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Expected Value Expansions in Random Subgraphs with Applications to Network Reliability

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Subgraph expansions are commonly used in the analysis of reliability measures of a failure-prone graph. We show that these expansions are special cases of a general result on the expected value of a random variable defined on a partially ordered set; when applied to random subgraphs, the general result defines a natural association between graph functions. As applications, we consider several graph invariants that measure the connectivity of a graph: the number of connected vertex sets of size k, the number of components of size k, and the total number of components. The expected values of these invariants on a random subgraph are global performance measures that generalize the ones commonly studied. Explicit results are obtained for trees, cycles, and complete graphs. Graphs which optimize these performance measures over a given class of graphs are studied

1. Introduction

A random subgraph of a graph G is defined by imposing a probability measure on the subgraphs of G. Any real-valued function defined on the subgraphs of G is then a random variable, in which case the expected value may be a parameter of interest. Most measures of the reliability of a failure-prone graph fit into this general context. In the network reliability model, G is used to represent either a communications network or a network of components; the vertices of G represent either sites or terminals and the edges of G represent either communications links or components. A general probability measure, interpreted in the reliability context, allows both edge and vertex failures and allows statistical dependence between element failures. However, it is usually assumed that only the edges (or only the vertices) fail, and it is also assumed that elements of a given type (either edges or vertices) fail independently with the same probability. These restrictive assumptions have been overused in the literature, since they are sometimes not realistic and sometimes not necessary.

The most studied performance measure for a failure-prone graph is the probability that some distinguished vertex set U of the graph remains connected after the element failures.

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This measure is called the *U-terminal* reliability; references [5], [9], and [19] are good surveys of this subject. The special cases which have received the most attention are when the set consist of two vertices, the *two-terminal* reliability, and when the set consists of all of the vertices, the *all-terminal* reliability. The all-terminal reliability is a global reliability measure. Another global performance measure is the *pair-connected* reliability or the *resilience* of the graph: the expected number of pairs of vertices that remain connected after the element failures (see for example [1]–[4], [8], [10], and the references in these papers). There have been two main lines of investigation in network reliability. One involves computational issues such as complexity questions, explicit formulas, reduction and factoring results. The other line involves the study of optimally reliable graphs within a given class of graphs.

In many cases, the analysis of performance measures involves expansions over subgraphs of the failure-prone graph, in a certain way. One of the main points of this paper is to show that these subgraph expansions are all special cases of one very simple, general result in which the expected value of a graph function can be expressed in terms of another graph function. The natural home for this result is the Möbius inversion theory of partially ordered sets, and it is interesting beyond the application to reliability because it defines a natural correspondence between graph functions. As applications, we will consider several graph invariants that describe the connectivity of a graph: the number of connected vertex sets of size k, the number of components of size k, and the total number of components. The expected values of these invariants on a failure-prone graph are performance measures that generalize some considered in the literature. We will obtain a number of results relating these graph invariants, their associated invariants, and their expected values. We will give some explicit results for trees, cycles, and complete graphs. Finally, we will consider optimal graphs for these performance measures in the model in which elements fail independently and elements of a given type fail with the same probability. Our analysis will provide insight into the fact that, for many global performance measures, there is no uniformly optimal graph.

2. Probability measures on partially ordered sets

In this section, we will review the partial sum expansion for a real-valued function defined on a finite partially ordered set and the Möbius inversion formula. This expansion is useful for representing the expected value of a random variable when the poset is endowed with a probability measure.

Definition. Let S be a finite set, partially ordered by a relation $\preceq$, and having a zero element 0 and a unity element 1 (thus $0 \preceq s \preceq 1$ for every $s \in S$). For $s, t \in S$ with $s \preceq t$, the subsets $[s,t]$, (s,t), $[s,t)$, and $(s,t]$ are defined in the usual way. For example, $[s,t) = \{u \in S : s \preceq u \prec t\}$.

There is no loss in generality in assuming the existence of zero and unity elements, since they can always be appended. The following results are standard in the theory of finite posets (see [6]).

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Lemma 2.1. *If $X : S \to \mathbb{R}$, then there exists a unique function $X^- : S \to \mathbb{R}$ such that*

$$X(t) = \sum_{s \in [0,t]} X^-(s) \text{ for } t \in S. \tag{2.1}$$

□

Note that (2.1) determines X^- recursively since

$$X^-(t) = X(t) - \sum_{s \in [0,t)} X^-(s) \text{ for } t \in S. \tag{2.2}$$

The operator that takes X into X^- is linear and is a *downward difference operator.*

Example 2.1. *Let S be a chain (that is, $\preceq$ is a total order) whose elements are $s_0 \prec s_1 \prec \cdots \prec s_n$, then $X^-(s_0) = X(s_0)$ and $X^-(s_i) = X(s_i) - X(s_{i-1})$ for $i = 1, 2 \ldots, n$. In this case, the downward difference operator is the ordinary backward difference operator.*

Definition. The *Möbius function* μ is defined inductively on the set of ordered pairs $\{(s,t) : s, t \in S, s \preceq t\}$ as follows:

$$\begin{aligned} \mu(s,s) &= 1 \text{ for } s \in S, \\ \mu(r,t) &= -\sum_{s \in [r,t)} \mu(r,s) \text{ for } r, t \in S \text{ with } r \prec t. \end{aligned}$$

Lemma 2.2. *If $X : S \to \mathbb{R}$ then*

$$X^-(t) = \sum_{s \in [0,t]} \mu(s,t) X(s) \text{ for } t \in S. \tag{2.2}$$

□

Lemma 2.2 gives the *Möbius inversion formula.*

If P is a probability measure on (the subsets of) S, then of course P is completely determined by its probability mass function $s \mapsto P\{s\}$. Let E denote the expected value operator associated with P. On the probability space (S, P), a function $X : S \to \mathbb{R}$ can be thought of as a random variable and the following lemma gives a simple expansion of the expected value of X in terms of X^-. This expansion has been derived in a number of special cases in network reliability models and is the basis for a good deal of reliability analysis. However, the simplicity and generality of the result seems to have been overlooked.

Theorem 2.1. *If $X : S \to \mathbb{R}$ and P is a probability measure on S, then*

$$E(X) = \sum_{s \in S} X^-(s) P[s, 1]. \tag{2.3}$$

Conversely, if $X, Y : S \to \mathbb{R}$ and

$$E(X) = \sum_{s \in S} Y(s) P[s, 1] \tag{2.4}$$

for every probability measure P on S, then $Y = X^-$.

