16

Fourier Methods

In the previous chapter, the fermionic signature operator and the unregularized fermionic projector were constructed abstractly. We now turn to the question how to compute them in the Minkowski vacuum. This question can be addressed and answered with Fourier methods. Since these techniques are frequently used and of independent interest, we introduce them from a general perspective before entering the proof of the mass oscillation properties and the construction of the fermionic signature operator. More details can be found in [71, 81].

16.1 The Causal Green's Operators

We already encountered Green's operators in Chapter 13 when solving the Cauchy problem with methods of hyperbolic partial differential equations (see Theorem 13.4.3). In Minkowski space, these Green's operators can be computed in more detail with Fourier methods. Our starting point is the definition of the *Green's operator* $s_m(x,y)$ of the vacuum Dirac equation by the distributional equation

$$(\mathrm{i}\partial\!\!\!/_x - m)\, s_m(x,y) = \delta^4(x-y)\,, \tag{16.1}$$

where $\delta^4(x,y)$ denotes the four-dimensional δ distribution. Taking the Fourier transform of (16.1),

$$s_m(x,y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, s_m(k)\, \mathrm{e}^{-\mathrm{i}k(x-y)}, \tag{16.2}$$

where $x, y \in \mathscr{M}$ are spacetime points, k is the four-momentum and $k(x-y)$ denotes the Minkowski inner product; we obtain the algebraic equation

$$(k\!\!\!/ - m)\, s_m(k) = \mathbb{1}\,. \tag{16.3}$$

Multiplying by $k\!\!\!/ + m$ and using the identity $(k\!\!\!/ - m)(k\!\!\!/ + m) = k^2 - m^2$, one sees that if $k^2 \neq m^2$, the matrix $k\!\!\!/ - m$ is invertible. If conversely $k^2 = m^2$, we have $(k\!\!\!/ - m)^2 = -2m(k\!\!\!/ - m)$, which shows that the matrix $k\!\!\!/ - m$ is diagonalizable with eigenvalues $-2m$ and zero. Since the Dirac matrices (1.26) are trace-free, we have $\mathrm{Tr}(k\!\!\!/ - m) = -4m$. It follows that the matrix $k\!\!\!/ - m$ has a two-dimensional kernel if k is on the mass shell. This shows that the Green's operator of the Dirac equation is *not unique*. If we add to it any vector in the kernel of $k\!\!\!/ - m$, that is, if we add to it a solution of the homogeneous Dirac equation, it still satisfies the defining equation (16.1) (for details, see [14]).

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A convenient method for solving Eq. (16.3) for $s_m(k)$ is to use a $\pm i\varepsilon$-regularization on the mass shell. Common choices are the *advanced* and the *retarded* Green's functions, which are defined by

$$\begin{aligned} s_m^\vee(k) &= \lim_{\varepsilon \searrow 0} \frac{\not{k} + m}{k^2 - m^2 - \mathrm{i}\varepsilon k^0} \qquad \text{and} \\ s_m^\wedge(k) &= \lim_{\varepsilon \searrow 0} \frac{\not{k} + m}{k^2 - m^2 + \mathrm{i}\varepsilon k^0}\,, \end{aligned} \tag{16.4}$$

respectively (with the limit $\varepsilon \searrow 0$ taken in the distributional sense). Computing their Fourier transform (16.2), one sees that they are *causal* in the sense that their supports lie in the upper and lower light cone, respectively,

$$\operatorname{supp} s_m^\vee(x,.) \subset J_x^\vee\,, \qquad \operatorname{supp} s_m^\wedge(x,.) \subset J_x^\wedge\,. \tag{16.5}$$

Mathematically, the formulas in (16.4) define the Green's operators in momentum space as tempered distributions. Taking their Fourier transform according to (16.2), the advanced and retarded Green's operators are tempered distributions in the variable $\xi := y - x$. We also regard these distributions as integral kernels of corresponding operators on the wave functions, that is,

$$(s_m(\psi))(x) := \int_{\mathscr{M}} s_m(x,y)\, \psi(y)\, \mathrm{d}^4 y\,. \tag{16.6}$$

We thus obtain operators

$$s_m^\wedge, s_m^\vee \,:\, C_0^\infty(\mathscr{M}, S\mathscr{M}) \to C_{\mathrm{sc}}^\infty(\mathscr{M}, S\mathscr{M})\,. \tag{16.7}$$

Here, $C_0^\infty(\mathscr{M}, S\mathscr{M})$ denote the smooth functions with compact support in $\mathscr{M}$, taking values in the spinors, and C_{sc}^∞ denotes the smooth functions with spatially compact support.

16.2 The Causal Fundamental Solution and Time Evolution

We now state a few properties of the Green's operators and explain why they are useful. The considerations in this section are valid more generally in the presence of an external potential. Then, the defining equation of the Green's operator (16.1) is modified similar to (1.39) to

$$(\mathrm{i}\not{\partial}_x + \mathcal{B} - m)\, s_m(x,y) = \delta^4(x-y)\,, \tag{16.8}$$

where $\mathcal{B}$ is again a multiplication operator satisfying the symmetry condition (1.38). Then, the existence of Green's operators can no longer be proven by Fourier transformation. Instead, one can use methods of hyperbolic PDEs as introduced in Chapter 13 (see Section 13.6). Here, we do not assume that the reader is familiar with these methods. Instead, we simply assume that we are given advanced and retarded Green's operators.

