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On Cheng's eigenvalue comparison theorem

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Abstract

We observe that Cheng's Eigenvalue Comparison Theorem for normal geodesic balls [**4**] is still valid if we impose bounds on the mean curvature of the distance spheres instead of bounds on the sectional and Ricci curvatures. In this version, there is a weak form of rigidity in case of equality of the eigenvalues. Namely, equality of the eigenvalues implies that the distance spheres of the same radius on each ball has the same mean curvature. On the other hand, we construct smooth metrics g_κ on $[0, r] \times \mathbb{S}^3$, non-isometric to the standard metric can_κ of constant sectional curvature κ, such that the geodesic balls $B_{g_\kappa}(r) = ([0, r] \times \mathbb{S}^3, g_\kappa)$, $B_{\text{can}_\kappa}(r) = ([0, r] \times \mathbb{S}^3, \text{can}_\kappa)$ have the same first eigenvalue, the same volume and the distance spheres $\partial B_{g_\kappa}(s)$ and $\partial B_{\text{can}_\kappa}(s)$, $0 < s \leqslant r$, have the same mean curvatures. In the end, we apply this version of Cheng's Eigenvalue Comparison Theorem to construct examples of Riemannian manifolds M with arbitrary topology with positive fundamental tone $\lambda^*(M) > 0$ extending Veeravalli's examples, [**7**].

1. *Introduction*

Let $B_M(r)$ be a geodesic ball with radius r in a complete Riemannian n-manifold M and denote by $\lambda_1(B_M(r))$ its first Dirichlet eigenvalue. And by $\lambda_1(B_{\mathbb{M}^n(\kappa)}(r))$ the first Dirichlet eigenvalue of the geodesic ball $B_{\mathbb{M}^n(\kappa)}(r)$ of radius r in the simply connected n-space form $\mathbb{M}^n(\kappa)$ of constant sectional curvature κ.

Cheng in [**4**], using a result of Barta [**1**], proved that if the sectional curvature of M is bounded above $K_M \leqslant \kappa$ and $r < \min\{\text{inj}(p), \pi/\sqrt{\kappa}\}$, $(\pi/\sqrt{\kappa} = \infty \text{ if } \kappa \leqslant 0)$ then $\lambda_1(B_M(r)) \geqslant \lambda_1(B_{\mathbb{M}^n(\kappa)}(r))$. And if the Ricci curvature of M is bounded below $Ric_M \geqslant (n-1)\kappa$ then the reverse inequality $\lambda_1(B_M(r)) \leqslant \lambda_1(B_{\mathbb{M}^n(\kappa)}(r))$ holds for $r < \text{inj}(p)$. It is implicit in Cheng's proof that $\lambda_1(B_M(r)) = \lambda_1(B_{\mathbb{M}^n(\kappa)}(r))$ implies that the geodesic balls $B_M(p, r)$ and $B_{\mathbb{M}^n(\kappa)}(r)$ are isometric. It should be remarked that in [**5**], choosing a suitable test function for the Rayleigh quotient, Cheng proved that $Ric_M \geqslant (n-1)\kappa$, implies $\lambda_1(B_M(r)) \leqslant \lambda_1(B_{\mathbb{M}^n(\kappa)}(r))$ for every $r > 0$, with equality if and only if the geodesic balls $B_M(r)$ and $B_{\mathbb{M}^n(\kappa)}(r)$ are isometric and $r < \text{inj}(p)$.

In this paper we observe that Cheng's proof of the eigenvalue inequalities using Barta's theorem in [**4**] is valid under weaker geometric hypotheses. This is, let $B_M(r)$ and $B_{\mathbb{M}^n(\kappa)}(r)$ be within the cut locus of their centers and let $(t, \theta) \in (0, r] \times \mathbb{S}^{n-1}$ be geodesic coordinates for $B_M(r)$ and $B_{\mathbb{M}^n(\kappa)}(r)$. Let $H_M(t, \theta)$ and $H_{\mathbb{M}^n(\kappa)}(t, \theta) = H_{\mathbb{M}^n(\kappa)}(t)$ be the mean curvatures

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of the distance spheres $\partial B_M(t)$ and $\partial B_{\mathbb{M}^n(\kappa)}(t)$ at (t,θ) with respect to the unit vector field $-\partial/\partial t$. Then the following version of Cheng's Eigenvalue Comparison Theorems is true.

THEOREM 1·1. *If $H_M(s,\theta) \geqslant H_{\mathbb{M}^n(\kappa)}(s)$ for all $s \in (0,r]$ and all $\theta \in \mathbb{S}^{n-1}$ then*

$$\lambda_1(B_M(r)) \geqslant \lambda_1(B_{\mathbb{M}^n(\kappa)}(r)). \tag{1·1}$$

If $H_M(s,\theta) \leqslant H_{\mathbb{M}^n(\kappa)}(s)$ for all $s \in (0,r]$ and all $\theta \in \mathbb{S}^{n-1}$ then

$$\lambda_1(B_M(r)) \leqslant \lambda_1(B_{\mathbb{M}^n(\kappa)}(r)). \tag{1·2}$$

Equality in (1·1) *or in* (1·2) *holds if and only if $H_M(s,\theta) = H_{\mathbb{M}^n(\kappa)}(s)$ for all $s \in (0,r]$ and for all $\theta \in \mathbb{S}^{n-1}$.*

