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Density of half-horocycles on geometrically infinite hyperbolic surfaces

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Abstract. On the unit tangent bundle of a hyperbolic surface, we study the density of positive orbits $(h^s v)_{s\geq 0}$ under the horocyclic flow. More precisely, given a full orbit $(h^s v)_{s\in\mathbb{R}}$, we prove that under a weak assumption on the vector v, both half-orbits $(h^s v)_{s\geq 0}$ and $(h^s v)_{s\leq 0}$ are simultaneously dense or not in the non-wandering set $\mathcal{E}$ of the horocyclic flow. We give also a counterexample to this result when this assumption is not satisfied.

1. *Introduction*

On the unit tangent bundle of a finite-volume hyperbolic surface, Hedlund [**H**] proved that all positive non-periodic orbits $(h^s v)_{s\geq 0}$ of the horocyclic flow are dense.

On the other hand, there exists today a complete geometric criterion (see [**Da**] for a general result and full references) that tells us whether, on any hyperbolic surface, a full horocycle $(h^s v)_{s\in\mathbb{R}}$ is dense or not in the non-wandering set $\mathcal{E}$ of the horocyclic flow. It is dense if and only if v is *horospherical* (see §3).

When such a full horocycle is dense, a natural question from a dynamical point of view is whether both half-horocycles $(h^s v)_{s\geq 0}$ and $(h^s v)_{s\leq 0}$ are dense or not.

Hedlund had a partial positive answer, on the so-called 'surfaces of the first kind' for *radial vectors*, that is, vectors v whose backward geodesic orbit returns infinitely often to a compact set. (We consider backward geodesics when we study the unstable horocyclic flow.)

In [**Scha**], we solved the problem on hyperbolic surfaces whose fundamental group is finitely generated, and we proved that the answer is always positive (both positive and negative horocyclic orbits are simultaneously dense or not), except in the case (trivial obstruction, see Figure 1) of vectors v such that one half-horocycle is dense in $\mathcal{E}$, and the other eventually leaves all compact sets.

In this work, we obtain an almost complete answer to this question, on any (infinitely generated) oriented hyperbolic surface.

Let us introduce some definitions and notations. Let S be a connected oriented hyperbolic surface. It can be written $S = \Gamma\backslash\mathbb{D}$, where Γ is a discrete group of isometries of

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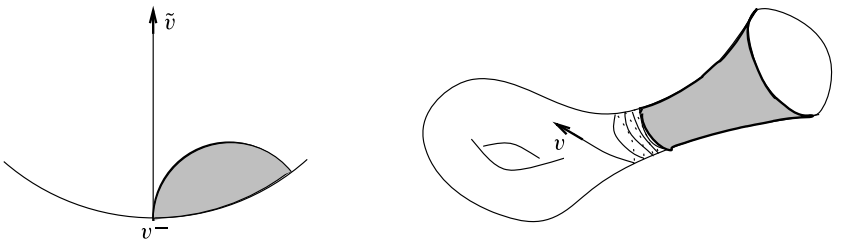


FIGURE 1. A vector v whose positive horocycle $(h^s v)_{s\geq 0}$ is not dense in $\mathcal{E}$.

the hyperbolic disc $\mathbb{D}$, and its unit tangent bundle T^1S can be identified with the quotient $\Gamma\backslash T^1\mathbb{D}$ of the unit tangent bundle of the hyperbolic disc by Γ. Let $\pi : T^1S \to S$ be the canonical projection. If $u \in T^1S$, and $\widetilde{u}$ is a lift of u to $T^1\mathbb{D}$, we denote by $u^- \in S^1$ (respectively, u^+) the negative (respectively, positive) endpoint of the geodesic $(g^t\widetilde{u})_{t\in\mathbb{R}}$ defined by $\widetilde{u}$ on the boundary at infinity $S^1 = \partial\mathbb{D}$. The limit set $\Lambda_\Gamma \subset S^1$ is the set $\overline{\Gamma \cdot o}\backslash\Gamma \cdot o$ of limit points of an orbit. In this work, we assume Γ to be non-elementary, that is, not virtually abelian. This is equivalent to saying that the limit set Λ_Γ is infinite.

We endow the boundary S^1 with its counterclockwise orientation.

A horocycle is a Euclidean circle of $\mathbb{D}$, tangent to $\partial\mathbb{D}$. An unstable horocycle is the set of unit vectors orthogonal to a given horocycle, and pointing outwards. We study here the *unstable horocyclic flow*. This flow $(h^s)_{s\in\mathbb{R}}$ acts on $T^1\mathbb{D}$; its orbits are unstable horocycles, and the map h^s turns vectors a distance $|s|$ (for the induced Riemannian distance) on the unstable horocycle in the clockwise direction. This flow induces on T^1S the unstable horocycle flow of T^1S.

A vector $v \in T^1S$ is *radial* if its geodesic orbit $(g^{-t}v)_{t\geq 0}$ returns infinitely often to a compact set (here, $(g^t)_{t\in\mathbb{R}}$ denotes the geodesic flow, which moves vectors a distance t along the geodesic line that they define). It is well known that if v is radial, its horocycle $(h^s v)_{s\in\mathbb{R}}$ is dense in the non-wandering set $\mathcal{E}$ of the horocyclic flow, that is, the set $\mathcal{E}$ of vectors of T^1S such that all neighbourhoods V satisfy $h^{s_n}V \cap V \neq \emptyset$ for a sequence $s_n \to +\infty$. (This is the set which contains the interesting dynamics.)

Hedlund's result extends to the following theorem.

THEOREM 1.1. *Let S be a non-elementary oriented hyperbolic surface. Let $v \in T^1S$ be a radial vector. Then its positive half-horocycle $(h^s v)_{s\geq 0}$ is dense in the non-wandering set $\mathcal{E}$ of the horocyclic flow if and only if v^- is not the first endpoint (in the counterclockwise direction) of an interval of $S^1\backslash\Lambda_\Gamma$.*

The above result is stated and proved because its proof is simple and short, and uses simplified versions of Hedlund's ideas. But I prove below a much more general result.

THEOREM 1.2. *Let S be a non-elementary oriented hyperbolic surface. Let v be a vector whose full horocycle $(h^s v)_{s\in\mathbb{R}}$ is dense in $\mathcal{E}$, and such that there exist two constants $\Lambda > 0$ and $0 < \alpha_0 \leq \pi/2$, such that the geodesic ray $(\pi(g^{-t}v))$, $t \geq 0$, intersects infinitely many closed geodesics of length at most Λ, with an angle of intersection at least α_0. Then both half-orbits $(h^s v)_{s\geq 0}$ and $(h^s v)_{s\leq 0}$ are dense in $\mathcal{E}$.*

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In the assumptions of this theorem, the geodesic ray $(\pi(g^{-t}v))_{t\geq 0}$ can intersect (with an angle of intersection bounded from below) infinitely many times finitely many closed geodesics, or infinitely many distinct closed geodesics.

A radial vector satisfies this assumption, because closed geodesics are dense in $\mathcal{E}$, and if v is radial, $(g^{-t}v)$ returns infinitely often to a closed ball of $\mathcal{E}$. One can show that the assumption of Theorem 1.2 is therefore satisfied, by using for example a pants decomposition of the surface.

It is worth observing that a very close assumption is used in a recent work of Sarig [**Sa**] on the horocyclic flow.

We can ask whether this result is optimal or not. We construct a counterexample to a completely general statement.