The *causal fundamental solution* k_m is defined as the difference between the advanced and the retarded Green's operator,

$$k_m(x,y) := \frac{1}{2\pi \mathrm{i}}\, (s_m^\vee(x,y) - s_m^\wedge(x,y))\,. \tag{16.9}$$

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It is a distribution that is causal in the sense that it vanishes if x and y have spacelike separation. Moreover, it is a distributional solution of the homogeneous Dirac equation,

$$(\mathrm{i}\partial\!\!\!/_x + \mathcal{B} - m)\, k_m(x,y) = 0\,. \tag{16.10}$$

The unique solvability of the Cauchy problem allows us to introduce the time evolution operator of the Dirac equation as follows. Solving the Cauchy problem with initial data at time t and evaluating the solution at some other time t' gives rise to a mapping

$$U^{t',t} \,:\, \mathcal{H}_t \to \mathcal{H}_{t'}\,, \tag{16.11}$$

referred to as the *time evolution operator*. Since the scalar product (15.2) is time independent, the operator $U^{t',t}$ is unitary. Moreover, using that the Cauchy problem can be solved forward and backward in time, the unitary time evolution operators form a representation of the group $(\mathbb{R},+)$. More precisely,

$$U^{t,t} = \mathbb{1} \qquad \text{and} \qquad U^{t'',t'}\, U^{t',t} = U^{t'',t}\,. \tag{16.12}$$

Proposition 13.6.1 immediately gives the following representation of $U^{t',t}$:

$$\big(U^{t',t}\,\psi|_t\big)(\vec{y}) = \int_{\mathbb{R}^3} U^{t',t}(\vec{y},\vec{x})\; \psi(t,\vec{x})\; \mathrm{d}^3x\,, \tag{16.13}$$

where the kernel $U^{t',t}(\vec{x},\vec{y})$ is defined as

$$U^{t',t}(\vec{y},\vec{x}) = 2\pi\, k_m\big((t',\vec{y}),(t,\vec{x})\big)\,\gamma^0\,. \tag{16.14}$$

16.3 Proof of the Weak Mass Oscillation Property in the Minkowski Vacuum

In the remainder of this chapter, we return to the Dirac equation in Minkowski space (16.1). An external potential will be considered in the next chapter.

The mass oscillation property in the Minkowski vacuum can be proved using Fourier methods. Here, we shall give two different approaches in detail. The method of the first proof (in this section) is instructive because it gives an intuitive understanding of "mass oscillations." However, this method only yields the weak mass oscillation property. The second proof (Section 16.4) is more abstract but also gives the strong mass oscillation property.

We again consider the foliation $\mathcal{N}_t = \{(t,\vec{x})\,|\,\vec{x}\in\mathbb{R}^3\}$ of constant time Cauchy hypersurfaces in a fixed reference frame $(t,\vec{x})$ and a variable mass parameter m in the interval $I=(m_L,m_R)$ with $m_L, m_R > 0$. The families of solutions $\psi=(\psi_m)_{m\in I}$ of the Dirac equations $(i\partial\!\!\!/ - m)\psi_m = 0$ are contained in the Hilbert space $(\mathcal{H},(.|.))$ with scalar product (15.13). The subspace $\mathcal{H}^\infty\subset\mathcal{H}$ in Definition 15.2.1 is chosen as

$$\mathcal{H}^\infty = C^\infty_{\mathrm{sc},0}(\mathscr{M}\times I, S\mathscr{M}) \cap \mathcal{H}\,. \tag{16.15}$$

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For what follows, it is convenient to work with the Fourier transform in space, that is,

$$\hat{\psi}(t,\vec{k}) = \int_{\mathbb{R}^3} \psi(t,\vec{x})\, \mathrm{e}^{-\mathrm{i}\vec{k}\vec{x}}\, \mathrm{d}^3x\,, \qquad \psi(t,\vec{x}) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3k}{(2\pi)^3}\, \hat{\psi}(t,\vec{k})\, \mathrm{e}^{\mathrm{i}\vec{k}\vec{x}}\,. \tag{16.16}$$

Then, a family of solutions $\psi \in \mathcal{H}^\infty$ has the representation

$$\hat{\psi}_m(t,\vec{k}) = c_+(\vec{k},m)\, \mathrm{e}^{-\mathrm{i}\omega(\vec{k},m)\, t} + c_-(\vec{k},m)\, \mathrm{e}^{\mathrm{i}\omega(\vec{k},m)\, t} \quad \text{for all } m \in I, \tag{16.17}$$

with suitable spinor-valued coefficients $c_\pm(\vec{k},m)$ and $\omega(\vec{k},m) := \sqrt{|\vec{k}|^2 + m^2}$. Integrating over the mass parameter, we obtain a superposition of waves oscillating at different frequencies. Intuitively speaking, this leads to destructive interference for large t, giving rise to decay in time. This picture can be made precise using integration by parts in m, as we now explain. Integrating (16.17) over the mass by applying the operator $\mathfrak{p}$, (15.20), we obtain

$$\begin{aligned}
\mathfrak{p}\hat{\psi}(t,\vec{k}) &= \int_I \left(c_+\, \mathrm{e}^{-\mathrm{i}\omega t} + c_-\, \mathrm{e}^{\mathrm{i}\omega t}\right) \mathrm{d}m \\
&= \int_I \frac{\mathrm{i}}{t\, \partial_m \omega} \left(c_+\, \partial_m \mathrm{e}^{-\mathrm{i}\omega t} - c_-\, \partial_m \mathrm{e}^{\mathrm{i}\omega t}\right) \mathrm{d}m \\
&= -\frac{\mathrm{i}}{t} \int_I \left[\partial_m\Big(\frac{c_+}{\partial_m\omega}\Big)\, \mathrm{e}^{-\mathrm{i}\omega t} - \partial_m\Big(\frac{c_-}{\partial_m\omega}\Big)\, \mathrm{e}^{\mathrm{i}\omega t}\right] \mathrm{d}m
\end{aligned} \tag{16.18}$$