It is well known that $K_M \leqslant \kappa$ implies that $H_M(s,\theta) \geqslant H_{\mathbb{M}^n(\kappa)}(s)$ and $Ric_M \geqslant (n-1)\kappa$ implies that $H_M(s,\theta) \leqslant H_{\mathbb{M}^n(\kappa)}(s)$. However, the converse statements are not true in general. In Section 3, we construct smooth metrics on $\mathbb{R}^n$ with radial sectional curvatures bounded below $K(\partial/\partial s, v) > \kappa$ outside a compact set with $H_M(s,\theta) \geqslant H_{\mathbb{M}(\kappa)}(s)$ for all $s \geqslant 0$, see Example 3.1. In a second example, we construct smooth metrics g_κ on $[0,r]\times\mathbb{S}^3$, non-isometric to the standard metric can_κ of constant sectional curvature κ, such that the geodesic balls $B_{g_\kappa}(r) = ([0,r]\times\mathbb{S}^3, g_\kappa)$, $B_{\mathrm{can}_\kappa}(r) = ([0,r]\times\mathbb{S}^3, \mathrm{can}_\kappa)$ have the same first eigenvalue, the same volume and the geodesic spheres $\partial B_{g_\kappa}(s)$ and $\partial B_{\mathrm{can}_\kappa}(s)$, $0 < s \leqslant r$, have the same mean curvatures, see Example 3.2. On the other hand, if the metric of a Riemannian n-manifold restricted to a geodesic ball $B_M(r)$ of radius r is rotationally symmetric, i.e. its expression in geodesic coordinates is given by $ds^2 = dt^2 + f^2(t)d\theta^2$, $f(0) = 0$, $f'(0) = 1$, $f(t) > 0$ for $t > 0$ then the equality of eigenvalues $\lambda_1(B_M(r), ds^2) = \lambda_1(B_{\mathbb{M}^n(\kappa)}(r), \mathrm{can}_\kappa)$ implies that the balls $B_M(r)$ and $B_{\mathbb{M}^n(\kappa)}(r)$ are isometric, see Remark 3.

In Section 2 we present the proof of Theorem 1·1. In Section 3 we construct examples mentioned above. In Section 4 we present few corollaries of Theorem 1·1 that can be viewed as generalizations of Cheng's Eigenvalue Comparison Theorems. As applications of these corollaries we construct complete Riemannian manifolds with arbitrary topology, with variable sign sectional curvature and with positive fundamental tone. Compare with Veeravalli's examples [**7**].

2. *Proof of Theorem* 1·1

A powerful tool to obtain bounds for the first Dirichlet eigenvalue of smooth bounded domains in Riemannian manifolds is the following theorem due to Barta [**1**].

THEOREM 2·1 (Barta). *Let $\Omega \subset M$ be a domain with compact closure and smooth boundary $\partial\Omega$. Let $f \in C^2(\Omega) \bigcap C^0(\overline{\Omega})$ with $f > 0$ in Ω and $f|\partial\Omega = 0$. If $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of Ω, then*

$$\sup_\Omega \left(-\frac{\triangle f}{f} \right) \geqslant \lambda_1(\Omega) \geqslant \inf_\Omega \left(-\frac{\triangle f}{f} \right). \tag{2·1}$$

Moreover, each inequality of (2·1) *is strict unless f is a positive first eigenfunction.*

Remark 1. To prove the lower inequality in (2·1) it is necessary only to have $f > 0$.

Let $u\colon B_{\mathbb{M}^n(\kappa)}(r) \to \mathbb{R}$ be a positive first Dirichlet eigenfunction. It is well known that u is radial function, $u(t,\theta) = u(t)$, $u'(t) \leqslant 0$ and satisfies the following differential equation,

$$u''(s) + (n-1)\frac{C_\kappa}{S_\kappa}(s)\,u'(s) + \lambda_1\big(B_{\mathbb{M}^n(\kappa)}(r)\big)u(s) = 0, \;\; s \in [0,r] \tag{2·2}$$

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where

$$S_\kappa(t) = \begin{cases} \dfrac{\sin(\sqrt{\kappa}\,t)}{\sqrt{\kappa}}, & \text{if } \kappa > 0 \\ t, & \text{if } \kappa = 0 \\ \dfrac{\sinh(\sqrt{-\kappa}\,t)}{\sqrt{-\kappa}}, & \text{if } \kappa < 0 \end{cases} \tag{2·3}$$

and $C_\kappa(t) = S'_\kappa(t)$. Observe that $u(t,\theta) = u(t)$ also defines a smooth function on $B_M(r)$, (called transplanted function), with gradient $\operatorname{grad} u = u'\,\partial/\partial t$, where $\partial/\partial t$ is the unit vector field normal to the distance spheres $\partial B_M(t)$ pointing outward. Therefore

$$\triangle u = \operatorname{div}\left(u'\frac{\partial}{\partial t}\right) = \left\langle \operatorname{grad} u', \frac{\partial}{\partial t}\right\rangle + u'\operatorname{div}\left(\frac{\partial}{\partial t}\right) = u'' + u'\operatorname{Trace}\left(\xi \to \nabla_\xi \frac{\partial}{\partial t}\right). \tag{2·4}$$

Here $\xi \to \nabla_\xi \partial/\partial t$ is the *Weingarten map* of the distance spheres. Its trace is the mean curvature H_M. By (2·4) we have at any point (t,θ) of $B_M(r)$ that

$$\begin{aligned} -\frac{\triangle u}{u}(t,\theta) &= \left[-\frac{u''}{u} - \frac{u'}{u}\operatorname{div}\left(\frac{\partial}{\partial t}\right)\right](t,\theta) \\ &= \left[(n-1)\frac{C_\kappa}{S_\kappa}(t) - H_M(t,\theta)\right]\frac{u'}{u}(t) + \lambda_1\big(B_{\mathbb{M}^n(\kappa)}(r)\big). \end{aligned} \tag{2·5}$$

Observing that $H_{\mathbb{M}^n(\kappa)} = (n-1)C_\kappa/S_\kappa$ we have by Barta's Theorem (2·1) that

$$\sup_{(t,\theta)}\left[(H_{\mathbb{M}^n(\kappa)} - H_M)\frac{u'}{u}\right] + \lambda_1\big(B_{\mathbb{M}^n(\kappa)}(r)\big) = \sup_{(t,\theta)}\left(-\frac{\triangle u}{u}\right) \geqslant \lambda_1(B_M(r)) \tag{2·6}$$

and

$$\lambda_1(B_M(r)) \geqslant \inf_{(t,\theta)}\left(-\frac{\triangle u}{u}\right) = \inf_{(t,\theta)}\left[\big(H_{\mathbb{M}^n(\kappa)} - H_M\big)\frac{u'}{u}\right] + \lambda_1\big(B_{\mathbb{M}^n(\kappa)}(r)\big). \tag{2·7}$$

