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Edge Distribution of Graphs with Few Copies of a Given Graph

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We show that if a graph contains few copies of a given graph, then its edges are distributed rather unevenly.

In particular, for all $\varepsilon > 0$ and $r \geqslant 2$, there exist $\xi = \xi(\varepsilon, r) > 0$ and $k = k(\varepsilon, r)$ such that, if n is sufficiently large and $G = G(n)$ is a graph with fewer than ξn^r r-cliques, then there exists a partition $V(G) = \cup_{i=0}^{k} V_i$ such that

$$|V_i| = \lfloor n/k \rfloor \quad \text{and} \quad e(W_i) < \varepsilon |V_i|^2$$

for every $i \in [k]$.

We deduce the following slightly stronger form of a conjecture of Erdős.

For all $c > 0$ and $r \geqslant 3$, there exist $\xi = \xi(c, r) > 0$ and $\beta = \beta(c, r) > 0$ such that, if n is sufficiently large and $G = G(n, \lceil cn^2 \rceil)$ is a graph with fewer than ξn^r r-cliques, then there exists a partition $V(G) = V_1 \cup V_2$ with $|V_1| = \lfloor n/2 \rfloor$ and $|V_2| = \lceil n/2 \rceil$ such that

$$e(V_1, V_2) > (1/2 + \beta) e(G).$$

1. Introduction

Our graph-theoretic notation is standard (*e.g.*, see [6]). In particular, $G(n)$ stands for a graph of order n and $G(n, m)$ is a graph of order n and size m. Given two graphs H and G, we write $k_H(G)$ for the number of induced copies of H in G and $k_r(G)$ for $k_{K_r}(G)$. If $U \subset V(G)$, we write $e(U)$ for $e(G[U])$ and $k_r(U)$ for $k_r(G[U])$. We write $\Gamma(u)$ for the set of neighbours of a vertex u and set $d(u) = |\Gamma(u)|$. A partition $V = \cup_{i=0}^{k} V_i$ is called *equitable* if $|V_0| < k$ and $|V_1| = \cdots = |V_k|$.

In [9] Erdős raised the following problem (see also [5], p. 363).

Conjecture 1.1. *Let $c > 0$. Suppose $G = G(n, \lceil cn^2 \rceil)$ is such that*

$$e(W) \geqslant (c/4 + o(1))n^2$$

for every $W \subset V(G)$ with $|W| = \lfloor n/2 \rfloor$. Then, for every fixed r and sufficiently large n, the graph G contains K_r.

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This conjecture was proved in [12], where, in fact, we proved the following more general result.

Theorem 1.2. *For every $c > 0$ and $r \geqslant 3$, there exists $\beta = \beta(c,r) > 0$ such that, for every K_r-free graph $G = G(n, \lceil cn^2 \rceil)$, there exists a partition $V(G) = V_1 \cup V_2$ with $|V_1| = \lfloor n/2 \rfloor$, $|V_2| = \lceil n/2 \rceil$, and*

$$e(V_1, V_2) > (1/2 + \beta)e(G). \tag{1.1}$$

Somewhat later Kohayakawa and Rödl [10] gave another proof of Conjecture 1.1 based on Szemerédi's Regularity Lemma; however, it seems that their method does not imply Theorem 1.2.

Observe that inequality (1.1) gives a lower bound on the MaxCut function for dense K_r-free graphs; it differs significantly from those found in [1], [2] and [4]. Another result in this vein is stated in Theorem 3.1.

In this note we extend Theorem 1.2 using the following theorem.

Theorem 1.3. *For all $0 < \varepsilon < 1$, $r \geqslant 2$, there exist $\rho = \rho(\varepsilon, r) > 0$ and $k = k(\varepsilon, r)$ such that, if n is sufficiently large and $G = G(n)$ is a graph with $k_r(G) < \rho n^r$, then there exists an equitable partition $V(G) = \cup_{i=0}^{k} V_i$ such that*

$$e(V_i) < \varepsilon |V_i|^2$$

for every $i \in [k]$.

Theorem 1.3 in turn is implied by the following general statement.