(we do not get boundary terms because $\psi \in \mathcal{H}^\infty$ has compact support in m). With $\partial_m\omega = m/\omega$, we conclude that

$$\mathfrak{p}\hat{\psi}(t,\vec{k}) = -\frac{\mathrm{i}}{t} \int_I \left[\partial_m\Big(\frac{\omega\, c_+}{m}\Big)\, \mathrm{e}^{-\mathrm{i}\omega t} - \partial_m\Big(\frac{\omega\, c_-}{m}\Big)\, \mathrm{e}^{\mathrm{i}\omega t}\right] \mathrm{d}m\,. \tag{16.19}$$

Since the coefficients $c_\pm$ depend smoothly on m, the resulting integrand is bounded uniformly in time, giving a decay at least like $1/t$, that is, $|\mathfrak{p}\hat{\psi}(t,\vec{k})| \lesssim 1/t$. Iterating this procedure, one even can prove decay rates $\lesssim 1/t^2, 1/t^3, \ldots$ The price one pays is that higher and higher powers in ω come up in the integrand, which means that in order for the spatial Fourier integral to exist, one needs a faster decay of $c_\pm$ in $|\vec{k}|$. Expressed in terms of the initial data, this means that every factor $1/t$ gives rise to an additional spatial derivative acting on the initial data. This motivates the following basic estimate.

Lemma 16.3.1 *For any $\psi \in \mathcal{H}^\infty$, there is a constant $C = C(m_L)$ such that*

$$\big\|(\mathfrak{p}\psi)|_t\big\|_t \leq \frac{C\,|I|}{1+t^2} \sup_{m\in I} \sum_{b=0}^{2} \big\|(\partial_m^b \psi_m)|_{t=0}\big\|_{W^{2,2}}\,, \tag{16.20}$$

where $\|.\|_t$ is the norm corresponding to the scalar product

$$(.|.)_t := 2\pi \int_{\mathbb{R}^3} \prec .\, |\gamma^0\, .\, \succ_{\vec{x}}\, \mathrm{d}^3x \;:\; L^2(\mathcal{N}_t, S\mathscr{M}) \times L^2(\mathcal{N}_t, S\mathscr{M}) \to \mathbb{C} \tag{16.21}$$

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(which is similar to (15.2)*, but now applied to wave functions that do not need to be solutions), and* $\|.\|_{W^{2,2}}$ *is the spatial Sobolev norm*

$$\|\phi\|^2_{W^{2,2}} := \sum_{\alpha \text{ with } |\alpha| \leq 2} \int_{\mathbb{R}^3} |\nabla^\alpha \phi(\vec{x})|^2 \, \mathrm{d}^3x \,, \tag{16.22}$$

where α *is a multi-index.*

The absolute value in (16.22) is the norm $|\,.\,| := \sqrt{\prec .|\gamma^0 .\succ}$ on the spinors. If we again identify all spinor spaces in the Dirac representation with $\mathbb{C}^4$, this simply is the standard Euclidean norm on $\mathbb{C}^4$.

The proof of this lemma will be given later in this section. Before, we infer the weak mass oscillation property.

Corollary 16.3.2 *The vacuum Dirac operator* $i\partial\!\!\!/$ *in Minkowski space has the weak mass oscillation property with domain* (16.15)*.*

Proof For every $\psi, \phi \in \mathcal{H}^\infty$, the Schwarz inequality gives

$$\begin{aligned} |<\mathfrak{p}\psi|\mathfrak{p}\phi>| &= \frac{1}{2\pi} \left| \int_{-\infty}^\infty \left((\mathfrak{p}\psi)|_t \,\middle|\, \gamma^0 \, (\mathfrak{p}\phi)|_t \right)_t \, \mathrm{d}t \right| \\ &\leq \int_{-\infty}^\infty \left\| (\mathfrak{p}\psi)|_t \right\|_t \left\| (\mathfrak{p}\phi)|_t \right\|_t \, \mathrm{d}t \,. \end{aligned} \tag{16.23}$$

Applying Lemma 16.3.1 together with the estimate

$$\begin{aligned} \left\| (\mathfrak{p}\phi)|_t \right\|_t^2 &= \iint_{I \times I} \left(\phi_m|_t \,\middle|\, \phi_{m'}|_t \right)_t \, \mathrm{d}m \, \mathrm{d}m' \\ &\leq \frac{1}{2} \iint_{I \times I} \left(\|\phi_m\|_m^2 + \|\phi_{m'}\|_{m'}^2 \right) \mathrm{d}m \, \mathrm{d}m' = |I| \, \|\phi\|^2 \,, \end{aligned} \tag{16.24}$$

we obtain inequality (15.21) with

$$c = C \, |I|^{\frac{3}{2}} \sup_{m \in I} \sum_{b=0}^2 \|\partial_m^b (\psi_m)|_{t=0}\|_{W^{2,2}} \int_{-\infty}^\infty \frac{1}{1+t^2} \, \mathrm{d}t < \infty \,. \tag{16.25}$$

The identity (15.22) follows by integrating the Dirac operator by parts,

$$\begin{aligned} <\mathfrak{p}T\psi|\mathfrak{p}\phi> = <\mathfrak{p}\mathcal{D}\psi|\mathfrak{p}\phi> = <\mathcal{D}\mathfrak{p}\psi|\mathfrak{p}\phi> &= \int_{\mathscr{M}} \prec \mathcal{D}\mathfrak{p}\psi|\mathfrak{p}\phi\succ_x \, \mathrm{d}^4x \\ \overset{(\star)}{=} \int_{\mathscr{M}} \prec \mathfrak{p}\psi|\mathcal{D}\mathfrak{p}\phi\succ_x \, \mathrm{d}^4x &= <\mathfrak{p}\psi|\mathcal{D}\mathfrak{p}\phi> = <\mathfrak{p}\psi|\mathfrak{p}T\phi> \,. \end{aligned} \tag{16.26}$$

