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Stability of accessibility

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Abstract. We give a partial answer to a question posed by C. Pugh and M. Shub in [**PS2**]. A partially hyperbolic diffeomorphism $f \in Diff^2(M)$ is accessible if every two points of M can be connected by a path which is piecewise in a stable or an unstable leaf. We prove that accessibility is open in $Diff^2(M)$ among the partially hyperbolic diffeomorphisms with one-dimensional center bundle.

1. *Results, definitions and preliminaries*

1.1. *Introduction.* Let M be a compact, connected, smooth n-dimensional ($n \geq 3$) Riemannian manifold and $f : M \to M$ a C^2 partially hyperbolic diffeomorphism. To my knowledge, partial hyperbolicity was first introduced in [**HPS1**]. It is a generalization of the Anosov diffeomorphism: f has contracting and expanding directions with non-complementary dimensions. Topological transitivity of a pair of foliations, or accessibility as it is called in all the recent articles about stable ergodicity of partially hyperbolic diffeomorphisms (see e.g. [**BW, GPS, NT, PS2**]), was first introduced in [**BP**] for the purpose of studying the topological transitivity of these diffeomorphisms. [**PS2**] use it as a technical tool which permits the application of the Hopf argument for the purpose of proving the ergodicity of a partial hyperbolic diffeomorphism. The motivation for this work was found in the article of Plante [**Plan**], which defines joint integrability, a notion closely related to accessibility and which is stable under perturbation as pointed out in [**BPW**]. Stability results on transitivity or ergodicity strongly depend on the assumption that accessibility is stable. This question was posed in [**PS2**], and we answer it in the case where the center direction is not necessarily integrable but has dimension 1. Note that recently Dolgopyat and Wilkinson have proved in [**DW**] that stable accessibility is C^1-open and dense in the space of all C^r ($r \geq 1$) partially hyperbolic diffeomorphisms.

THEOREM 1. (Stability of accessibility) *Accessibility is C^1-open among the C^2 partially hyperbolic diffeomorphisms whose center direction is one-dimensional.*

Let λ be the Lebesgue measure on M. Here are two corollaries that we will prove in §5. Theorem (1.1) of [**Brin**] implies the following.

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COROLLARY 1. (Stable transitivity) *Let $f \in Diff^2(M)$ be partially hyperbolic with one-dimensional center bundle. If f is accessible and belongs to $Diff^2_\lambda(M)$ (i.e. preserves the Lebesgue measure on M) then it is stably transitive: there is a C^1-neighborhood $\mathcal{N}_1$ of f in $Diff^2_\lambda(M)$ whose elements are transitive.*

Remark. f preserving the Lebesgue measure, which is positive on open sets, implies that the non-wandering set of f, $\Omega(f)$, is the whole manifold M.

Corollary (a) of [**PS2**] implies the following.

COROLLARY 2. (Stable ergodicity) *Let $f \in Diff^2(M)$ be partially hyperbolic with one-dimensional center bundle. If $f \in Diff^2_\lambda(M)$ is accessible, stably dynamically coherent and center bunched then it is stably ergodic: there is a C^1-neighborhood $\mathcal{N}_2$ of f in $Diff^2_\lambda(M)$ whose elements are ergodic.*

We will define the center bunching condition and dynamical coherence in §5.

Remark. We would like to point out that in Corollary 2, stable transitivity is a direct consequence of Corollary 1 rather than of the ergodicity: due to [**Brin**], stable transitivity is part of the hypotheses of [**PS1**] and [**PS2**].

The proof of Theorem 1 relies on the observation that the notion of accessibility has a local expression, non-joint integrability, which provides an open and branched accessible set, say U_f, and it is stable under perturbation. It also has a global expression, the (u, s)-arcs connecting a point in M to a point in U_f, which is ϵ-stable under perturbation.

1.2. *Definitions.*

Definition 1. (Partial hyperbolicity) $f \in Diff^2(M)$ is partially hyperbolic if the tangent bundle of M is a Tf-invariant direct sum

$$TM = E^u \oplus E^c \oplus E^s$$

with at least two of the sub-bundles non-trivial, and there exist constants

$$0 < \mu \leq \nu < \gamma \leq 1$$

such that, for all $p \in M$ and for all $v \in T_pM$:

(1) $v \in E^s_p \Longrightarrow \mu\|v\|_p \leq \|T_pfv\|_{f(p)} \leq \nu\|v\|_p$;

(2) $v \in E^c_p \Longrightarrow \gamma\|v\|_p \leq \|T_pfv\|_{f(p)} \leq \gamma^{-1}\|v\|_p$;

(3) $v \in E^u_{f(p)} \Longrightarrow \mu\|v\|_{f(p)} \leq \|(T_pf)^{-1}v\|_p \leq \nu\|v\|_{f(p)}$;

where $\|\cdot\|$ is the Riemannian metric.

Let f be partially hyperbolic; then E^u and E^s are tangent to unique foliations $\mathcal{W}^u$ and $\mathcal{W}^s$ which have C^1 leaves, but are only Hölder continuous. Furthermore, partial hyperbolicity is stable under perturbation in $Diff^2(M)$ (see [**HPS2**]).

Let d be the Riemannian metric on M. Define for $x \in M$, $\delta > 0$,

$$B(x, \delta) = \{y \in M; d(x, y) < \delta\}.$$

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We shall use the following notation:

$$\mathcal{W}^u_l(x) = \{y \in \mathcal{W}^u(x);\, d^u(x, y) \leq l\},$$

where d^u is the Riemannian metric induced on a leaf of the foliation $\mathcal{W}^u$; similarly for the stable and the center foliations.

In the following theorem, we regroup the regularity and perturbation results concerning partially hyperbolic diffeomorphisms.

THEOREM 2. (Regularity and stability of invariant manifolds) *Let $f \in Diff^2(M)$ be partially hyperbolic with constants μ, ν and γ satisfying for f the same requirements as in the preceding definition. We have the following.*

(1) *The bundles E^u, E^c and E^s are θ-Hölder, where $\theta = \ln(\nu/\gamma)/\ln(\mu)$.*
(2) *For all $l > 0$ and all $a \in \{u, s\}$, the $\mathcal{W}^a_l(x)$ vary θ-Hölder continuously as $C^{1+\theta}$ embedded discs.*

Furthermore, let $\epsilon > 0$ and $l > 0$ be given. Then there exists a C^1-neighborhood $\mathcal{V}$ of f in $Diff^2(M)$ such that every $g \in \mathcal{V}$ verifies the following.

