THE REVIEW OF SYMBOLIC LOGIC
Volume 6, Number 1, March 2013

FORMALIZATION, PRIMITIVE CONCEPTS, AND PURITY

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Abstract. We emphasize the role of the choice of vocabulary in formalization of a mathematical area and remark that this is a particular preoccupation of logicians. We use this framework to discuss Kennedy's notion of 'formalism freeness' in the context of various schools in model theory. Then we clarify some of the mathematical issues in recent discussions of purity in the proof of the Desargues proposition. We note that the conclusion of 'spatial content' from the Desargues proposition involves arguments which are algebraic and even metamathematical. Hilbert showed that the Desargues proposition implies the coordinatizing ring is associative, which in turn implies the existence of a three-dimensional geometry in which the given plane can be embedded. With W. Howard we give a new proof, removing Hilbert's 'detour' through algebra, of the 'geometric' embedding theorem.

Finally, our investigation of purity leads to the conclusion that even the introduction of explicit definitions in a proof can violate purity. We argue that although both involve explicit definition, our proof of the embedding theorem is pure while Hilbert's is not. Thus the determination of whether an argument is pure turns on the content of the particular proof. Moreover, formalizing the situation does not provide a tool for characterizing purity.

Mathematical logic formalizes normal mathematics. We analyze the meaning of 'formalize' in that sentence and use this analysis to address several recent questions. In Section 1 we establish some precise definitions to formulate our discussion and we illustrate these notions with some examples of David Pierce. This enables us to describe a variant on Kennedy's notion of formalism freeness and connect it with recent developments in model theory. In Section 2 we discuss the notion of purity in geometric reasoning based primarily on the papers of Hallett (2008) and Arana & Mancosu (2012). In the brief Section 3, we discuss differences in algebraic and geometric proof. In particular, we consider the ability of algebraic proofs to avoid special cases. This theme is reinforced by the many special cases in the Appendix written with William Howard of a geometric proof (differing from Levi, 1939) of Hilbert's theorem that a Desarguesian projective plane can be embedded in three-space.

Suppose there is some area of mathematics that we want to clarify. There are five components of a formalization. The first four—1) specification of primitive notions, 2) specifications of formulas and 3) their truth, and 4) proof—provide the setting for studying a particular topic. 5) is a set of axioms that pick out the actual subject area. We will group these five notions in various ways through the paper to make certain distinctions.

Our general argument is: while formalization is the key tool for the general foundational analysis and has had significant impact as a mathematical tool[1] there are specific problems

Received: August 22, 2011.

[1] Examples include the theory of computability, Hilbert's 10th problem, the Ax–Kochen theorem, o-minimality and Hardy fields, Hrushovski's proof of the geometric Mordell–Lang theorem, and current work on motivic integration.

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doi:10.1017/S1755020312000263

https://doi.org/10.1017/S1755020312000263 Published online by Cambridge University Press

in mathematical logic (Section 1.3) and philosophy (Section 2) where 'formalism-free' methods are essential. In the case of logic, we describe the long standing contrast in model theory between formal definability of a class and definition by closure under mathematical operations. In philosophy, we note that such natural formal notions as extension by explicit definition can introduce impurity. And yet, such extensions are natural mathematical constructions. We argue that the purity or impurity of such an argument must be decided on the contentual grounds of the specific case.

We adapt the term 'formalism-freeness' from Kennedy (To appear). In order to clarify this concept, we first describe our notion of a formalization of a mathematical topic.[2] This involves not only the usual components of a formal system, specification of ground vocabulary, well-formed formulas, and proof but also a semantics. From a model-theoretic standpoint the semantic aspect has priority over the proof aspect. The topic could be all mathematics via, for example, a set-theoretic formalization. But our interest is more in the local foundations of, say, plane geometry. In any case, a mathematical topic is a collection of concepts and the relations between them.

It is not accidental that 'formalization' rather than 'formal system' is being defined. The relation between intuitive conceptions about some area of mathematics (geometry, arithmetic, Diophantine equations, set theory) and a formal system describing this area is central to our concerns. The first step in a formalization is to list the intuitive concepts which are the subject of the formalization. The second is to list the key relations the investigator finds among them.

DEFINITION 0.1. *We see a* full formalization *as involving the following components.*

1. *Vocabulary: specification of primitive notions.*
2. *Logic*
   - *(a) Specify a class*[3] *of well-formed formulas.*
   - *(b) Specify truth of a formula from this class in a structure.*
   - *(c) Specify the notion of a formal deduction for these sentences.*
3. *Axioms: specify the basic properties of the situation in question by sentences of the logic.*

We adopt Kennedy's term 'formalism-freeness from Kennedy (To appear). Formalism-freeness is a matter of degree. It involves suppressing some of the components of Definition 0.1. The most extreme version reduces to a semantic study (involving only a rough version of 1 and some of 2b): we can determine whether a structure is a member of a class of structures that are being studied. This determination will often be expressed in terms of the closure of the class under certain mathematical operations.[4] We think of the specification Definition 0.1 as being in informal set theory; the usual mathematicians' "could formalize in ZFC" applies. In the critique of some problems from a calculus text (Example 1.2) we will see how crucial item 1 is to avoid confusions in a situation where there is no thought of a formal language as in 2a). An even more basic example

[2] In general, context and topic are synonyms in this paper. See Section 2.3.

[3] For most logics there are only a set of formulas, but some infinitary languages have a proper class of formulas.

[4] See Section 1.3.

https://doi.org/10.1017/S1755020312000263 Published online by Cambridge University Press

of the importance of fixing vocabulary is the confusion between whether the minus sign represents binary subtraction or the unary operation of taking the additive inverse.

This semantic approach both represents a method of doing mathematics in the first section and a tool, we argue an essential tool, for analysis of the notion of purity in the second.

I thank Paolo Mancosu for introducing me to the study of purity, Juliette Kennedy for the notion of 'formalism-free' and acknowledge helpful comments from Andrew Arana, Phil Ehrlich, Bill Howard, Juliette Kennedy, and Victor Pambuccian.

§1. Formalization: vocabulary, logic, and proof. In Section 4.1 of Arana & Mancosu (2012), Arana and Mancosu draw an important distinction between 'intuitive' and 'formal' content. We expand a bit on that distinction. First we stress that the 'intuitive content of a proposition' is often ill-defined; there is some background context of the proposition that needs to be clarified. As a simple example, consider 'Are there infinitely many primes?'. At the most basic level, one might take this to mean 'prime natural numbers'. But it is not clear whether these are numbers in the structure $(\mathbb{N}, \times, 1)$, $(\mathbb{N}, \times, 1, <)$, $(N, +, \times, 0, 1)$, or many other choices including $(N, |, 1, S, <)$; see Arana (2011) for a number of variants. A natural response from a mathematician might be, 'In what ring?'.

So we speak not of formalizing a proposition but of formalizing a topic or context—those words are interchangeable. A first step towards formalization is to further specify the intuitive content. What concepts are involved? By the *intuitive content* of a *context* or a *topic* we mean a certain collection of concepts and an understanding of how they are related. The daily work of a mathematician is to take such situations and write down more precise descriptions. Formalization is a very stringent scheme for writing such a description. In our description of formalization, Section 1.2, we lay out one schema for making such descriptions.[5] As the examples indicate, the intuitive content is determined by an individual (or community) understanding of a complex of concepts; it is not a property of a proposition.