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Proof. If $X : S \to \mathbb{R}$ and P is a probability measure on S then, from the definition of expected value and from Lemma 2.1,

$$\begin{aligned} E(X) &= \sum_{t \in S} X(t) P\{t\} = \sum_{t \in S} \sum_{s \in [0,t]} X^-(s) P\{t\} \\ &= \sum_{s \in S} \sum_{t \in [s,1]} X^-(s) P\{t\} = \sum_{s \in S} X^-(s) P[s,1]. \end{aligned}$$

Hence (2.3) holds. Conversely, suppose $X, Y : S \to \mathbb{R}$ and (2.4) holds for every probability measure P on S. If $t \in S$ and P is point-mass at t then (2.4) gives

$$X(t) = \sum_{s \in [0,t]} Y(s).$$

Hence by Lemma 2.1, $Y = X^-$. □

Similarly, for any $X : S \to \mathbb{R}$, there exists a unique $X^+ : S \to \mathbb{R}$ such that

$$X(s) = \sum_{t \in [s,1]} X^+(t) \text{ for } s \in S, \tag{2.5}$$

$$X^+(s) = X(s) - \sum_{t \in (s,1]} X^+(t) \text{ for } s \in S. \tag{2.6}$$

The operator that takes X into X^+ is also linear and is an *upward difference* operator. Of course the upward difference X^+ can also be thought of as a downward difference relative to the reverse of the given partial order, so any result for downward differences will have an analogue for upward differences. In particular, an analogous Möbius function and inversion formula could be given for X^+. If P is a probability measure on S then

$$E(X) = \sum_{t \in S} X^+(t) P[0,t]. \tag{2.7}$$

Moreover, X^+ is the only function satisfying (2.7) for every probability measure P.

3. Graph functions and random subgraphs

We will consider undirected graphs without loops, but perhaps with multiple edges. If G is a graph, let $V(G)$ denote the vertex set of G, $n(G)$ the number of vertices in G, and $m(G)$ the number of edges in G. Let $\emptyset$ denote the 'empty' graph which has no vertices or edges. If G is a subgraph of H, we write $G \preceq H$ and the segments $[G,H]$, $[G,H)$, $(G,H]$ and (G,H) are defined as in Section 2. If $G \preceq H$, let G_H denote the corresponding induced subgraph which contains all edges in H that connect vertices in $V(G)$. Reference [14] is a general reference on graph theory.

For any graph H, the subgraph relation $\preceq$ is a partial order on $[\emptyset, H]$, the collection of subgraphs of H, with $\emptyset$ as the zero element and H as the unity element. A random subgraph of a given graph H is described by a probability measure P_H defined on the sample space $[\emptyset, H]$. In the reliability context, a random subgraph of H is used to model a failure-prone version of H. Note that, in general, both edges and vertices can fail, with different probabilities, and element failures may be statistically dependent. If only edges

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can fail, then P_H is concentrated on *spanning* subgraphs of H and if only vertices can fail then P_H is concentrated on *induced* subgraphs of H.

Example 3.1. *Let H be a graph and let $p, \rho \in [0,1]$. For $G \preceq H$ define*

$$P_{H,p,\rho}(G) = p^{n(G)}(1-p)^{n(H)-n(G)}\rho^{m(G)}(1-\rho)^{m(G_H)-m(G)}. \tag{3.1}$$

In the reliability context, each vertex of H fails with probability $1-p$, each edge of H fails with probability $1-\rho$, and all element failures are independent. Thus, we will refer to this as the model with *independent, identically distributed* (IID) element failures; p and ρ are the parameters. Note that the segment $[G,H]$ is the event that all elements of G (and perhaps other elements) are operational and hence

$$P_{H,p,\rho}[G,H] = p^{n(G)}\rho^{m(G)}. \tag{3.2}$$

The segment $[\emptyset, G]$ is the event that each vertex in H that is not in G, and each edge in G_H that is not in G (and perhaps other elements) have failed, so

$$P_{H,p,\rho}[\emptyset, G] = (1-p)^{n(H)-n(G)}(1-\rho)^{m(G_H)-m(G)}. \tag{3.3}$$

If $p = 1$, we have the IID edge-failure model with parameter ρ, and if $\rho = 1$, we have the IID vertex-failure model with parameter p.

Suppose that H is a graph and $X : [\emptyset, H] \to \mathbb{R}$. The results of Section 2 apply, so, by (2.1) and (2.2),

$$X(G) = \sum_{F \in [\emptyset, G]} X^-(F) \text{ for } G \preceq H, \tag{3.4}$$

$$X^-(G) = X(G) - \sum_{F \in [\emptyset, G)} X^-(F) \text{ for } G \preceq H. \tag{3.5}$$

Note that $X^-(G)$ depends only on the subgraphs of G and hence is independent of H. In particular, if X is a graph invariant, then so is X^-, and thus the downward difference operator is a natural correspondence between graph invariants. Using standard results in [6], it is easy to determine the Möbius function for the subgraph relation. If $F \preceq G$ and if the vertices in $V(G) - V(F)$ are isolated, then

$$\mu(F,G) = (-1)^{m(G)-m(F)}(-1)^{n(G)-n(F)};$$

otherwise $\mu(F,G) = 0$. Thus, X^- can be given explicitly in terms of X using the Möbius inversion formula in Lemma 2.2.

In contrast, the definition of the upward difference X^+ depends on the underlying graph H. Thus (2.5) and (2.6) become

$$X(F) = \sum_{G \in [F,H]} X^+(G,H), \text{ for } F \preceq H, \tag{3.6}$$

$$X^+(F,H) = X(F) - \sum_{G \in (F,H]} X^+(G,H) \text{ for } F \preceq H. \tag{3.7}$$

The Möbius function corresponding to the upward difference would also depend on H.

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It is important to note that upward and downward differences depend on the partial order. For example, if the induced or spanning subgraph relation were used instead of the general subgraph relation, then the upward and downward difference of a function would change accordingly.

If P_H is a probability measure on $[\emptyset, H]$, then, by (2.3) and (2.7),

$$E_H(X) = \sum_{G \in [\emptyset, H]} X^-(G) P_H[G, H] = \sum_{G \in [\emptyset, H]} X^+(G, H) P_H[\emptyset, G]. \tag{3.8}$$

4. Graph invariants and reliability measures

In this section we will study a number of graph functions which in some sense measure the connectivity of a graph. When these functions are viewed as random variables defined for a given failure-prone graph, their expected values give performance measures.

