar{y}))\,v^+_{\text{opt}}(\bar{y},\alpha_{\text{opt}}(\bar{y})),$$

where

$$a_{i,j} = \frac{\delta_{i,j}(1+|\nabla\alpha_{\text{opt}}(\bar{y})|^2) - \partial_{x_i}\alpha_{\text{opt}}\,\partial_{x_j}\alpha_{\text{opt}}}{(1+|\nabla\alpha_{\text{opt}}(\bar{y})|^2)^{\frac{3}{2}}}.$$

In fact, under some assumptions on $a_{i,j}$ (see, for instance, [6]), it is possible to prove a higher regularity on $\alpha_{\text{opt}}$. But it is not clear if the coefficients $a_{i,j}$ satisfy the required assumptions or not, and it is even unclear whether, in higher dimensions $d > 2$, we can prove that the optimal free export/import region $E^{\pm}_{\text{opt}}$ is smooth, as $E^{\pm}_{\text{opt}}$ may not be, for instance, $\Omega$-convex when $d > 2$.

The following example shows that, in general, the optimal free export set $E^+_{opt}$ is not $C^2$ on the boundary of $\text{spt}(f)$ if $f \notin C(\Omega)$.

***Example 3.12.1*** Let $\Omega = \bar{B}(0,2)$ and let $f$ be a nonnegative density such that $f = 1$ on $[-1,1]\times[-1,1]$. Assume $g = 1$ on $\partial\Omega$. Fix $\lambda > 0$, so that problem (1.4) is equivalent to minimizing

$$\min\big\{\lambda\ \text{Per}(E) - |E\cap([-1,1]\times[-1,1])| : E\subset\Omega\big\}.$$

Let $\kappa$ be the curvature of $\partial\Omega$; then we know that $\kappa = \frac{1}{\lambda}$ on $]-1,1[\times]-1,1[$. If $\lambda < \frac{1}{2}$, then it is clear that the optimal set $E_{\text{opt}} \neq \emptyset$. Yet we know that $E_{\text{opt}}$ is $C^1$. This implies that there is part of $E_{\text{opt}}$ inside $]-1,1[\times]-1,1[$ and the curvature of this part is $\frac{1}{\lambda}$. However, we can see easily that $E_{\text{opt}}\cap\partial(]-1,1[\times]-1,1[) \neq \emptyset$. Consequently, $\partial E_{\text{opt}}$ is not $C^2$.

## References

[1] G. Buttazzo, G. Carlier, and S. Guarino Lo Bianco, *Optimal regions for congested transport*. ESAIM Math. Model. Numer. Anal. 49(2015), 1607–1619. http://dx.doi.org/10.1051/m2an/2015022

https://doi.org/10.4153/S0008439520000788 Published online by Cambridge University Press

[2] S. Dweik, *Optimal transportation with boundary costs and summability estimates on the transport density*. J. Convex Anal. **25**(2018), 135–160.
[3] S. Dweik, *Weighted Beckmann problem with boundary costs*. Quart. Appl. Math. **76**(2018), 601–609. http://dx.doi.org/10.1090/qam/1512
[4] S. Dweik and F. Santambrogio, *Summability estimates on transport densities with Dirichlet regions on the boundary via symmetrization techniques*. ESAIM Control Optim. Calc. Var. **24**(2018), 1167–1180. http://dx.doi.org/10.1051/cocv/2017018
[5] S. Dweik, N. Ghoussoub, and A. Z. Palmer, *Optimal controlled transports with free end times subject to import/export tariffs*. J. Dyn. Control Syst. **26**(2020), 481–507. http://dx.doi.org/10.1007/s10883-019-09458-1
[6] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*. 2nd ed., Grundlehren der Mathematischen Wissenschaften, 224, Springer-Verlag, New York/Berlin, 1983. http://dx.doi.org/10.1007/978-3-642-61798-0
[7] A. Henrot and M. Pierre, *Variation et optimisation de formes. Une analyse géométrique.* Mathématiques & Applications, 48, Springer-Verlag, Berlin, 2005. http://dx.doi.org/10.1007/3-540-37689-5
[8] J. M. Mazon, J. Rossi, and J. Toledo, *An optimal transportation problem with a cost given by the euclidean distance plus import/export taxes on the boundary*. Rev. Mat. Iberoam. **30**(2014), no. 1, 277–308. http://dx.doi.org/10.4171/RMI/778
[9] F. Santambrogio, *Optimal transport for applied mathematicians*. In: Progress in nonlinear differential equations and their applications, 87, Birkhäuser Basel, 2015. http://dx.doi.org/10.1007/978-3-319-20828-2

*Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada*
*e-mail*: dweik@math.ubc.ca

https://doi.org/10.4153/S0008439520000788 Published online by Cambridge University Press