In Arana & Mancosu (2012), formal content is defined somewhat vaguely as the 'inferential role of a statement'. Here we will say the *formal content* of a collection of statements Φ is the collection of their models. Of course, by the completeness theorem, we could equally well say the logical consequences of Φ.

We distinguish two degrees of formalization. We'll call the first framework Euclid–Hilbert and the second Hilbert–Gödel–Tarski. In the first we have the notions of axioms, definitions, proofs, and, with Hilbert, models. This Hilbert is the author of the Grundlagen. We could treat formal content as consequences or the collection of models of consequences in this framework but it would be more by convention than a theorem. For Euclid–Hilbert logic is a means of proof. For Hilbert–Gödel–Tarski, logic is a mathematical subject. This Hilbert is the founder of proof theory. There are now explicit rules of proof and after Gödel and Tarski proofs within set theory of the completeness theorem. So our identification of the two notions of formal content is a theorem. A full formalization requires the second understanding. But as we'll point out at several points in this paper, the Euclid–Hilbert understanding is adequate for most of the philosophical points of this paper.

[5] Note that our focus on formal languages is a definite choice; Turing's 'formalization' of the notion of computability goes in a very different direction.

https://doi.org/10.1017/S1755020312000263 Published online by Cambridge University Press

The process of specification of intuition forces clarification of the relations between the concepts. Thus the notion that through any line and any point not on that line there is a *unique* line parallel to the given line is a natural intuition about geometry. One of Euclid's great contributions is to realize that this intuition might require a separate declaration; it took 2000 years to confirm his doubt.

We will elaborate on our definition of 'formalization' in Section 1.2 and on 'formal content' in Sections 2.1 and 2.3. But first in Section 1.1. we emphasize the significance of the first item: vocabulary. We elaborate on 'formalism-freeness' in Section 1.3.

1.1. Vocabulary. We establish some specific notations which emphasize some distinctions between the mindsets of logicians, in particular, model theorists, and 'normal' mathematicians.

DEFINITION 1.1. *1. A* vocabulary τ *is a list of function, constant, and relation symbols.*

2. A τ-structure $\mathcal{A}$ *is a set with an interpretation of each symbol in* τ.

Specifying a vocabulary (signature, similarity type)[6] is only one aspect of the notion of a formal system. But it is a crucial one and one that is often overlooked by non-logicians. From the standpoint of formalization, fixing the vocabulary is singling out the 'primitive concepts'. This choice is a first step in formalization. Considerable reflection from both mathematical and philosophical standpoints may be involved in the choice. For example, suppose one wants to study 'Napoleon's theorem' that the lines joining the midpoints of any quadrilateral are a parallelogram. At first sight, one might think the key notions (and therefore primitive concepts) are quadrilateral and parallelogram. But experience[7] even before Euclid showed that the central basic notions for studying the properties of quadrilaterals are point, line, incidence, and parallel[8] and the delineation of types of quadrilaterals is by explicit definition.

The choice of primitive notions for a topic is by no means unique. For example, formulated in a vocabulary with only a binary function symbol, the theory of groups needs $\forall\exists$-axioms and groups are not closed under subalgebra. Adding a constant for the identity and a unary function for inverse, turns groups into a universally axiomatized class. Alternatively, groups can be formulated with one ternary relation as the only symbol in the vocabulary. The three resulting theories are pairwise bi-interpretable.

It is a commonplace in model theory that just specifying a vocabulary means little. For example in the vocabulary with a single binary relation, I can elect to formalize either linear order or successor (by axioms asserting the relation is the graph of a unary function). Thus, while I here focus on the choice of relation symbols—their names mean nothing; the older usage of signature or similarity type might be more neutral. The actual collection of

[6] Years ago Tarski used the phrase *similarity type* for essentially this notion; sometimes it is called the *signature*. These notions (a specification of arities of a sequence of arities) are one level of abstraction higher than vocabulary. But they have the same effect in distinguishing the syntactic from the semantic and we will use vocabulary. Still another 'synonym' is language. We explain in Section 1.3 why we try to avoid this word.

[7] Experience includes determining which definitions and postulates best conform to intuition and which allow a smooth development of the subject. Well-known precursors of Euclid include Eudoxus, Theudius, and Hermotimus (Euclid, 1956).

[8] I note that alternatives basing geometry on regions rather than line for epistemological reasons were advanced by, for example, Whitehead and Leśniewski, but that is not relevant to this point.

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structures under consideration is determined in a formal theory—by sentences in the logic. In the formalism-free approaches discussed in Section 1.3 the specification is in normal mathematical language. Having fixed a vocabulary with one binary relation, we say, for example, 'Let $\boldsymbol{K}$ be the class of well-orderings of type $< \lambda$ such that ....'

But while axioms are necessary to determine the meanings of the relations in a vocabulary; the mere specification of the vocabulary provides important information. Pierce (2011) has pointed out the following example of mathematicians' lack of attention to vocabulary specification.

EXAMPLE 1.2 (Pierce). *Spivak's Calculus book (Spivak, 1980) is, one of the most highly regarded texts in late twentieth century United States. It is more rigorous than the usual Calculus I textbooks. Problems 9–11 on page 30 of Spivak (1980) ask the students to prove the following are equivalent conditions on* $\mathbb{N}$*, the natural numbers. This assertion is made without specifying the vocabulary that is intended for* $\mathbb{N}$*. In fact,* $\mathbb{N}$ *is described as the counting numbers,*

$$1, 2, 3, \ldots.$$

1 **induction** *(*$1 \in X$ *and* $k \in X$ *implies* $k + 1 \in X$*) implies* $X = N$.

2 **well-ordered** *Every nonempty subset has a least element.*

3 **strong induction** *(*$1 \in X$ *and for every* $m < k$, $m \in X$ *implies* $k \in X$*) implies* $X = N$.

As Pierce points out, this doesn't make sense: 1 is a property of a unary algebra; 2 is a property of ordered sets (and doesn't imply the others even in the language of ordered algebras); 3 is a property of an ordered unary algebras. For example, 2) is satisfied by any well-ordered set while the intent is that the model should have order type ω.

It is instructive to consider what proof might be intended for 1) implies 3). Here is one possibility. Let X be a nonempty subset of N. Since every element of N is a successor (Look at the list!), the least element of X^c must be $k+1$ for some $k \in X$. But the existence of such a k contradicts property 1). There are two problems with this 'proof'. The first problem is that there is no linear order mentioned in the formulation of 1). The second is, "what does it mean to 'look at the list'?" These objections can be addressed. Assuming that $\mathbb{N}$ has a discrete linear order satisfying $(\forall x)(\forall y)[x \leq y \vee y + 1 \leq x]$ and that the least element is the only element which is not a successor resolves the problem. This assertion follows informally (semantically) if one considers the natural numbers as a subset of the linearly ordered set of reals.

As Pierce notes (Pierce, 2011), a fundamental difficulty in Spivak's treatment is the failure to distinguish the truth of each of these properties on the appropriate expansion of (N, S) and a purported equivalence of the properties—*which can make sense only if the properties are expressed in the same vocabulary*.

But in another sense the problem is the distinction between Hilbert's axiomatic approach and the more naturalistic approach of Frege. I'll call Pierce's characterization of Spivak's situation, Pierce's paradox. It will recur; Pierce writes:

> Considered as axioms in the sense of Hilbert, the properties are not meaningfully described as equivalent. But if the properties are to be understood just as properties of the numbers that we grew up counting, then it is also meaningless to say that the properties are equivalent: they are just properties of those numbers.