In $(\star)$, we used that the Dirac operator is symmetric with respect to the inner product $<.|.>$. Moreover, we do not get boundary terms because of the time decay in Lemma 16.3.1. □

The remainder of this section is devoted to the proof of Lemma 16.3.1. Using the result of Proposition 13.6.1, we can express the solution ψ_m of the Cauchy problem in terms of the causal fundamental solution k_m. In order to bring k_m into a more explicit form, we use (16.9) together with formulas for the advanced and retarded Green's operators. Indeed, these Green's operators are the multiplication operators

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in momentum space (16.4) (with the limit $\varepsilon \searrow 0$ taken in the distributional sense, and where the vector k is the four-momentum). We thus obtain in momentum space

$$\begin{aligned} k_m(p) &= \frac{1}{2\pi \mathrm{i}}\,(\not{p}+m)\,\lim_{\varepsilon\searrow 0}\left[\frac{1}{p^2-m^2-\mathrm{i}\varepsilon p^0}\;-\;\frac{1}{p^2-m^2+\mathrm{i}\varepsilon p^0}\right] \\ &= \frac{1}{2\pi \mathrm{i}}\,(\not{p}+m)\,\lim_{\varepsilon\searrow 0}\left[\frac{1}{p^2-m^2-\mathrm{i}\varepsilon}\;-\;\frac{1}{p^2-m^2+\mathrm{i}\varepsilon}\right]\epsilon(p^0) \end{aligned} \tag{16.27}$$

(where for notational clarity, we denoted the momentum variables by p, and ϵ is the *sign function* $\epsilon(x) = 1$ if $x > 0$ and $\epsilon(x) = -1$ otherwise). Employing the distributional equation

$$\lim_{\varepsilon\searrow 0}\left(\frac{1}{x-\mathrm{i}\varepsilon}-\frac{1}{x+\mathrm{i}\varepsilon}\right) = 2\pi\mathrm{i}\,\delta(x)\,, \tag{16.28}$$

we obtain the simple formula

$$k_m(p) = (\not{p}+m)\,\delta(p^2-m^2)\,\epsilon(p^0)\,. \tag{16.29}$$

It is convenient to transform spatial coordinates of the time evolution operator to momentum space. First, in the Minkowski vacuum, the time evolution operator can be represented as in (16.13) with an integral kernel $U^{t,t'}(\vec{y},\vec{x})$, which depends only on the difference vector $\vec{y}-\vec{x}$. We set

$$U^{t,t'}(\vec{k}) := \int_{\mathbb{R}^3} U^{t,t'}(\vec{y},0)\,\mathrm{e}^{-\mathrm{i}\vec{k}\vec{y}}\;\mathrm{d}^3y\,. \tag{16.30}$$

Combining (16.14) with (16.29) yields

$$U^{t,t'}(\vec{k}) = \int_{-\infty}^{\infty} (\not{k}+m)\,\gamma^0\,\delta(k^2-m^2)\big|_{k=(\omega,\vec{k})}\,\epsilon(\omega)\,\mathrm{e}^{-\mathrm{i}\omega(t-t')}\;\mathrm{d}\omega\,. \tag{16.31}$$

Carrying out the ω-integral, we get

$$U^{t,t'}(\vec{k}) = \sum_{\pm} \Pi_\pm(\vec{k})\,\mathrm{e}^{\mp\mathrm{i}\omega(t-t')}\,, \tag{16.32}$$

where we set

$$\Pi_\pm(\vec{k}) := \pm\frac{1}{2\omega(\vec{k})}\,(\not{k}_\pm+m)\,\gamma^0 \tag{16.33}$$

$$\text{with}\qquad \omega(\vec{k}) = \sqrt{|\vec{k}|^2+m^2}\qquad\text{and}\qquad k_\pm = (\pm\omega(\vec{k}),\vec{k})\,.$$

Moreover, applying Plancherel's theorem, the scalar product (15.2) can be written in momentum space as

$$(\psi_m\,|\,\phi_m)_m = (2\pi)^{-2}\int_{\mathbb{R}^3} \prec\hat{\psi}_m(t,\vec{k})\,|\,\gamma^0\,\hat{\phi}_m(t,\vec{k})\succ\;\mathrm{d}^3k\,. \tag{16.34}$$

The unitarity of the time evolution operator in position space implies that the matrix $U^{t,t'}(\vec{k})$ is unitary (with respect to the scalar product $\langle .,.\rangle_{\mathbb{C}^2} \equiv \prec .\,|\gamma^0\,.\succ$), meaning that its eigenvalues are on the unit circle and the corresponding eigenspaces are orthogonal. It follows that the operators $\Pi_\pm(\vec{k})$ in (16.32)

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are the orthogonal projection operators to the eigenspaces corresponding to the eigenvalues $\mathrm{e}^{\mp i\omega(t-t')}$, that is,

$$\gamma^0 \Pi_s^* \gamma^0 = \Pi_s \quad \text{and} \quad \Pi_s(\vec{k})\, \Pi_{s'}(\vec{k}) = \delta_{s,s'}\, \Pi_s(\vec{k}) \tag{16.35}$$

for $s, s' \in \{+,-\}$ (these relations can also be verified by straightforward computations using (16.33); see Exercise 16.8).

The next two lemmas involve derivatives with respect to the mass parameter m. For clarity, we again denote the m-dependence of the operators by the subscript m.