Since $u'/u \leqslant 0$ we have that, $H_M \geqslant H_{\mathbb{M}^n(\kappa)} \Rightarrow \inf[(H_{\mathbb{M}^n(\kappa)} - H_M)\frac{u'}{u}] \geqslant 0$ and $\lambda_1(B_M(r)) \geqslant \lambda_1(B_{\mathbb{M}^n\kappa}(r))$. And $H_M \leqslant H_{\mathbb{M}^n(\kappa)} \Rightarrow \sup[(H_{\mathbb{M}^n(\kappa)} - H_M)\frac{u}{u}] \leqslant 0$ and $\lambda_1(B_M(r)) \leqslant \lambda_1(B_{\mathbb{M}^n\kappa}(r))$.

If $H_M \geqslant H_{\mathbb{M}^n(\kappa)}$ and $\lambda_1(B_M(r)) = \lambda_1(B_{\mathbb{M}^n\kappa}(r))$ we have that $\lambda_1(B_M(r)) = \inf_{(t,\theta)}(-\frac{\triangle u}{u})$. As observed in Barta's Theorem, the transplanted function u is a positive eigenfunction of $B_M(r)$ and from (2·5) we have that

$$\left[(n-1)\frac{C_\kappa}{S_\kappa} - H_M\right]\frac{u'}{u}(t,\theta) = 0$$

for all $t \in (0,r]$ and $\theta \in \mathbb{S}^{n-1}$. Since $u'/u < 0$ in $(0,r]$ then $(n-1)\frac{C_\kappa}{S_\kappa} - H_M = 0$ in all points (t,θ) with $t \in (0,r]$. By continuity $(n-1)\frac{C_\kappa}{S_\kappa} - H_M = 0$ on the whole ball with radius r. The equality case under $H_M \leqslant H_{\mathbb{M}^n(\kappa)}$ is treated similarly. This finishes the proof of Theorem 1·1.

3. *Examples*

In this section we construct examples of metrics regarding certain aspects of Cheng's Eigenvalue Comparison Theorem. In the first example, we construct a family of smooth complete rotationally symmetric metrics $ds^2 = dt^2 + f^2(t)d\theta^2$ on $\mathbb{R}^n = [0,\infty) \times \mathbb{S}^{n-1}$

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with radial sectional curvature $K_{(\mathbb{R}^n,ds^2)} > \kappa$ outside a compact set such that the mean curvatures of the distance spheres satisfy $H_{(\mathbb{R}^n,ds^2)}(t,\theta) \geqslant H_{(\mathbb{M}^n(\kappa),\mathrm{can}_\kappa)}(t,\theta)$. Likewise, we construct smooth complete rotationally symmetric metrics on $\mathbb{R}^n$ with radial sectional curvature $K_{(\mathbb{R}^n,ds^2)} < \kappa$ outside a compact set such that the mean curvatures of the distance spheres satisfy $H_{(\mathbb{R}^n,ds^2)}(t,\theta) \leqslant H_{(\mathbb{M}^n(\kappa),\mathrm{can}_\kappa)}(t,\theta)$.

In a second example, we consider a smooth metric $dt^2 + a^2(t)dx^2 + b^2(t)dy^2 + c^2(t)dz^2$ on $[0,r] \times \mathbb{S}^3$, where dx, dy and dz is the dual co-frame of a global frame X, Y, Z on $\mathbb{S}^3$ satisfying $[X,Y] = 2Z$, $[Y,Z] = 2X$, $[Z,X] = 2Y$ and $a, b, c\colon [0,r] \to \mathbb{R}$ are smooth functions, positive in $(0,r]$, satisfying the initial data $a(0) = b(0) = c(0) = 0$ and $a'(0) = b'(0) = c'(0) = 1$. Choosing $a(t) = b(t) = c(t) = S_\kappa(t)$ we obtain the standard metric can_κ of constant sectional curvature κ. Here, we choose $a(t) = S_\kappa^2(t)/t^2$, $b(t) = t$, $c(t) = S_\kappa(t)$ to obtain a smooth metric with the following properties.

(i) The set $[0,r] \times \mathbb{S}^3$ endowed with the metrics can_κ, g_κ are geodesic balls of radius r, i.e. $B_{\mathrm{can}_\kappa}(r) = ([0,r] \times \mathbb{S}^3, \mathrm{can}_\kappa)$ and $B_{g_\kappa}(r) = ([0,r] \times \mathbb{S}^3, g_\kappa)$.
(ii) $B_{\mathrm{can}_\kappa}(r)$ and $B_{g_\kappa}(r)$ are not isometric if $\kappa \neq 0$.
(iii) $B_{\mathrm{can}_\kappa}(r)$ and $B_{g_\kappa}(r)$ have the same first eigenvalue $\lambda_1(B_{\mathrm{can}_\kappa}(r)) = \lambda_1(B_{g_\kappa}(r))$.
(iv) $B_{\mathrm{can}_\kappa}(r)$ and $B_{g_\kappa}(r)$ have the same volume $\mathrm{vol}_{\mathrm{can}_\kappa}(B_{\mathrm{can}_\kappa}(r)) = \mathrm{vol}_{g_\kappa}(B_{g_\kappa}(r))$.
(v) The distance spheres $(\partial B(t), \mathrm{can}_\kappa)$ and $(\partial B(t), g_\kappa)$ have the same mean curvatures $H_{g_\kappa}(t,x) = H_{\mathrm{can}_\kappa}(t,x) = C_\kappa(t)/S_\kappa(t)$ for every $t \in (0,r]$ and $x \in \mathbb{S}^3$.