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Note that this distinction about vocabulary is prior to distinctions between first- and second-order logic. We stated the difficulty in the purported equivalence of 1) and 2) in terms of second-order logic. But the same anomaly would arise if PA (with a schema of first-order induction) were compared with 'every definable set has a least element'.

Pierce's paradox is fundamentally a semantic remark. Two sentences are equivalent if they have the same models; this makes no sense if they do not have same vocabularies or at least are viewed as sentences for a vocabulary that contains the symbols from each sentence. It might have been more precise for Pierce to say 'trivial' rather than meaningless. In the second and third cases enumerated by Pierce, it is clear that as sentences in the language $(S, <)$ they are simply not equivalent. And trivially they are both true in $(N, S, <)$. It can be objected that it makes sense to prove one property of a given structure $\mathcal{A}$ implies the truth of another on $\mathcal{A}$ using properties of $\mathcal{A}$. That seems a normal enough mathematical strategy. But consider the case at hand: on $(N, <, S)$, well-order implies induction (i.e., order type ω). Why? Because it is a property of N that the order type is ω. But this seems to me to be just the type of argument I attribute to Spivak a few paragraphs up; it is hard to find a nontrivial phrasing of it.

We introduced vocabulary in Definition 0.1 as 'the specification of primitive notions'. Thus the choice of the vocabulary is the fundamental step in the formalization process. The vocabulary should focus attention on the concepts seen as most basic.[9] Thus in algebraic geometry, systems of equations (perhaps of high degree in several variables) pick out the geometric relations that are the main subject of study.

As the Spivak example illustrates, the specification of a vocabulary can be rather fluid. Much of this paper concerns the role of explicit definition.

DEFINITION 1.3. *For any theory T in vocabulary τ and formula $\phi(\mathbf{x})$, if we expand τ to τ' by adding a new relation symbol, R_ϕ, and the axiom*

$$(\forall \mathbf{x})[\phi(\mathbf{x}) \leftrightarrow R_\phi(\mathbf{x})]$$

we say R_ϕ is explicitly defined *in T.*

At first sight it seems extension by explicit definition is totally harmless. No new concept can be introduced;[10] it is just a kind of abbreviation. And if no new properties that are not provable in the base theory are introduced, this step should surely not infringe on the purity of an argument. It is a 'logical' not a 'mathematical' step. However, we will see that if arbitrary explicit definitions are allowed in a proof, the notion of such a proof being pure is almost meaningless.

It is natural when formalizing to try to find a minimal set of 'primitive concepts'. This was a frequent theme of Tarski.[11] And arguments for the naturality of various notions are part of the justification for a particular choice. However, there are other considerations.

[9] We don't attempt to analyze the meaning of 'basic' or 'natural' concept; we just rely on the usual understandings in mathematical practice. A more detailed analysis would distinguish between the criteria of 'reflecting intuition' and 'provides a clear framework'.

[10] We think of the concepts formalizable in a theory as exactly those expressed by the formulas. Making an explicit definition is focusing attention an existing concept, not adding a new one. See the discussion in Section 2.1.

[11] See Givant & Tarski (1999) for a summary of Tarski's work on this area for geometry.

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Explicit definition can reduce complexity in a way that can be measured formally—by reducing quantifier complexity. In developing a first-order theory to describe a given mathematical structure or class of structures, a model theorist would like to have the resulting theory T admit elimination of quantifiers (every formula $\phi(\mathbf{x})$ is equivalent in T to a formula $\psi(\mathbf{x})$ which has no quantifiers[12]) and, as we stressed above in discussing the process of formalization, be formulated with concepts which are basic and central to the topic under study. Sometimes, with judicious choice, a small family of well-understood predicates can be added that suffice for the quantifier elimination. Two examples are Pressburger arithmetic where unary predicates for divisibility by n are added and Macintyre's celebrated proof of quantifier elimination for the p-adic numbers; he adds predicates $P_n(x)$ meaning x is an nth power.

In contrast, the quantifier elimination can be easily established by fiat Morley (1965a) and Keisler (1960): add explicit definitions for each formula $\phi(\mathbf{x})$, a predicate $R_\phi(\mathbf{x})$ and an axiom $R_\phi(\mathbf{x}) \leftrightarrow \phi(\mathbf{x})$. But now the vocabulary contains many essentially incomprehensible relations.[13] The definition of infinitely many formulas by recursive introduction of quantifiers of indefinite length has been done in one step by fiat. There is no attempt to ground the meaning of each formula from the primitive concepts. This defeats the goal of quantifier elimination in studying specific theories: every definable relation is a Boolean combination of *well-understood* relations. This quantifier elimination by fiat is a powerful tool at the foundation of stability theory. And it provides positive results for specific theories. But it loses the analysis of definable sets in terms of the primitive notions.

1.2. Formalization. We first expand a bit on what we mean by a full formalization. Definition 1.4 summarizes the notion of a *model-theoretic logic* $\mathcal{L}$ defined in Barwise & Feferman (1985). This paper will only use some specific examples of such logics including, first order, second order, $L_{\omega_1,\omega}$ etc..

DEFINITION 1.4. *A logic contains certain logical vocabulary: connectives, quantifiers, and a set of variables. For each (nonlogical) vocabulary τ, the collection of $\mathcal{L}(\tau)$-formulas is defined inductively in the natural way. An $\mathcal{L}(\tau)$-formula with no free variables is called an $\mathcal{L}(\tau)$-sentence.*

Thus, a logic *$\mathcal{L}$ is the pair $(\mathcal{L}, \models_{\mathcal{L}})$ such that $\mathcal{L}(\tau)$ is a collection of τ-sentences and for each $\phi \in \mathcal{L}(\tau)$ and each τ-structure $\mathcal{A}$, $\mathcal{A} \models_{\mathcal{L}} \phi$ is defined in the natural inductive way.*

Crucially, 'the natural inductive definition' implies that the truth of a τ-sentence in a τ-structure M depends only on the isomorphism type of M. Thus the entire framework of 'model-theoretic' logics depends on what Burgess (2010) has called 'indifferentism to

[12] Sometimes, the specification that ψ have the same free variables as ϕ is omitted. But this seems wrong-headed. The actual equivalence is then $\phi(\mathbf{x}) \leftrightarrow (\forall \mathbf{y} \psi(\mathbf{x}, \mathbf{y}))$. And the negation of ϕ is equivalent to an existential formula. So this is an equivalence to a Δ_0 formulas and holds only on models of T. Quantifier elimination in our sense means that each formula has unique interpretation on each substructure of any model of T.

[13] Of course, it may be possible to unpack individual formulas. But a usually finite (or at least given by a regular list of concepts with clear meaning) has been replaced a set of abbreviations. Any formula with more than three alternations of quantifiers is hard to comprehend; here many such are introduced simultaneously.

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identity': ignoring a specific set-theoretic construction of the model. This is one aspect of formalism freeness; but much more is entailed.[14]

Definition 1.4 yields two natural notions of implication.

DEFINITION 1.5. *For any* $\phi, \psi \in \mathcal{L}(\tau)$*, we say* ϕ logically implies ψ *and write* $\phi \models_{\mathcal{L}} \psi$ *if for every* τ*-structure* M*,* $M \models_{\mathcal{L}} \phi$ *then* $M \models_{\mathcal{L}} \psi$.