Lemma 16.3.3 *The time evolution operator in the vacuum satisfies the relation*

$$(t-t')\, U_m^{t,t'}(\vec{k}) = \frac{\partial}{\partial m} V_m^{t,t'}(\vec{k}) + W_m^{t,t'}(\vec{k})\,, \tag{16.36}$$

where

$$V_m^{t,t'}(\vec{k}) = \sum_{\pm} \frac{\mathrm{i}}{2m}\, (\not{k}_\pm + m)\gamma^0\, \mathrm{e}^{\mp \mathrm{i}\omega(t-t')}, \tag{16.37}$$

$$W_m^{t,t'}(\vec{k}) = \sum_{\pm} \frac{\mathrm{i}}{2} \Big(\frac{\not{k}_\pm \gamma^0}{m^2} \mp \frac{1}{\omega} \Big)\, \mathrm{e}^{\mp \mathrm{i}\omega(t-t')}\,. \tag{16.38}$$

The operators $V_m^{t,t'}$ and $W_m^{t,t'}$ are estimated uniformly by

$$\|V_m^{t,t'}(\vec{k})\| + \|W_m^{t,t'}(\vec{k})\| \leq C \left(1 + \frac{|\vec{k}|}{m}\right), \tag{16.39}$$

where the constant C is independent of m, $\vec{k}$, t and t' (and $\|\,.\,\|$ is any norm on the 2×2-matrices).

Proof First, we generate the factor $t - t'$ by differentiating the exponential in (16.32) with respect to ω,

$$(t-t')\, U_m^{t,t'}(\vec{k}) = \sum_{\pm} \Pi_\pm(\vec{k}) \Big(\pm \mathrm{i} \frac{\partial}{\partial \omega}\, \mathrm{e}^{\mp \mathrm{i}\omega(t-t')} \Big). \tag{16.40}$$

Next, we want to rewrite the ω-derivative as a derivative with respect to m. Taking the total differential of the dispersion relation $\omega^2 - |\vec{k}|^2 = m^2$ for fixed $\vec{k}$, one finds that

$$\frac{\partial}{\partial \omega} = \frac{\omega}{m}\, \frac{\partial}{\partial m}\,. \tag{16.41}$$

Hence,

$$\begin{aligned}
(t-t')\, U_m^{t,t'} &= \sum_{\pm} \Pi_\pm \Big(\pm \mathrm{i}\, \frac{\omega}{m} \frac{\partial}{\partial m}\, \mathrm{e}^{\mp \mathrm{i}\omega(t-t')} \Big) \\
&= \frac{\partial}{\partial m} \sum_{\pm} \Big(\pm \mathrm{i} \frac{\omega}{m}\, \Pi_\pm\, \mathrm{e}^{\mp \mathrm{i}\omega(t-t')} \Big) - \sum_{\pm} \left(\frac{\partial}{\partial m} \Big[\pm \mathrm{i} \frac{\omega}{m}\, \Pi_\pm \Big] \right) \mathrm{e}^{\mp \mathrm{i}\omega(t-t')}\,.
\end{aligned} \tag{16.42}$$

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Computing the operators in the round brackets using (16.33) gives the identities (16.37) and (16.38). Estimating these formulas, one obtains bounds that are at most linear in $|\vec{k}|$, proving (16.39). □

This method can be iterated to generate more factors of $t - t'$. In the next lemma, we prove at least quadratic decay in time. For later use, it is preferable to formulate the result in position space.

Lemma 16.3.4 *The time evolution operator in the vacuum has the representation*

$$U_m^{t,t'} = \frac{1}{(t-t')^2} \left(\frac{\partial^2}{\partial m^2} A_m^{t,t'} + \frac{\partial}{\partial m} B_m^{t,t'} + C_m^{t,t'} \right), \tag{16.43}$$

with operators

$$A_m^{t,t'}, B_m^{t,t'}, C_m^{t,t'} \;:\; W^{2,2}(\mathcal{N}_{t'}, S\mathscr{M}) \to L^2(\mathcal{N}_t, S\mathscr{M}), \tag{16.44}$$

which are bounded uniformly in time by

$$\|A_m^{t,t'}(\phi)\|_t + \|B_m^{t,t'}(\phi)\|_t + \|C_m^{t,t'}(\phi)\|_t \leq c\, \|\phi\|_{W^{2,2}}, \tag{16.45}$$

where c is a constant that depends only on m.

Proof A straightforward computation using exactly the same methods as in Lemma 16.3.3 yields the representation

$$(t-t')^2\, U_m^{t,t'}(\vec{k}) = \frac{\partial^2}{\partial m^2} A_m^{t,t'}(\vec{k}) + \frac{\partial}{\partial m} B_m^{t,t'}(\vec{k}) + C_m^{t,t'}(\vec{k}), \tag{16.46}$$

where the operators $A_m^{t,t'}$, $B_m^{t,t'}$ and $C_m^{t,t'}$ are bounded by

$$\|A_m^{t,t'}(\vec{k})\| + \|B_m^{t,t'}(\vec{k})\| + \|C_m^{t,t'}(\vec{k})\| \leq \frac{C}{m} \left(1 + \frac{|\vec{k}|}{m} + \frac{|\vec{k}|^2}{m^2} \right), \tag{16.47}$$

with a numerical constant $C > 0$. We remark that, compared to (16.36), the right-hand side of (16.47) involves an additional $1/m$. This prefactor is necessary for dimensional reasons because the additional factor $t - t'$ in (16.46) (compared to (16.36)) brings in an additional dimension of length (and in natural units, the factor $1/m$ also has the dimension of length). The additional summand $|\vec{k}|^2/m^2$ in (16.47) can be understood from the fact that applying (16.41) generates a factor of ω/m, which for large $|\vec{k}|$ scales like $|\vec{k}|/m$.