Remark 2. The referee kindly pointed out to us that in the first version of Example 3.2, the metric we constructed on $[0,r] \times \mathbb{S}^n$ were not complete. To have a complete metric with properties (i-v) we took the warping functions we used in the original version combined with these type of metrics on $[0,r] \times \mathbb{S}^3$ we learned from Perelman [**6**].

Example 3.1

Let $\mathbb{R}^n = [0,\infty) \times \mathbb{S}^{n-1}$ with the metric $ds^2 = dt^2 + f^2(t)d\theta^2$, $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$. Set $\psi_\kappa(t) = (-f'S_\kappa + fS_\kappa')(t)$, where $'$ means differentiation with respect to t. The radial sectional curvature of $(\mathbb{R}^n, ds^2)$ is bounded above by κ if and only if $\psi_\kappa'(t) \leqslant 0$. The mean curvatures of $\partial B_{\mathbb{R}^n}(t)$ and $\partial B_{\mathbb{M}^n(\kappa)}(t)$ satisfies $H_{\mathbb{R}^n}(t,\theta) \geqslant H_{\mathbb{M}^n(\kappa)}(t)$ if and only if $\psi_\kappa(t) \leqslant 0$. From $\psi_\kappa(t) = (-f'S_\kappa + fS_\kappa')(t)$ we have that $\psi_\kappa(0) = \psi_\kappa'(0) = 0$. Observe that for $t \neq 0$

$$\frac{\psi_\kappa}{S_\kappa^2} = \frac{-f'S_\kappa + fS_\kappa'}{S_\kappa^2} = -\left(\frac{f}{S_\kappa}\right)'$$

Moreover, $\lim_{t\to 0} \frac{\psi_\kappa}{S_\kappa^2}(t) = 0$. Integrating $(f/S_\kappa)'$ between ϵ and t we obtain

$$\int_\epsilon^t \left(\frac{f}{S_\kappa}\right)'(s)\,ds = \frac{f}{S_\kappa}(t) - \frac{f}{S_\kappa}(\epsilon) = -\int_\epsilon^t \frac{\psi_\kappa}{S_\kappa^2}(s)ds.$$

Since $\lim_{\epsilon\to 0} \frac{f}{S_\kappa}(\epsilon) = 1$ we have that

$$f(t) = S_\kappa(t) - S_\kappa(t) \cdot \int_0^t \psi_\kappa(s)/S_\kappa^2(s)ds. \tag{3·1}$$

Let $\psi_\kappa\colon [0,\infty) \to \mathbb{R}$ be a smooth function satisfying $\psi_\kappa(0) = \psi_\kappa'(0) = 0$, $\psi(t) \leqslant 0$, $|\int_{[0,\infty)} \psi_\kappa(s)/S_\kappa^2(s)ds| < \infty$ and such that $\psi_\kappa'(t) > 0$ for $t > 1$. This gives a complete smooth metric $ds^2 = dt^2 + f^2(t)d\theta$ on $\mathbb{R}^n$ with radial sectional curvature $K_{(\mathbb{R}^n,ds^2)} > \kappa$ outside the ball $B_{(\mathbb{R}^n,ds^2)}(1)$ and such that the distance spheres $\partial B_{(\mathbb{R}^n,ds^2)}(t)$ have mean curvature $H_{(\mathbb{R}^n,ds^2)}(t,\theta) \geqslant H_{\mathbb{M}^n(\kappa)}(t,\theta)$.

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If we take $\psi_\kappa\colon[0,\infty)\to\mathbb{R}$ to be a smooth function satisfying $\psi_\kappa(0)=\psi_\kappa'(0)=0$, $\psi(t)\geqslant 0$, $|\int_{[0,\infty)}\psi_\kappa(s)/S_\kappa^2(s)ds|<1$ and such that $\psi_\kappa'(t)<0$ for $t>1$ we obtain a smooth metric with sectional curvature $K_{(\mathbb{R}^n,ds^2)}<\kappa$ outside a compact set with $H_{(\mathbb{R}^n,ds^2)}(t,\theta)\leqslant H_{\mathbb{M}^n(\kappa)}(t,\theta)$.

Remark 3. For rotationally symmetric metrics $ds^2=dt^2+f^2(t)d\theta^2$, the equality $H_{(\mathbb{R}^n,ds^2)}(t,\theta)=H_{(\mathbb{M}(\kappa),\mathrm{can}_\kappa)}(t)$ for all $t\in(0,r]$ implies that $ds^2=dt^2+S_\kappa^2 d\theta^2$. Since $H_{(\mathbb{R}^n,ds^2)}(t)=(n-1)(f'/f)(t)=(n-1)(C_\kappa/S_\kappa)(t)=H_{\mathbb{M}(\kappa)}(t)$ implies that $\psi_\kappa(t)=0$ for all $t\in[0,r]$ and by (3·1) $f\equiv S_\kappa$.