THEOREM 1.3. *There exist non-elementary oriented hyperbolic surfaces containing vectors v such that $(h^s v)_{s\geq 0}$ is dense in $\mathcal{E}$, $(h^s v)_{s\leq 0}$ is not dense in $\mathcal{E}$, v^- is not the endpoint of an interval of $S^1\backslash\Lambda_\Gamma$, and $(g^{-t}v)_{t\geq 0}$ intersects infinitely many closed geodesics of length going to infinity.*

Depending on the examples, the intersection angle of $(g^{-t}v)_{t\geq 0}$ with these geodesics can be uniformly bounded away from zero, or go to zero.

Section 2 is devoted to preliminaries, and the rest of the paper to proofs of the three results above.

2. *Preliminaries*

Hyperbolic geometry. The hyperbolic disc $\mathbb{D} = D(0, 1)$ is endowed with the hyperbolic metric $4dx^2/(1-|x|^2)^2$. Let o be the origin of the disc, and $\pi : T^1\mathbb{D} \to \mathbb{D}$ the canonical projection. The boundary at infinity of $\mathbb{D}$ is $S^1 = \partial\mathbb{D}$. We denote by d both Riemannian distances on $\mathbb{D}$ and $T^1\mathbb{D}$.

The Busemann cocycle is the continuous map defined on $S^1 \times \mathbb{D}^2$ by

$$\beta_\xi(x, y) = \lim_{z\to\xi}(d(x, z) - d(y, z)).$$

The map

$$v \in T^1\mathbb{D} \mapsto (v^-, v^+, \beta_{v^-}(\pi(v), o)) \in (S^1 \times S^1)\backslash\Delta \times \mathbb{R},$$

where $v^\pm$ are the endpoints of the geodesic $(g^t v)_{t\in\mathbb{R}}$ in S^1, and Δ is the diagonal of $S^1 \times S^1$, is a homeomorphism. In the following, we will identify $T^1\mathbb{D}$ with the set of *Hopf coordinates* $(S^1 \times S^1)\backslash\Delta \times \mathbb{R}$.

The classical identification of $\mathbb{D}$ with $\mathbb{H} = \mathbb{R} \times \mathbb{R}_+^*$ through the homography $z \mapsto i(1+z)/(1-z)$ allows us to identify the group of isometries preserving orientation of $\mathbb{D}$ with $\mathrm{PSL}(2, \mathbb{R})$ acting by homographies on $\mathbb{H}$. This action extends to a simply transitive action on $T^1\mathbb{D}$ (or $T^1\mathbb{H}$). We therefore identify $T^1\mathbb{D}$ with $\mathrm{PSL}(2, \mathbb{R})$. In the Hopf coordinates, an element $\gamma \in \mathrm{PSL}(2, \mathbb{R})$ acts as follows:

$$\gamma \cdot (v^-, v^+, t) = (\gamma \cdot v^-, \gamma \cdot v^+, t + \beta_{v^-}(o, \gamma^{-1} \cdot o)).$$

If $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ is a discrete group, we identify the unit tangent bundle $T^1S = \Gamma\backslash T^1\mathbb{D}$ of $\Gamma\backslash\mathbb{D}$ with the quotient $\Gamma\backslash((S^1 \times S^1)\backslash\Delta \times \mathbb{R})$.

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The *limit set* Λ_Γ of the group is defined as $\Lambda_\Gamma = \overline{\Gamma \cdot o} \backslash \Gamma \cdot o \subset S^1$. It is also the smallest non-empty closed Γ-invariant subset of S^1. We will often use the minimality of the action of Γ on Λ_Γ: for all $\xi \in \Lambda_\Gamma$, $\Gamma \cdot \xi$ is dense in Λ_Γ.

A point $\xi \in \Lambda_\Gamma$ is *radial* if it is the limit of a sequence $(\gamma_n \cdot o)$ of points of $\Gamma \cdot o$ that stay at bounded hyperbolic distance of the geodesic ray $[o, \xi)$ from o to ξ. We denote by Λ_{rad} the *radial limit set*. The set of points of Λ_Γ that are fixed by a hyperbolic isometry (see definition below) of Γ is included in Λ_{rad}.

A *horocycle* of $\mathbb{D}$ is a Euclidean circle tangent to S^1; it is also a level set of a Busemann function. A *horoball* is a Euclidean disk bounded by a horocycle. A point $\xi \in \Lambda_\Gamma$ is *horospherical* if all horoballs centred at ξ contain infinitely many points of the orbit $\Gamma \cdot o$. In particular, Λ_{rad} is included in the set of horospherical limit points, denoted by Λ_{hor}.

An isometry of $\mathrm{PSL}(2, \mathbb{R})$ is said to be *hyperbolic* if it has exactly two fixed points on S^1, *parabolic* if it fixes exactly one point of S^1, and *elliptic* in other cases. We denote by $\Lambda_{\mathrm{p}} \subset \Lambda_\Gamma$ the set of *parabolic* limit points, that is, the fixed points of a parabolic isometry of Γ.

Any oriented hyperbolic surface can be written as the quotient $S = \Gamma \backslash \mathbb{D}$ of $\mathbb{D}$ by a discrete subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$ without elliptic element, and its unit tangent bundle $T^1 S = \Gamma \backslash T^1 \mathbb{D}$ is identified with $\Gamma \backslash \mathrm{PSL}(2, \mathbb{R})$.

In this note, we will always assume Γ to be *non-elementary*, that is, $\#\Lambda_\Gamma = +\infty$.

When S is compact, $\Lambda_\Gamma = \Lambda_{\mathrm{rad}} = S^1$. The surface is *convex-cocompact* when Γ is finitely generated and contains only hyperbolic isometries. In this case, $\Lambda_\Gamma = \Lambda_{\mathrm{rad}}$ is strictly included in S^1, and Γ acts cocompactly on the set $(\Lambda_\Gamma \times \Lambda_\Gamma) \backslash \Delta \times \mathbb{R} \subset T^1 \mathbb{D}$. When S has finite volume, all ends are cusps isometric to $\{z \in \mathbb{H}, |z| > 1\} / \langle z \mapsto z + 1 \rangle$, and $\Lambda_\Gamma = \Lambda_{\mathrm{rad}} \sqcup \Lambda_{\mathrm{p}} = S^1$.

Geodesic and horocyclic flows. A hyperbolic geodesic of $\mathbb{D}$ is a diameter or half-circle orthogonal to S^1. A vector $v \in T^1 \mathbb{D}$ is tangent to a unique geodesic, and orthogonal to exactly two horocycles containing its basepoint, tangent to S^1 respectively at v^+ and v^-. The set of vectors $w \in T^1 \mathbb{D}$ such that $w^- = v^-$ and whose basepoint belongs to the horocycle tangent to S^1 at v^- and containing $\pi(v)$ is the *strong unstable horocycle*, or strong unstable manifold of v. We denote it by $W^{su}(v) = \{h^s v, s \in \mathbb{R}\}$. The *strong stable horocycle* is defined similarly.