Theorem 1.4. *For all $0 < \varepsilon < 1$, $r \geqslant 2$, there exist $\xi = \xi(\varepsilon, r) > 0$ and $l = l(\varepsilon, r)$ such that, if H is a graph of order r, G is a graph of sufficiently large order n, and $k_H(G) < \xi r n^r$, then there exists an equitable partition $V(G) = \cup_{i=0}^{l} V_i$ such that, for every $i \in [l]$,*

$$\text{either } e(V_i) < \varepsilon |V_i|^2 \quad \text{or} \quad e(V_i) > (1 - \varepsilon)|V_i|^2.$$

Theorem 1.4 strengthens earlier results of Rödl (Theorem 1, [14]) and of Kohayakawa and Rödl (Theorem 18, [10]); at the same time, as suggested by a referee, with some additional work Theorem 1.4 can be deduced from these results. A self-contained proof of Theorem 1.4 can be found also in [13]. Unfortunately all known proofs of Theorem 1.4 are based on Szemerédi's Regularity Lemma and the bound on $l(\varepsilon, r)$ – a tower of size at least $1/\varepsilon$ – is too large for many applications.

The situation with Theorem 1.3 seems somewhat better: although it can be derived from Theorem 1.4, we shall prove it without using the Regularity Lemma; moreover, we show that $k(\varepsilon, r) = O(\varepsilon^{-r+1})$.

It should be noted that Alon, Fisher, and Newman [3] proved a result similar to Theorem 1.4 for bipartite graphs. Their proof does not use the Regularity Lemma and gives a polynomial bound on the size of the partition.

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Using Theorem 1.3 we extend Theorem 1.2 and, for that matter, Theorem 20 of [10], as follows.

Theorem 1.5. *For all $c > 0$ and $r \geqslant 3$, there exist $\xi = \xi(c,r) > 0$ and $\beta = \beta(c,r) > 0$ such that, for n sufficiently large and every graph $G = G(n, \lceil cn^2 \rceil)$, if $k_r(G) < \xi n^r$ then there exists a partition $V(G) = V_1 \cup V_2$ with $|V_1| = \lfloor n/2 \rfloor$, $|V_2| = \lceil n/2 \rceil$, and*

$$e(V_1, V_2) > (1/2 + \beta)e(G). \tag{1.2}$$

The rest of this note is organized as follows. We prove Theorem 1.3 in Section 2, and in Section 3 we prove Theorem 1.5. Some further details can be found in [13].

2. Proof of Theorem 1.3

We shall make use of the following key lemma.

Lemma 2.1. *Let $s \geqslant 1$, $r \geqslant 2$ be integers and $0 < \varepsilon < 1$. For every graph G with*

$$|G| \geqslant \frac{r-1}{\varepsilon^{r-2}} s \text{ and } k_r(G) \leqslant \varepsilon \frac{s^r}{r!},$$

there exists $U \subset V(G)$ with $|U| = s$, and $e(U) < \varepsilon s^2$.

Proof. We shall use induction on r. If $r = 2$, there is nothing to prove since $e(G) = k_2(G) \leqslant (\varepsilon/2)s^2$ implies that $e(U) < \varepsilon s^2$ for any $U \subset V(G)$. Now, let $r > 2$ and assume that the assertion holds for smaller values of r; set

$$M = \left\{ u : u \in V(G),\ d(u) \geqslant \frac{r-2}{\varepsilon^{r-3}} s \right\}.$$

Suppose first that $|M| \geqslant s$. Since

$$rk_r(G) = \sum_{u \in V(G)} k_{r-1}(\Gamma(u)) \geqslant \sum_{u \in M} k_{r-1}(\Gamma(u)),$$

there exists $u \in M$ with

$$k_{r-1}(\Gamma(u)) \leqslant \frac{rk_r(G)}{|M|} \leqslant \varepsilon \frac{s^{r-1}}{(r-1)!}.$$

Hence, in view of $d(u) \geqslant \frac{r-2}{\varepsilon^{r-3}} s$, the induction assumption implies that $e(U) < \varepsilon s^2$ for some $U \in \Gamma(u)$ with $|U| = s$, completing the proof if $|M| \geqslant s$.

If $|M| \leqslant s - 1$ then

$$|G - M| - 1 \geqslant \frac{r-1}{\varepsilon^{r-2}} s - s > \frac{r-2}{\varepsilon^{r-2}} s > \frac{\Delta(G-M)}{\varepsilon},$$

and so

$$e(G - M) \leqslant \frac{|G-M|}{2} \Delta(G-M) \leqslant \varepsilon \binom{|G-M|}{2}.$$

Thus, $e(U) \leqslant \varepsilon \binom{s}{2} < \varepsilon s^2$ for some $U \subset V(G-M)$ with $|U| = s$, completing the proof. □