Example 3.2

Let $\{X,Y,Z\}$ be a globally defined frame on $\mathbb{S}^3$ with $[X,Y]=2Z$, $[Y,Z]=2X$, $[Z,X]=2Y$ and let dx, dy and dz be its dual co-frame. Consider on $[0,r]\times\mathbb{S}^3$ the following metric $dt^2+a^2(t)dx^2+b^2(t)dy^2+c^2(t)dz^2$, where $a,b,c\colon[0,r]\to\mathbb{R}$ are smooth functions, positive on $(0,r]$, with $a(0)=b(0)=c(0)=0$ and $a'(0)=b'(0)=c'(0)=1$. Here dt is the dual co-vector of the radial vector field T. Observe that vector field T commutes with X,Y,Z, i.e. $[T,X]=[T,Y]=[T,Z]=0$. As mentioned before, taking $a(t)=b(t)=c(t)=S_\kappa(t)$ we obtain the standard metric can_κ of constant sectional curvature κ. Consider the orthonormal frame $E_1=T$, $E_2=\frac{1}{a}\cdot X$, $E_3=\frac{1}{b}\cdot Y$, $E_4=\frac{1}{c}\cdot Z$ on $[0,r]\times\mathbb{S}^3$. It is easy to show that

$$\nabla_{E_1}E_1=0,\ \nabla_{E_2}E_2=-\frac{a'}{a}E_1,\ \nabla_{E_3}E_3=-\frac{b'}{b}E_1,\ \nabla_{E_4}E_4=-\frac{c'}{c}E_1.$$

The Laplace operator is given by

$$\triangle=\sum_{i=1}^4\left(E_i^2-\nabla_{E_i}E_i\right)=E_1^2+\left(\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}\right)E_1+\sum_{i=2}^4E_i^2, \tag{3·2}$$

where $E_i^2=E_iE_i$. Moreover, the sectional curvatures of the planes generated by $\{E_1,E_2\}$, $\{E_1,E_3\}$ at any point $(t,p)\in(0,r]\times\mathbb{S}^3$ are respectively given by

$$K(E_1,E_2)=-\frac{a''}{a},\ K(E_1,E_3)=-\frac{b''}{b}.$$

In our example, we set $a(t)=S_\kappa^2(t)/t$, $b(t)=t$, $c(t)=S_\kappa(t)$, where we are assuming that $r<\pi/\sqrt{\kappa}$ if $\kappa>0$. We obtain a smooth metric g_κ on $[0,r]\times\mathbb{S}^3$ that clearly is non-isometric to can_κ if $\kappa\neq 0$. For instance, at $(t,p)\in(0,r]\times\mathbb{S}^3$ we have for $\kappa<0$ that $K(E_1,E_2)=2\kappa-\frac{2}{t^2}+\frac{4\sqrt{-\kappa}\coth(\sqrt{-\kappa}\,t)}{t}-2\coth^2(\sqrt{-\kappa}\,t)$ and $K(E_1,E_3)=0$. For $\kappa>0$ we have that $K(E_1,E_2)=2\kappa-\frac{2}{t^2}+\frac{4\sqrt{\kappa}\cot(\sqrt{\kappa}\,t)}{t}-2\cot^2(\sqrt{\kappa}\,t)$ and $K(E_1,E_3)=0$. Moreover, $[0,r]\times\mathbb{S}^3$ is the (closed) geodesic ball of radius r centered at $0=\{0\}\times\mathbb{S}^3$ with respect to both metrics can_κ, g_κ, since the function $\rho\colon[0,r]\times\mathbb{S}^3\to\mathbb{R}$ given by $\rho(t,x)=t$ is the distance function to 0 for both metrics. The Laplace operators $\triangle_{g_\kappa}$, $\triangle_{\mathrm{can}_\kappa}$ of g_κ and can_κ given by (3·2) are given by

$$\begin{aligned}\triangle_{g_\kappa}&=T^2+3\frac{C_\kappa}{S_\kappa}T+\frac{t^2}{S_\kappa^4}X^2+\frac{1}{t^2}Y^2+\frac{1}{S_\kappa^2}Z^2\\ \triangle_{\mathrm{can}_\kappa}&=T^2+3\frac{C_\kappa}{S_\kappa}T+\frac{1}{S_\kappa^2}X^2+\frac{1}{S_\kappa^2}Y^2+\frac{1}{S_\kappa^2}Z^2.\end{aligned} \tag{3·3}$$

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Let $u: B_{\mathrm{can}_\kappa}(r) \to \mathbb{R}$ be a positive first eigenfunction of $B_{\mathrm{can}_\kappa}(r) = ([0, r] \times \mathbb{S}^3, \mathrm{can}_\kappa)$. This means that u satisfies the following boundary value problem

$$\begin{cases} \triangle_{\mathrm{can}_\kappa} u + \lambda_1(B_{\mathrm{can}_\kappa}(r))\, u = 0 & \text{in } B_{\mathrm{can}_\kappa}(r) \\ u = 0 & \text{on } \partial B_{\mathrm{can}_\kappa}(r). \end{cases} \tag{3·4}$$

It is well known, see [**3**], that u is radial, i.e. $u(t, x) = u(t)$. From (3·3) we see that the Laplace operators $\triangle_{g_\kappa}$ and $\triangle_{\mathrm{can}_\kappa}$ coincide on the set of smooth radial functions defined on $[0, r]\times\mathbb{S}^3$. Thus we have $\triangle_{g_\kappa} u = \triangle_{\mathrm{can}_\kappa} u$. This implies that u satisfies the following boundary value problem

$$\begin{cases} \triangle_{g_\kappa} u + \lambda_1(B_{\mathrm{can}_\kappa}(r))\, u = 0 & \text{in } B_{g_\kappa}(r) \\ u = 0 & \text{on } \partial B_{g_\kappa}(r). \end{cases} \tag{3·5}$$

This shows that u is an eigenfunction of $B_{g_\kappa}(r)$. Since u is positive then it is a first eigenfunction and $\lambda_1(B_{\mathrm{can}_\kappa}(r)) = \lambda_1(B_{g_\kappa}(r))$ is the first eigenvalue. The mean curvatures of the distance spheres $\partial B_{g_\kappa}(t)$, $\partial B_{\mathrm{can}_\kappa}(t)$ is just $\triangle_{g_\kappa}\rho(t, x) = \triangle_{\mathrm{can}_\kappa}\rho(t, x) = (C_\kappa/S_\kappa)(t)$. Observe that $\det(g_\kappa)_{ij} = \det(\mathrm{can}_\kappa)_{ij}$

4. *Generalizations of Theorem* 1·1

In this section we derive few corollaries of the proof of Theorem 1·1 that can be viewed as generalizations of Cheng's Eigenvalue Comparison Theorems. The first generalization we are going to consider is the following.