The *geodesic flow* $(g^t)_{t \in \mathbb{R}}$ acts on $T^1 \mathbb{D}$ by moving a vector v a distance t along the geodesic that it defines. In the identification of $T^1 \mathbb{D}$ with $\mathrm{PSL}(2, \mathbb{R})$, this flow corresponds to the right action by multiplication of the one-parameter subgroup

$$\left\{ a_t := \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, t \in \mathbb{R} \right\}.$$

The *unstable horocyclic flow* $(h^s)_{s \in \mathbb{R}}$ acts on $T^1 \mathbb{D}$ by moving a vector v a distance $|s|$ along its strong unstable horocycle. There are two possible orientations for such a flow, and we choose the orientation which corresponds to the right action by multiplication by the one-parameter subgroup

$$\left\{ n_s := \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}, s \in \mathbb{R} \right\}$$

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on PSL(2, $\mathbb{R}$). This flow makes the vectors turn around their strong unstable horocycle so that the orbit $\{h^s v,\ s \in \mathbb{R}\}$ is equal to the full strong unstable horocycle.

Moreover, for all $s \in \mathbb{R}$ and $t \in \mathbb{R}$, these geodesic and horocyclic flows satisfy the following fundamental relation:

$$g^t \circ h^s = h^{se^t} \circ g^t. \tag{2.1}$$

Remark 2.1. With our choice of orientation of S^1, when $s \to +\infty$, if $u \in T^1\mathbb{D}$ and $u_s^+ \in S^1$ is the positive endpoint of the geodesic determined by $h^s u$, then u_s^+ converges to u^-, with $u_s^+ \geq u^-$ in the counterclockwise orientation of S^1.

These two right actions commute with the left action by multiplication of PSL(2, $\mathbb{R}$) on itself, so that they are well defined on the quotient on $T^1S \simeq \Gamma\backslash\mathrm{PSL}(2, \mathbb{R})$.

Definition 2.2. Let $(\phi^t)_{t\in\mathbb{R}}$ be a flow acting by homeomorphisms on a topological space X. The *non-wandering set* of ϕ is the set of points $x \in X$ such that for all neighbourhoods V of x, there exists a sequence $t_n \to +\infty$ such that $\phi^{t_n} V \cap V \neq \emptyset$.

The first part of the following lemma is due to Eberlein [**E1**], and the second part is proved in [**Scha**].

LEMMA 2.3. *The non-wandering set of the geodesic flow acting on T^1S is*

$$\Omega := \Gamma\backslash((\Lambda_\Gamma \times \Lambda_\Gamma)\backslash\Delta \times \mathbb{R}).$$

The non-wandering set of the horocyclic flow acting on T^1S is

$$\mathcal{E} := \Gamma\backslash((\Lambda_\Gamma \times S^1)\backslash\Delta \times \mathbb{R}).$$

Moreover,

$$\mathcal{E} = \bigcup_{s\in\mathbb{R}} h^s\Omega \quad \textit{and} \quad \mathcal{E} = \overline{\bigcup_{s\geq 0} h^s\Omega}.$$

Recall that $W^{su}(v) = \{h^s v,\ s \in \mathbb{R}\}$ is compact if and only if $v^- \in \Lambda_\mathrm{p}$, and dense in $\mathcal{E}$ if and only if $v^- \in \Lambda_\mathrm{hor}$. Denote by $W_+^{su}(v) = \{h^s v,\ s \geq 0\}$ the positive half-horocycle of v.

We assume in the following that S^1 is oriented in the counterclockwise direction.

Funnels. Recall (see, for example, [**Scha**]) the following standard facts.

Remark 2.4. If the surface S viewed as $\Gamma\backslash\mathbb{H}$ has a *funnel* isometric to $\{z \in \mathbb{H}, \mathrm{Re}(z) \geq 0\}/\langle z \mapsto az\rangle$, for some fixed $a > 1$, the geodesic $\mathrm{Re}(z) = 0$ induces on the quotient a closed geodesic bounding the funnel. A geodesic intersecting this closed geodesic and entering in the funnel will never leave the funnel. In particular, the limit set Λ_Γ viewed in $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ does not intersect $\mathbb{R}_+^*$.

A horocycle centred in $\mathbb{R}_+^*$, viewed on the quotient on S, will stay in the funnel except at most during a finite interval of time. A horocycle centred at 0, viewed on S, will have a side which will never enter the funnel, and the other side which will stay in the funnel and never go outside.

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FACT 2.5. *In the case of a geometrically finite hyperbolic surface, the endpoints of an interval of $S^1\backslash\Lambda_\Gamma$ are hyperbolic; they are the endpoints of the axis of a lift of the closed geodesic which bounds the funnel. This is not necessarily the case on a general hyperbolic surface*†.

FACT 2.6. *If $v^- \in \Lambda_\Gamma$ is the first endpoint of an interval of $S^1\backslash\Lambda_\Gamma$, then $W^{su}_+(v) = \{h^s v,\ s \geq 0\}$ is not dense in $\mathcal{E}$ (see Figure 1).*

3. *Proof of Theorem 1.1*

Right horocyclic points and vectors. If $v \in T^1\mathbb{D}$, let $v^\pm$ be its endpoints in $\partial\mathbb{D}$, $\mathrm{Hor}(v) \subset \mathbb{D}$ the horoball centred at v^- and containing its base point $\pi(v)$. The *right horoball* $\mathrm{Hor}^+(v) \subset \mathrm{Hor}(v)$ is the set of basepoints of vectors of

$$\bigcup_{t\geq 0}\bigcup_{s\geq 0} h^s g^{-t} v = \bigcup_{t\geq 0}\bigcup_{s\geq 0} g^{-t} h^s v$$

(see relation (2.1)). Similarly, the *left horoball* $\mathrm{Hor}^-(v)$ is defined as the other side of $\mathrm{Hor}(v)$, that is, $\mathrm{Hor}^-(v) = \overline{\mathrm{Hor}(v)\backslash\mathrm{Hor}^+(v)}$ is the set of basepoints of vectors of

$$\bigcup_{t\geq 0}\bigcup_{s\leq 0} h^s g^{-t} v = \bigcup_{t\geq 0}\bigcup_{s\leq 0} g^{-t} h^s v.$$

Definition 3.1. If $v \in T^1\mathbb{D}$ and $\alpha > 0$, the *cone* of width α around v is the set $\mathcal{C}(v, \alpha)$ of points $x \in \mathrm{Hor}(v)$ at (hyperbolic) distance at most α of the geodesic ray $(g^{-t}v)_{t\geq 0}$. It is the intersection of $\mathrm{Hor}(v)$ with a Euclidean cone.

Definition 3.2. A vector $v \in T^1 S$ is *right horocyclic* if it admits a lift $\tilde{v} \in T^1\mathbb{D}$, such that for all $\alpha > 0$ and $D > 0$, the orbit $\Gamma \cdot o$ intersects the right part of the horoball $\mathrm{Hor}^+(g^{-D}\tilde{v})$ minus the cone $\mathcal{C}(g^{-D}\tilde{v}, \alpha)$.

A point $\xi \in \Lambda_\Gamma$ is *right horocyclic* if there exists a right horocyclic vector $v \in T^1 S$ such that $\xi = v^-$ (see Figure 2).

Recall that a vector v is horospherical if all horoballs $\mathrm{Hor}(g^{-D}v)$ contain infinitely many points of $\Gamma \cdot o$.

Recall the following proposition.

PROPOSITION 3.3. [**Scha**] *Let S be a hyperbolic surface. A vector $v \in T^1 S$ is right horocyclic if and only if $(h^s v)_{s\geq 0}$ is dense in $\mathcal{E}$.*

Recall also the following result.