Alternatively, $\mathcal{L}$ *may be assigned a collection of logical axioms and deduction rules giving rise to the natural notion of* ψ *can be* deduced *from* ϕ*,* $\phi \vdash_{\mathcal{L}} \psi$.

Deduction is a syntactic notion. We divided the notion of formalization into five components: 1) specification of primitive notions, 2a) specifications of formulas and 2b) their truth, 2c) proof, and 3) axioms. The standard account of a formal system (e.g., Smullyan, 1961) includes 2a), 2c), and 3) but not 2b). From our standpoint and that of Barwise & Feferman (1985), 2a), and 2b) are basic; 2c) may or may not exist.

In Definition 0.1, the logic does not depend on the particular area of mathematics. In earlier days, from a more general logical perspective, the vocabulary might be universal, containing infinitely many n-ary relations for each n. In contrast, we seek primitive terms which pick out the most basic concepts of the field in question and axioms which in Hilbert's sense give us an implicit definition of the area. Thus, we can formalize concepts such as real closed fields (RCF) or algebraic geometry[15] or set theory without reference to the construction of specific models.[16] Our treatment of the primitive terms is analogous to the treatment of the element relation in set theory. But this analysis is relevant to either traditional (global) or local foundations. For any particular area of mathematics, one can lay out the primitive concepts involved and choose a logic appropriate for expressing the important concepts and results in the field. While in the last quarter century mathematical logic has primarily focused on first-order logic as the tool, we discuss some alternatives in Section 1.3.

1.3. Formalism freeness (mathematical properties). We have defined formalization. What do we mean by 'formalism-freeness'? Kennedy begins (Kennedy, To appear) with

> That mathematics is practiced in what one might call a formalism free manner has always been the case – and remains the case. Of course, no

[14] It is ironic that although model theory is based on the notion of identifying structures up to isomorphism with respect to the given vocabulary, Shelah has proved major results by deliberately ignoring this convention. For example, J.H.C. Whitehead asked whether whenever B is an abelian group and $f : B \mapsto A$ is a surjective group homomorphism whose kernel is isomorphic to the group of integers $\mathbb{Z}$, then $B \approx \mathbb{Z} \otimes A$. This question is entirely about abelian groups. Nevertheless, Shelah (Eklof, 1976) shows the answer is independent of the axioms of ZFC by representing a Whitehead group of cardinality $\aleph_1$ as a structure with universe $\aleph_1$ and identifies invariants depending on both the group operation and the ϵ-relation on $\aleph_1$. A further example of this use of the explicit construction of a model is Shelah's use of Ehrenfeuht–Mostowski models in an expanded language to construct many nonisomorphic models of first-order theories under various instability hypotheses. Still another is the Ehrlich's (2001) use of Conway's surreal numbers to investigate real closed fields. These examples illustrate the power that derives from being careful about vocabulary and in contrast to the usual indifferentism, paying careful attention to how a structure is built in an expanded language (essentially set theory in the first and third case, the Skolem theory with order in the second).

[15] See the discussion of Zariski geometries in Section 1.3.

[16] Geometry and analysis are presented in this way in, for example, Spivak (1980); Hilbert (1971); Heyting (1963).

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> one would have thought to put it this way prior to the emergence of the foundational formal systems in the late nineteenth and early twentieth centuries; . . .

Kennedy goes on to explore what Gödel (1946) calls the search for absolute notions of computability, provability, and definability. We try to distinguish such practice from the use of logical methods (either for metamathematical studies or as tools for proving mathematical results in the normal sense (in number theory or geometry or graph theory or . . .). Our topic differs from Kennedy's by focusing specifically on the development of (formal?) theories for parts of mathematics. In discussing formalism-freeness, we are not arguing against formal methods; we are just pointing to some limitations.

Formalism free is a matter of degree. It involves dropping some of the components of Definition 0.1. Clause 1 of Definition 0.1 is sacrosanct. We deal only with structures of a fixed vocabulary. In Kennedy's work on the notion of definition she says that a class of structures has been given a *Formalism independent* definition if the class can be defined in several different logics. We are investigating the interaction between proof and definition here and make a more stringent requirement on definitions. We call a definition *formalism free* if it is given semantically without any formal distinction between syntax and a semantics. For example, ' a group is a set and a binary function satisfying . . . '. We call an investigation *formalism free* if it is a semantic study. Thus there are two aspects: definability and investigation.

Here are three illustrative examples. We say the notion of 'computability' is 'formalism independent' or formalism-free in Kennedy's terminology because of the known equivalence among the Turing, Markov, λ-calculus, Gödel characterizations. Only the Gödel characterization as Δ_1^0-definable functions in number theory is completely formal in the sense defined here.

Secondly, as Kennedy has pointed out,[17] if L is constructed by adding relatively definable sets then the usual L is obtained by several different logics; for example, either by iterating first-order definitions or weak second-order logic definitions. This represents formalism independence. But the definition of L as the closure of the ordinals under the Gödel functions provides a fully formalism-free definition.[18]

In a third direction, consider the notion of a polynomial. In a standard high school Algebra I book (Educational Development Corporation, 2009) a *polynomial* is defined (with no fanfare of syntax and semantics) as a sum of *monomials* where a monomial is earlier defined as a product of variables raised to nonnegative integer powers and a (usually real) number coefficient. In this style of development a *polynomial function* is map (e.g., from $\Re^2 \mapsto \Re$ defined by a polynomial). The interpretation of the formal language in the structure that is fundamental to Tarski's definition of truth is made matter of factly in elementary algebra. In this formulation the ring of polynomials is smoothly defined in a concrete way in the style of universal algebra (Grätzer, 1968). In contrast, the definition of a ring of polynomials in Lang's Algebra book (Lang, 1964) is much more abstract but

[17] This is a sampling from a much more detailed account in Kennedy (To appear).

[18] Although Gödel's construction takes place in models of set theory, whose vocabulary contains only ϵ, this choice of vocabulary is in the background for the definition of L. Crucially, L is defined by a normal mathematical construction. The eight Gödel functions are defined (really in naive set theory) and $L_{\alpha+1}$ is the closure of L_α under these functions (Jech, 1978). In fact, the first-order definability of the Gödel functions plays a crucial role in the proofs but is suppressed in describing the structure L.

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avoids even a glimpse of the syntax–semantic distinction. Then polynomial functions are defined by a composition of functions. In our terminology, both of these approaches are formalism-free.

We turn to the historical origins of this distinction in model theory. Tarski begins his fundamental work on definable subsets of the real numbers (Tarski, 1931) by saying, 'Mathematicians, in general, do not like to deal with the notion of definability; their attitude toward this notion is one of distrust and reserve'. He goes on to focus the cause of this attitude on the failure to fully specify dependence of definability on the choice of primitive terms and particular model.[19] But he goes on to try to refute this view (Tarski, 1931):

> I believe that I have found a general method to construct a rigorous metamathematical definition of this notion. Moreover, by analyzing the definition obtained it proves possible[20] to replace it with a definition solely in mathematical terms. Under this new definition the notion of definability does not differ from other mathematical notions and need not arouse either fears or doubts; it can be discussed entirely within the domain of normal mathematical reasoning.