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Proof of Theorem 1.3. We may and shall assume that $\varepsilon < (r-1)^{-1/(r-2)}$. Set

$$\delta = \frac{\varepsilon}{6},$$
$$\rho = \rho(\varepsilon, r) = \frac{\delta^{r^2-r+1}}{(r-1)^r r!},$$
$$k = k(\varepsilon, r) = \left\lfloor (1-\delta)\frac{r-1}{\delta^{r-1}} \right\rfloor,$$

and suppose $k_r(G) \leqslant \rho n^r$. We shall construct the partition of $V(G)$ in two steps: (a) we find a partition $V(G) = \cup_{i=0}^{k} U_i$ such that $|U_0| < 2\delta n$, $|U_1| = \cdots = |U_k|$, and $e(U_i) < \delta |U_i|^2$ for all $i \in [k]$; (b) we evenly distribute all but $n - k\lfloor n/k \rfloor$ vertices of U_0 among the sets $U_1, \ldots, U_k$.

Set

$$s = \left\lceil \frac{\delta^{r-1}}{r-1} n \right\rceil,$$

and observe that if $G' \subset G$ then

$$k_r(G') \leqslant k_r(G) \leqslant \frac{\delta^{r^2-r+1}}{(r-1)^r r!} n^r \leqslant \delta \frac{1}{r!} \left(\frac{\delta^{r-1}}{r-1} n \right)^r \leqslant \delta \frac{s^r}{r!}.$$

We select the sets $U_1, \ldots, U_p$ as follows. If

$$|G - U_1 - \cdots - U_j| < \frac{r-1}{\delta^{r-2}} s, \tag{2.1}$$

set $p = j$ and stop the selection; otherwise Lemma 2.1 implies that there exists

$$U \subset V(G - U_1 - \cdots - U_j)$$

with $|U| = s$ and $e(U) < \delta s^2$. Set $U_{j+1} = U$ and continue the selection process. Since the process stops only if (2.1) holds, we see that

$$\begin{aligned} n - sp &< \frac{r-1}{\delta^{r-2}} \left\lceil \frac{\delta^{r-1}}{r-1} n \right\rceil = \frac{r-1}{\delta^{r-1} n} \left\lceil \frac{\delta^{r-1}}{r-1} n \right\rceil \delta n \\ &< \frac{r-1}{\delta^{r-1} n} \left(\frac{\delta^{r-1}}{r-1} n + 1 \right) \delta n < \delta n + \frac{r-1}{\delta^{r-2}}. \end{aligned}$$

Consequently,

$$\begin{aligned} p &> \frac{(1-\delta)n - \frac{r-1}{\delta^{r-2}}}{s} = \frac{(1-\delta)n - \frac{r-1}{\delta^{r-2}}}{\lceil \frac{\delta^{r-1}}{r-1} n \rceil} > \frac{(1-\delta)n - \frac{r-1}{\delta^{r-2}}}{\frac{\delta^{r-1}}{r-1} n + 1} \\ &= \frac{(1-\delta) - \frac{r-1}{\delta^{r-2} n}}{\frac{\delta^{r-1}}{r-1} + \frac{1}{n}} = \left((1-\delta)\frac{r-1}{\delta^{r-1}} \right) \left(\frac{1 - \frac{r-1}{(1-\delta)\delta^{r-2} n}}{1 + \frac{r-1}{\delta^{r-1} n}} \right) \end{aligned}$$

and hence, for n large, it follows that $p \geqslant k$. Setting

$$U_0 = V(G) \setminus \left(\cup_{i=1}^{k} U_i \right),$$

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in view of $\delta < (r-1)^{-1/(r-2)}$, we see that

$$\begin{aligned}|U_0| = n - ks &= n - \left\lfloor (1-\delta)\frac{r-1}{\delta^{r-1}} \right\rfloor \left\lceil \frac{\delta^{r-1}n}{r-1} \right\rceil \\ &< n - \left((1-\delta)\frac{r-1}{\delta^{r-1}} - 1 \right) \frac{\delta^{r-1}n}{r-1} \\ &\leqslant n - (1-\delta)n + \frac{\delta^{r-1}n}{r-1} \leqslant 2\delta n,\end{aligned}$$

completing step (a).