Translating this result to position space and keeping in mind that the vector $\vec{k}$ corresponds to the derivative $-\mathrm{i}\vec{\nabla}$, we obtain the result. □

Proof of Lemma 16.3.1. First of all, the Schwarz inequality gives

$$\big\| (\mathfrak{p}\psi)|_t \big\|_t \leq \int_I \|\psi_m\|_m \, \mathrm{d}m \leq \sqrt{|I|}\, \|\psi\|. \tag{16.48}$$

Thus, it remains to show the decay for large t, that is,

$$\big\| (\mathfrak{p}\psi)|_t \big\|_t \leq \frac{C\,|I|}{t^2} \sup_{m \in I} \sum_{b=0}^{2} \|\partial_m^b (\psi_m)|_{t=0}\|_{W^{2,2}}. \tag{16.49}$$

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We apply Lemma 16.3.4 and integrate by parts in m to obtain

$$\begin{aligned}(\mathfrak{p}\psi)|_t &= \int_I U_m^{t,0}\, \psi_m|_{t=0}\, \mathrm{d}m \\ &= \frac{1}{t^2} \int_I \left(\partial_m^2 A_m^{t,0} + \partial_m B_m^{t,0} + C_m^{t,0}\right) \psi_m|_{t=0}\, \mathrm{d}m \\ &= \frac{1}{t^2} \int_I \Big(A_m^{t,0}\, (\partial_m^2 \psi_m|_{t=0}) - B_m^{t,0}\, (\partial_m \psi_m|_{t=0}) + C_m^{t,0}\, \psi_m|_{t=0} \Big)\, \mathrm{d}m\,. \end{aligned} \tag{16.50}$$

Taking the norm and using (16.45) gives (16.49). □

We finally note that the previous estimates are not optimal for two reasons. First, the pointwise quadratic decay in (16.20) is more than what is needed for the convergence of the integral in (16.25). Second and more importantly, the Schwarz inequality (16.23) does not catch the optimal scaling behavior in $\vec{k}$. This is the reason why the constant in (15.21) involves derivatives of ψ_m (cf. (16.25)), making it impossible to prove the inequality (15.23), which arises in the strong mass oscillation property. In order to improve the estimates, one needs to use Fourier methods both in space and time, as will be explained in the next section.

16.4 Proof of the Strong Mass Oscillation Property in the Minkowski Vacuum

Theorem 16.4.1 *The vacuum Dirac operator in Minkowski space has the strong mass oscillation property with domain* (16.15).

Our proof relies on a Plancherel argument in spacetime. It also provides an alternative method for establishing the weak mass oscillation property.

Proof of Theorem 16.4.1. Let $\psi = (\psi_m)_{m \in I} \in \mathcal{H}^\infty$ be a family of solutions of the Dirac equation for a varying mass parameter in the Minkowski vacuum. Using Proposition 13.6.1, one can express ψ_m in terms of its values at time $t = 0$ by

$$\psi_m(x) = 2\pi \int_{\mathbb{R}^3} k_m(x, (0, \vec{y}))\, \gamma^0\, \psi_m|_{t=0}(\vec{y})\, \mathrm{d}^3 y\,. \tag{16.51}$$

We now take the Fourier transform, denoting the four-momentum by k. Using (16.29), we obtain

$$\begin{aligned}\psi_m(k) &= 2\pi k_m(k)\, \gamma^0 \hat{\psi}_m^0(\vec{k}) \\ &= 2\pi\, \delta(k^2 - m^2)\, \epsilon(k^0)\, (\not{k} + m)\, \gamma^0 \hat{\psi}_m^0(\vec{k})\,, \end{aligned} \tag{16.52}$$

where $\hat{\psi}_m^0(\vec{k})$ denotes the spatial Fourier transform of $\psi_m|_{t=0}$ (in order to avoid an ambiguity of notation, the hat of the Fourier transform in spacetime was omitted). Obviously, this is a distribution supported on the mass shell. In particular, it is not square integrable over $\mathbb{R}^4$.

Integrating over m, we obtain the following function

$$(\mathfrak{p}\psi)(k) = 2\pi\, \chi_I(m)\, \frac{1}{2m}\, \epsilon(k^0)\, (\not{k} + m)\, \gamma^0 \hat{\psi}_m^0(\vec{k})\Big|_{m=\sqrt{k^2}}\,, \tag{16.53}$$

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where m now is a function of the momentum variables. Since the function $\psi_m|_{t=0}$ is compactly supported and smooth in the spatial variables, its Fourier transform $\hat{\psi}^0_m(\vec{k})$ has rapid decay. This shows that the function (16.53) is indeed square integrable. Using Plancherel, we see that condition (a) in Definition 15.2.2 is satisfied. Moreover, the operator T is simply the operator of multiplication by $\sqrt{k^2}$, so that condition (b) obviously holds. This again shows the weak mass oscillation property.

In order to prove the strong mass oscillation property, we need to compute the inner product $<\mathfrak{p}\psi|\mathfrak{p}\phi>$. To this end, we first write this inner product in momentum space as

$$\begin{aligned} &<\mathfrak{p}\psi|\mathfrak{p}\phi> \\ &= \int \frac{\mathrm{d}^4k}{(2\pi)^4} 4\pi^2\, \chi_I(m)\, \frac{1}{4m^2}\, \prec (\not{k}+m)\,\gamma^0 \hat{\psi}^0_m(\vec{k}) \,|\, (\not{k}+m)\,\gamma^0 \hat{\phi}^0_m(\vec{k}) \succ \Big|_{m=\sqrt{k^2}} \\ &= \int \frac{\mathrm{d}^4k}{4\pi^2}\, \chi_I(m)\, \frac{1}{2m}\, \prec \gamma^0 \hat{\psi}^0_m(\vec{k}) \,|\, (\not{k}+m)\,\gamma^0 \hat{\phi}^0_m(\vec{k}) \succ \Big|_{m=\sqrt{k^2}} . \end{aligned} \tag{16.54}$$