If G is a graph, let $c(G) = m(G) - n(G) + 1$. If G is connected then $c(G)$, the *cyclomatic number* of G, is the dimension of the cycle space of G. Let $e(G)$ denote the number of end-points of G (vertices of degree 1). The path on $n \geqslant 1$ vertices will be denoted L_n, the complete graph on $n \geqslant 1$ vertices by K_n, and the cycle on $n \geqslant 3$ vertices by C_n. Also, we will adopt the usual conventions for binomial coefficients: $\binom{a}{b} = 0$ if $b < 0$ or $b > a$.

Definition. Let H be a graph and $U \subseteq V(H)$. For $G \preceq H$, let $I_U(G) = 1$ if $U \subseteq V(G)$ and the vertices in U are connected in G, and let $I_U(G) = 0$ otherwise.

On the random subgraph $\big([\emptyset, H], P_H\big)$, I_U is the indicator random variable that takes the value 1 if U is a connected set of vertices in the random subgraph and takes the value 0 otherwise. In particular, if $u \in V$, then $I_{\{u\}}$ is the indicator variable that takes the value 1 if u has not failed. If $u, v \in V$ with $u \neq v$ then $I_{\{u,v\}}$ is the indicator variable that takes the value 1 if u and v are connected vertices in the random subgraph of H. Generally, if U has at least two elements and U_2 denotes the collection of two-element subsets of U then

$$I_U = \prod_{\{u,v\} \in U_2} I_{\{u,v\}},$$

although of course this is not usually a minimal factoring. Note that $E_H(I_U) = P_H(I_U = 1)$ is the probability that U is a connected set of vertices in the random subgraph of H. This is the U-terminal reliability of H. In particular, $E_H(I_{\{u,v\}})$ is the two-terminal reliability of H, with terminals u and v, and $E_H(I_{V(H)})$ is the all-terminal reliability of H.

Definition. For $k = 1, 2, \ldots$, and for any graph G, let $X_k(G)$ denote the number of connected vertex sets of size k in G; let $Y_k(G)$ denote the number of components of size k in G; and let $Z(G)$ denote the total number of components in G.

Note that X_k, Y_k, and Z are graph invariants, since no distinguished vertices are referenced. On the random subgraph $([\emptyset, H], P_H)$, X_k, Y_k, and Z are random variables that give, respectively, the number of connected vertex sets of size k, the number of

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components of size k, and the total number of components in the random subgraph. Note that $X_1(G) = n(G)$, the number of vertices in G. On $([\emptyset, H], P_H)$, X_1 is the number of surviving vertices in the random subgraph of H, so for the IID model of Example 3.1, X_1 has the binomial distribution with parameters $n(H)$ and p. On $([\emptyset, H], P_H)$, X_2 is the number of pairs of vertices which are connected in the random subgraph of H, so $E_H(X_2)$ is the pair-connected reliability of H. Finally, $X_{n(H)} = I_{V(H)}$ on $[\emptyset, H]$ so $E_H(X_{n(H)})$ is the all-terminal reliability of H. In general, $E_H(X_k)$, $E_H(Y_k)$, and $E_H(Z)$ are global performance measures of of H, although only the special cases mentioned have received much attention. Of course, the general results in Section 3 and in particular (3.4)–(3.8) hold for these graph functions.

There are simple, linear relationships between these graph functions and hence also their upward differences, downward differences, and expected values on $([\emptyset, H])$. These relationships are analogous to Sylvester's formula and the sieve formula in combinatorics (see [6]).

Theorem 4.1. *For $G \preceq H$ and $k = 1, 2, \ldots,$*

$$X_k(G) = \sum \{I_U(G) : U \subseteq V(G), |U| = k\}, \tag{4.1}$$

$$X_k(G) = \sum_{j=k}^{n(G)} \binom{j}{k} Y_j(G), \tag{4.2}$$

$$Y_k(G) = \sum_{j=k}^{n(G)} (-1)^{j-k} \binom{j}{k} X_j(G), \tag{4.3}$$

$$Z(G) = \sum_{j=1}^{n(G)} Y_j(G) = \sum_{j=1}^{n(G)} (-1)^{j+1} X_j(G). \tag{4.4}$$

Proof. Equations (4.1), (4.2) and the first equation in (4.4) follow immediately from the definitions. Equation (4.3) follows from (4.2), and the second equation in (4.4) follows from (4.3). □

Corollary 4.1. *Equations* (4.1)–(4.4) *hold with each graph function replaced by its expected value on $([\emptyset, H], P_H)$, and with $V(G)$ replaced by $V(H)$ in* (4.1) *and $n(G)$ replaced by $n(H)$ in* (4.2)–(4.4).

Proof. These results follow directly from Theorem 4.1 and the linearity of expected value. □

Corollary 4.2. *Equations* (4.1)–(4.4) *hold with each graph function replaced by its downward difference. Similarly, equations* (4.1)–(4.4) *hold with each graph function replaced by its upward difference defined relative to H.* □

One of the goals of this section is to study the downward differences of these graph functions because they are themselves measures of graph connectivity and, as noted above,

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their definitions do not depend on the underlying graph H. Also, the downward differences are fundamental in the study of the expected values that are performance measures.

In the study of U-terminal reliability, Satyanarayana [16, 17] showed that (in our notation)

$$E_H(I_U) = \sum_{G \in [\emptyset, H]} D_U(G) P_H[G, H]. \tag{4.5}$$

He called the coefficient $D_U(G)$ the *U-domination* of G and was able to describe it in graph-theoretic terms. However, since (4.5) holds for all probability measures P_H, it follows from Theorem 2.1 that $D_U = I_U^-$, and hence this result is simply a special case of the downward difference expansion of expected value (3.8). Moreover, in Theorem 4.2 below we give a very simple combinatorial proof of Satyanarayana's graph-theoretic interpretation of the coefficient. This leads in turn to an interpretation of X_k^-.

Definition. A tree T is a *U-tree* if $U \subseteq V(T)$ and T is a minimal graph connecting the vertices in U. A *U-formation* of a graph G is a set of U-trees whose union is G; the formation is *odd* if the number of U-trees in the set is odd and *even* if the number of U-trees in the set is even.