Vaught (1986) summarizes the definition nicely:

> In the same paper Tarski (1931), Tarski first made a definition of truth, or rather of the related notion of definability. Moreover, he did so without mentioning a formal language, by a method to which he returned in Tarski (1952). Indeed, he defined the class of definable relations over $(\Re, +, *)$, for example, to be the smallest class of finitary relations over $\Re$ containing the ternary relation $(x, y, z) : z = x + y$ and $(x, y, z) : z = x.y$ and closed under the Boolean operations and the "geometrical" operation of projection.

van den Dries took up this idea and called a class of sets closed, as Vaught describes, a Tarski System in van den Dries (1986), which is one of the origins of the notion of o-minimality. In the introduction to van den Dries (1999) Tarski's name has disappeared in this context but several examples of Tarski systems are given to motivate the general definition of an o-minimal structure.[21]

In the opening paragraph of what might be viewed as the founding paper of model theory (Tarski, 1954), Tarski writes,

> Every set Σ of sentences determines uniquely a class $\boldsymbol{K}$ of mathematical systems. . . . Among questions which arise naturally in the study of these notions, the following may be mentioned: Knowing some structural (formal) properties of a set Σ of sentences, what conclusions can we

[19] See also Feferman (2008) where Feferman discusses Tarski's attitude toward mathematics and metamathematics. The distinction between formal/logical I discuss here is a separate issue from the controversy Feferman addresses over Tarski's concept of a 'logical notion'.

[20] He says 'with some reservations'. They are that there is no uniform definition but only one formulas of quantifier rank n for each n.

[21] In fact the syntactical notions are not introduced. On page 3 of van den Dries (1999) a set $A \subset M^n$ is said to be *definable* if it is in one of these collections $S_n \subset \mathcal{P}(M^n)$, closed under the natural operations. This is then an example of formalism freeness as we discuss in Section 1.3.

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> draw concerning the mathematical properties of the correlated set of models? Conversely, . . .

Tarski takes his readers to understand the notion of 'mathematical property'. His sample theorems make his meaning plain—the ordinary stuff of mathematics: subalgebra, homomorphisms, direct products, etc. Tarski gives a number of examples of answers to questions of this sort; two are:

Tarski A class $\boldsymbol{K}$ of structures in a finite relational language is axiomatized by a set of universal sentences if and only if $\boldsymbol{K}$ is closed under isomorphism, substructure and if for every finite substructure B of a structure A, $B \in \boldsymbol{K}$ then $A \in \boldsymbol{K}$.

Birkhoff A class $\boldsymbol{K}$ of algebras is axiomatized by a set of equations if and only it it is closed under homomorphism, subalgebra, and direct product.

Thus formal properties of the axioms for a class of models are shown equivalent to the mathematical property of the class being closed under certain basic mathematical operations. This notion of 'mathematical property' is similar to that which Kennedy (To appear) traces as the notion of 'formalism freeness' in the works of Gödel. She writes, 'one can think of indifferentism[22] or formalism-freeness . . . as the simple preference for semantic methods, that is methods *which do not involve or require the specification* of a logic– at least not *prima facie*'. We hope that the distinctions in Section 1.2[23] can clarify this notion. It is tempting to speak of language here. We have avoided the word 'language' because of its several usages in the context at hand. One may speak of the 'language of rings' meaning $+, \times, 0, 1$. A different and more specific version is for this phrase to imply that these operations obey the axioms of ring theory; see Section 2.3. In a third version, one speaks of the language of first-order logic, meaning the collection of formulas generated from a vocabulary (equivalently signature) by the finitary propositional connectives and existential and universal quantification. Kennedy is making this distinction. That is, a formalism-free approach would take language in the first sense, not the second or third. An inquiry can be 'formalism-free' while being very careful about the vocabulary but eschewing a choice of logic (in the sense of Definition 1.4) and in particular any notion of formal proof. Thus it studies mathematical properties in the sense we quoted from Tarski above.

It is in this sense that certain recent work of Zilber and Shelah can be seen as developing a formalism-free approach to model theory. Both Zilber's notions of a quasiminimal excellent class (Zilber, 2005) and of a Zariski geometry (Hrushovski & Zilber, 1993), and Shelah's concept of an Abstract Elementary Class (Shelah, 2009) give axiomatic but mathematical definitions of classes of structures in a vocabulary τ. That is, the axioms are not properties expressed in some formal language based on τ but are mathematical properties of the class of τ-structures and some relations on it.

These are three examples where difficulties in the normal (from a logician's viewpoint) syntactic approach to a problem led to more semantic methods. The notion of abstract elementary class arose from the difficulties of dealing with the syntax of infinitary logics.

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[22] I regard Burgess's notion of indifferentism to identity on page 9 of Burgess (2010) as a component of formalism freeness. Indifferentism seems to me to refer to working with structures up to isomorphism rather than caring about the set-theoretic construction. Here we take that modus operandi for granted and consider how one is to describe the connection between various structures.

[23] Our articulation of them here was partially motivated by Kennedy's work.

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The failure of compactness leads to an inability to develop the usual model-theoretic tools of stability theory. And underlying similarities among different logics are obscured by technicalities about the meaning of sentence. In contrast the examples of Zilber and Hrushovski–Zilber arise from specific mathematical problems: the desire to understand the complex numbers with exponentiation and investigating the failure of the first-order model-theoretic formalization to catch a key idea of algebraic geometry.

In Shelah's case, the basic relation is a notion of 'strong submodel' relating the members of the class. Quasiminimal excellent classes require a combinatorial geometry on each model which has certain connections with the basic vocabulary.

The notion of abstract elementary class arose from the difficulties of dealing with the syntax of infinitary logics. The axiomatic properties of 'strong submodel' summarize the properties of elementary submodel in first order and various infinitary logics as well as some unexpected contexts. An abstract elementary class (AEC) $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ is a collection of structures[24] for a fixed vocabulary τ and a relation of 'strong substructure', $\prec_{\boldsymbol{K}}$ among the members of $\boldsymbol{K}$. Examples of 'strong substructure' might be 'substructure', or elementary substructure, or elementary in $L_{\omega_1,\omega}$. But these various syntactically based properties are replaced by mathematical properties of $\prec_{\boldsymbol{K}}$. First the class is closed under unions of increasing $\prec_{\boldsymbol{K}}$-chains and more subtly the union of a $\prec_{\boldsymbol{K}}$-chain is a strong submodel of any M which is an extension of each element in the chain. From a category-theoretic standpoint this just means the class is closed under colimits.[25] There is a more technical condition which is the semantic version of the Tarski–Vaught condition for elementary submodel. Finally, countable Lowenheim Skolem number means, every infinite subset is contained in a submodel of the same size.[26]

In fact, these classes can be defined in another 'formalism-free' way using purely categorical terms; see Kirby: "Abstract Elementary Categories" (Kirby, 2008) or Lieberman: "AECs as accessible categories" (Lieberman, 2011). The connection with logic is at first only motivational. The AEC notion was developed to simplify the study of infinitary logics by generalizing some of the crucial properties and avoiding syntactical complications. The crucial Löwenheim Skolem property is derived from thinking of $\prec$ as a kind of elementary submodel. But there is no explicit syntax[28] and no notion of a definable set.

In contrast, Zilber's notion of a *quasiminimal excellent class* (Zilber, 2005) was developed to provide a smooth framework for proving the categoricity in all uncountable powers of Zilber's pseudoexponential field. This example itself can be axiomatized in a standard model-theoretic framework in $L_{\omega_1,\omega}(Q)$. But this is *not* the way the construction is made. The structure is patterned on (and conjecturally isomorphic to) to the complex exponential field $(\mathbb{C}, +, \times, e^x)$. For technical reasons having to do with the simplicity

[24] Naturally we require that both $\boldsymbol{K}$ and $\prec_{\boldsymbol{K}}$ are closed under isomorphism.