PROPOSITION 3.4. [**Scha**] *Let S be an oriented hyperbolic surface. If $p \in \Omega$ is a periodic vector for the geodesic flow, then its positive half-horocycle $(h^s(p))_{s\geq 0}$ is dense in the non-wandering set $\mathcal{E}$ of the horocyclic flow if and only if p^- is not the first endpoint of an interval of $S^1\backslash\Lambda_\Gamma$.*

† We can consider the group $\Gamma = \langle \alpha^n h \alpha^{-n},\ n \in \mathbb{Z}\rangle$, where h and α are two hyperbolic isometries generating a Schottky group, and $\alpha \notin \Gamma$, so that its fixed points α^- and α^+ are the endpoints of an interval of $S^1\backslash\Lambda_\Gamma$, but it does not correspond to a funnel on the quotient surface. We are indebted to M. Peigné for this example.

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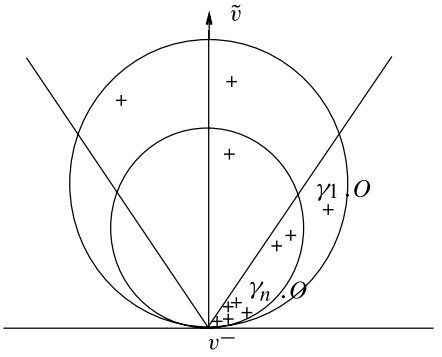


FIGURE 2. Lift of a right horocyclic vector.

THEOREM 3.5. (Hedlund [**H**, Theorem 4.2]) *Let $S = \Gamma\backslash\mathbb{D}$ be an oriented hyperbolic surface of the first kind, that is, such that $\Lambda_\Gamma = S^1$. Let $v \in T^1S$ be such that $(g^{-t}v)_{t\geq 0}$ returns infinitely often to a compact set, that is, such that $v^- \in \Lambda_{\mathrm{rad}}$. Then its positive half-horocycle $(h^s v)_{s\geq 0}$ is dense in T^1S.*

3.1. *Proof of Theorem 1.1.* If v is a radial vector, and v^- is the first endpoint of an interval of $S^1\backslash\Lambda_\Gamma$, using Lemma 2.3, we see that for $s > 0$ large enough, $h^s v$ will leave the non-wandering set $\mathcal{E}$ of the horocyclic flow, and never return to $\mathcal{E}$. Thus it cannot be dense in $\mathcal{E}$.

We will prove that a radial vector $v \in T^1S$ such that v^- is not the first endpoint of an interval of $S^1\backslash\Lambda_\Gamma$ is right horocyclic. To simplify notation, without ambiguity we shall write v for the lift of v to $T^1\mathbb{D}$. Let $R > 0$ and $D > 0$ be two sufficiently large constants. Let us prove that

$$\#\Gamma \cdot o \cap \mathrm{Hor}^+(g^{-D}v)\backslash\mathcal{C}(v, R) = +\infty.$$

• As v is radial, there exist R_0 and a sequence $\gamma_n \cdot o \to v^-$ such that $d(\gamma_n \cdot o, (v^-, v]) \leq R_0$. Let $g^{-t_n}v$, $t_n \to +\infty$, be a sequence of vectors at distance at most R_0 from $\gamma_n \cdot o$. Applying the isometry γ_n^{-1} to $g^{-t_n}v$, we get a sequence of vectors $\gamma_n^{-1}g^{-t_n}v$ on the unit tangent bundle $T^1B(o, R_0)$ of the ball $B(o, R_0)$. Up to a subsequence, we can assume that this sequence converges to a vector $w_\infty \in T^1B(o, R_0)$, with endpoints $w_\infty^\pm \in \partial\mathbb{D}$.

Note that (see Figure 3) $\gamma_n^{-1} \cdot v^- \to w_\infty^-$, and that the half-horocycle of $\mathbb{D}$, $\pi(\gamma_n^{-1} \cdot \bigcup_{s\geq 0} h^s v)$, converges (in the Hausdorff topology of closed subsets of $\overline{\mathbb{D}}$) to the half-circle $[w_\infty^-, w_\infty^+]$ of the boundary, oriented in the counterclockwise direction.

• As v^- is not the first endpoint of an interval of $S^1\backslash\Lambda_\Gamma$, and the γ_n preserve orientation, w_∞^- is not the first endpoint of an interval of $S^1\backslash\Lambda_\Gamma$.

• Choose a hyperbolic isometry $h \in \Gamma$, with fixed points $h^\pm \in \Lambda_\Gamma$, both in the interval $[w_\infty^-, w_\infty^+]$, on the right-hand side of w_∞^-, close to w_∞^-. The axis of h projects on S to a closed geodesic. Let $D_0 > 0$ be the distance from this geodesic to the projection of the origin o of the disk $\mathbb{D}$.

Up to conjugation of h by another hyperbolic isometry which has its fixed points on the right-hand side of w_∞^-, we can assume that the axis (h^-, h^+) of the isometry h is at distance at least $2R + R_0 + D_0$ of the geodesic (w_∞^-, w_∞^+).

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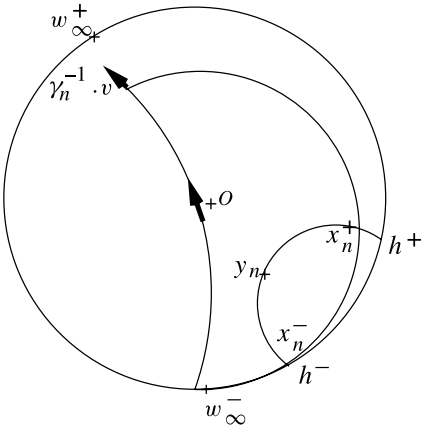


FIGURE 3. Proof of Theorem 1.1.

• As the half-horocycle $\pi(\bigcup_{s\geq 0} h^s \gamma_n^{-1} v)$ converges to $[w_\infty^-, w_\infty^+]$ (in the Hausdorff topology on closed subsets of $\overline{\mathbb{D}}$) for n large enough, this half-horocycle intersects the geodesic (h^-, h^+) at two points x_n^-, x_n^+, with $d(x_n^-, x_n^+) \to \infty$.

We see also that if y_n is the point of the geodesic segment $[x_n^-, x_n^+]$ at the largest distance of the horocycle $\pi(\bigcup_{s\geq 0} h^s \gamma_n^{-1} v)$, then this distance from y_n to this horocycle goes to infinity when $n \to +\infty$.

But this point y_n is at distance at most D_0 of a point, say $g_n \cdot o$, of the orbit $\Gamma \cdot o$. In other words, for n large enough, $g_n \cdot o$ belongs to the horoball $\mathrm{Hor}^+(g^{-D}\gamma_n^{-1} v)$.

The point y_n lies on (h^-, h^+), and is therefore at a distance at least $2R + R_0 + D_0$ from the geodesic (w_∞^-, w_∞^+). We deduce that $g_n \cdot o$ is at a distance at least $2R$ from this geodesic. For n large enough, $g_n \cdot o$ is at a distance at least R from the geodesic $(\gamma_n^{-1} \cdot v^-, \gamma_n^{-1} \cdot v^+)$.

• Now apply the isometry γ_n: denote $h_n^\pm = \gamma_n \cdot h^\pm$, where $h_n = \gamma_n \circ h \circ \gamma_n^{-1}$ is the corresponding hyperbolic isometry. The points $\gamma_n \cdot g_n \cdot o$, for n large enough, are all in the horoball $\mathrm{Hor}^+(g^{-D} v)$ but not in the cone $\mathcal{C}(v, R)$. This is the desired result.