Note that a tree T is a U-tree if and only if $U \subseteq V(T)$ and U contains all end-points of T.

Theorem 4.2. *If G is a graph then I_U^- is the number of odd U-formations of G minus the number of even U-formations of G.*

Proof. Let $N_U(G)$ denote the number of odd U-formations of G minus the number of even U-formations of G. If G contains n U-trees then

$$\sum_{F \in [\emptyset, G]} N_U(F) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k}.$$

The second sum is 0 if $n = 0$ and 1 if $n \geqslant 1$. In either case it follows that

$$\sum_{F \in [\emptyset, G]} N_U(F) = I_U(G).$$

By the uniqueness of the downward difference (Lemma 2.1), it follows that $N_U = I_U^-$. □

The domination coefficients $I_U^-(G)$ have been described in somewhat more topological terms in [17]; the sign of $I_U^-(G)$ is $(-1)^{c(G)}$ and the magnitude of $I_U^-(G)$ is the number of orientations of G that can be written as acyclic unions of oriented U-trees in a certain sense.

There have been a number of recent papers on the characterization and computation of domination coefficients, both in this setting and for more general reliability structures. See, for example, [7], [12], [18], [20], [21], and the references in these papers. In [19], the Möbius inversion formula is used for the computation of two-terminal reliability when elements fail independently, but with different probabilities.

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By combining Theorem 4.2 and Corollary 4.2, we can now obtain a graph-theoretic interpretation of the invariant X_k^-.

Definition. For $k \geqslant 1$, a *k-formation* of a graph G is a U-formation with $|U| = k$.

Note that two different k-formations of G may consist of the same set of trees, but corresponding to different subsets U.

Corollary 4.3. *For $k = 1, 2, \ldots$ and any graph G, $X_k^-(G)$ is the number of odd k-formations minus the number of even k-formations of G.* □

From the remarks following Theorem 4.2, note that the sign of $X_k^-(G)$ is the number of orientations of G that can be represented as acyclic unions of oriented U-trees (in the precise sense given in [17]), summed over all U with $|U| = k$.

From previous results on domination and from Corollaries 4.2 and 4.3, the values of X_k^-, Y_k^-, and Z^- can be obtained for some very simple graphs. Additional computational results for X_2^-, which correspond to pair-connected reliability, will be given in the next section.

Theorem 4.3. *For a graph G and for $k = 1, 2, \ldots$, $X_k^-(G) \neq 0$ if and only if there exists $U \subseteq V(G)$ such that $|U| = k$ and every edge of G lies on some U-tree in G. Equivalently, $X_k^-(G) \neq 0$ if and only if G is connected and there exists $U \subseteq V(G)$ such that $|U| = k$ and, for any cut-point v of G, every component of $G - v$ has a vertex in U.*

Proof. By Corollary 4.2, equation (4.1) holds with I_U replaced by I_U^- and X_k by X_k^-. But I_U^- is the U-domination coefficient, so the theorem follows immediately from results in [18]. □

Note in particular that $X_1^-(K_1) = 1$ and $X_1^-(G) = 0$ for any other graph G. By Corollary 4.2, equations (4.2) and (4.3) hold with the invariants replaced by their downward differences. It follows that if $X_j^-(G) = 0$ for $j \geqslant k$ then $Y_k^-(G) = 0$, and if $X_j^-(G) = 0$ for $j \geqslant 1$ then $Z^-(G) = 0$. In particular, $Y_k^-(G) = 0$ and $Z^-(G) = 0$ if G is disconnected.

Theorem 4.4. *If T is a tree and $k = 1, 2, \ldots$ then*

$$X_k^-(T) = \binom{n(T) - e(T)}{k - e(T)}, \tag{4.6}$$

$$Y_k^-(T) = (-1)^{n(T)-k}\binom{e(T)}{n(T) - k}, \tag{4.7}$$

$$Z^-(T) = \begin{cases} 1 & \text{if } T = L_1, \\ -1 & \text{if } T = L_2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.8}$$

Proof. Let T be a tree with n vertices and e end-points. Note that $X_k^-(T)$ is the number

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of ways in which T can be represented as a U-tree with $|U| = k$. As noted above, in order for T to be a U-tree, the end-points of T must be in U, but the remaining $k - e$ vertices of U can be chosen from the remaining $n - e$ vertices of T in any manner. Hence (4.6) holds. Next, by Corollary 4.2 and (4.6),

$$\begin{aligned} Y_k^-(T) &= \sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} \binom{n-e}{j-e} \\ &= \frac{(-1)^k}{k!} \sum_{i=(k-e)\vee 0}^{n-e} (-1)^{i+e} (i+e)_k \binom{n-e}{i}, \end{aligned} \tag{4.9}$$

where $\vee$ is the maximum operator and $(i+e)_k$ denotes the usual falling factorial. But

$$\begin{aligned} \sum_{i=0}^{n-e} (-1)^{i+e} \binom{n-e}{i} x^{i+e} &= (-x)^e (1-x)^{n-e} \\ &= \sum_{i=n-e}^{n} (-1)^n \binom{e}{i-n+e} (x-1)^i. \end{aligned} \tag{4.10}$$

Comparing the second sum in (4.9) and the first sum in (4.10), we see that the former is the kth derivative of the latter at $x = 1$. Using the second sum in (4.10) to compute the derivative gives (4.7). Finally, from Corollary 4.2 and (4.7),

$$Z^-(T) = \sum_{j=1}^{n} (-1)^{n-j} \binom{e}{n-j}. \tag{4.11}$$

If $n = 1$ and $e = 0$ (so that $T = L_1$) then the sum in (4.11) is 1. If $n = 2$ and $e = 2$ (so that $T = L_2$), the sum in (4.11) is -1. If $n \geqslant 3$ (and hence $e < n$), the sum is 0. □

From (4.7), note that $|Y_k^-(T)|$ is the number of vertex sets of T of size k containing all of the non-end-points. An interesting open problem is to give a graph-theoretic interpretation of the invariants Y_k^- and Z^-.