(3) *g is partially hyperbolic; its constants are ϵ-close to ν, γ and μ, and g satisfies the preceding two statements for a Hölder constant θ' ϵ-close to θ.*
(4) *The angles between the distributions are uniformly ϵ-close: for all $a \in \{u, s\}$ and all $p \in M$, $\measuredangle(E^a_f(p), E^a_g(p)) \leq \epsilon$.*
(5) *The Hausdorff distance between the foliations is uniformly ϵ-small: for all $a \in \{u, s\}$ and all $p \in M$, $d_{\mathcal{H}}(\mathcal{W}^a_{f,l}(p), \mathcal{W}^a_{g,l}(p)) \leq \epsilon$.*

Proof. The first two statements are proved in Corollary (2.1) of [**BP**] and Theorem (IV.1) of [**Shub**]. The other statements are consequences of Theorem (6.1) in [**HPS2**]. □

For $S \subset M$ and $\delta > 0$ let

$$Sat^u(S) = \bigcup_{x \in S} \mathcal{W}^u(x),$$

and

$$Sat^u_\delta(S) = \bigcup_{x \in S} \mathcal{W}^u_\delta(x),$$

be the unstable saturate (resp. the unstable saturate with size δ) of the set S; dually, we can define the stable saturate.

We will define accessibility following [**Brin**].

Definition 2. (Accessibility) Let $x, y \in M$. Then x and y are accessible if there exist an integer $N = N(x, y) \geq 0$, a sequence $x_0 = x, x_1, \ldots, x_{N-1}$ and $x_N = y$ of points in M, the path of accessibility connecting x and y, such that:

$$x_i \in \mathcal{W}^{a_{i-1}}_{l_{i-1}}(x_{i-1}), \quad \forall i = 1, \ldots, N,$$

where for all $i = 0, \ldots, N-1$, $a_i = a_i(x, y) \in \{u, s\}$, and $l_i = l_i(x, y) > 0$. $S \subset M$ is accessible if every two points of S are accessible. f is accessible if M is accessible.

Accessibility is an equivalence relation, so that we write '$\sim$'.

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Definition 3. (Accessibility class) Let $S \subset M$, the accessibility class of S is: $\mathcal{C}(S) = \{y \in M; \exists x \in S : x \sim y\}$.

We will need the following related notion.

Definition 4. ((N, l)-branched accessibility) $S \subset M$ is (N, l)-branched accessible if it is accessible and for all pair of points (x, y) in S, there exists a path of accessibility connecting x and y having at most N branches of induced length at most l:

$$N(x, y) \leq N \quad \text{and} \quad l_i(x, y) \leq l \quad \forall i = 0, \ldots, N(x, y) - 1.$$

1.3. *Holonomy and joint integrability.* Let $f \in Diff^2$ be partially hyperbolic. There exists a smooth distribution E^0 such that

$$TM = E^u \oplus E^0 \oplus E^s.$$

E^0 can be chosen arbitrarily close to E^c, and it is tangent to a smooth foliation, say $\mathcal{W}^0$. Note that neither E^0 nor $\mathcal{W}^0$ has a dynamical behavior related to f: the only property they share is being transversal to the stable and unstable direction. Furthermore, we can always find a neighborhood of f in $Diff^2(M)$, say $\mathcal{N} = \mathcal{N}_{f,E^0}$, such that these properties are satisfied by the stable and unstable direction of the elements of $\mathcal{N}$.

Remark. We will see that this construction dramatically generalizes the recent articles concerning accessibility (see [**NT, PS2**]) and using the dynamical coherence assumption. Note that [**DW**] uses a construction similar to ours.

We have the following generalization of Theorem (1.1) of [**Plan**].

PROPOSITION 1. (Product neighborhood) *There exists $\delta_0 > 0$ independent of $x \in M$ such that for $\delta \leq \delta_0$ the maps*

$$G : \mathcal{W}^u_\delta(x) \times Sat^s_\delta(\mathcal{W}^0_\delta(x)) \to M,$$
$$H : Sat^u_\delta(\mathcal{W}^0_\delta(x)) \times \mathcal{W}^s_\delta(x) \to M,$$

given by

$$G(y, z) = Sat^s_{\delta_0}(\mathcal{W}^0_{\delta_0}(y)) \cap \mathcal{W}^u_{\delta_0}(z), \quad H(y, z) = \mathcal{W}^s_{\delta_0}(y) \cap Sat^u_{\delta_0}(\mathcal{W}^0_{\delta_0}(z)),$$

are defined and are homeomorphisms onto their respective images.

These images are called product neighborhoods of x.

PROPOSITION 2. (Holonomy) *There exists $0 < \delta_1 < \delta_0$ such that for all $0 < \delta < \delta_1$, $x \in M$ and $y \in \mathcal{W}^u_\delta(x)$, the map*

$$h^u = h^u_{(x,y,\delta)} : Sat^s_\delta(\mathcal{W}^0_\delta(x)) \to Sat^s_{\delta_0}(\mathcal{W}^0_{\delta_0}(y))$$

defined by $h^u(z) = \mathcal{W}^u_{\delta_0}(z) \cap Sat^s_{\delta_0}(\mathcal{W}^0_{\delta_0}(y))$ is a homeomorphism onto its image. Dually, we can define the stable holonomy, h^s.

This is a consequence of the transversality of the continuous foliations. Now we can define joint integrability in a slightly different way than in [**Plan**].

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Definition 5. (Joint integrability) Take $\delta_1 > 0$ defined in Proposition 2. f is jointly integrable if for all $\delta \in (0, \delta_1)$, $x \in M$, $y \in \mathcal{W}^u_\delta(x)$ and $z \in \mathcal{W}^s_\delta(x)$, the intersection $\mathcal{W}^s_{\delta_0}(y) \cap \mathcal{W}^u_{\delta_0}(z)$ is a single point.

Note that this definition is symmetric on the stable and unstable foliations. We will see at the end of this section that our definition of joint integrability is equivalent to that of [**Plan**].

Such f will satisfy a kind of local integrability: in the next section we will be able to define $\mathcal{W}^{us}$, a Hölder continuous foliation by C^1 leaves. Our main interest is in the negation of this propriety. For the sake of completeness we give its definition.

Definition 6. (Non-joint integrability) Take $\delta_1 > 0$ defined in Proposition 2. f is non-jointly integrable if there exists δ in $(0, \delta_1)$ such that there exist $x \in M$, $y \in \mathcal{W}^u_\delta(x)$ and $z \in \mathcal{W}^s_\delta(x)$ satisfying: $\mathcal{W}^s_{\delta_0}(y) \cap \mathcal{W}^u_{\delta_0}(z) = \emptyset$.

Such f will satisfy a kind of local accessibility: in the next section we will see that it enables us to construct a class of open sets U_f which are branched accessible.