Reparametrizing the k^0-integral as an integral over m, we obtain

$$\begin{aligned} <\mathfrak{p}\psi|\mathfrak{p}\phi> = \frac{1}{4\pi^2} \int_I \mathrm{d}m \int_{\mathbb{R}^3} \frac{\mathrm{d}^3k}{2\,|k^0|} \\ \times\; \prec \gamma^0 \hat{\psi}^0_m(\vec{k}) \,|\, (\not{k}+m)\,\gamma^0 \hat{\phi}^0_m(\vec{k}) \succ \Big|_{k^0=\pm\sqrt{|\vec{k}|^2+m^2}} . \end{aligned} \tag{16.55}$$

Estimating the inner product with the Schwarz inequality and applying Plancherel's theorem, one finds

$$\begin{aligned} |<\mathfrak{p}\psi|\mathfrak{p}\phi>| &\leq \frac{1}{4\pi^2} \int_I \mathrm{d}m \int_{\mathbb{R}^3} \|\hat{\psi}^0_m(\vec{k})\|\, \|\hat{\phi}^0_m(\vec{k})\|\; \mathrm{d}^3k \\ &\leq 2\pi \int_I \|\psi_m\|_m\, \|\phi_m\|_m\; \mathrm{d}m\,. \end{aligned} \tag{16.56}$$

Thus, the inequality (15.23) holds. □

Apart from completing the proof of the strong mass oscillation property, the computation in the above proof also tells us what the fermionic signature operator is. In order to see this, we return to the formula (16.55). Applying Plancherel's theorem and using (15.2), we conclude that

$$<\mathfrak{p}\psi|\mathfrak{p}\phi> = \int_I (\psi^0_m \,|\, \mathcal{S}_m\, \phi^0_m)_m\; \mathrm{d}m\,, \tag{16.57}$$

where $\mathcal{S}_m$ is the multiplication operator in momentum space

$$\mathcal{S}_m(\vec{k}) := \sum_{k^0=\pm\omega(\vec{k})} \frac{\not{k}+m}{2\,\omega(\vec{k})}\,\gamma^0 = \frac{\vec{k}\vec{\gamma}+m}{\omega(\vec{k})}\,\gamma^0\,. \tag{16.58}$$

Comparing (16.57) with (15.31), one sees that the matrix $\mathcal{S}_m(\vec{k})$ is indeed the fermionic signature operator, considered as a multiplication operator in momentum space. By direct computation, one verifies that the matrix $\mathcal{S}_m(\vec{k})$ has

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eigenvalues ± 1 (here one can use that $\mathcal{S}_m = \Pi_+ - \Pi_-$ with $\Pi_\pm$ as introduced in (16.33)).

16.5 Exercises

Exercise 16.1 This exercise recalls basics on the principal value in one dimension

$$\frac{1}{2} \lim_{\varepsilon \searrow 0} \left(\frac{1}{x - \mathrm{i}\varepsilon} + \frac{1}{x + \mathrm{i}\varepsilon} \right) =: \frac{\mathrm{PP}}{x} \,. \tag{16.59}$$

(a) Repeat the method in Exercise 2.23 to show that the limit of the left-hand side of (16.59) exists for any $\eta \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$. Derive a corresponding estimate that shows that PP is a well-defined tempered distribution.

(b) Show that for any $\eta \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$,

$$\mathrm{PP}(\eta) = \lim_{\varepsilon \searrow 0} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{\eta(x)}{x} \, dx \,. \tag{16.60}$$

Exercise 16.2 The goal of this exercise is to justify that the one-dimensional relations

$$\lim_{\varepsilon \searrow 0} \left(\frac{1}{x - \mathrm{i}\varepsilon} - \frac{1}{x + \mathrm{i}\varepsilon} \right) = 2\pi i \, \delta(x), \tag{16.61}$$

$$\frac{1}{2} \lim_{\varepsilon \searrow 0} \left(\frac{1}{x - \mathrm{i}\varepsilon} + \frac{1}{x + \mathrm{i}\varepsilon} \right) =: \frac{\mathrm{PP}}{x} \tag{16.62}$$

can be used in the four-dimensional setting to obtain the identity

$$\lim_{\varepsilon \searrow 0} \frac{1}{r^2 + (\varepsilon + \mathrm{i}t)^2} = \lim_{\varepsilon \searrow 0} \frac{1}{r^2 - t^2 + \mathrm{i}\varepsilon t} = -\frac{\mathrm{PP}}{\xi^2} - \mathrm{i}\pi \, \delta(\xi^2) \, \epsilon(\xi^0) \,. \tag{16.63}$$

(a) Let T be a distribution on $\mathbb{R}$, $\Omega \subset \mathscr{M}$ an open subset of Minkowski space and $f : \Omega \to \mathbb{R}$ a smooth function with nowhere vanishing gradient. Show that the relation

$$\big(f^* T\big)(\eta) := T\big(\phi_f(\eta)\big) \,, \qquad \eta \in C_0^\infty(\Omega), \tag{16.64}$$

with

$$\phi_f(\eta)(t) := \frac{\partial}{\partial t} \int_\Omega \Theta\big(t - f(x)\big) \, \eta(x) \, \mathrm{d}^4 x \tag{16.65}$$

(where Θ is the Heaviside function) defines f^*T as a distribution on Ω (this is the so-called *pullback* of T under f; for details, see [89, Section 7.2]).