Theorem 4.5. *For the cycle C_n on $n \geqslant 3$ vertices,*

$$X_k^-(C_n) = -(k-1)\binom{n}{k}, \quad k = 1, 2, \ldots, \tag{4.12}$$

$$Y_k^-(C_n) = \begin{cases} 0, & k = 1, 2, \ldots, n-2, \\ n, & k = n-1, \\ 1-n, & k = n, \end{cases} \tag{4.13}$$

$$Z^-(C_n) = 1.$$

Proof. First note that (4.12) is correct if $k = 1$ or if $k > n$, so suppose that $2 \leqslant k \leqslant n$. Fix a cyclic orientation of C_n. A U-tree is a directed path from $u \in U$ to $v \in U$ such that the directed path from v to u meets no other vertices of U. Hence, if U is a vertex set of size k, then there are k U-trees. Any collection of two or more of these U-trees is a

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U-formation of C_n and therefore, by Corollary 4.3,

$$X_k^-(C_n) = \binom{n}{k} \sum_{j=2}^{k} (-1)^{j+1} \binom{k}{j} = -(k-1)\binom{n}{k}.$$

Next, by Corollary 4.2 and by (4.12),

$$Y_k^-(C_n) = \sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} \left[-(j-1)\binom{n}{j} \right],$$

which gives the result in (4.13). Finally, by Corollary 4.2 and (4.13), $Z^-(C_n) = n-(n-1) = 1$ □

Theorem 4.6. *For the complete graph K_n on $n \geqslant 2$ vertices,*

$$\begin{aligned} X_k^-(K_n) &= (-1)^{\binom{n-1}{2}} \binom{n}{k} (k-1)(n-2)!, \quad k = 1, 2, \ldots, \qquad (4.14) \\ Y_k^-(K_n) &= \begin{cases} 0, & k = 1, 2, \ldots, n-2, \\ (-1)^{\binom{n-1}{2}+1} n(n-2)!, & k = n-1, \\ (-1)^{\binom{n-1}{2}} (n-1)!, & k = n, \end{cases} \\ Z^-(K_n) &= (-1)^{\binom{n-1}{2}+1} (n-2)!. \end{aligned}$$

Proof. In [17] it is shown that, if U is a vertex set with k elements, then (in our notation)

$$I_U^-(K_n) = (-1)^{\binom{n-1}{2}} (k-1)(n-2)!.$$

It follows from Corollary 4.2 that (4.14) holds. Note that, interestingly,

$$X_k^-(K_n) = (-1)^{\binom{n-1}{2}+1} (n-2)! X_k^-(C_n),$$

and hence, by Corollary 4.2,

$$\begin{aligned} Y_k^-(K_n) &= (-1)^{\binom{n-1}{2}+1} (n-2)! Y_k^-(C_n), \\ Z^-(K_n) &= (-1)^{\binom{n-1}{2}+1} (n-2)! Z^-(C_n). \end{aligned}$$

The remaining results in Theorem 4.6 now follow immediately from Theorem 4.5. □

5. The IID model

In this section, we consider the IID probability model with parameters p and ρ, as defined in Example 3.1. The primary goal of this section is the study of the global performance measure $E_{H,p,\rho}(X_k)$. This generalizes, in two ways, the measures that have been most commonly studied. The expected value includes pair-connected reliability ($k = 2$) and all-terminal reliability ($k = n(H)$), and the probability model includes the IID edge-failure model ($p = 1$) and the IID vertex-failure model ($\rho = 1$).

Several authors have studied polynomial expansions of performance measures in the IID edge-failure model, and to a lesser extent in the IID vertex-failure model. These expansions are all special cases of the following general result for the expected value of

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a graph function over a random subgraph, which relates the coefficients in the expansion to the downward and upward differences of the graph function.

Theorem 5.1. *Let* $X : [\emptyset, H] \to \mathbb{R}$. *Then*

$$\begin{aligned} E_{H,p,\rho}(X) &= \sum_{i=0}^{n(H)} \sum_{j=0}^{m(H)} X^-(H,i,j) p^i \rho^j \\ &= \sum_{i=0}^{n(H)} \sum_{j=0}^{m(H)} X^+(H,i,j)(1-p)^i (1-\rho)^j, \end{aligned}$$

where

$$\begin{aligned} X^-(H,i,j) &= \sum \{X^-(G) : G \preceq H, n(G) = i, m(G) = j\}, \\ X^+(H,i,j) &= \sum \{X^+(G,H) : G \preceq H, n(G) = n(H) - i, m(G) = m(G_H) - j\}. \end{aligned}$$

Proof. These results follow from (3.2), (3.3), (3.8) and a summation interchange. □

Theorem 5.1 holds of course for the invariants X_k, Y_k, and Z. Using summation interchange and Corollary 4.2, it is simple to see that, for fixed i and j, the coefficients $I_U^-(G,i,j)$, $X_k^-(G,i,j)$, $Y_k^-(G,i,j)$ and $Z^-(G,i,j)$ are related in a way analogous to equations (4.1)–(4.4). A similar statement holds for the coefficients with + superscripts.

If $n(G) = n(H)$ then $G_H = H$, and therefore

$$X^+(H,0,0) = X^+(H,H) = X(H). \tag{5.1}$$

Also, from Theorem 5.1 and the binomial expansion, the sequences $X^-(H,i,j)$ and $X^+(H,i,j)$ for $i = 0,1,\ldots,n(H)$ and $j = 0,1\ldots,m(H)$ are inverse-pairs in the combinatorial sense [11], related by

$$X^-(H,i,j) = (-1)^{i+j} \sum_{k=i}^{n(H)} \sum_{l=j}^{m(H)} \binom{k}{i} \binom{l}{j} X^+(H,k,l).$$

Corollary 5.1. *If* $E_{H,1,\rho}(X)$ *is expanded as a polynomial in powers of* ρ, *then the coefficient of* ρ^j *is*

$$\sum \{X^-(G) : G \preceq H, m(G) = j\}.$$

If $E_{H,1,\rho}(X)$ *is expanded as a polynomial in powers of* $1-\rho$, *then the coefficient of* $(1-\rho)^j$ *is*

$$\sum \{X^+(G,H) : G \preceq H, n(G) = n(H), m(G) = m(H) - j\}.$$ □

Corollary 5.2. *If* $E_{H,p,1}(X)$ *is expanded as a polynomial in powers of* p, *then the coefficient of* p^i *is*

$$\sum \{X^-(G) : G \preceq H, n(G) = i\}.$$

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If $E_{H,p,1}(X)$ is expanded as a polynomial in powers of $1-p$, then the coefficient of $(1-p)^i$ is

$$\sum\{X^+(G,H) : G \preceq H, n(G) = n(H) - i, m(G) = m(G_H)\}. \qquad \square$$

Note that if $G \preceq H$ and $m(G) = m(H)$ then necessarily $n(G) = n(H)$ and $G = H$. Hence

$$X^-(H, n(H), m(H)) = X^-(H).$$

Thus, if $E_H(X)$ is known for (p, ρ) in a rectangle, then $X^-(H)$ can be recovered as the coefficient of $p^{n(H)}\rho^{m(H)}$. Similarly, in the IID edge-failure model ($p = 1$), $X^-(H)$ is the coefficient of $\rho^{m(H)}$.