4. *And more generally*

Before presenting the proof of Theorem 1.2, let us recall some classical lemmas of hyperbolic geometry. We refer to [**G-Ha**] or [**C-D-P**], for example.

LEMMA 4.1. *Let (a, b, c) be a hyperbolic triangle (possibly infinite). If the angle at a is greater than $\alpha_0 > 0$, there exists a constant $C(\alpha_0) > 0$, such that:*

(1) $d(a, b) + d(b, c) - C(\alpha_0) \leq d(b, c) \leq d(a, b) + d(a, c)$;

(2) *the distance from a to $[b, c]$ is smaller than $C(\alpha_0)$.*

The converse to this lemma is also true.

LEMMA 4.2. *Let $k > 0$. There exist constants $\alpha(k) > 0$, $d(k) > 0$ and $C(k) = C(\alpha(k))$ such that if (a, b, c) is a hyperbolic triangle (possibly infinite) such that $d(a, [b, c]) \leq k$*

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and $d(b, c) \geq d(k)$, then the angle at the vertex a of the triangle (a, b, c) is greater than or equal to $\alpha(k)$, and

$$d(a, b) + d(b, c) - C(k) \leq d(b, c) \leq d(a, b) + d(a, c).$$

Let us state another lemma which will be useful in the following.

LEMMA 4.3. *There exists a constant $\delta > 0$, such that for all $\xi \in \partial\mathbb{D}$ and $p, q \in \mathbb{D}$ such that $\beta_\xi(p, q) = 0$, there exists an 'interior triangle' α, β, γ satisfying $\alpha \in (\xi, q]$, $\beta \in (\xi, p]$, $\gamma \in [p, q]$, $d(\alpha, \beta) \leq \delta$, $d(\alpha, \gamma) \leq \delta$, $d(\beta, \gamma) \leq \delta$. Moreover, $\beta_\xi(\alpha, \beta) = 0$, $d(\alpha, q) = d(\gamma, q)$, $d(\beta, p) = d(\gamma, p)$.*

In this situation, we also have $d(p, q) = 2d(p, \gamma) = 2d(p, \beta) = 2d(q, \gamma) = 2d(q, \alpha)$.

Let us now prove Theorem 1.2.

Proof. Let us prove first that one of the two sides $(h^s v)_{s\geq 0}$ or $(h^s v)_{s\leq 0}$ is dense in $\mathcal{E}$, or equivalently that v^- is left or right horocyclic. As $(h^s v)_{s\in\mathbb{R}}$ is dense in $\mathcal{E}$, the point $v^- \in \Lambda_\Gamma$ is horospherical. If v^- were horocyclic but neither left nor right horocyclic, by definition, the horoball $\mathrm{Hor}(\widetilde{v})$ minus a certain cone $\mathcal{C}(\widetilde{v}, R)$ would contain no element of $\Gamma \cdot o$. This implies that $\Gamma \cdot o \cap \mathcal{C}(\widetilde{v}, R)$ is infinite, so that v^- is radial. As v^- is in the limit set, it cannot be the endpoint of two distinct connected components of $S^1\backslash\Lambda_\Gamma$. But Theorem 1.1 implies in this case that one of the two sides $(h^s v)_{s\geq 0}$ or $(h^s v)_{s\leq 0}$ is dense in $\mathcal{E}$.

Assume therefore that $(h^s v)_{s\geq 0}$ (for example) is dense, or equivalently that v^- is right-horocyclic, and let us prove that it is also left-horocyclic, or equivalently that $(h^s v)_{s\leq 0}$ is dense. The idea is as follows.

Let γ be a hyperbolic isometry coresponding to one of the closed geodesics, of length at most Λ, that are intersected by $(g^{-t} v)_{t\geq 0}$. This isometry leaves the orbit $\Gamma \cdot o$ globally invariant. Given two constants $D > 0$ and $R > 0$, this isometry, iterated in a convenient way, should send a point of $\Gamma \cdot o \cap \mathrm{Hor}^+(g^{-D} v)\backslash\mathcal{C}(v, R)$ to $\Gamma \cdot o \cap \mathrm{Hor}^-(g^{-D+\mathrm{const}} v)\backslash\mathcal{C}(v, R - \mathrm{const}')$, for some constants const and const$'$ depending on D, R, but not on γ. This will allow us to prove that there exist infinitely many points of $\Gamma \cdot o$ inside $\mathrm{Hor}^-(g^{-D'} v)\backslash\mathcal{C}(v, R')$, for all D', $R' > 0$.

Let us introduce some notation. We lift v to $T^1\mathbb{D}$, still denoting it by v. Assume that $\gamma^- \geq v^- \geq \gamma^+$, on the circle oriented in the counterclockwise direction, so that γ will roughly move points from Hor^+ to Hor^-, and not in the other direction. Assume therefore that v^- is right horocyclic, and let us show that it is left horocyclic.

Fix first D, R large compared to all constants appearing in the statement and in all lemmas used in the proof: the bound Λ on the lengths of closed geodesics that are intersected, the angle α_0 which is a lower bound for the angle of intersection between $(g^{-t} v)$ and these closed geodesics, and the constants $C(\alpha_0)$ of Lemma 4.1 and $C(k)$, $d(k)$ of Lemma 4.2 for $k = C(\pi/2)$ above, and the constant $C_2(\alpha_0)$ which appears below.

Choose a point $y_0 \in \Gamma \cdot o$ in $\mathrm{Hor}^\pm(g^{-D} v)\backslash\mathcal{C}(v, R)$, and x_0 its projection on the axis (γ^-, γ^+). Let $x_n = \gamma^n(x_0)$ and $y_n = \gamma^n(y_0)$.

Consider, finally, an isometry γ such that the axis (γ^-, γ^+) of γ induces on S a closed geodesic intersecting $(g^{-t} v)_{t\geq 0}$, such that the angle on $\mathbb{D}$ between the axis of γ

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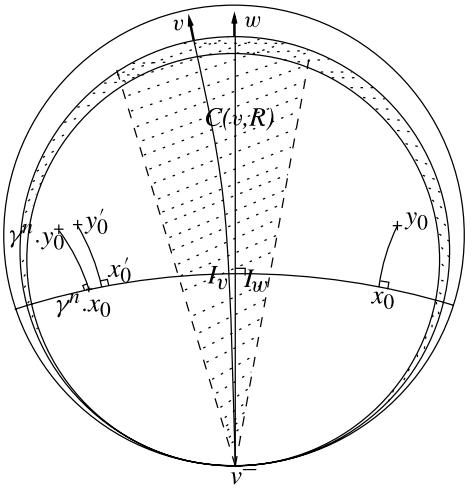


FIGURE 4. Proof of Theorem 1.2.

and $(g^{-t}v)$ is at least α_0, and such that y_0 belongs to the bounded connected component of $\mathrm{Hor}(v)\backslash(\gamma^-, \gamma^+)$. As there exist infinitely many such geodesics (γ^-, γ^+) arbitrarily far from y_0, we shall assume that the distance from y_0 to (γ^-, γ^+) is greater than the constant $d(k)$ of Lemma 4.2, for $k = C(\pi/2)$, the constant given by Lemma 4.1.

Denote by $w = h^s v$ the vector of the horocycle of v such that $(g^{-t}w)$ intersects (γ^-, γ^+) orthogonally. Denote respectively by I_v and I_w the intersection points of $(g^{-t}v)$ (respectively $(g^{-t}w)$) with (γ^-, γ^+).