(b) Choosing Ω as the half-space in the future, $\Omega = \{x \in \mathscr{M}, x^0 > 0\}$, one can rewrite the expression on the left-hand side of (16.63) as

$$\lim_{\varepsilon \searrow 0} \frac{1}{r^2 - t^2 + \mathrm{i}\varepsilon} \,. \tag{16.66}$$

Use (a) to conclude that this expression is a well-defined distribution for any $\varepsilon > 0$. Show that the limit $\varepsilon \searrow 0$ exists in the distributional sense.

(c) Repeating the procedure of (b) for the half-space in the past, one obtains a distribution on $\mathscr{M} \setminus \{t = 0\}$. Show that this distribution coincides with the

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limit in (16.63). *Hint:* Similar to Exercise 2.23, one can estimate the behavior at the origin with Lebesgue's dominated convergence theorem.

Exercise 16.3 This exercise is devoted to the *advanced Green's operator* $s_m^\vee$.

(a) Assume that $m > 0$. Show that the limit $\nu \searrow 0$ in (16.4) exists in the distributional sense.

(b) Show that the limit $\nu \searrow 0$ in (16.4) also exists in the massless case $m = 0$ and that

$$\lim_{m \searrow 0} s_m^\vee(k) = s_0^\vee(k) \qquad \text{as a distribution} . \tag{16.67}$$

Hint: Proceed similarly to Exercise 16.2.

(c) Consider the Fourier integral in the q^0-variable

$$\int_{-\infty}^{\infty} \frac{1}{q^2 - m^2 - \mathrm{i}\nu q^0} \, \mathrm{e}^{\mathrm{i}q^0 t} \, \mathrm{d}q^0 . \tag{16.68}$$

Show with residues that this integral vanishes for sufficiently small ν if $t < 0$.

(d) Argue with Lorentz invariance to prove the left-hand side of (16.5).

Exercise 16.4 Modifying the location of the poles in (16.4) gives rise to the distribution

$$s_m^F(k) := \lim_{\nu \searrow 0} \frac{\not{k} + m}{k^2 - m^2 + \mathrm{i}\nu} . \tag{16.69}$$

This is the well-known *Feynman propagator*, which is often described intuitively by saying that "positive frequencies move to the future and negative frequencies move to the past." Make this sentence precise by a computation similar to that in Exercise 16.3 (c).

Exercise 16.5 Given $\omega \in \mathbb{R}$, we consider the ordinary differential operator $\mathcal{D} = \mathrm{i}\partial_t + \omega$.

(a) Construct the advanced and retarded Green's operators, which satisfy in analogy to (16.8) the equation

$$\mathcal{D}_t s(t, t') = \delta(t - t') . \tag{16.70}$$

(b) Compute the resulting causal fundamental solution according to (16.9). How is it related to the time evolution operator $U^{t,t'}$? On which Hilbert space does the time evolution operator act as a unitary operator?

Exercise 16.6 Consider the massless Dirac equation $\mathcal{D}\psi = 0$ in the two-dimensional spacetime cylinder $\mathbb{R} \times S^1$, that is,

$$\mathcal{D} = \mathrm{i} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_t + \mathrm{i} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_\varphi , \tag{16.71}$$

with $t \in \mathbb{R}$ and $\varphi \in (0, 2\pi)$.

(a) Choose the spin inner product such that the Dirac matrices become symmetric. What is the resulting spacetime inner product $<.|.>$? What is the scalar product $(.|.)$?

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(b) Employ for $k \in \mathbb{Z}$ the separation ansatz

$$\psi(t,\varphi) = \mathrm{e}^{-\mathrm{i}k\varphi}\chi(t) \qquad \text{with} \qquad \chi(t) \in \mathbb{C}^2\,. \tag{16.72}$$

Derive the resulting ordinary differential equation for χ. Compute the time evolution operator for this ODE. *Hint:* Use the result of Exercise 16.5.

(c) Use a Fourier series decomposition in order to deduce a series representation of the time evolution operator of the Dirac operator on $\mathbb{R} \times S^1$. Try to carry out the infinite series to obtain a closed expression for $U^{t,t'}$. How can one see finite propagation speed?

Exercise 16.7 As in Exercise 16.6, we consider the two-dimensional massless Dirac equation.

(a) Adapt the formulas for the advanced and retarded Green's operators in momentum space to the two-dimensional massless case.
(b) Compute the Fourier transform to obtain $s^\vee(x,y)$ and $s^\wedge(x,y)$.
(c) Use the result of (b) to compute the causal fundamental solution and the time evolution operator.
(d) How can one see finite propagation speed? How is the obtained formula related to the formula in Exercise 16.6 (c)?

Exercise 16.8 Verify the relations (16.35) by direct computation starting from the definition (16.33).

Exercise 16.9 Verify by formal computation that in the Minkowski vacuum, the fundamental solution k_m and the Green's operator s_m defined by

$$s_m := \frac{1}{2}\left(s_m^\vee + s_m^\wedge\right), \tag{16.73}$$

satisfy the distributional relations in the mass parameters m and m'

$$k_m\, k_{m'} = \delta(m-m')\, p_m, \tag{16.74}$$

$$k_m\, s_{m'} = s_{m'}\, k_m = \frac{\mathrm{PP}}{m-m'}\, k_m\,, \tag{16.75}$$

where PP denotes the principal part, and p_m is the distribution

$$p_m(k) = (\not{k} + m)\, \delta(k^2 - m^2)\,. \tag{16.76}$$

Hint: By a "formal computation," we mean that you do not need to evaluate weakly in the mass with test functions.

Exercise 16.10 Proceed similarly to Exercise 16.9 to derive a relation for the operator product $s_m^\vee s_{m'}^\vee$. Derive the relation

$$s_m\, s_{m'} = \frac{\mathrm{PP}}{m-m'}\,(s_m - s_{m'}) + \pi^2\, \delta(m-m')\, p_m\,. \tag{16.77}$$

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