Recall from Corollary 4.3 that the magnitude of $X_2^-(H)$ is the number of ways in which H can be represented as an oriented acyclic union of paths, and hence is an interesting invariant. Moreover, $X_2^-(H)$ is the coefficient of $\rho^{m(H)}$ in $E_{H,1,\rho}(X_2)$ (pair-connected reliability with IID edge failures). In [2], the pair-connected reliability was computed explicitly for certain classes of graphs which arise in applications, and hence X_2^- can be computed for these graphs, leading to the following results:

Example 5.1. *If W_{n+1} denotes the wheel on $n+1$ vertices (obtained by connecting each vertex of the cycle C_n by a new edge to a common, new vertex), then*

$$X_2^-(W_{n+1}) = (-1)^n \left(n^2 - n + n^2(n^2-1)/12\right).$$

If F_{n+1} denotes the fan on $n+1$ vertices (obtained by connecting each vertex of the path L_n by a new edge to a common, new vertex), then

$$X_2^-(F_{n+1}) = (-1)^{n-1} \left(n(n^2-1)/6 + n\right).$$

If D_{n+1} denotes the diamond on $n+1$ vertices (obtained by connecting each vertex of the star S_n on n vertices by a new edge to a common, new vertex), then

$$X_2^-(D_{n+1}) = (-1)^{n-1}(n^2 + n - 1).$$

Polynomial expansions of performance measures have been used to compare the measures for different graphs, and for the study of optimal graphs within a given class of graphs, primarily in the IID edge-failure model. The analysis usually hinges on a graph-theoretic interpretation of the coefficients in the expansion. We will now generalize some of the results that have been obtained by considering $E_{H,p,\rho}(X_k)$ in the IID model. The following two theorems identify some of the coefficients.

Theorem 5.2. *Let H be a graph and let $k = 1, 2, \ldots$. Then*

$$X_k^-(H,i,j) = Y_k^-(H,i,j) = 0 \text{ if } i < k \text{ or } j < i-1. \tag{5.2}$$

Also, $X_k^-(H,k,k-1) = Y_k^-(H,k,k-1)$ and the common value is the number of trees of order k in H.

Proof. If $n(G) < k$ then G cannot be represented as a union of U-trees with $|U| = k$, so $X_k^-(G) = 0$ by Corollary 4.3. If $j \geqslant k$ then $X_j^-(G) = 0$, so, by Corollary 4.2, $Y_k^-(G) = 0$ also.

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By Theorem 4.3, $X_k^-(G) = Y_k^-(G) = 0$ if $m(G) < n(G) - 1$ (since G would be disconnected). By Corollary 4.3, if $G \preceq H$ and $n(G) = k$, $m(G) = k - 1$, then $X_k^-(G) = 1$ if G is a tree and $X_k^-(G) = 0$ otherwise. Also, $X_j^-(G) = 0$ if $j > k$, so $X_k^-(G, k, k-1) = Y_k^-(G, k, k-1)$. □

Corollary 5.3. *Let G and H be graphs and $k = 1, 2, \ldots$. If there exist $p > 0$ and $\epsilon > 0$ such that*

$$E_{H,p,\rho}(X_k) \geqslant E_{G,p,\rho}(X_k) \text{ for } 0 < \rho < \epsilon, \tag{5.3}$$

then

$$X_k^-(H, k, k-1) \geqslant X_k^-(H, k, k-1). \tag{5.4}$$

The statement is also true with X_k replaced by Y_k throughout.

Proof. By Theorems 5.1 and 5.2, if $E_{H,p,\rho}(X_k)$ or $E_{H,p,\rho}(Y_k)$ is expanded as a polynomial in powers of p and ρ, the smallest order term is $p^k \rho^{k-1}$ with coefficient $X_k^-(H, k, k-1) = Y_k^-(H, k, k-1)$. All other terms have order at least k in p and at least k in ρ. The same comments apply to G, so the result follows. □

Corollary 5.3 has a partial converse. If strict inequality holds in (5.4) then, for each $p > 0$, strict inequality holds in (5.3) for some $\epsilon > 0$ (which may depend on p).

In particular, Corollary 5.3 applies in the IID edge-failure model. When $k = 2$ (pair-connected reliability), the lowest order coefficient in the expansion is $X_2^-(H, 2, 1) = m(H)$, the number of edges in H. When $k = n(H)$ (all-terminal reliability), the lowest order coefficient in this expansion is

$$X_{n(H)}^-(H, n(H), n(H) - 1),$$

the number of spanning trees in H.

Definition. Let H be a graph having a component of order at least $k \geqslant 2$. Let $\lambda_k(H)$ denote the smallest number of edges whose deletion disconnects a component of H of order at least k.

If H is connected and has at least k vertices, then $\lambda_k(H) = \lambda(H)$, the edge connectivity of H.

Theorem 5.3. *Let H be a graph having a component of order at least $k \geqslant 2$. If $0 < j < \lambda_k(H)$ then*

$$X_k^+(H, 0, j) = Y_k^+(H, 0, j) = 0.$$

Moreover,

$$\begin{aligned} X_k^+(H, 0, \lambda_k(H)) &= \sum_{G \in \mathscr{G}_k} (X_k(G) - X_k(H)), \\ Y_k^+(H, 0, \lambda_k(H)) &= \sum_{G \in \mathscr{G}_k} (Y_k(G) - Y_k(H)), \end{aligned}$$

where $\mathscr{G}_k$ is the set of all subgraphs G obtained from H by deleting k edges.