LEMMA 4.4. *If the angle at I_v between (γ^-, γ^+) and $(g^{-t}v)_{t\geq 0}$ is bounded from below by α_0, the distance between I_v and I_w is bounded (from above) by $C(\alpha_0)$.*

Proof. Consider the infinite triangle v^-, I_v, I_w. By Lemma 4.1, we get $d(I_v, (v^-, I_w)) \leq C(\alpha_0)$. As the angle at I_w between (I_v, I_w) and (v^-, I_w) is equal to $\pi/2$, this distance equals $d(I_v, I_w)$. The lemma is proven. □

LEMMA 4.5. *Let x_0 be the projection of y_0 on (γ^-, γ^+), and $x_n = \gamma^n x_0$, $y_n = \gamma^n y_0$. There exist constants $C_2(\alpha_0)$ and $R(\alpha_0)$, depending only on α_0, such that if $d(x_n, I_w) \geq R(\alpha_0)$, and $d(x_n, I_v) \geq R(\alpha_0/4)$, then*

$$d(y_0, x_0) + d(x_n, I_w) - C_2(\alpha_0) \leq d(y_n, (g^{-t}w)_{t\geq 0}) \leq d(y_0, x_0) + d(x_n, I_w),$$
$$d(y_0, x_0) + d(x_n, I_v) - C_2(\alpha_0) \leq d(y_n, (g^{-t}v)_{t\geq 0}) \leq d(y_0, x_0) + d(x_n, I_v).$$

As a consequence,

$$|d(y_n, (g^{-t}w)_{t\in\mathbb{R}}) - d(y_n, (g^{-t}v)_{t\in\mathbb{R}})| \leq d(I_v, I_w) + C_2(\alpha_0) \leq C_1(\alpha_0) + C_2(\alpha_0).$$

Proof. The third line follows directly from the first two lines and Lemma 4.4. It is therefore enough to prove the first two lines of inequalities.

In both cases, the right inequality follows immediately from the standard triangle inequality and from the fact that $d(x_n, y_n) = d(x_0, y_0)$.

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The triangle y_n, x_n, I_w has a right angle at x_n, so that by Lemma 4.1,

$$d(y_n, I_w) \geq d(y_n, x_n) + d(x_n, I_w) - C(\pi/2) = d(y_0, x_0) + d(x_n, I_w) - C(\pi/2).$$

In the same way, $d(y_n, I_v) \geq d(y_0, x_0) + d(x_n, I_v) - C(\alpha_0)$.

We still have to prove that the distance from y_n to $(g^{-t}w)_{t\geq 0}$ (respectively $(g^{-t}v)_{t\geq 0}$), is, up to uniform constants, realised by $d(y_n, I_w)$ (respectively $d(y_n, I_v)$).

Let p be the projection of y_n on $(g^{-t}w)_{t\in\mathbb{R}}$. Assume that $d(x_n, I_w) \geq C(\alpha_0/4)$. Lemma 4.1 in the triangle y_n, I_w, x_n implies that the angle at I_w between $[I_w, x_n]$ and $[I_w, y_n]$ is bounded from above by $\alpha_0/4$.

Then the angle at I_w between $[I_w, p]$ and $[I_w, y_n]$ is larger than $\pi/2 - \alpha_0/4 > 0$. Lemma 4.1 in the triangle (p, I_w, y_n) therefore gives $d(y_n, p) \geq d(y_n, I_w) + d(I_w, p) - C(\pi/2 - \alpha_0/4)$. Thus, we have proved that

$$\begin{aligned} d(y_n, (g^{-t}w)_{t\geq 0}) = d(y_n, p) \geq d(y_n, I_w) - C(\pi/2 - \alpha_0/4) \geq d(y_0, x_0) \\ + d(x_n, I_w) - C(\pi/2) - C(\pi/2 - \alpha_0). \end{aligned}$$

The same reasoning, replacing I_w by I_v, gives

$$d(y_n, (g^{-t}v)_{t\geq 0}) \geq d(x_0, y_0) + d(x_n, I_v) - C(\alpha_0) - C(3\alpha_0/4).$$

This concludes the proof of the lemma. □

Let us now conclude the proof of the theorem. Recall that $y_0 \in \Gamma \cdot o$ belongs to $\mathrm{Hor}^+(g^{-D}v)\backslash\mathcal{C}(v, R)$, and that we want to show that for a suitable n, $\gamma^n \cdot y_0$ belongs to $\mathrm{Hor}^-(g^{-D\pm\mathrm{const}}v)\backslash\mathcal{C}(v, R \pm \mathrm{const}')$. Recall also that w is the vector on $(h^s v)_{s\in\mathbb{R}}$ such that (v^-, w^+) intersects orthogonally (γ^-, γ^+).

Let y_0' and x_0' be respectively the images of y_0 and x_0 under the symmetry of the axis (v^-, w^+). As the iterates $\gamma^n \cdot x_0$ are its translates from a distance $l(\gamma) \leq \Lambda$ on the axis (γ^-, γ^+), there exists $n \geq 1$, such that $d(\gamma^n x_0, x_0') \leq \Lambda$, and $d(\gamma^n x_0, x_0) \geq d(x_0', x_0)$. By symmetry,

$$d(y_0', \partial\mathrm{Hor}(v)) = d(y_0, \partial\mathrm{Hor}(v)) \quad \text{and} \quad d(y_0', (g^{-t}v)_{t\geq 0}) = d(y_0, (g^{-t}v)_{t\in\mathbb{R}}).$$

Write $\mathcal{H} = \mathrm{Hor}(v)$ and compare first $d(y_0, \partial\mathcal{H}) \geq D$ with $d(y_n, \partial\mathcal{H})$. By symmetry, of course, $d(y_0', \partial\mathcal{H}) \geq D$. As $x_n = \gamma^n \cdot x_0$ and x_0' are at distance at most Λ, we deduce that $|d(x_n, \partial\mathcal{H}) - d(x_0', \partial\mathcal{H})| \leq \Lambda$. We will try to bound from below $d(y_n, \partial\mathcal{H}) - d(y_0', \partial\mathcal{H})$ by a uniform constant. This will imply $d(y_n, \partial\mathcal{H}) \geq D - cste$ for D large enough.

Denote respectively by q_n and q' the projections of $y_n = \gamma^n \cdot x_0$ and y_0' on $\partial\mathcal{H}$. If $d(y_n, \partial\mathcal{H}) \geq d(y_0', \partial\mathcal{H}) \geq D$, it is perfect. Assume that $d(y_n, \partial\mathcal{H}) \leq d(y_0', \partial\mathcal{H})$, and let us prove that this distance cannot be too small compared to D.