In [**Plan**], joint integrability is defined using the unstable holonomy and the dynamical coherence satisfied by Anosov flows: the corresponding definition for a partially hyperbolic diffeomorphism is not straightforwardly symmetric on stable and unstable holonomies. We verify the equivalence between our definition of joint integrability and Plante's.

LEMMA 1. *Let $f \in Diff^2(M)$ be partially hyperbolic and jointly integrable. Then for all $0 < \delta < \delta_1$, $x \in M$, $y \in \mathcal{W}^u_\delta(x)$ and $z \in \mathcal{W}^s_\delta(x)$, we have*

$$h^u(\mathcal{W}^s_\delta(x)) \subset \mathcal{W}^s_{\delta_0}(y),$$

and dually for the stable holonomy. Conversely, suppose $f \in Diff^2(M)$ is partially hyperbolic and satisfies, for all $0 < \delta < \delta_1$, $x \in M$, $y \in \mathcal{W}^u_\delta(x)$,

$$h^u(\mathcal{W}^s_\delta(x)) \subset \mathcal{W}^s_{\delta_0}(y);$$

then f is jointly integrable.

Proof. Suppose f is partially hyperbolic and jointly integrable. Let z be in $\mathcal{W}^s_\delta(x)$. Then by Proposition 2,

$$h^u(z) = \mathcal{W}^u_{\delta_1}(z) \cap Sat^s_{\delta_0}(\mathcal{W}^0_{\delta_0}(y)).$$

Joint integrability and the holonomy being injective imply

$$\{h^u(z)\} = \mathcal{W}^u_{\delta_1}(z) \cap \mathcal{W}^s_{\delta_0}(y) \subset \mathcal{W}^s_{\delta_0}(y),$$

hence $h^u(\mathcal{W}^s_\delta(x)) \subset \mathcal{W}^s_{\delta_0}(y)$.

Conversely, let f be partially hyperbolic and satisfy Plante's definition of joint integrability, i.e. for all $0 < \delta < \delta_1$, for all x in M and y in $\mathcal{W}^u_\delta(x)$: $h^u(\mathcal{W}^s_\delta(x)) \subset \mathcal{W}^s_{\delta_0}(y)$. Take x in M, y in $\mathcal{W}^u_\delta(x)$ and z in $\mathcal{W}^s_\delta(x)$. We have $h^u(z) \in \mathcal{W}^s_{\delta_0}(y)$, therefore $\mathcal{W}^s_{\delta_1}(y) \cap \mathcal{W}^u_{\delta_1}(z) = \{h^u(z)\}$. The intersection being a single point, f is jointly integrable. □

Remark. In the following, we will use mostly the holonomy definition of joint integrability.

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Standing hypothesis. In the following we assume that f is partially hyperbolic and has a one-dimensional center bundle, $\dim(E^c) = 1$. We suppose

$$\dim(E^u) = u \quad \text{and} \quad \dim(E^s) = s, \quad \text{hence: } u + 1 + s = n.$$

2. *Non-joint integrability and accessibility*

2.1. *Local accessibility.*

Here we construct the set U_f. Let $f \in Diff^2(M)$ be partially hyperbolic and non-jointly integrable. We take δ, x, y and z as in Definition 6.

We first show that an open set of points in $\mathcal{W}^0_\delta(y)$ is accessible to x and then to y:

LEMMA 2. *There exists an open set $O_1 \subset M$ such that: $O_2 = O_1 \cap \mathcal{W}^0_\delta(y) \neq \emptyset$ and every point of O_2 is accessible to x. O_2 is $(6, \delta_0)$-branched accessible.*

Proof. By non-joint integrability of f we have: $h^u(\mathcal{W}^s_\delta(x)) \not\subset \mathcal{W}^s_{\delta_0}(y)$. $\mathcal{W}^s_{\delta_0}(y)$ is a connected subset of the C^1 leaf $\mathcal{W}^s(y)$. $h^u(\mathcal{W}^s_\delta(x))$ is an image by the homeomorphism h^u of such an object; it is a connected set. For every point z in $h^u(\mathcal{W}^s_\delta(x) \cap Sat^s_\delta(\mathcal{W}^0_\delta(x))) - \mathcal{W}^s_{\delta_0}(y) \cap h^u(Sat^s_\delta(\mathcal{W}^0_\delta(x)))$, continuity implies that there exists an open neighborhood of z in M, say V_z, such that

$$\forall p \in V_z \cap h^u(\mathcal{W}^s_\delta(x) \cap Sat^s_\delta(\mathcal{W}^0_\delta(x))), \quad p \notin \mathcal{W}^s_{\delta_0}(y) \cap h^u(Sat^s_\delta(\mathcal{W}^0_\delta(x))).$$

We can choose z and V_z so that the intersection

$$Sat^s_{\delta_0}(V_z \cap h^u(\mathcal{W}^s_\delta(x))) \cap \mathcal{W}^0_{\delta_0}(y)$$

is a non-empty open set for the induced topology of the leaf $\mathcal{W}^0_{\delta_0}(y)$. More precisely, $V_z \cap h^u(\mathcal{W}^s_\delta(x))$ is a non-empty, connected set (we can choose V_z connected). The stable foliation being continuous, sliding along the stable foliation is a continuous operation, hence $Sat^s_{\delta_0}(V_z \cap h^u(\mathcal{W}^s_\delta(x))) \cap \mathcal{W}^0_{\delta_0}(y)$ is connected: $\mathcal{W}^0$ being one-dimensional, it could be a point or an interval. Non-joint integrability implies that it cannot always be a point: if for all the points z and the sets V_z we have been considering, the set $Sat^s_{\delta_0}(V_z \cap h^u(\mathcal{W}^s_\delta(x))) \cap \mathcal{W}^0_{\delta_0}(y)$ is a point, then continuity of the holonomy implies that this point is unique; it is y and we have $h^u(\mathcal{W}^s_\delta(x)) \subset \mathcal{W}^s_{\delta_0}(y)$, contradicting the non-joint integrability assumption. We take O_2 to be a non-empty set in this interval. The other statements of the lemma are obvious. □

Remarks. Note that without our assumption on the dimension of the center bundle, the preceding proof would not work. In this article, the other results not relying on Lemma 2 are still true for $\dim(E^c) \geq 1$.

Lemma 2 shows exactly the '4-branched accessibility' appearing in [**BW**] and [**PS2**] where we have $\mathcal{W}^0$ instead of $\mathcal{W}^c$: two points of a $\mathcal{W}^0$-leaf are connected by a 4-branched path of accessibility.