[25] Category theoretic colimits are model-theoretic direct limits.

[26] An abstract elementary class (AEC) $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ is a collection of structures[27] for a fixed vocabulary τ that satisfy the following, where $A \prec_{\boldsymbol{K}} B$ means in particular A is a substructure of B:

1. If $A, B, C \in \boldsymbol{K}$, $A \prec_{\boldsymbol{K}} C$, $B \prec_{\boldsymbol{K}} C$ and $A \subseteq B$ then $A \prec_{\boldsymbol{K}} B$;
2. Closure under direct limits of $\prec_{\boldsymbol{K}}$-embeddings;
3. Downward Löwenheim–Skolem. If $A \subset B$ and $B \in \boldsymbol{K}$ there is an A' with $A \subseteq A' \prec_{\boldsymbol{K}} B$ and $|A'| \leq |A| = \mathrm{LS}(\boldsymbol{K})$.

[28] Shelah's presentation theorem, discussed below, shows there is an 'implicit' syntax.

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of dealing with relational languages, the vocabulary is taken to include all polynomially definable sets as basic predicates. The fundamental result that a quasiminimal excellent class is categorical in all uncountable powers can be presented in a formalism-free way. The key point is that there are no axioms in the object language of the general quasiminimal excellence theorem; there are only statements about the combinatorial geometry determined by what are in the application the $(L_{\omega_1,\omega}(Q))$-definable sets. But there is a formal corollary. Kirby (Theorem 5.5 of Kirby, 2010) proves: Any quasiminimal excellent class with an infinite dimensional model is the class of models of sentence in $L_{\omega_1,\omega}(Q)$. Analogously to the results of and Tarski and Birkhoff that opened this subsection, this result has the form: Any class which satisfies specified 'mathematical properties' has a formal definition of a specified sort.

Hrushovski & Zilber (1993) introduced Zariski geometries partly in an attempt to remedy a notorious gap in the model-theoretic study of algebraic geometry.[29] Algebraic geometry is concerned with the solution of systems of *equations*. But from a model-theoretic standpoint, there is no way to distinguish among definable sets, 'all definable sets are equal'. In particular, the class of definable sets is closed under negation; equations and inequations have the same status. But from the perspective of algebraic geometry, 'some definable sets are more equal than others'. Systems of equations (varieties) are the objects of true interest. Hrushovski and Zilber remedy this situation by introducing a topology. The definition of a Zariski geometry[30] (Zilber, 2010) concerns the relations between a family of topologies (with a dimension) on the sets D^n for a fixed D. Generalizing the case of algebraic geometry the closed sets should be given by conjunction of equations. The main result of Hrushovski & Zilber (1993) is that every Zariski geometry satisfying sufficiently strong semantic conditions can in fact be realized as a finite cover of an algebraic curve.[31]

The classes of Zilber and Shelah are presented 'mathematically': by properties of the class of models that are not connected to truth of formal sentences. But Zilber's quasiminimal excellent classes are definable in $L_{\omega_1,\omega}(Q)$ (Kirby, 2010)[32] and this axiomatization was the explicit goal of the project.

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[29] Experts may correctly point out that the immediate impetus was to find a correct version of Zilber's conjecture that all strongly minimal sets were 'set-like', 'group-like', or 'field-like'. That is true but does not affect the explanation here.

[30] A collection of sets S_n is defined in each D^n which is closed under Boolean operations and the projection of a set in S_{n+1} to D^n is in S_n. In contrast to Footnote 21, the topology is not definable; the topology discriminates between equations and their negations.

[31] A finite cover is a quotient where the inverse image of each point is finite. Here is the precise theorem.

THEOREM 1.6 (Hrushovski–Zilber). *If M is an ample Noetherian Zariski structure then there is an algebraically closed field K, a quasi-projective algebraic curve $C_M = C_M(K)$ and a surjective map*

$$p : M \mapsto C_M$$

of finite degree such that for every closed $S \subseteq M^n$, the image $p(S)$ is Zariski closed in C_M^n (in the sense of algebraic geometry); if $\hat{S} \subseteq C_M^n$ is Zariski closed, then $p^{-1}(\hat{S})$ is a closed subset of M^n (in the sense of the Zariski structure M).

[32] Kirby's formulation makes the clear distinction between the 'mathematical' and 'logical' descriptions; Zilber blurs the distinction in the original paper (Zilber, 2005).

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To clarify the distinctions between 'formalism-free' and logical treatments, we provide some more examples of results in the study of AEC. Here is an example of a 'purely semantical' theorem. WGCH abbreviates the assertion, for all λ, $2^{\lambda} < 2^{\lambda^+}$.

THEOREM 1.7. *[WGCH] Let* $\boldsymbol{K}$ *be an abstract elementary class (AEC). Suppose* $\lambda \geq \mathrm{LS}(\boldsymbol{K})$ *and* $\boldsymbol{K}$ *is* λ*-categorical. If amalgamation fails in* λ *there are* 2^{λ^+} *models in* $\boldsymbol{K}$ *of cardinality* $\kappa = \lambda^+$.

Shelah's Theorem 1.7 is proved (e.g., Chapter 17 of Baldwin, 2009) in a 'formalism-free' manner; there is no mention of syntax.

Shelah's celebrated 'presentation theorem' (Shelah, 1983; Baldwin, 2009) changes the role of logic from a motivation (AEC are supposed to abstract the properties of classes defined in various infinitary logics.) to a tool. The presentation theorem asserts that an AEC with arbitrarily large models can be defined as the reducts of models of a first-order theory which omit a family of types. But Morley (1965b) had calculated the Hanf number for such syntactically defined classes. Thus, using the syntactical presentation as a tool, Shelah obtains a purely semantic theorem: if an AEC with Löwenheim number $\aleph_0$ has a model of cardinality $\beth_{\omega_1}$ it has arbitrarily large models.

When spelled out, the syntactic condition in the presentation theorem is a set of sentences in roughly Tarski's sense. We have discussed several modern examples of Tarski's consideration of a duality between description in a formal language and mathematical description. In the case of Kirby's axiomatization of quasiminimal excellent classes and Shelah's presentation theorem we gain a firmer grasp on a certain class of models by seeing a specific logic in which they are definable. The Hrushovski–Zilber analysis provides a syntactical analysis of the relations on a model, provided these relations satisfy a set of semantic conditions. In addition, via the presentation theorem, we are able to deduce purely semantical conclusions passing through the syntactic representation. Notably, the vocabulary arising in the presentation theorem arises naturally only as a tool to prove that theorem. There is no apparent connection of each symbol of the resulting vocabulary with any basic mathematical properties of the abstract elementary class in question.

§2. Formalization and purity in geometry. In this section, we use our earlier analysis of vocabulary as a tool for considering the question of whether/how formalization can be used to capture the notion of 'purity of method'. Our attention was drawn to this topic by the papers of Hallett (2008) and Arana & Mancosu (2012). Our first goal here was to use our perspective of the role of vocabulary to set out the mathematical issues involved in the study of Desargues theorem in a somewhat different way and to make some comment on how this affects purity concerns. This analysis is based rather directly on a first-order axiomatization of geometry. Consideration in terms of Manders (2008) 'diagram proofs' would raise other pertinent issues.