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Proof. Suppose that $j < \lambda_k(H)$. By definition, $X_k^+(H,0,j)$ is the sum of $X_k^+(F,H)$ over all subgraphs F of H which are obtained by deleting j edges of H. If $F \preceq G \prec H$, then, by definition of $\lambda_k(H)$, $X_k(G) = X_k(H)$ and hence it follows inductively that $X_k^+(G,H) = 0$. Hence $X_k^+(H,0,j) = 0$. It then follows that $X_k^+(H,0,\lambda_k(H))$ has the representation given in the theorem. The proof for Y_k^+ is similar. □

Corollary 5.4. *Let G and H be graphs, each having a component of order at least k. Suppose that, for some $\epsilon > 0$,*

$$E_{H,1,\rho}(X_k) \geqslant E_{G,1,\rho}(X_k) \text{ for } \rho > 1-\epsilon. \tag{5.5}$$

Then

$$X_k(H) \geqslant X_k(G). \tag{5.6}$$

If equality holds in (5.6)*, then*

$$\lambda_k(H) \geqslant \lambda_k(G). \tag{5.7}$$

If equality holds in (5.6) *and* (5.7)*, then*

$$X_k^+(H,0,\lambda_k(H)) \geqslant X_k^+(G,0,\lambda_k(G)). \tag{5.8}$$

Proof. By Theorems 5.1 and 5.3, if $E_{H,1,\rho}(X_k)$ is expanded as a polynomial in powers of $1-\rho$, then the constant term is $X_k(H)$, and the next term is the one of order $\lambda_k(H)$, with coefficient $X_k^+(H,0,\lambda_k(H))$. The same comments apply to G so the results follow. □

Corollary 5.4 has a partial converse. If strict inequality holds in (5.6), or equality in (5.6) and strict inequality in (5.7), or equality in (5.6) and (5.7) and strict inequality in (5.8), then there exists $\epsilon > 0$ such that, for each $\rho \in (1-\epsilon, 1)$, there exists $\delta > 0$ (which may depend on ρ) such that $E_{H,p,\rho}(X_k) > E_{G,p,\rho}(X_k)$ for $p \in (1-\delta, 1]$.

If T is a tree and $U \subseteq V(T)$ then there is a unique U-tree in T. Hence the following result follows from (4.1) and Corollary 4.3.

Theorem 5.4. *Let T be a tree and let $k \leqslant n(T)$. For $k \leqslant i \leqslant n(T)$, $X_k^-(T,i,i-1)$ is the number of vertex sets U such that $|U| = k$ and the U-tree in T has i vertices; $X_k^-(T,i,j) = 0$ otherwise.* □

From Theorems 5.1 and 5.4 it follows that

$$E_{T,p,\rho}(X_k) = \sum_{i=k}^{n(T)} X_k^-(T,i,i-1) p^i \rho^{i-1}. \tag{5.9}$$

Therefore

$$E_{T,p,\rho}(X_k) = pE_{T,1,p\rho}(X_k) = \frac{1}{\rho} E_{T,p\rho,1}(X_k).$$

Thus, for trees, the polynomial for the full IID model can be obtained from the polynomials for the IID edge-failure or vertex-failure models. From Theorem 5.4, note that $X_2^-(T,i,i-1)$ is the number of pairs of vertices in T at distance i. This result was obtained in [1] for the edge-failure model ($p = 1$) and in [3] for the vertex-failure model ($\rho = 1$).

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Example 5.2. *Let S_n denote the star on $n \geqslant 2$ vertices. For any vertex set U consisting of k end-points, the U-tree has $k+1$ vertices. Therefore, by Theorem* 5.4,

$$E_{S_n,p,\rho}(X_k) = \binom{n-1}{k-1} p^k \rho^{k-1} + \binom{n-1}{k} p^{k+1} \rho^k, \quad k = 2, \ldots, n.$$

Example 5.3. *Let L_n denote the path on $n \geqslant 2$ vertices. If a vertex set U of size k $(2 \leqslant k \leqslant n)$ has a U-tree with i vertices $(k \leqslant i \leqslant n)$ then the pair of points in U with maximum distance must have distance $i-1$. There are $n-i+1$ pairs of points at distance $i-1$ and, for each such pair, there are $\binom{i-1}{k-2}$ vertex sets containing this pair of points but no pair with greater distance. Therefore*

$$E_{L_n,p,\rho}(X_k) = \sum_{i=k}^{n} (n-i+1) \binom{i-2}{k-2} p^i \rho^{i-1}.$$

We now consider the existence of graphs that maximize $E_{G,p,\rho}(X_k)$ for G in some class of graphs and for all values of p and ρ. The following theorem generalizes results for pair-connected reliability ($k = 2$) in the IID edge-failure model ($p = 1$) [1] and the IID vertex-failure model ($\rho = 1$) [3], and shows that, among trees, the star is best and the path is worst for any value of k. Of course, for all-terminal reliability, all trees of order n are equally reliable.

Theorem 5.5. *If T is any tree on $n \geqslant 2$ vertices and $1 \leqslant k \leqslant n$ then, for any $p, \rho \in [0,1]$,*

$$E_{L_n,p,\rho}(X_k) \leqslant E_{T,p,\rho}(X_k) \leqslant E_{S_n,p,\rho}(X_k). \tag{5.10}$$

Proof. By (5.9), for any tree T with n vertices,

$$E_{T,p,\rho}(X_k) = p \sum_{i=k}^{n} X_k^-(T,i,i-1)(p\rho)^{i-1}.$$

For $k \leqslant j \leqslant n$, we denote the partial sums of the coefficients in this polynomial as follows:

$$A(T,k,j) = \sum_{i=k}^{j} X_k^-(T,i,i-1).$$

To prove the theorem, it suffices to show that

$$A(L_n,k,j) \leqslant A(T,k,j) \leqslant A(S_n,k,j) \text{ for } k \leqslant j \leqslant n. \tag{5.11}$$

Note that $A(T,k,j)$ is the number of vertex sets U in T with $|U| = k$ whose U-tree in T has j or fewer vertices. For $u \in V(T)$, we will also denote by $A(T,u,k,j)$ the number of vertex sets U in T with $u \in U$, $|U| = k$, whose U-tree in T has j or fewer vertices. Suppose that T' is another tree with n vertices and with $u' \in V(T')$. To show that $A(T,u,k,j) \leqslant A(T',u',k,j)$, it suffices to find a bijection between $V(T)$ and $V(T')$, mapping u to u', such that, for any $U \subseteq V(T)$ with $u \in U$, the subtree of T determined by U has at least as many vertices as the subtree in T' determined by the image U'. We