LEMMA 3. (Local accessibility) *If $f \in Diff^2(M)$ is partially hyperbolic and non-jointly integrable then there exists some non-empty branched accessible open set.*

Proof. Consider the set $Sat^u_\delta(Sat^s_\delta(O_2))$, with O_2 given by Lemma 2. Now, Proposition 1 implies that this set contains a non-empty product neighborhood, U_f, which is $(8, \delta_0)$-branched accessible: every point of U_f is $(4, \delta_0)$-branched accessible to x. Note that this construction is not canonical. □

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2.2. *A local expression of accessibility.* Here we prove the following proposition.

PROPOSITION 3. *If $f \in Diff^2(M)$ is partially hyperbolic and accessible then f is non-jointly integrable.*

Proof. Suppose that f is jointly integrable. For the sake of simplicity, we will assume that

$$u = s = 1.$$

Let $0 < \delta_1 < \delta_0$ be the constants introduced in Propositions 1 and 2. The proof works along the same lines in the higher dimensional case. We will proceed in several steps.

Step 1. Let $p \in M$. We introduce a special chart at the point p:

LEMMA 4. *There exist constants ϵ_2 and ϵ_1 satisfying*

$$0 < \epsilon_1 < \epsilon_2 < \delta_1$$

such that, for all $p \in M$, there exist an open neighborhood of p, say $\mathcal{U}$, and a diffeomorphism $\phi_p = \phi : (-2, 2)^3 \to \mathcal{U}$ such that:

(1) $\phi(0, 0, 0) = p$,
(2) $B(p, \epsilon_1) \subset \mathcal{U} \subset B(p, \epsilon_2)$,
(3) *for all* $(x, y) \in (-2, 2)^2$, $\phi(\{x\} \times \{y\} \times (-2, 2)) = \mathcal{W}^0_{loc}(\phi(x, y, 0)) \cap \mathcal{U}$,
(4) $\phi((-2, 2) \times \{0\} \times \{0\}) = \mathcal{W}^u_{loc}(p) \cap \mathcal{U}$, *and*
(5) $\phi(\{0\} \times (-2, 2) \times \{0\}) = \mathcal{W}^s_{loc}(p) \cap \mathcal{U}$.

Note that the choice of the constants ϵ_2 and ϵ_1 can be made uniformly by compactness of M.

Step 2. We define for $a = u, s$ and $q \in (-2, 2)^3$ the leaves

$$\mathcal{F}^a(q) = \phi^{-1}(\mathcal{W}^a_{loc}(\phi(q)) \cap \mathcal{U}).$$

For $z \in (-2, 2)$, let

$$\mathcal{S}^{s,u}_z = \bigcup_{q \in \mathcal{F}^s(0,0,z)} \mathcal{F}^u(q),$$

$$\mathcal{S}^{u,s}_z = \bigcup_{q \in \mathcal{F}^u(0,0,z)} \mathcal{F}^s(q),$$

$$\mathcal{F}^{us}(q) = \mathcal{F}^u(\mathcal{F}^s(q)) \cap \mathcal{F}^s(\mathcal{F}^u(q)).$$

In Lemma 4, we can always choose the constants ϵ_1 and ϵ_2 small enough so that by joint integrability we have

$$\mathcal{S}_z = \mathcal{S}^{u,s}_z \cap (-1, 1)^3 = \mathcal{S}^{s,u}_z \cap (-1, 1)^3 = \mathcal{F}^{us}(q) \cap (-1, 1)^3,$$

and the smooth foliation by the verticals, the foliation $\mathcal{F}^u$ and the foliation $\mathcal{F}^s$ are transversal foliations: that will allow us to define the respective holonomies (see Proposition 2).

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Step 3. Let $(Y, Z) \in (-1/2, 1/2)^2$ be fixed. $\mathcal{F}^u(0, Y, Z)$ is the graph of a C^1 application, say $g^u(\cdot, Y, Z)$, where g^u is a θ-Hölder continuous application

$$g^u : (-1, 1) \times (-1/2, 1/2)^2 \to (-1, 1)^3$$

given by $(x, Y, Z) \mapsto (x, \beta^u(x, Y, Z), \gamma^u(x, Y, Z))$, such that

$$(x, Y, Z) \mapsto \frac{\partial g^u}{\partial x}(x, Y, Z)$$

is defined and θ-Hölder continuous: this means that the unstable foliation is uniformly smooth in the sense of [**Jour**]. Note that

$$\forall (Y, Z) \in (-1/2, 1/2)^2, \quad g^u(0, Y, Z) = (0, Y, Z).$$

Dually, there exists a θ-Hölder continuous application

$$g^s : (-1/2, 1/2) \times (-1, 1) \times (-1/2, 1/2) \to (-1, 1)^3$$

given by $(X, y, Z) \mapsto (\alpha^s(X, y, Z), y, \gamma^s(X, y, Z))$, such that

$$(X, y, Z) \mapsto \frac{\partial g^s}{\partial y}(X, y, Z)$$

is defined and θ-Hölder continuous and $\mathcal{F}^s(X, 0, Z)$ is the graph of $g^s(X, \cdot, Z)$. Note that the stable foliation is uniformly smooth and we have

$$\forall (X, Z) \in (-1/2, 1/2)^2, \quad g^s(X, 0, Z) = (X, 0, Z).$$

Step 4. Recall $\mathcal{S}_z = \mathcal{S}_z^{u,s} \cap (-1, 1)^3$. We have chosen ϵ_2 small enough so that the following holonomy is defined and a homeomorphism onto its image:

$$h^s_{(y,Z)} : \ Sat^0(\mathcal{F}^u(0, 0, Z)) \to (-1, 1) \times \{y\} \times (-1, 1).$$

Here, Sat^0 is the saturate by the vertical foliation

$$\mathcal{F}^0(x, y, z) = \{x\} \times \{y\} \times (-1, 1).$$

Let Π be the projection $(x, y, z) \mapsto \Pi(x, y, z) = (x, y, 0)$.

The holonomy being uniformly continuous we can choose ϵ_2 small enough so that for every $Z \in (-1/2, 1/2)$ and $(x, y) \in (-1/4, 1/4)^2$ there exists a point p in $\mathcal{F}^u(0, 0, Z)$ (the holonomy being bijective, this point is unique) such that $(x, y, 0) = \Pi \circ h^s_{(y,Z)}(p)$. Consider $h^s_{(0,Z)}(p) = (X^s(x, y, Z), 0, Z^s(x, y, Z))$. By construction, $\mathcal{S}_Z$ is the graph of

$$\gamma_Z : \ (-1/4, 1/4)^2 \to (-1, 1)$$

defined by $\gamma_Z(x, y) = \gamma^s(X^s(x, y, Z), y, Z^s(x, y, Z))$.