To complete step (b) let $U_0 = \cup_{i=0}^{k} W_i$ be an arbitrary equitable partition and set $t = \lfloor |U_0|/k \rfloor$; observe that $|W_1| = \cdots = |W_k| = t$ and $t < 2\delta n/k$. Set $V_0 = W_0$ and $V_i = U_i \cup W_i$ for every $i \in [q]$. Then, for every $i \in [q]$,

$$\begin{aligned}e(V_i) = e(U_i) + e(U_i, W_i) + e(W_i) &< \delta s^2 + ts + \frac{t^2}{2} \\ &< \delta s^2 + \frac{2\delta ns}{k} + \frac{2\delta^2 n^2}{k^2} \\ &\leqslant \delta s^2 + \frac{2\delta^r ns}{(1-\delta)(r-1)} + \frac{2\delta^{2r} n^2}{(1-\delta)^2 (r-1)^2} \\ &\leqslant \delta s^2 + \frac{2\delta}{(1-\delta)} s^2 + \frac{2\delta^2}{(1-\delta)^2} s^2 \\ &\leqslant \left(\delta + 4\frac{\delta}{1-\delta} \right) s^2 < \left(\frac{1}{6} + 4\frac{1/6}{1-1/6} \right) \varepsilon s^2 < \varepsilon |V_i|^2.\end{aligned}$$

This completes the proof of Theorem 1.3. □

3. Bipartitions of low density

In this section we deduce Theorem 1.5. We first state and prove a preliminary result that may be of interest in itself. Although this result is close to the study of judicious bipartitions of dense graphs with few r-cliques, it significantly differs from known general results as in [2], [7] and [8].

Theorem 3.1. *For all $r \geqslant 3$, $c > 0$ and $\varepsilon > 0$, there exist $\varrho = \varrho(\varepsilon, c, r) > 0$ and $\gamma = \gamma(\varepsilon, c, r) > 0$ such that, for every graph $G = G(n, \lceil cn^2 \rceil)$ of sufficiently large order n, if $k_r(G) < \varrho n^r$ then there exists a partition $V(G) = V_1 \cup V_2$ such that*

$$e(V_1) < \varepsilon |V_1|^2 \quad \text{and} \quad e(V_2) < (c-\gamma)|V_2|^2. \tag{3.1}$$

Proof. Let $\rho(\varepsilon, r)$ and $k(\varepsilon, r)$ be as defined in Theorem 1.3. Set

$$\begin{aligned}\sigma &= \min\{c/4, \varepsilon\}, \\ \varrho &= \varrho(\varepsilon, c, r) = \rho(\sigma, r), \\ k &= k(\sigma, r), \\ \gamma &= \gamma(\varepsilon, c, r) = \frac{c - 2\sigma}{(k-1)^2},\end{aligned} \tag{3.2}$$

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and suppose that $G = G(n, \lceil cn^2 \rceil)$ and $k_r(G) < \varrho n^r$. If n is sufficiently large, we apply Theorem 1.3, and find an equitable partition $V(G) = \cup_{i=0}^{k} W_i$ such that $|W_i| = \lfloor n/k \rfloor$ and $e(W_i) \leqslant \sigma \lfloor n/k \rfloor^2$ for every $i \in [k]$. Set $V' = V \setminus W_0$. We may and shall assume that W_1 satisfies

$$e(W_1, V' \setminus W_1) = \max_{i \in [k]} \{e(W_i, V' \setminus W_i)\}.$$

Set $V_1 = W_1$ and $V_2 = V \setminus W_1$. We shall prove that the partition $V(G) = V_1 \cup V_2$ satisfies (3.1). Indeed, we immediately have $e(V_1) < \sigma |V_1|^2 \leqslant \varepsilon |V_1|^2$; thus, all we have to prove is that, for n sufficiently large, $e(V_2) < (c - \gamma)|V_2|^2$, that is to say,

$$e(V \setminus W_1) < (c - \gamma)\left(n - \left\lfloor \frac{n}{k} \right\rfloor\right)^2. \tag{3.3}$$

We have

$$\begin{aligned} e(V') &= \sum_{i=1}^{k} e(W_i) + \frac{1}{2} \sum_{i=1}^{k} e(W_i, V' \setminus W_i) \leqslant \sum_{i=1}^{k} e(W_i) + \frac{k}{2} e(W_1, V' \setminus W_1) \\ &\leqslant \sigma k \left\lfloor \frac{n}{k} \right\rfloor^2 + \frac{k}{2} e(W_1, V' \setminus W_1), \end{aligned}$$

and, in view of

$$e(V') \geqslant e(V) - \sum_{u \in W_0} d(u) > e(V) - kn,$$

we find that

$$e(W_1, V' \setminus W_1) \geqslant \frac{2e(V')}{k} - 2\sigma \left\lfloor \frac{n}{k} \right\rfloor^2 > \frac{2e(V)}{k} - 2n - 2\sigma \left\lfloor \frac{n}{k} \right\rfloor^2.$$