Consider the metric $ds^2 = dt^2 + f^2(t)d\xi^2$ on $\mathbb{R}^m = [0, \infty)\times\mathbb{S}^{m-1}$, where $f: [0, \infty) \to \mathbb{R}$ is a smooth function satisfying $f(0) = 0$, $f'(0) = 1$, $f(t) > 0$ for $t \in (0, \infty)$. Let $B_{(M,g)}(r)$ be a normal geodesic ball of an n-dimensional complete Riemannian manifold (M, g). Let $(t, \theta) \in (0, r] \times \mathbb{S}^{n-1}$ be geodesic coordinates for $B_{(M,g)}(r)$ and $(t, \xi) \in (0, r] \times \mathbb{S}^{m-1}$ be geodesic coordinates for $B_{(\mathbb{R}^m, ds^2)}(r)$.

Let $H_{(M,g)}(t, \theta)$ and $H_{(\mathbb{R}^m,ds^2)}(t, \xi) = (m - 1)(f'/f)(t)$ be the mean curvatures of the distance spheres $\partial B_{(M,g)}(t)$ and $\partial B_{(\mathbb{R}^m,ds^2)}(t)$ at the points (t, θ) and (t, ξ) with respect to the unit vector field $-\partial/\partial t$.

COROLLARY 4·1. *If* $H_{(M,g)}(t, \theta) \geqslant H_{(\mathbb{R}^m,ds^2)}(t)$, $\forall\, t \in (0, r]$ *and* $\theta \in \mathbb{S}^{n-1}$ *then*

$$\lambda_1\big(B_{(M,g)}(r)\big) \geqslant \lambda_1\big(B_{(\mathbb{R}^m,ds^2)}(r)\big). \tag{4·1}$$

If $H_M(t, \theta) \leqslant H_{\mathbb{R}^m}(t)$, $\forall\, t \in (0, r]$ *and* $\theta \in \mathbb{S}^{n-1}$ *then*

$$\lambda_1(B_{(M,g)}(r)) \leqslant \lambda_1(B_{(\mathbb{R}^m,ds^2)}(r)). \tag{4·2}$$

Equality in (4·1) *or* (4·2) *holds if and only if* $n = m$ *and* $H_M(t, \theta) = H_{\mathbb{R}^n}(t)$, $\forall\, t \in (0, r]$ *and* $\theta \in \mathbb{S}^{n-1}$.

The proof that first eigenfunctions of geodesic balls the space forms of constant sectional curvature are radial, presented in [**3**], in fact shows that first eigenfunctions $u: B_{(\mathbb{R}^m,ds^2)}(r) \to \mathbb{R}$ are radial, $u(t, \xi) = u(t)$. Transplant u to $B_{(M,g)}(r)$ setting $u(t, \theta) = u(t)$. Computing the Laplacian of u we have,

$$-\frac{\triangle_g u}{u} = -\frac{u''}{u} - \frac{u'}{u}\mathrm{div}_g\left(\frac{\partial}{\partial t}\right) = \big[H_{\mathbb{R}^m}(t) - H_{(M,g)}(t, \theta)\big] + \lambda_1\big(B_{(\mathbb{R}^m,ds^2)}(r)\big).$$

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Applying Barta's theorem we have the statements of Corollary 4·1. In case of equality $\lambda_1(B_{(M,g)}(r)) = \lambda_1(B_{(\mathbb{R}^m,ds^2)}(r))$ we have that the transplanted function u is a positive eigenfunction of $B_{(M,g)}(r)$ and $H_{\mathbb{R}^m}(t) = H_{(M,g)}(t,\theta)$. To show that $m = n$ we proceed as follows. Let p be the center of the ball $B_{(M,g)}(r)$. For fixed $\theta \in \mathbb{S}^{n-1} \subset T_pM$, let τ_t denote parallel translation by t units along the unique minimal geodesic γ_θ satisfying $\gamma_\theta(0) = p$ and $\gamma'_\theta(0) = \theta$. For $\eta \in \theta^\perp \subset T_pM$ set $\mathcal{R}\eta = \tau_{-t}\{R(\gamma'_\theta(t), \tau_t\eta)\gamma'_\theta(t)\}$, where R is the Riemannian curvature tensor and set $\mathcal{A}(t,\theta)$ the path of linear transformations of $\theta^\perp$ satisfying $\mathcal{A}'' + \mathcal{R}\mathcal{A} = 0$ with initial conditions $\mathcal{A}(0,\theta) = 0$, $\mathcal{A}'(0,\theta) = I$. The Riemannian metric of M on the geodesic ball $B_{(M,g)}(r)$ is expressed by

$$ds^2(\exp t\theta) = dt^2 + |\mathcal{A}(t,\theta)d\theta|^2.$$

Set $\sqrt{G}(t,\theta) = det\mathcal{A}(t,\theta)$. The mean curvature $H_{(M,g)}(t,\theta)$ of the geodesic sphere $\partial B_{(M,g)}(t)$ at a point (t,θ) (with respect to $-\partial/\partial t$) is given by $\sqrt{G}'(t,\theta)/\sqrt{G}(t,\theta)$. Moreover for small t we have the Taylor expansions $\sqrt{G}(t,\theta) = t^{n-1}(1 - t^2Ric(\theta,\theta)/6 + O(t^3))$. See [**3**, pages 316–317]. Thus,

$$\frac{\sqrt{G}'(t,\theta)}{\sqrt{G}(t,\theta)} = \frac{(n-1) - (n+1)t^2Ric(\theta,\theta)/6 + O(t^3)}{t(1 - t^2Ric(\theta,\theta)/6 + O(t^3))}. \tag{4·3}$$