Now recall that by joint integrability we have $\mathcal{S}_Z = \mathcal{S}_Z^{s,u} \cap (-1, 1)^3$ so that a dual construction gives $\gamma_Z(x, y) = \gamma^u(x, Y^u(x, y, Z), Z^u(x, y, Z))$, where

$$\Pi \circ h^u_{(x,Z)} \circ (h^u)^{-1}_{(0,Z)}(0, Y^u(x, y, Z), Z^u(x, y, Z)) = (x, y, 0).$$

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Step 5. We are going to apply Theorem 1 of [**Jour**] to prove that $\mathcal{S}_Z$ is C^1. The projection to the horizontal plane of the stable and unstable foliations restricted to $\mathcal{S}_Z$ give rise to two continuous and transverse foliations which are uniformly smooth. Hence, by definition, γ_Z is uniformly smooth along the leaves of these foliations. More precisely, fix z_0; the projection of the unstable leaf in $\mathcal{S}_{z_0}$ passing through the point (x_0, y_0) is given by the equation

$$y = \beta^u(x, Y_0^u, Z_0^u),$$

where $Y_0^u = Y^u(x_0, y_0, z_0)$ and $Z_0^u = Z^u(x_0, y_0, z_0)$. The height of the unstable leaf above the projection is given by

$$\gamma_{z_0}(x, y) = \gamma_{z_0}(x, \beta^u(x, Y_0^u, Z_0^u)) = \gamma^u(x, Y_0^u, Z_0^u),$$

which is uniformly smooth.

A dual result holds for the projection of the stable leaves through $\mathcal{S}_Z$. Therefore, the main theorem of [**Jour**] implies that γ_Z is smooth.

A standard argument implies the continuity of the application γ defined by $\gamma(x, y, z) = \gamma_z(x, y)$: it gives a foliation box following [**HPS2**]. Thus we have a continuous local foliation by C^1 local leaves. Furthermore, for $L > 0$ and $p \in M$ let

$$\mathcal{W}_L^{us}(p) = Sat^u(\mathcal{W}_L^s(p)) \cap Sat^s(\mathcal{W}_L^u(p)).$$

Note that this set is accessible. To summarize, in this step we have constructed at p a continuous chart.

LEMMA 5. *There exist constants $0 < \epsilon_1 < \epsilon_2 < \delta_1$ and $L > 0$ such that, for all p in M, there exist an open neighborhood $\mathcal{V}$ of p and a homeomorphism*

$$\psi_p = \psi : (-1, 1)^3 \to \mathcal{V}$$

such that $B(p, \epsilon_1) \subset \mathcal{V} \subset B(p, \epsilon_2)$ and the restriction of ψ to $(-1, 1)^2 \times \{z\}$ is a C^1-diffeomorphism satisfying:

$$\mathcal{W}_L^{us}(\psi(x, y, z)) \cap B(p, \epsilon_1) \subset \psi((-1, 1)^2 \times \{z\}) \subset \mathcal{W}_L^{us}(\psi(x, y, z)) \cap B(p, \epsilon_2).$$

Step 6. Compactness implies that we can cover M by a finite family of these charts, say $\{\psi_i, \mathcal{V}_i\}_{i=1,\ldots,K}$, where $\mathcal{V}_i = \psi_i((-1, 1)^3)$ is an open neighborhood of p_i. We will show that this is a compatible family of charts.

LEMMA 6. *Let $1 \le i_1, i_2 \le K$ and $-1 < z_1 < 1$ be such that*

$$\psi_{i_1}((-1, 1)^2 \times \{z_1\}) \cap \mathcal{V}_{i_2} \neq \emptyset.$$

Then there exists a unique z_2 in $(-1, 1)$ such that the set

$$\psi_{i_1}^{-1}(\psi_{i_1}((-1, 1)^2 \times \{z_1\}) \cap \psi_{i_2}((-1, 1)^2 \times \{z_2\}))$$

is non-empty and relatively open in $(-1, 1)^2 \times \{z_1\}$.

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Let i_1, i_2 and z_1 be as above. The existence of z_2 is provided by Lemma 5. Take z_2 and z_2' such that there exist p and q satisfying

$$p \in \psi_{i_1}((-1,1)^2 \times \{z_1\}) \cap \psi_{i_2}((-1,1)^2 \times \{z_2\})$$

and

$$q \in \psi_{i_1}((-1,1)^2 \times \{z_1\}) \cap \psi_{i_2}((-1,1)^2 \times \{z_2'\}).$$

Note that $d(p,q) < \epsilon_2 < \delta_1 < \delta_0$. Hence we can apply Proposition 1 and there exists a unique point w in a δ_1-product neighborhood of p and q such that: $\mathcal{W}^s_{\delta_0}(q) \cap Sat^u_{\delta_0}(\mathcal{W}^0_{\delta_0}(p)) = \{w\}$. Recall that we have p and q in $\psi_{i_1}((-1,1)^2 \times \{z_1\})$ so that by joint integrability the intersection between $\mathcal{W}^s_{\delta_0}(q)$ and $\mathcal{W}^u_{\delta_0}(p)$ is non-empty: an element of this intersection need not be in $\psi_{i_1}((-1,1)^2 \times \{z_1\})$, but by Proposition 1 it is in a δ_1-product neighborhood of p and q and it is unique, hence this point is w. This implies $z_2 = z_2'$ because p and q belong to $\mathcal{V}_{i_2}$. The intersection $\psi_{i_1}((-1,1)^2 \times \{z_1\}) \cap \psi_{i_2}((-1,1)^2 \times \{z_2\})$ is relatively open because by Lemma 5 it is the intersection of two C^1 local manifolds.

Step 7. We need the following definition.

Definition 7. Let p, q in M and $R \geq 0$ be an integer; p and q are R-chained if there exists a sequence $\mathcal{V}_{i_0}, \ldots, \mathcal{V}_{i_R}$ and a sequence $z_{i_k} \in (-1,1)$ such that
(1) $p \in \psi_{i_0}((-1,1)^2 \times \{z_{i_0}\})$ and $q \in \psi_{i_R}((-1,1)^2 \times \{z_{i_R}\})$,
(2.) $\psi_{i_k}((-1,1)^2 \times \{z_{i_k}\}) \cap \psi_{i_{k+1}}((-1,1)^2 \times \{z_{i_{k+1}}\}) \neq \emptyset$.

For $R \geq 1$, let

$$\mathcal{C}^{us}_R(p) = \{q \in M;\ q \text{ is at most } R\text{-chained to } p\}.$$

By a standard argument, one can show that for p in M and an integer $R \geq 0$, there exist an integer P, integers $1 \leq i_1, \ldots, i_P \leq K$ and a sequence $z_{i_1}, \ldots, z_{i_P}$ in $(-1,1)$ such that

$$\mathcal{C}^{us}_R(p) = \bigcup_{l=0}^{P} \psi_{i_l}((-1,1)^2 \times \{z_{i_l}\}),$$

and by Lemma 6, the intersections

$$\psi_{i_k}((-1,1)^2 \times \{z_{i_k}\}) \cap \psi_{i_l}((-1,1)^2 \times \{z_{i_l}\})$$

are either open (for the induced topology on the leaf) or empty.

