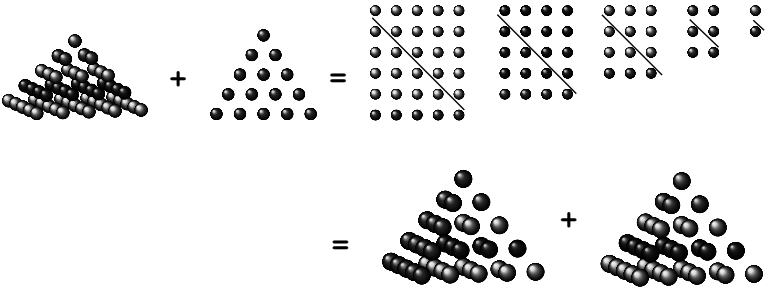
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100.36 Another look at higher vertical motion

Introduction

Suppose a projectile is fired vertically into space with initial height h and velocity v_0. Assuming that the acceleration due to gravity is the constant g and there is no air resistance, the motion is governed by

$$\frac{d^2y}{dt^2} = -g. \tag{1}$$

The velocity v and the distance above the ground y are given by

$$\frac{dy}{dt} = -gt + v_0, \text{ and } y = -\tfrac{1}{2}gt^2 + v_0t + h.$$

Here we assume that the projectile does not go very high, so that we can have a constant gravitational force. But how high is 'not very high'? This question cannot be answered by the above analysis. In this paper we will discover how to replace the differential equation (1) by a more accurate relation

$$\frac{d^2y}{dt^2} = -g + \left(\frac{h}{R}\right)\frac{2gy}{h}. \tag{2}$$

In (2), R is the radius of the planet or moon near the surface of which the motion of the particle mass m is taking place and g is the acceleration at the surface of this heavenly body. Because R is very large, the dimensionless factor $\frac{h}{R}$ is very small. This small correction allows us to calculate the motion of the projectile when it reaches heights that are a bit too high for (1) to be satisfactory. We will solve (2) and obtain an improvement in the velocity as

$$\frac{dy}{dt} = -gt + v_0 + \frac{h}{R}\left(2gt + \frac{v_0g}{h}t^2 - \frac{g^2}{3h}t^3\right) \tag{3}$$

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and an improved displacement as $y = y_1(t) + \frac{h}{R}y_2(t)$, or explicitly,

$$y = -\frac{g}{2}t^2 + v_0 t + h + \frac{h}{R}\left(gt^2 + \frac{v_0 g}{3h}t^3 - \frac{g^2}{12h}t^4\right). \tag{4}$$

While these relations are still not exact, they are better. From these we can determine that the old relations have a one percent error at a height of 48 miles above the surface of the earth. Since even a very small heavy particle will experience significant air resistance over a 48 mile drop, the earth is a poor choice for our study. For the rest of the paper we will refer to the 'moon' as the heavenly body on which our motion is taking place since there is no air there.

In a recent paper [1] the author obtained (2) and (3). The purpose of this paper is to show that these results can be obtained in a much simpler way.

The equation of motion

The exact differential equation that governs vertical motion is

$$m\frac{d^2r}{dt^2} = -\frac{GmM}{r^2}, \tag{5}$$

where m is the mass of the particle in motion and M is the mass of the moon. The variable r is the distance from the centre of the moon to the particle. If R is the radius of the moon we have $r = R + y$ and so

$$\frac{d^2(R + y)}{dt^2} = -\frac{GM}{(R + y)^2} = -\frac{GM}{R^2\left(1 + \frac{y}{R}\right)^2}.$$

Using the first order approximation $\frac{a}{(1 + \varepsilon)^2} \approx a(1 - 2\varepsilon)$, this last result for small values of $\frac{y}{R}$ is after simplification

$$\frac{d^2y}{dt^2} - 2\frac{GM}{R^2}\frac{y}{R} = -\frac{GM}{R^2}.$$

At the surface of the moon we have the acceleration $g = \frac{GM}{R^2}$, so our equation of motion is conveniently written as

$$\frac{d^2y}{dt^2} - \left(\frac{h}{R}\right)\frac{2gy}{h} = -g. \tag{6}$$

Solving the equation of motion

Mathematically we must solve the differential equation (6) subject to the initial conditions that when $t = 0, y = h$ and $\frac{dy}{dt} = v_0$. We are looking for an approximate solution of the form $y = y_1(t) + \frac{h}{R}y_2(t)$ where