However, careful analysis (Section 2.4) of Hilbert's embedding theorem shows that even explicit definition can introduce impurities. This emphasizes the impotence of purely formal methods (as logical and strong logical purity discussed in Section 2.3) to characterize purity. We see that to diagnose the impurity of Hilbert's embedding theorem one must consider the content of what is in fact a 'formalism-free' proof in the sense of Section §1. The Appendix provides what I argue is a pure proof.)

We discuss the relation between the choice of vocabulary and basic axioms and the 'content' of a subject in Section 2.1. In Section 2.2 we lay out some of the mathematical

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background in terms of formal axiomatic systems to clarify the relation between the Desargues proposition in affine and projective geometry. Section 2.3 analyzes several attempts to define the notion of purity and concludes that formalization can highlight impure methods but does not give a criteria for 'purity'. In Section 2.4, we come to grips with the essence of Hilbert's work on Desargues' theorem and assert the impurity of his argument that the three-dimensional proof is impure.

The discussion of purity below will make clear that the context of a proposition is crucial for the issues we are discussing. But the notions of what geometry actually is have changed radically over the centuries. The distinction between the coordinate geometry of Descartes and synthetic geometry is crucial for our purposes. As Arana & Mancosu (2012) makes clear the study of geometry in the late nineteenth century involved multiple dimensions, surfaces of various curvature, coordinate and coordinate-free approaches. These distinctions remain today although perhaps fields of mathematics are laid out differently: algebraic geometry, real algebraic geometry, differential geometry, topology, etc. Hilbert's work on Desargues theorem led directly to one subfield of geometry; the coordinatization theorems which are a main focus of this paper led to the study of projective planes over usually finite fields as a distinct area of mathematics. In such major sources for this area as (Artin, 1957; Hughes & Piper, 1973; Dembowski, 1977) the proof of the Desargues theorem in 3-D geometry is not central, if it appears at all. Rather, the importance of Desargues theorem is an indicator of the algebraic properties of the coordinatizing field. The latter two books connect the projective geometries with combinatorics. There are important links with the statistical field of experimental design.

2.1. Content and vocabulary. Given some area of mathematics, for example, plane geometry, a formalization of the area has three essential components: the choice of primitive notions (vocabulary), the choice of logic,[33] and the choice of basic assumptions (axioms). In this section, we restrict to first-order logic.

The discussion of purity in geometry frequently (Arana & Mancosu, 2012; Hallett, 2008) draws on the following remark of Hilbert in his Lectures on Geometry of 1898/99 (page 316 of Hilbert, 2004, translation from Arana & Mancosu, 2012).

> This theorem gives us an opportunity now to discuss an important issue. *The content of Desargues' theorem belongs completely to planar geometry; for its proof we needed to use space.* Therefore we are for the first time in a position to put into practice *a critique of means of proof*. In modern mathematics such criticism is raised very often, where the aim is to preserve *the purity of method*, i.e. to prove the theorem using means that are suggested by the content of the theorem.

We stress that the task is to 'critique a means of proof'. We are given both conclusion and some premises. Both the premises and the intermediary steps are at issue. And the issue is 'method'. It is not the mere existence of a valid implication; there can be pure and impure proofs of the same implication.

A key issue is whether concepts are introduced in the proof that are not inherent 'in the statement of the problem'. But 'in the statement of the problem' is incomplete without

[33] The choice of logic includes three of the components we discussed in Section 1: formulas, truth, proof.

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some specification of the context. In Hilbert's approach such a specification is made by asserting axioms for the geometry which 'implicitly define' the primitive concepts. We follow that line here.

We somewhat refine below the 'standard account of definability' from Suppes (1956) which argues that a proper definition must satisfy the eliminability and noncreative criteria.[34] As pointed out in Shoenfield (1967), an extension by explicit definitions is conservative over the base theory, that is, noncreative. We will justify below the need to allow adding to the vocabulary of a given formalization (even in pure proofs) in order to prove a result. The next examples show that axioms for additional relations can change content; we provide directly relevant examples in Section 2.3. The crux is the word 'content'. What is the content of a proposition? Surely, the content changes if the models in the base vocabulary in which the proposition holds change when additional axioms are added.

EXAMPLE 2.1. *1. A vocabulary to study equivalence relations contains only* $=$ *and a binary relation symbol* E_1*. There we can assert by a theory* T_1 *that* E_1 *is an equivalence relation. But suppose the vocabulary is expanded by a binary relation symbol* E_2 *and* T_1 *to a theory* T_2 *asserting that* E_2 *is also an equivalence relation and each* E_2 *class intersects each* E_1 *class in a single element. Now all the reducts of models of the full theory are equivalence relations in which all equivalence classes have the same size.*

Example 2.1.1 makes the point very clearly. Surely the first theory tells us exactly what an equivalence relation is. And the concept of equivalence relation entails nothing about the relative size of the equivalence classes. Specifically, the sentence:

$$[(\exists x)(\forall y)(xE_1y \rightarrow x = y)] \rightarrow [(\forall x)(\forall y)(xE_1y \rightarrow x = y)]$$

is a consequence of the added information about E_2*.*

2. *Consider the proposition* $x \cdot x \cdot x = x$ *in the class of Abelian groups formulated in a vocabulary with a binary operation* $\cdot$ *and a constant symbol* 1*. Now expand the vocabulary to a vocabulary for fields by adding new symbols* $+$ *and* 0*; add the axioms for fields and the axiom* $x + x + x = 0$*.*

   The meaning of the expression $x \cdot x \cdot x = x$ *is very different in an arbitrary Abelian group (where there may be elements of arbitrary order) than it is in a field of characteristic 3 (where it is a law of the multiplicative group).*

3. *Consider the class of linear orders in a vocabulary with a single binary relation symbol* $<$*. Add an operation symbol* $+$ *and constant* 0 *and assert the structure is an ordered Abelian group. The original class contained* $2^{\aleph_0}$ *countable models; but only* $\aleph_1$ *can be expanded to a group. See page 207 of Hodges (1987).*

Of course, our assertion that content changes if the class of models changes is based on the Hilbertian notion that the primitive concepts are implicitly defined by the axioms. One might think of content in a more Fregean way; the axioms are describing geometry. The danger of such a position is falling victim to Pierce's paradox. If we are speaking of a fixed geometry, the various properties are just true; one cannot be meaningfully said to strictly imply the other without access to countermodels. I don't see how one can back off from

[34] Suppes attribution of these criteria to Leśnewieski is vigorously contested in Urbanik & Hämäri (2012).

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describing a geometry to describing a family of geometries without embracing Hilbert.[35] For Hilbert, the axioms are not arbitrary; they are developed to clarify intuition. But the geometries we are able to study formally are whatever happens to satisfy the axioms.

The sense in which the Arana–Mancosu attempt to found 'purity' on the 'intuitive content' of geometry falls victim to Pierce's paradox. They argue in Section 4.5 of Arana & Mancosu(2012) 'against construing purity in terms of formal content'. In fact we agree that 'purity' concerns cannot be fully resolved in terms of formal content. But we object to portions of their justification. On page 237, they write 'Formal content entails, however, a radical contextualism regarding the content of statements like Desargues theorem'. We argue at several points in this paper for the necessity of describing a context rather than a particular statement that has radically different meanings in different contexts.[36] But our complaint here is. Arana–Mancosu reject the Hilbertian view that represents the first horn of the Pierce paradox. But now compare the theorem of Pappus with the theorem of Desargues. They are both true in 'intuitive geometry'. And Pappus implies Desargues by strictly geometric arguments.[37] The two propositions are not equivalent but we can only see this by adapting Hilbert's view of axioms admitting multiple interpretations.