The triangle (y_n, x_n, v^-) has an angle greater than $\pi/2$ at x_n (because x_n is the projection of y_n on (γ^-, γ^+) which intersects $[y_n, v^-)$). By Lemma 4.1, $d(x_n, [y_n, v^-)) \leq C(\pi/2)$. If we consider the triangle (q_n, x_n, v^-), we deduce in particular that the distance from y_n to the side $[q_n, x_n]$ satisfies $d(y_n, [q_n, x_n]) \leq C(\pi/2)$. We assumed that the distance $d(y_n, x_n) = d(y_n, (\gamma^-, \gamma^+))$ is large compared to $k(C(\pi/2))$. Lemma 4.2 therefore gives

$$d(x_n, y_n) + d(y_n, q_n) - C(k(C(\pi/2))) \leq d(x_n, q_n) \leq d(x_n, y_n) + d(y_n, q_n).$$

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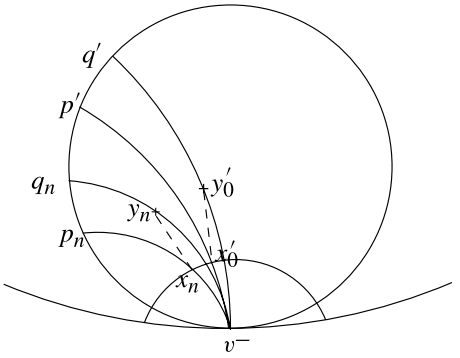


FIGURE 5. Proof of Theorem 1.2.

The same reasoning applies if we replace x_n, y_n, q_n by x_0', y_0', q'. We therefore deduce (using the fact that $d(x_n, y_n) = d(x_0', y_0')$) that

$$d(y_n, \partial\mathcal{H}) - d(y_0', \partial\mathcal{H}) = d(y_n, q_n) - d(y_0', q') \geq d(x_n, q_n) - d(x_0', q') - C(k(C(\pi/2))).$$

On the other hand, we know that $|d(x_n, \partial\mathcal{H}) - d(x_0', \partial\mathcal{H})| \leq \Lambda$. We therefore have to compare $d(x_n, q_n)$ with $d(x_n, \partial\mathcal{H})$ on the one hand, and $d(x_0', q')$ with $d(x_0', \partial\mathcal{H})$ on the other hand. We clearly have $d(x_n, q_n) \geq d(x_n, \partial\mathcal{H})$. Let us show that $d(x_0', q)$ is close to $d(x_0', \partial\mathcal{H})$.

Apply Lemma 4.3 in the triangle (p', q', v^-). In the notation of that lemma, the point x_0' lies on $(v^-, p']$. If $x_0' \in [\beta, v^-)$, we deduce easily that $|d(x_0', p') - d(x_0', q)| \leq \delta$. If x_0' were on $]\beta, p']$, there would exist a point of (γ^-, γ^+) inside the triangle (α, β, γ) closer to y_0' than to x_0', which would contradict the definition of x_0'.

All the above imply that

$$|d(x_0', \partial\mathcal{H}) - d(x_0', q')| = |d(x_0', p') - d(x_0', q')| \leq \delta.$$

Putting all these inequalities together, we have proved that

$$\begin{aligned} d(y_n, \partial\mathcal{H}) - d(y_0', \partial\mathcal{H}) &\geq d(x_n, \partial\mathcal{H}) - d(x_0', \partial\mathcal{H}) - \delta - C(k(C(\pi/2))) \\ &\geq -\Lambda - \delta - C(k(C(\pi/2))). \end{aligned}$$

In particular, we deduce the desired result, that is, that for all $D > 0$ large enough, if $d(y_0, \partial\mathcal{H}) \geq D$, then $d(y_n, \partial\mathcal{H}) \geq D - \Lambda - \delta - C(k(C(\pi/2)))$.

We still have to understand the distance from $y_n = \gamma^n \cdot y_0$ to the geodesic $(g^{-t}v)_{t\geq 0}$. We assumed that $d(y_0, (g^{-t}v)_{t\geq 0}) \geq R$. If y_0' is farther than y_0 from this geodesic, we have $d(y_0', (g^{-t}v)_{t\in\mathbb{R}}) \geq R$. In the other case, this means that y_0' is closer to $(g^{-t}v)_{t\geq 0}$ than to $(g^{-t}w)_{t\geq 0}$. By symmetry and by Lemma 4.5,

$$\begin{aligned} d(y_0', (g^{-t}w)_{t\geq 0}) &= d(y_0, (g^{-t}w)_{t\geq 0}) \\ &\geq d(y_0, (g^{-t}v)_{t\geq 0}) - C(\alpha_0) - C_2(\alpha_0) \geq R - C(\alpha_0) - C_2(\alpha_0). \end{aligned}$$

By Lemma 4.5,

$$d(y_n, (g^{-t}v)_{t\geq 0}) \geq d(y_n, (g^{-t}w)_{t\geq 0}) - C(\alpha_0) - C_2(\alpha_0).$$

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By definition of y_0' and y_n, y_n is farther than y_0' from $(g^{-t}w)_{t\in\mathbb{R}}$. We deduce from the foregoing that $d(y_n, (g^{-t}v)_{t\geq 0}) \geq R - 2C(\alpha_0) - 2C_2(\alpha_0)$.

In other words, we associated to the point y_0 of $\Gamma\cdot o \cap \mathrm{Hor}^+(g^{-D}v)\backslash\mathcal{C}(v, R)$ another point $\gamma^n y_0$ which belongs to $\mathrm{Hor}^-(g^{-D+\Lambda+\delta+C(k(C(\pi/2)))}v)\backslash\mathcal{C}(v, R - 2C(\alpha_0) - 2C_2(\alpha_0))$.

As v was assumed to be right horocyclic, we can choose successively, for $k = 1, 2, \ldots$, infinitely many such points $y_0^{(k)}$ in $\Gamma\cdot o \cap \mathrm{Hor}^+(g^{-D}v)\backslash\mathcal{C}(v, R)$ and construct for each of them another point in $\Gamma\cdot o \cap \mathrm{Hor}^-(g^{-D+\mathrm{const}}v)\backslash\mathcal{C}(v, R - \mathrm{const}')$. We can choose a sequence of points $(y_0^{(k)})_{k\in\mathbb{N}}$ on the right-hand side in such a manner that their 'images' on the left are pairwise distinct (for example, by letting the distance from $y_0^{(k)}$ to the boundary $\partial\mathrm{Hor}(v)$ go to infinity with k).

In particular, as v is right horocyclic, this proves that it is also necessarily left horocyclic. In view of Proposition 3.3, it implies that under the assumptions of Theorem 1.2, $(h^s v)_{s\geq 0}$ and $(h^s v)_{s\leq 0}$ are simultaneously dense in $\mathcal{E}$, and concludes the proof of Theorem 1.2. □

5. *A counterexample*

We prove the following result.

THEOREM 1.3. *There exist non-elementary oriented hyperbolic surfaces whose unit tangent bundle contains vectors v such that $(h^s v)_{s\geq 0}$ is dense in $\mathcal{E}$, $(h^s v)_{s\leq 0}$ is not dense in $\mathcal{E}$, v^- is not the endpoint of an interval of $S^1\backslash\Lambda_\Gamma$, and $(g^{-t}v)_{t\geq 0}$ intersects infinitely many closed geodesics with length going to infinity, with an angle which can, depending on the surface, go to 0, or be uniformly bounded from below.*

The idea of the construction is as follows. Fix $v^- = \infty$, $v^+ = 0$, and study the orbit of $o = i$.

Choose on $\mathbb{R}_+$ half-circles C_n^+, so that C_n^+ is tangent to $C_{n\pm 1}^+$, their Euclidean radius is bounded, let us say equal to 1, and they are centred at $2n + 1$, $n \geq 0$, $n \to +\infty$. Choose on $\mathbb{R}_-$ half-circles C_n^-, centred at $-x_n$, so that C_n^+ is tangent to $C_{n\pm 1}^+$, of radius $R_n \to +\infty$. By an immediate induction, we get $x_1 = R_1$, and $x_n = \sum_{k=0}^{n-1} 2R_k + R_n$. Choose hyperbolic isometries γ_n of translation length going to $+\infty$, of fixed points $\gamma_n^- = 2n + 1$ and $\gamma_n^+ = x_n$, which send C_n^+ onto C_n^-.