Step 8. In the preceding step, we have seen that $\mathcal{C}^{us}_R(p)$ is an immersed C^1 2-manifold: it is nowhere dense in M. Consider

$$\mathcal{W}^{us}(p) = \bigcup_{R \geq 1} \mathcal{C}^{us}_R(p).$$

By the Baire Category theorem, this set is nowhere dense in M. Take q in M. By accessibility, there exists an (N,l)-branched accessible path connecting p and q. Hence, we can find an integer R such that $q \in \mathcal{C}^{us}_R(p)$. Therefore, we have $M = \bigcup_{R \geq 1} \mathcal{C}^{us}_R(p)$, which leads to a contradiction. We have proved that f is non-jointly integrable. □

This non-trivial result is quite satisfactory: it means that when we have accessibility, which is a global notion, then somewhere in the manifold we have a local branched accessible set whose size we can control.

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2.3. *A uniform expression of accessibility.* Here we prove an important consequence of Lemma 3 and Proposition 3.

PROPOSITION 4. *If $f \in Diff^2(M)$ is partially hyperbolic and accessible then there exist a positive integer N and $l > 0$ such that f is (N, l)-branched accessible.*

Proof. Proposition 3 implies f is jointly integrable. Therefore, we can apply Lemma 3, which provides a non-empty open set U_f which is $(8, \delta_0)$-branched accessible. Let y be in U_f and be fixed. Consider x in M. By accessibility there exists an (N, l)-branched accessible path connecting y and x; N and l depend on x. Let $x_0 = y$, $x_1, \ldots, x_{N-1}$, $x_N = x$, $\{a_i\}_{i=0,\ldots,N-1}$ and $\{l_i\}_{i=0,\ldots,N-1}$ be its accessibility sequences. We have the following.

LEMMA 7. *There exists a sequence of non-empty open sets $\{V_i\}_{i=0,\ldots,N}$ such that $V_0 = U_f$, for all $i = 0, \ldots, N$, $x_i \in V_i$ and for all $i = 1, \ldots, N$, $V_i \subset Sat^{a_{i-1}}_{2l_{i-1}}(V_{i-1})$.*

Observe that every point in V_N is $(N, 2l)$-branched accessible to a point in V. More precisely, take $z \in V_i$: by the preceding lemma, there exists $z' \in V_{i-1}$ such that $z \in \mathcal{W}^{a_{i-1}}_{2l_{i-1}}(z')$. Iterating the process gives the accessibility sequence. At each point $x \in M$, we have constructed a non-empty open set $V_{N(x)}(x)$ which is $(N(x), 2l(x))$-branched accessible to U_f: this is an open covering of M. Being compact, there exists a finite subcover of M by such sets. □

Proof of Lemma 7. Consider $x \in M$ and $l > 0$. Let V be an open set containing x and $y \in \mathcal{W}^u_l(x)$. All we have to prove to get the lemma is that there exists an open set W such that $y \in W \subset Sat^u_{2l}(V)$. Take $0 < \delta < \delta_1$, such that $Sat^s_\delta(\mathcal{W}^0_\delta(x)) \subset V$. We can divide $\mathcal{W}^u_l(x)$ in unstable balls of size δ and apply Proposition 2 to define the holonomy at length l:

$$h^u : Sat^s_{\delta'}(\mathcal{W}^0_{\delta'}(x)) \to Sat^s_{\delta_0}(\mathcal{W}^0_{\delta_0}(y)),$$

where $0 < \delta' < \delta$ depends on the number of steps of the construction. Proposition 2 implies h^u is a homeomorphism onto its image: h^u is the composition of the holonomies at each subdivision of $\mathcal{W}^u_l(x)$. Hence, $h^u(Sat^s_{\delta'}(\mathcal{W}^0_{\delta'}(x)))$ is open in $Sat^s_{\delta_0}(\mathcal{W}^0_{\delta_0}(y))$, and there exists $\eta > 0$ such that

$$Sat^s_\eta(\mathcal{W}^0_\eta(y)) \subset h^u(Sat^s_{\delta'}(\mathcal{W}^0_{\delta'}(x))).$$

Theorem 2 implies that $\mathcal{W}^u_{2l}(z)$ depends continuously on z in $Sat^s_{\delta'}(\mathcal{W}^0_{\delta'}(x))$, so that we can choose $\eta > 0$ small enough for the product neighborhood of size η at y, say W, to be contained in $Sat^u_{2l}(\mathcal{W}^{cs}_{\delta'}(x))$. We have a dual statement for the stable saturate. □

3. *Perturbation results*

3.1. *Non-joint integrability.*

PROPOSITION 5. (Stability of non-joint integrability) *If $f \in Diff^2(M)$ is partially hyperbolic and non-jointly integrable then it is stably non-jointly integrable: there is a C^1-neighborhood $\mathcal{V}$ of f in $Diff^2(M)$ whose elements are non-jointly integrable.*

Furthermore, there exists an open branched accessible for f set, say U_f, such that there exist $x_f \in U_f$ and $\alpha_f > 0$ satisfying $B(x_f, \alpha_f) \subset U_f$ and such that we can find a

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C^1-neighborhood $\mathcal{V}$ such that for $g \in \mathcal{V}$, $B(x_f, \alpha_f) \subset U_g$, where U_g is an open branched accessible set for g.

Proof. Let f be partially hyperbolic and non-jointly integrable. There exist $0 < \delta < \delta_1$, $x, y \in M$, such that the holonomy

$$h^u_f : Sat^s_{f,\delta}(\mathcal{W}^0_\delta(x)) \to Sat^s_{f,\delta_0}(\mathcal{W}^0_{\delta_0}(y))$$

is defined and satisfies $h^u_f(x) = y$ and $h^u_f(\mathcal{W}^s_{f,\delta}(x)) \not\subset \mathcal{W}^s_{f,\delta_0}(y)$. Recall that in the first section, we have constructed a smooth distribution E^0 transversal to the stable and unstable distributions of f: g close to f satisfy the same property. We can always choose the constants δ_1 and δ_0 so that there exists a neighborhood of f in $Diff^2(M)$, say $\mathcal{N}_0$, such that g in $\mathcal{N}_0$ is partially hyperbolic and satisfies Proposition 2 with the same constants as f and the same foliation $\mathcal{W}^0$. Take $g \in \mathcal{N}_0$. The holonomy

$$h^u_g : Sat^s_{g,\delta}(\mathcal{W}^0_\delta(x)) \to Sat^s_{g,\delta_0}(\mathcal{W}^0_{\delta_0}(y))$$

is defined, but in the general case $h^u_g(x) \neq y$. Recall from Theorem 2 that, for all $\epsilon > 0$, there exists a neighborhood of f in $Diff^2(M)$, say $\mathcal{N}_1 \subset \mathcal{N}_0$, such that for all $g \in \mathcal{N}_1$

$$d_{\mathcal{H}}(\mathcal{W}^s_{g,\delta_0}(h^u_g(x)), \mathcal{W}^s_{f,\delta_0}(y)) \leq \epsilon, \quad \text{and} \quad d_{\mathcal{H}}(h^u_g(\mathcal{W}^s_{g,\delta}(x)), h^u_f(\mathcal{W}^s_{f,\delta}(x))) \leq \epsilon.$$