Hence,

$$\begin{aligned} e(V \setminus W_1) &= e(V) - e(W_1) - e(W_1, V \setminus W_1) \leqslant e(V) - e(W_1, V' \setminus W_1) \\ &\leqslant \frac{k-2}{k} e(V) + 2n + 2\sigma \left\lfloor \frac{n}{k} \right\rfloor^2 \\ &= \frac{k-2}{k} \lceil cn^2 \rceil + 2n + 2\sigma \left\lfloor \frac{n}{k} \right\rfloor^2. \end{aligned} \tag{3.4}$$

Assume for a contradiction that there are arbitrary large values of n for which (3.3) is false; in view of (3.4), this implies that

$$(c - \gamma)\left(n - \left\lfloor \frac{n}{k} \right\rfloor\right)^2 \leqslant \frac{k-2}{k} \lceil cn^2 \rceil + 2n + 2\sigma \left\lfloor \frac{n}{k} \right\rfloor^2.$$

Dividing both sides by n^2 and taking the limit, we find that

$$(c - \gamma)\left(1 - \frac{1}{k}\right)^2 \leqslant \frac{k-2}{k} c + \frac{2\sigma}{k^2},$$

and hence

$$c - 2\sigma \leqslant \gamma (k-1)^2,$$

a contradiction to (3.2). The proof is completed. □

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Proof of Theorem 1.5. Let $\gamma(\varepsilon, c, r)$ and $\varrho(\varepsilon, c, r)$ be as defined in Theorem 1.5. Set $\varepsilon = c$, $\beta(c, r) = \gamma(c, c, r)$, $\xi(c, r) = \varrho(c, c, r)$. If the conditions of Theorem 1.5 hold then Theorem 3.1 implies that

$$e(V_1) < c|V_1|^2 \quad \text{and} \quad e(V_2) < (c - \beta)|V_2|^2$$

for some bipartition $V(G) = V_1 \cup V_2$. Setting $s = |V_1|$, we have

$$\begin{aligned} cn^2 \leqslant e(G) &\leqslant c|V_1|^2 + |V_1||V_2| + (c - \beta)|V_2|^2 \\ &< cs^2 + s(n - s) + (c - \beta)n^2 \leqslant sn + (c - \beta)n^2 \end{aligned}$$

and hence, $s > \beta n$. In [12], for every graph $G = G(n, m)$, the function

$$\Phi(G, t) = \min_{U \subset V(G),\, |U| = t} \left\{ \frac{e(U)}{t} + \frac{e(V \setminus U)}{n - t} - \frac{m}{n} \right\}$$

was introduced; in particular, it was shown that if $1 \leqslant t \leqslant \lfloor n/2 \rfloor$ then

$$\Phi\left(G, \left\lfloor \frac{n}{2} \right\rfloor\right) \leqslant \frac{t}{n - t} \Phi(G, k).$$

Hence,

$$\Phi\left(G, \left\lfloor \frac{n}{2} \right\rfloor\right) < \frac{\beta}{1 - \beta} \left((c - \beta)s + (c - \beta)(n - s) - \frac{\lceil cn^2 \rceil}{n} \right) \leqslant \frac{-\beta^2}{1 - \beta} n,$$

and so there is a bipartition $V(G) = V_1 \cup V_2$ with $|V_1| = \lfloor n/2 \rfloor$, $|V_2| = \lceil n/2 \rceil$, such that

$$\frac{e(V_1)}{\lfloor n/2 \rfloor} + \frac{e(V_2)}{\lceil n/2 \rceil} - \frac{e(V_1) + e(V_2) + e(V_1, V_2)}{n} = \Phi\left(G, \left\lfloor \frac{n}{2} \right\rfloor\right) < \frac{-\beta^2}{1 - \beta} n.$$

Inequality (1.2) follows easily. □

4. Final remarks

The most interesting open problem related to the topic of this note seems to be to find a proof of Theorem 1.4 without using Szemerédi's Regularity Lemma; presumably $l(\varepsilon, r)$ and n can be bounded by polynomials in $1/\varepsilon$. The crucial point here is to extend Lemma 1.3 for induced subgraphs. Such an extension would be very close to the conjecture of Erdős and Hajnal stating that any H-free graph of order n contains a clique or an independent set of order n^c for some $c = c(|H|) > 0$.

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