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$y_1(t) = -\frac{g}{2}t^2 + v_0 t + h$ and $y_2(t)$ is unknown. Now (6) reads

$$\frac{d^2y_1(t)}{dt^2} + \frac{h}{R}\frac{d^2y_2(t)}{dt^2} - \left(\frac{h}{R}\right)\frac{2g}{h}\left(y_1(t) + \frac{h}{R}y_2(t)\right) = -g.$$

But $\frac{d^2y_1(t)}{dt^2} = -g$ so this last equation reduces to

$$\frac{d^2y_2(t)}{dt^2} - \frac{2g}{h}\left(y_1(t) + \frac{h}{R}y_2(t)\right) = 0.$$

Since $\frac{h}{R}$ is very small this reduces to

$$\frac{d^2y_2(t)}{dt^2} = \frac{2g}{h}y_1(t) = -\frac{g^2}{h}t^2 + \frac{2gv_0}{h}t + 2g.$$

Integrating twice with respect to t we get

$$y_2(t) = -\frac{g^2}{12h}t^4 + \frac{gv_0}{3h}t^3 + gt^2 + c_1t + c_2.$$

The initial conditions show that $c_1 = c_2 = 0$ and we have achieved (4).

An example

Suppose we drop our particle from height h with no initial speed. Then $v_0 = 0$ in (4) and we get

$$y = h - \frac{g}{2}\left(1 - \frac{2h}{R}\right)t^2 - \frac{g^2}{12R}t^4.$$

This is a simple quadratic equation in t^2. We can solve for the time it takes for our particle to reach the ground by setting $y = 0$ to obtain

$$\left(\frac{h}{R}\right)\frac{g^2}{12h}t^4 + \frac{g}{2}\left(1 - \frac{2h}{R}\right)t^2 - h = 0. \tag{7}$$

When $\frac{h}{R}$ is very small the approximate solution is $t^2 = \frac{2h}{g}$, so we seek an approximate solution of the form

$$t^2 = \frac{2h}{g} + \frac{h}{R}u, \tag{8}$$

where u is to be found.

Substituting (8) into (7) we get

$$\left(\frac{h}{R}\right)\frac{g^2}{12h}\left(\frac{2h}{g} + \frac{h}{R}u\right)^2 + \frac{g}{2}\left(1 - \frac{2h}{R}\right)\left(\frac{2h}{g} + \frac{h}{R}u\right) - h = 0.$$

This simplifies to

$$\frac{g}{12h}\left(\frac{2h}{g} + \frac{h}{R}u\right)^2 + \frac{u}{2} - \left(\frac{2h}{g} + \frac{h}{R}u\right) = 0.$$

We can now ignore the terms with the small factor $\frac{h}{R}$ and get $u = \frac{10h}{3g}$.

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Therefore from (8) we have

$$t^2 = \frac{2h}{g}\left(1 + \frac{5h}{3R}\right),$$

or taking square roots and using the first order approximation

$$t \approx \sqrt{\frac{2h}{g}}\left(1 + \frac{5h}{6R}\right). \tag{9}$$

The time to reach the ground assuming a constant force is $t \approx \sqrt{\frac{2h}{g}}$. This last relation tells us that the relative error is $\frac{5h}{6R}$. If the earth had no atmosphere, then a one percent error would occur when the height h was 48 miles!

Reference

1. T. J. Osler, A higher look at vertical motion, *The Mathematical Scientist*, **39** (2014) pp. 138-142.

10.1017/mag.2016.123

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100.37 Generalising an SMC problem

The Senior Mathematical Challenge (SMC) is a multiple-choice competition for senior school students. (For more information, see [1].)

Question 21 from the SMC of November 2015 was the following problem (where the names of points have been changed and the multiple-choice options have been removed).

Problem

The diagram shows a triangle ABC. The sides AB, BC and CA have lengths 4, 3 and 2 respectively. The lines GMD, IMF and EMH are drawn parallel to the sides of triangle ABC so that DE, FG and HI are of equal length. What is the length of DE?

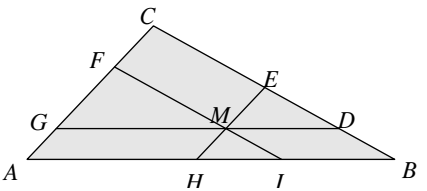


FIGURE 1

In this Note we derive results for a general triangle, not just one with sides 2, 3 and 4. More importantly, we generalise the problem by considering what happens when the point M is replaced by a triangle.

The reader might like to try the above problem before reading on.

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