So choosing ϵ small enough, the preceding inequalities imply

$$h^u_g(\mathcal{W}^s_{g,\delta}(x)) \not\subset \mathcal{W}^s_{g,\delta_0}(h^u_g(x)).$$

That means the elements of $\mathcal{N}_1$ are non-jointly integrable.

By Theorem 2 the set $O_{2,f}$ constructed in Lemma 2 depends continuously on the diffeomorphism f.The same is valid for the set $U_f = Sat^u_\delta(Sat^s_\delta(O_{2,f}))$ constructed in Lemma 3. Hence, for every $\epsilon > 0$, there exists a neighborhood of f, say $\mathcal{V}$, such that for $g \in \mathcal{V}$, $d_H(U_f, U_G) \leq \epsilon$. The second part of the lemma is straightforward. □

3.2. *Branched accessibility.*

PROPOSITION 6. (ϵ-branched accessibility) *Let $f \in Diff^2(M)$ be partially hyperbolic and (N, l)-branched accessible. Then, for all small $\epsilon > 0$ and $x \in M$, there exists $\mathcal{V}_\epsilon$ a C^1-neighborhood of f in $Diff^2(M)$ whose elements are ϵ-$(N, 2l)$-branched accessible to x: for all g in $\mathcal{V}_\epsilon$, for all y in M, y is $(N, 2l)$-branched accessible (for g) to a point in $B(x, \epsilon)$.*

Proof. Let N, l be positive integers, $f \in Diff^2(M)$ be partially hyperbolic and (N, l)-branched accessible. We first introduce a kind of modulus of continuity. For $\eta > 0$, let

$$K_f(\eta) = \sup\{d_{\mathcal{H}}(\mathcal{W}^a_{f,2l}(p), \mathcal{W}^a_{f,2l}(q)); d(p, q) \leq \eta, a = u, s\}.$$

Continuity of the foliations implies that $\lim_{\eta \to 0} K_f(\eta) = 0$.

Take $x \in M$ and $\epsilon > 0$. For all $\xi > 0$, Theorem 2 gives a neighborhood $\mathcal{V}_\xi$ of f such that: for all $a \in \{u, s\}$, for all $p \in M$, $d_{\mathcal{H}}(\mathcal{W}^a_{f,2l}(p), \mathcal{W}^a_{g,2l}(p)) \leq \xi$. The triangle inequality gives for $g \in \mathcal{V}_\xi$: $K_g(\eta) \leq 2\xi + K_f(\eta)$, and hence we have $\lim_{\eta \to 0, g \to f} K_g(\eta) = 0$.

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Let $\{\eta_i(\xi)\}_{i=0,\dots,N}$ be a sequence defined by: $\eta_0 = 0$ and $\eta_i = \xi + K_g(\eta_{i-1})$ for $1 \leq i \leq N$. Observe that $\eta_i(\xi)$ depends on g but by the preceding inequality it is bounded by a function depending on ξ and f. Composing the limits, we have: for all $0 \leq i \leq N$, $\lim_{\xi \to 0, g \to f} \eta_i(\xi) = 0$. Therefore, there exists $\xi_0 = \xi_0(\epsilon) > 0$ such that for all $0 < \xi < \xi_0$, we can find a neighborhood $\mathcal{V}'_\xi$ of f such that for all $g \in \mathcal{V}'_\xi$, we have: $\eta_i(\xi) < \epsilon$ for $0 \leq i \leq N$ and

$$\forall a \in \{u, s\},\ \forall p \in M,\ d_{\mathcal{H}}(\mathcal{W}^a_{f,2l}(p), \mathcal{W}^a_{g,2l}(p)) \leq \xi.$$

Take such a ξ, and consider the neighborhood $\mathcal{V}'_\xi$ as above. Let g be in $\mathcal{V}'_\xi$ and $y \in M$. There exists an (N, l)-branched accessibility path for f connecting x to y: $x_0 = x$, x_1, $\dots$, x_{N-1} and $x_N = y$, $\{a_i\}_{i=0,\dots,N-1}$. We construct an $(N, 2l)$-branched accessibility path connecting y to a point in $B(x, \epsilon)$, say $\{y_i\}_{i=0,\dots,N}$. Let $y_N = y$. We have $d_{\mathcal{H}}(\mathcal{W}^{a_{N-1}}_{f,2l}(y_N), \mathcal{W}^{a_{N-1}}_{g,2l}(y_N)) \leq \xi$, hence there exists a point $y_{N-1} \in \mathcal{W}^{a_{N-1}}_{g,2l}(y_N)$ such that: $d(x_{N-1}, y_{N-1}) \leq \xi$.

Recall that $\eta_1(\xi) = \xi$. We show that we can iterate this construction. Suppose that for $i \in \{1, \dots, N\}$ and $j \geq i$, there exists a point

$$y_j \in B(x_j, \eta_{N-j}(\xi)) \cap \mathcal{W}^{a_j}_{g,2l}(y_{j+1}).$$

The triangle inequality gives

$$d_{\mathcal{H}}(\mathcal{W}^{a_{i-1}}_{2l}(y_i), \mathcal{W}^{a_{i-1}}_{2l}(x_{i-1})) \leq \xi + K_g(\eta_{N-i}(\xi)) = \eta_{N-i+1}(\xi).$$

Hence, there exists a point $y_{i-1} \in B(x_{i-1}, \eta_{N-i+1}(\xi)) \cap \mathcal{W}^{a_{i-1}}_{g,2l}(y_i)$. Therefore, the iteration gives us a point y_0 in $B(x_0, \eta_N(\xi))$ and an $(N, 2l)$-branched accessible path connecting y_0 and y: this propriety is satisfied by every point $y \in M$. □