A classical ping pong argument gives the following lemma.

LEMMA 5.1. *The $(\gamma_n)_{n\in\mathbb{N}}$ generate a discrete, free (Schottky) group with infinitely many generators.*

Remark 5.2. It is unclear whether the above group satisfies $\Lambda_\Gamma = S^1$. If it does not, it would be interesting to construct another counterexample satisfying $\Lambda_\Gamma = S^1$.

Now, it is easy to see that the vector v based at $o = i$, with $v^- = \infty$, $v^+ = 0$, cannot be left horospherical† because the orbit of $o = i$ does not intersect $\mathrm{Hor}^-(v)$ at all.

It is also clear that if $\xi \in \Lambda_\Gamma$, $\xi \neq v^\pm$, $\gamma_n\xi$ converges to v^- on the right when $n \to +\infty$, and $\gamma_n^{-1}\xi$ converges to v^- on the left. In other words, v^- is not on the boundary of $S^1\backslash\Lambda_\Gamma$.

† The terminology *left or right horocyclic* seems absurd in this example, but it is convenient for all points v^- on the real axis on the boundary at infinity.

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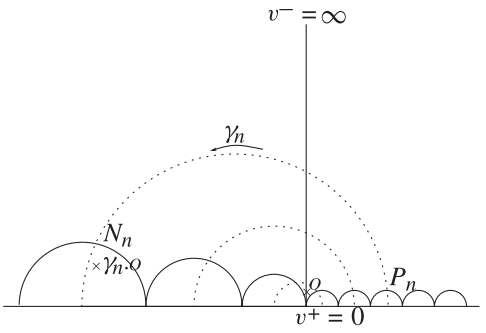


FIGURE 6. Proof of Theorem 1.3.

We just have to check that for a suitable choice of x_n and r_n, v^- is right horocyclic. To prove this, show that the vertical coordinate of $\gamma_n \cdot o$ goes to $+\infty$ (the horizontal coordinate goes to $-\infty$ by construction.)

Denote by z_n the point of coordinates $(2n+1, 1)$, and P_n the intersection point of the axis of γ_n and the circle C_n^+, that is, the intersection of both half circles of equations

$$(x - (2n+1))^2 + y^2 = 1$$

and

$$\left(x - \left(\frac{2n+1-x_n}{2}\right)\right)^2 + y^2 = \left(\frac{2n+1+x_n}{2}\right)^2.$$

Its coordinates are

$$\left(2n+1 - \frac{1}{x_n+2n+1}, \sqrt{1 - \frac{1}{(x_n+2n+1)^2}}\right),$$

so that the hyperbolic distance from P_n to z_n satisfies $d(P_n, z_n) \to 0$ when $n \to \infty$, and for n large enough (independently of the choice of x_n), $d(P_n, z_n) \leq 1$.

Furthermore, a classical computation gives $d(o, z_n) \sim 2 \ln n$ when $n \to +\infty$. Therefore, we have $d(o, P_n) \leq 3 \ln n$ when $n \to +\infty$.

The image of P_n under γ_n is the intersection point $N_n = \gamma_n \cdot P_n$ of the half-circle C_n^- and the axis of γ_n, whose equations are

$$(x + x_n)^2 + y^2 = r_n^2 \quad \text{and} \quad \left(x - \frac{2n+1-x_n}{2}\right)^2 + y^2 = \left(\frac{2n+1+x_n}{2}\right)^2.$$

The point N_n therefore has its coordinates equal to

$$\left(-x_n + \frac{r_n^2}{2n+1+x_n}, r_n\sqrt{1 - \frac{r_n^2}{(2n+1+x_n)^2}}\right).$$

Now, we know that the distance from $\gamma_n \cdot o$ to N_n is at most $3 \ln n$ for n large. We would like $\gamma_n \cdot o$ to have an imaginary part as large as possible. But

$$\mathrm{Im}(\gamma_n \cdot o) \geq r_n\sqrt{1 - \frac{r_n^2}{(2n+1+x_n)^2}} - 3 \ln n.$$

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Observe that $x_n + r_n = 2\sum_{k=1}^{n} r_k$. If we choose for all $k \geq 1$, $r_k = k$, we obtain $x_n + n = n(n+1)$, so that $x_n = n^2$, and

$$r_n\sqrt{1 - \frac{r_n^2}{(2n+1+x_n)^2}} = n\sqrt{1 - \frac{n^2}{(n+1)^4}} \sim n$$

when $n \to \infty$. In other words, this choice of $(r_n)_{n\geq 1}$ is convenient.

If we choose for all $k \geq 1$, $r_k = \alpha^k$, $\alpha > 1$, we get

$$x_n + \alpha^n = 2\alpha(\alpha^n - 1)/(\alpha - 1),$$

so that

$$x_n = \alpha^n \frac{\alpha+1}{\alpha-1} - \frac{2\alpha^{1-n}}{\alpha-1}.$$

Denote by y_n the vertical coordinate of N_n. We can see immediately that

$$y_n^2 \sim \alpha^{2n}\frac{4\alpha}{(\alpha+1)^2} \gg (3\ln n)^2.$$

Therefore, $\mathrm{Im}(\gamma_n \cdot o) \to +\infty$ when $n \to \infty$, and this choice of $(r_n)_{n\in\mathbb{N}}$ is also convenient.

Remark 5.3. It is clear by construction that on the unit tangent bundle of the quotient surface S, the geodesic $(g^{-t}v)_{t\geq 0}$ intersects infinitely many closed geodesics, of length going to $+\infty$, which are the projections on the surface of the axis of γ_n.

A computation shows that the angle θ_n between the geodesic $(v^-, v^+) = i\mathbb{R}$ and the axis of γ_n satisfies

$$\cos\theta_n = \frac{x_n - (2n+1)}{x_n + 2n + 1}.$$

In both examples above, we have $2n + 1 = o(x_n)$, whereas $\cos\theta_n \to 1$ and $\theta_n \to 0$.

In other words, the above example satisfies neither of the two assumptions of Theorem 1.2.

We can modify it so that it satisfies one of the two assumptions. The reader can easily check that if we keep the circles C_n^- unchanged, centred at $-x_n$, of radius $R_n \to +\infty$, but the circles C_n^+ are now centred at $+x_n$, and still of radius 1 (they are no longer tangent to one another), then the geodesic $(v^-, v^+) = i\mathbb{R}$ intersects the axis $(-x_n, x_n)$ of γ_n orthogonally. The distance from o to P_n is equivalent to $2\ln x_n$, so that $d(\gamma_n \cdot o, N_n) \sim 2\ln x_n$, whereas the vertical coordinate of the point N_n equals

$$y_n = r_n\sqrt{1 - \frac{r_n^2}{4x_n^2}}.$$

If $R_n = n$ and $x_n = n^2$, the imaginary part of $\gamma_n \cdot o$ goes to $+\infty$, so that the point $+\infty$ is still right horocyclic but not left horocyclic.

Furthermore, one can also add isometries that send circles of bounded height one to another, to 'fill the gaps' between the half-circles C_n^+. It is still not certain that this would give a limit set $\Lambda_\Gamma = S^1$, because of the unbounded radius of the circles C_n^-.

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