On the other hand the metric of $B_{(\mathbb{R}^m,ds^2)}(r)$ is given by $dt^2 + f^2(t)d\xi^2$, where $f(0) = 0$, $f'(0) = 1$. The mean curvature $H_{(\mathbb{R}^m,ds^2)}(t,\xi)$ of the geodesic sphere $\partial B_{(\mathbb{R}^m,ds^2)}(t)$ at a point (t,ξ) is given by $(m-1)f'(t)/f(t)$. The Taylor expansion of f is given by

$$f(t) = t + f''(0)t^2/2 + O(t^3).$$

Therefore,

$$(m-1)\frac{f'(t)}{f(t)} = (m-1)\frac{1 + f''(0)t + O(t^2)}{t(1 + f''(0)t/2 + O(t^2))} \tag{4·4}$$

Now, we have that $H_{(M,g)}(t,\theta) = H_{(\mathbb{R}^m,ds^2}(t)$ for all $t \in (0,r]$. Then

$$\frac{(n-1) - (n+1)t^2Ric(\theta,\theta)/6 + O(t^3)}{(1 - t^2Ric(\theta,\theta)/6 + O(t^3))} = (m-1)\frac{1 + f''(0)t + O(t^2)}{(1 + f''(0)t/2 + O(t^2))}. \tag{4·5}$$

Letting $t \to 0$ we have that $n = m$.

The proof of Theorem 1·1 yields another corollary when considering the incomplete cone over an $(n-1)$-dimensional compact Riemannian manifold (N, dh^2). The incomplete cone $C_f(N)$ over N is the Riemannian space $C(N) = (0,\infty) \times N$ with metric

$$ds_f^2 = dt^2 + f^2(t,x)\,dh^2,$$

where $f\colon [0,\infty) \times N \to \mathbb{R}$ is a smooth function satisfying $f(0,x) = 0$, $f'(0,x) = 1$, $f(t,x) > 0$ for all $t > 0$. The completed cone $\overline{C_f(N)} = C_f(N) \bigcup \{p\}$, $p = \{0\} \times N$. The Euclidean space $\mathbb{R}^m$ with metric $ds_g^2 = dt^2 + g^2(t)d\theta^2$ is the completed cone $\overline{C_g(\mathbb{S}^{m-1})}$. Recall that the fundamental tone $\lambda^*(\Omega)$ of an open set Ω of a Riemannian manifold is given by

$$\lambda^*(\Omega) = \inf\left\{\frac{\int_\Omega |\nabla f|^2}{\int_\Omega f^2},\ f \in H_0^1(\Omega), f \not\equiv 0\right\},$$

where $H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\varphi\|_\Omega^2 = \int_\Omega \varphi^2 + \int_\Omega |\nabla\varphi|^2$. When Ω is bounded (regardless its boundary) there is $u \in C^2(\Omega) \bigcap H_0^1(\Omega)$, $u > 0$

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in Ω, also called first eigenfunction, satisfying $\triangle u + \lambda^*(\Omega)u = 0$ in Ω. If $\partial\Omega$ is smooth then $u \in C^2(\Omega) \bigcap C^0(\overline{\Omega})$ with $u|\partial\Omega = 0$.

The next result compares the fundamental tone $\lambda^*(C_f(N)(r))$ of the the trunked cone $C_f(N)(r) = (0, r) \times N$ with the lowest Dirichlet eigenvalue $\lambda_1(B_{\mathbb{R}^m}(r))$ of the geodesic ball $(B_{\mathbb{R}^m}(r), ds_g^2)$.

COROLLARY 4·2. *Let $C_f(N)$ be an incomplete cone over a compact $(n-1)$-dimensional Riemannian manifold (N, dh^2) and $\mathbb{R}^m$ with metric $ds^2 = dt^2 + g^2(t)d\xi^2$. If*

$$(n-1)(f'/f)(t, x) \geqslant (m-1)(g'/g)(t), \tag{4·6}$$

for all $x \in N$ and all $t \in (0, r)$ where $'$ means the derivative with respect to the variable t, then

$$\lambda^*(C_f(N)(r)) \geqslant \lambda_1(B_{\mathbb{R}^n}(r)). \tag{4·7}$$

If (4·6) holds for all $t > 0$ then letting $r \to \infty$ we have that

$$\lambda^*(C_f(N)) \geqslant \lambda^*(\mathbb{R}^m).$$

To prove this corollary we need to apply the following version of Barta's Theorem for bounded open sets that was proved in [**2**, Proposition 2·1].

PROPOSITION 4·3. *Let Ω be a bounded domain in a smooth Riemannian manifold. Let $v \in C^2(\Omega) \bigcap C^0(\overline{\Omega})$, $v > 0$ in Ω and $v|\partial\Omega = 0$. Then*

$$\lambda^*(\Omega) \geqslant \inf_{\Omega}\left(-\frac{\triangle v}{v}\right). \tag{4·8}$$

Moreover, $\lambda^*(\Omega) = \inf_\Omega(-\frac{\triangle v}{v})$ if and only if $v = u$, where u is a positive eigenfunction of Ω, i.e. $\triangle u + \lambda^*(\Omega)u = 0$.

These ideas can be used to obtain examples of Riemannian manifolds M with arbitrary fundamental groups, variable sectional curvatures and with positive fundamental tone $\lambda^*(M) > 0$, extending Veeravalli's examples [**7**].

Let $M = \mathbb{R}^m \times N$ with the metric $ds^2 = dt^2 + f^2(t, \theta)d\theta^2 + g^2(t, \theta)dh^2$ where (N, dh^2) is a complete n-dimensional Riemannian manifold and $f, g: \mathbb{R}^n \to [0, \infty)$ are smooth functions satisfying $f(0, \theta) = 0$, $f'(0, \theta) = 1$, $f(t, \theta) > 0$ for $t > 0$ and $\theta \in \mathbb{S}^{m-1}$ and $g(t, \theta) > 0$ for all $(t, \theta) \in [0, \infty) \times \mathbb{S}^{m-1}$. Let $\Omega = B_{\mathbb{R}^m}(r) \times W \subset M$ where $B_{\mathbb{R}^m}(r) \subset \mathbb{R}^m$ is a ball with radius r and $W \subset N$ is a domain with compact closure and smooth boundary ∂W (possibly empty). Let $B_{\mathbb{M}^l(\kappa)}(r) \subset \mathbb{M}^l(\kappa)$ be a geodesic ball of radius r in the simply connected l-dimensional space form of constant sectional curvature κ with metric $dt^2 + S_\kappa^2(t)d\theta^2$.