4. *Proof of Theorem 1*

Let $f \in \mathit{Diff}^2(M)$ be partially hyperbolic and accessible. By Proposition 3, f is non-jointly integrable: there exists $0 < \delta < \delta_1 < \delta_0$ and $x, y \in M$ satisfying $h^u(\mathcal{W}^s_\delta(x)) \not\subset \mathcal{W}^s_{\delta_0}(y)$. By Lemma 3, there exists a non-empty open set U_f which is $(8, \delta_0)$-branched accessible. By Proposition 4, there exist a positive integer $N > 0$ and $l > 0$ such that f is (N, l)-branched accessible. Now, U_f being open, we can take $p \in U_f$ and $\alpha > 0$ such that $B(p, \alpha) \subset U_f$. By Proposition 5 there exists a C^1-neighborhood $\mathcal{V}$ of f in $\mathit{Diff}^2(M)$ such that: for all $g \in \mathcal{V}$, g is partially hyperbolic, non-jointly integrable and has a $(8, \delta_0)$-branched accessible set U_g containing $B(p, \alpha)$. Applying Proposition 6 to f and α, we obtain a neighborhood $\mathcal{V}_\alpha$ such that: for all $g \in \mathcal{V}_\alpha$, for all $q \in M$, q is $(N, 2l)$-branched accessible (for g) to a point in $B(p, \alpha)$. Take $g \in \mathcal{V} \cap \mathcal{V}_\alpha$, then g is accessible: every point $q \in M$ is $(N + 8, \delta_0 + 2l)$-branched accessible (for g) to p. □

Remark. We have just proved that an (N, l)-branched accessible f with one-dimensional center bundle is stably $(2N + 16, \delta_0 + 2l)$-branched accessible. This answers a question posed in the appendix of [**PS2**] concerning the possible growth of the number of branches to obtain stable branched accessibility.

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5. *Other results*

5.1. *Proof of Corollary 1.* Theorem (1.1) of [**Brin**] states the following.

THEOREM 3. (Brin) *Let $f \in Diff^2(M)$ be partially hyperbolic and accessible and suppose $\Omega(f) = M$. Then f is topologically transitive.*

Let $f \in Diff^2_\lambda(M)$ be partially hyperbolic with one-dimensional center bundle, suppose it is accessible and take $g \in Diff^2_\lambda(M)$ in the neighborhood $\mathcal{N}_0$ of f given by Theorem 1: g preserves λ implies $\Omega(g) = M$, and $g \in \mathcal{N}_0$ is accessible. Therefore, g is topologically transitive.

5.2. *Proof of Corollary 2.* Let f be partial hyperbolic. It is dynamically coherent if $E^u \oplus E^c$, E^c and $E^c \oplus E^s$ are tangent to continuous foliations with C^1 leaves, say $\mathcal{W}^{uc}$, $\mathcal{W}^c$ and $\mathcal{W}^{cs}$, such that: $\mathcal{W}^u$ and $\mathcal{W}^c$ subfoliate $\mathcal{W}^{uc}$, and $\mathcal{W}^c$ and $\mathcal{W}^s$ subfoliate $\mathcal{W}^{cs}$.

Remark. Theorem (2.2) of [**PS1**] states that if the foliation $\mathcal{W}^c$ is C^1 then f is dynamically coherent. A classical example of a stably dynamically coherent diffeomorphism is the time-one map of an Anosov flow.

Recall from [**PS2**] that f is center bunched if the ratio

$$\|T^c f\|/\mathbf{m}(T^c f)$$

is close to one (**m** is the conorm). For instance, the center bunching condition is: $\nu < \gamma^{2+2/\theta}$.

Corollary (a) of [**PS2**] states the following.

THEOREM 4. (Pugh–Shub) *Center bunched partially hyperbolic diffeomorphisms in $Diff^2_\lambda(M)$ which are stably dynamically coherent and stably accessible are stably ergodic.*

In Theorem 1, we have proved that stable accessibility is a consequence of accessibility (in the case of a one-dimensional center bundle).

5.3. *Structure of an accessibility class.* Following some ideas in [**Plan**] and [**BPW**], we give some simple properties of accessibility classes which motivated this work. We will show that in the case of the time-one map of an Anosov flow, accessibility and non-joint integrability are equivalent.

LEMMA 8. *Let $f \in Diff^2(M)$ be partially hyperbolic. If U is a non-empty open set, then $\mathcal{C}(U)$ is open.*

Proof. Let $x \in \mathcal{C}(U)$. There exists an (N, l)-branched accessible path connecting x to a point $y \in U$, U being open. Applying Lemma 7 to U, we get an open set $V \subset \mathcal{C}(U)$ containing x. □

The preceding result has the following nice consequence.

PROPOSITION 7. *Let Φ be an Anosov (not necessarily transitive) flow on M. The time-one map of Φ is non-jointly integrable if and only if it is accessible.*

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Proof. Lemma 3 gives a non-empty branched accessible open set, say U. Lemma 8 states that $\mathcal{C}(U)$ is open. We will show that its boundary is empty. M being connected implies $\mathcal{C}(U) = M$. U accessible implies that $\mathcal{C}(U)$ is accessible (in general, an accessibility class need not be accessible). The flow being continuous, there exists $\epsilon > 0$ such that: $\forall |t| \leq \epsilon, \phi^t \mathcal{C}(U) \cap \mathcal{C}(U) \neq \emptyset$. Hence, $\mathcal{C}(U)$ is accessible: for all $|t| \leq \epsilon, \phi^t \mathcal{C}(U) = \mathcal{C}(U)$. Take p in the boundary of $\mathcal{C}(U)$, then $p \notin \mathcal{C}(U)$ as $\mathcal{C}(U)$ is open. Let V be a product neighborhood of size $\epsilon/2$ of p. There exists a sequence in $\mathcal{C}(U)$ converging to p. Local product structure implies that there exists a sequence in $\mathcal{C}(U) \cap \mathcal{W}^c_{\epsilon/2}(p)$ converging to p. In the flow case, $\mathcal{W}^c(p)$ is the orbit of the flow at p. There exists $|t_0| < \epsilon/2$ such that: $q = \phi^{t_0}(p) \in \mathcal{C}(U) \cap \mathcal{W}^c_{\epsilon/2}(p)$. Continuity of the flow implies: $p = \phi^{-t_0}(q) \in \mathcal{C}(U)$, hence we get a contradiction.

Conversely, if ϕ^1 is accessible then Proposition 3 implies that it is non-jointly integrable. □

Remark. In the Anosov flow situation, much of the work has already been done by J. Plante in [**Plan**] and [**BPW**]: Proposition 3 is a generalization of Proposition (1.6) of [**Plan**] (which depends on a result in [**Anos**]). As we have seen, in the flow case, accessibility and non-joint integrability are equivalent, and so stable accessibility is a mere consequence of Proposition 5: with a different method, this has already been proved in [**GPS**] for the geodesic flow.

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