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will use this method to show that, for any tree T on n vertices and any $u \in V(T)$,

$$A(L_n, e, k, j) \leqslant A(T, u, k, j) \leqslant A(S_n, c, k, j) \text{ for } k \leqslant j \leqslant n, \tag{5.12}$$

where c is the centre of the star S_n and e is an end-point of the path L_n. First, root the star S_n at the centre c, the path L_n at the end-point e, and T at u. For each of these trees, order the vertices from 1 to n in a way that preserves distance to the root (thus, if v_j is farther from the root than v_i then $j > i$). The orderings define bijections between each pair of vertex sets. Now let $U \subseteq V(T)$ with $|U| = k$ and $u \in U$. Let U_1 be the corresponding subset of $V(S_n)$ and U_2 the corresponding subset of $V(L_n)$. The tree in T generated by U has at least k vertices; the tree in S_n generated by U_1 has exactly k vertices. If m is the number of the last vertex in U, then the tree in T generated by U has no more than m vertices; the tree in L_n generated by U_2 has exactly m vertices. Therefore (5.12) holds.

Suppose next that T is a tree on n vertices and T' is the tree obtained by joining a new vertex v to a vertex $u \in V(T)$ by a new edge. Let $U \subseteq V(T)$. The U-tree in T' is the same as the U-tree in T. If $u \in U$ then the $U \cup \{v\}$-tree in T' is the U-tree in T with v connected to u. Finally, if $u \notin U$ then the $U \cup \{v\}$ tree in T' is the $U \cup \{u\}$-tree in T with v connected to u. Hence

$$A(T', k, j) = A(T, k, j) + A(T, u, k-1, j-1) + A(T, u, k, j-1). \tag{5.13}$$

Finally, we will use induction on n to prove (5.11). First, (5.11) holds trivially when $n = 2$, since there is only one tree on 2 vertices. Thus, suppose that (5.11) holds for a given n. Let T' be a tree on $n+1$ vertices. Let v be an end-point of T' connected to a vertex u, and let T be the tree on n vertices obtained by deleting v. Note also that the star S_{n+1} can be obtained by joining a new vertex to the centre c of the star S_n, and that the path L_{n+1} can be obtained by joining a new vertex to an end-point e of the path L_n. From the induction hypothesis and (5.12) we have

$$\begin{aligned} A(L_n, k, j) &\leqslant A(T, k, j) \leqslant A(S_n, k, j), \\ A(L_n, e, k-1, j-1) &\leqslant A(T, u, k-1, j-1) \leqslant A(S_n, c, k-1, j-1), \\ A(L_n, e, k, j-1) &\leqslant A(T, u, k, j-1) \leqslant A(S_n, c, k, j-1). \end{aligned}$$

Adding through the inequalities and using (5.13) gives

$$A(L_{n+1}, k, j) \leqslant A(T', k, j) \leqslant A(S_{n+1}, k, j),$$

and the proof is complete. □

Now let $\mathcal{G}$ be a nonempty class of graphs and let $H \in \mathcal{G}$. If (5.3) is satisfied for each $G \in \mathcal{G}$, then, by Corollary 5.3, H must have the maximum number of trees of order k among all graphs in $\mathcal{G}$. On the other hand, if (5.5) is satisfied for each $G \in \mathcal{G}$ then, by Corollary 5.4, $X_k(H) \geqslant X_k(G)$ for each $G \in \mathcal{G}$, and $\lambda_k(H) \geqslant \lambda_k(G)$ must hold for all $G \in \mathcal{G}$ such that $X_k(H) = X_k(G)$. For most values of k the conditions for optimality in Corollaries 5.3 and 5.4 are inherently at odds with each other: if $\mathcal{G}$ is a sufficiently rich collection of graphs, then a graph $H \in \mathcal{G}$ which maximizes connectivity (see [13]) will not also maximize the number of trees of order k. For such a collection $\mathcal{G}$ there will not exist

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a graph H that maximizes $E_{H,p,\rho}(X_k)$ when ρ is small and also when p and ρ are both near 1.

Specifically, consider the class $\mathscr{G}_{n,m}$ of all unlabelled (n,m) graphs. When $k=2$ (pair-connected reliability) and $p=1$ (the IID edge-failure model), it is shown in [4] that, for $n \leqslant m \leqslant \binom{n}{2} - 2$, there does not exist a graph that is optimal for all values of ρ. When $k=2$ and $\rho=1$ (the IID vertex-failure model), it is shown in [3] that for some values of (n,m) there exists a graph that is optimal for all values of p, but for other values of (n,m) there does not exist such a graph. The situation is similar when $k=n$ and $p=1$ (all terminal reliability in the IID edge-failure model [15]). The cases $k=2$ and $k=n$ are a bit unusual. When $k=2$, Corollary 5.3 gives no useful information since the lowest order term in the expansion is m for all $G \in \mathscr{G}_{m,n}$. When $k=n$, the conditions for optimality in Corollaries 5.3 and 5.4 may not be inconsistent. For $n \geqslant 4$, $3 \leqslant k \leqslant n-1$, and $n \leqslant m \leqslant \binom{n}{2} - 2$, we conjecture that there does not exist a graph $H \in \mathscr{G}_{m,n}$ that satisfies conditions (5.3) and (5.5) for all $G \in \mathscr{G}_{m,n}$.

As an example, consider the problem of maximizing $E_{G,p,\rho}(X_k)$ over the class $\mathscr{G}_{n,n}$. It is shown in [15] that, when $k=n$ and $p=1$, the cycle C_n is optimal for all values of ρ. Suppose that $3 \leqslant k < n$. By Corollary 5.4, C_n is the unique optimal graph when $p=1$ and ρ is sufficiently close to 1. For $k \leqslant j < n$, let $G_j \in \mathscr{G}_{n,n}$ be the graph with a cycle of length j and $n-j$ end-points connected to a common point u on the cycle. There are j trees of order k contained in the j cycle of G_j. On the other hand, a tree of order k in G_j containing i of the end-points of G_j must contain a path through u in the j-cycle of G_j with order $k-i$. It follows that the number of trees of order k in G_j is

$$X_k^-(G_j,k,k-1) = j + \sum_{i=1}^{k-1}(k-i)\binom{n-j}{i}. \tag{5.14}$$

However, the cycle C_n has n trees of order k. The expression in (5.14) is larger than n so, by Corollary 5.3, it follows that C_n is not optimal when $p>0$ and ρ is small.

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