COROLLARY 4·4. *If $(m-1)\frac{f'}{f}(t, \theta) + n\frac{g'}{g}(t, \theta) \geqslant (l-1)\frac{S_\kappa'}{S_\kappa}(t)$ for all $t \in [0, r]$ and $\theta \in \mathbb{S}^{m-1}$ then*

$$\lambda_1(\Omega) \geqslant \lambda_1\big(B_{\mathbb{M}^l(\kappa)}(r)\big) + \inf_{(t,\theta)}\left[\frac{1}{g^2}\right] \cdot \lambda_1(W). \tag{4·9}$$

If $r = \infty$ and letting $W = N$ we have that

$$\lambda^*(M) \geqslant (l-1)^2\kappa^2/4 + \inf_{(t,\theta)}\left[\frac{1}{g^2}\right] \cdot \lambda^*(N).$$

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If $(m-1)\frac{f'}{f}(t,\theta)+n\frac{g'}{g}(t,\theta)\leqslant(l-1)\frac{S_\kappa'}{S_\kappa}(t)$ *for all* $t\in[0,r]$ *and* $\theta\in\mathbb{S}^{m-1}$ *then*

$$\lambda_1(\Omega)\leqslant\lambda_1\big(B_{\mathbb{M}^l(\kappa)}(r)\big)+\sup_{(t,\theta)}\left[\frac{1}{g^2}\right]\cdot\lambda_1(W). \tag{4·10}$$

If $r=\infty$, *and letting* $W=N$ *we have that*

$$\lambda^*(M)\leqslant(l-1)^2\kappa^2/4+\sup_{(t,\theta)}\left[\frac{1}{g^2}\right]\cdot\lambda^*(N).$$

Choose a positive function $\psi\colon\Omega\to\mathbb{R}$ given by $\psi(t,\theta,x)=u(t)\cdot v(x)$ where u and v are positive eigenfunctions of $B_{\mathbb{M}^l(\kappa)}(r)$ and W respectively, i.e. u satisfies the differential equation

$$\frac{\partial^2 u}{\partial t^2}(t)+(l-1)\frac{S_k'}{S_k}(t)\frac{\partial u}{\partial t}(t)+\lambda_1\big(B_{\mathbb{M}^l(\kappa)}(r)\big)\,u(t)=0 \tag{4·11}$$

with $u(0)=1$, $u'(0)=0$ and $v\colon W\to\mathbb{R}$ satisfies $\triangle_{dh^2}v+\lambda_1(W)v=0$ in W and $v|\partial W=0$. It is clear that $\psi\in C^2(\Omega)\bigcap C^0(\overline{\Omega_2})$ with $\psi>0$ in Ω and $\psi|\partial\Omega=0$. The Laplace operator of ds^2 is given by

$$\begin{aligned}\triangle_{ds^2}=\frac{\partial^2}{\partial t^2}+\left[(m-1)\,\frac{1}{f}\,\frac{\partial f}{\partial t}+n\,\frac{1}{g}\,\frac{\partial g}{\partial t}\right]\frac{\partial}{\partial t}+\frac{m-3}{f^3}\,d\theta^2\,(\nabla_{d\theta^2}f,\,\nabla_{d\theta^2}\cdot)\\+\frac{n}{gf^2}\,d\theta^2\,(\nabla_{d\theta^2}g,\,\nabla_{d\theta^2}\cdot)+\frac{1}{f^2}\triangle_{d\theta^2}+\frac{1}{g^2}\triangle_{dh^2}\end{aligned} \tag{4·12}$$

where $\nabla_{d\theta^2}$ and $\triangle_{d\theta^2}$ are respectively the gradients and the Laplacian of $\mathbb{S}^{m-1}$ and ∇_{dh^2} and $\triangle_{dh^2}$ are respectively the gradients and the Laplacian of N. Computing $-\triangle_{ds^2}\psi/\psi$ we have,

$$\begin{aligned}-\frac{\triangle_{ds^2}\psi}{\psi}&=-\frac{u''}{u}-(m-1)\,\frac{f'}{f}\,\frac{u'}{u}-n\,\frac{g'}{g}\,\frac{u'}{u}-\frac{1}{g^2}\,\frac{\triangle_{dh^2}v}{v}\\&=\lambda_1(B_{\mathbb{M}^s(\kappa)}(r))-\frac{u'}{u}\left((m-1)\,\frac{f'}{f}+n\,\frac{g'}{g}-(s-1)\frac{S_k'}{S_k}\right)+\frac{1}{g^2}\lambda_1(W).\end{aligned} \tag{4·13}$$

If $(m-1)\frac{f'}{f}+n\frac{g'}{g}\geqslant(s-1)\frac{S_k'}{S_k}$ then from (4·13)

$$\inf\left(-\frac{\triangle\psi}{\psi}\right)\geqslant\lambda_1\big(B_{\mathbb{M}^s(\kappa)}(r)\big)+\inf\frac{1}{g^2}\lambda_1(W).$$

If $(m-1)\frac{f'}{f}+n\frac{g'}{g}-(s-1)\frac{S_k'}{S_k}\leqslant 0$ then

$$\sup\left(-\frac{\triangle\psi}{\psi}\right)\leqslant\lambda_1\big(B_{\mathbb{M}^s(\kappa)}(r)\big)+\sup\frac{1}{g^2}\lambda_1(W),$$

since $-(u'/u)\geqslant 0$.

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