2.2. Projective and affine geometry. Before turning to our specific analysis of the purity of proof of the Desargues proposition we set some notation and clear away some extraneous matters. In this paper we discuss first-order axiomatizations of geometry. As in Hilbert (1971), we are trying to formalize the 'field of geometry (in the traditional sense)'. We make precise the distinction and connections between affine and projective geometry to clarify that while the two situations are distinct they behave the same with respect to purity of the Desargues proposition.

In this subsection we formulate some formal systems for geometry.We have slightly modified the statement of the axioms for projective planes from Heyting (1963) and for affine geometry from Hilbert (1971). A crucial distinction is that the axioms given here for a projective plane actually imply the structure is planar; any two lines intersect.

DEFINITION 2.2. *A projective geometry is a structure for a vocabulary with one binary relation R. We interpret the first coordinate to range over points and the second to range over lines. The axioms for a projective* plane *assert:*

1. *Any two lines intersect in a unique point.*
2. *Dually, there is a unique line through two given points.*
3. *There are four points with no three lying on a line.*

These axioms are far from complete; analogous axioms for an affine plane assert:

DEFINITION 2.3. 1. *There is a unique line through two given points.*
2. *There are four points with no three lying on a line.*

---

35 Smith (2010) ascribes priority for this viewpoint to Peano and especially Pieri.

36 Most glaringly, the Desargues proposition is contingent is two-space and true in 3-space.

37 This proof is a common exercise in undergraduate projective geometry courses. For example, in Vis (Unpublished notes) students are asked to draw and make connections between the diagrams for Pappus and Desargues. When writing the Grundlagen, Hilbert did not know this implication. It was proved by Hessenberg (1905). This raises the question of why Hilbert's concern is with the purity of Desargues rather than Pappus. Of course Hilbert's argument equally well shows that any Pappian plane is embeddable in 3-space. But the converse fails; not every subplane of 3-space is Pappian.

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We could extend Hilbert by passing from the informal axiomatization of the Grundlagen to fully formalized axiomatizations based on the conventions for first-order logic not fully established until some 30 to 40 years after his work on geometry. While we don't write out the axioms symbolically, the translation is clear. We could introduce predicates for points and lines or we could simply insist that the domain and range of the relation R do not intersect (a 0th axiom) and then define them to be respectively the set of points and the set of lines. Note that the axioms for spatial geometry in Heyting (1963) and Hilbert (1971) explicitly introduce a new primitive term: plane.

Following Henderson & Taimina (2005) by the 'high school parallel postulate' we mean the assertion: for any line ℓ and any point p not on the line, there *is a unique line* ℓ' through p and parallel to ℓ[38].

REMARK 2.4. *There is an easy translation between projective and affine geometry.*

Given a projective plane $\mathcal{P} = (\Pi, R)$, eliminate one line, ℓ, and all points that lie on it. Now two lines ℓ_1, ℓ_2 whose intersection point was on ℓ are parallel. It is easy to see that the affine plane satisfies the parallel postulate.

Similarly, suppose $\mathcal{A} = (\Pi, R)$ is an affine plane satisfying the high school parallel postulate. Add a new line ℓ_∞ and let all members of an equivalence class of parallel lines in Π intersect at a point on ℓ_∞; let these be the only points on ℓ_∞.

DEFINITION 2.5.
1. *The affine Desargues proposition asserts: if ABC and $A'B'C'$ are triangles with $AC \parallel A'C'$, $AB \parallel A'B'$ and $BC \parallel B'C'$ then AA', BB' and CC' are parallel or the three intersect in a single point.*
2. *The projective Desargues proposition asserts: if ABC and $A'B'C'$ are triangles such that the points of intersection of AC with $A'C'$, AB with $A'B'$ and BC with $B'C'$ are collinear then AA', BB' and CC' intersect in a point p.*

It is easy to check:

CLAIM 2.6. *Under correspondence in Remark 2.4, the affine plane satisfies affine Desargues if and only if the projective plane satisfies projective Desargues.*

Hilbert introduced a ternary betweenness relation to fill what he regarded as gaps in Euclid; this extension is essentially irrelevant to our discussion here. While betweenness is appropriate for affine geometry, to consider both affine and projective geometry requires the more general quaternary separation predicate introduced by Pasch (See, e.g., Arana & Mancosu, 2012.) representing that the four points are cyclically ordered. Unlike betweenness this relation is projectively invariant. For the axioms of this relation see, for example, Heyting (1963). Notably, neither betweenness nor cyclic order appears in Heyting (1963) until after the discussion of Desargues and coordinatization.

2.3. General schemes for characterizing purity. In this subsection, we discuss several suggestions for more clearly specifying the notion of a 'pure' proof, consider how they

[38] Euclid proves the existence of a parallel line on the basis of his first four axioms; in this context, the fifth postulate asserts uniqueness. In the context of projective plane geometry existence fails. See Henderson & Taimina (2005) for an amusing and informative account of professional confusion over the difference between existence and uniqueness of parallel lines and the actual statement of Euclid's fifth postulate.

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evaluate the purity of certain arguments, and then draw some conclusions about these specifications.

Detlefsen and Arana distinguish a notion of *topical purity*. Rephrasing their discussion in Section 3.5 of Detlefsen & Arana (2011), there are certain resources which *determine* a problem (for a given investigator). In mathematics, the determinants include, definitions, axioms concerning primitive terms, inferences. These are referred to as the 'commitments of the problem' and specify what we call the context of the problem. The topic of a problem is a set of commitments. 'A purity constraint restricts the resources available to solve a problem to those which determine it'. They then analyze the topical purity of a solution in terms of its stability under changes in the commitments. We want here to connect topical purity with several related notions considered by Arana (2008a, 2008b). Our general conclusion (as Arana's) is that it is not possible to translate the problem of purity into a proposition about formal systems in the most traditional sense; it is essential to retain a notion of 'meaning' in the discussion. Our argument can be seen as a model-theoretic analogue of the proof-theoretic discussion in Arana (2008b). Arana introduces a notion of logical purity in Arana (2008a).

DEFINITION 2.7 (Logical Purity A). *1. The axiom set S is logically minimal for P if $S \vdash P$ but there is no proper subset of S proves P.*

2. The proof of P is pure if it is a proof from an S which is logically minimal for P.

He points out that there are some obvious difficulties with this definition, since we could conjoin a set of axioms and get something that is logically minimal. Here is a more robust formulation.

DEFINITION 2.8 (Logical Purity B). *1. The axiom set S is fully logically minimal for P if $S \vdash P$ and there is no S' such that $S \vdash S'$, $S' \vdash P$ and $S' \nvdash S$.*

2. The proof of P is pure if it is a proof from an S which is fully logically minimal for P.

The difficulty with the second formulation is that it turns out to be an even stronger version of the following notion of Arana (2008a).

DEFINITION 2.9 (Strong logical purity). *The proof of P from S over T is strongly logically pure over some basis theory T if also $T \vdash P \rightarrow S$.*

As, S can only be minimal in the sense of Definition 2.8 if S is logically equivalent to P; otherwise choose P as the S' to show S is not minimal. Thus the existence of logically pure proof of P from S in the sense of Definition 2.8 requires that S and P are logically equivalent.

Strong logical purity has a long history including Sierpinski's equivalents of the Continuum hypothesis in the 20's, Rubin and Rubin's 101 equivalents of the axiom of choice and Friedman's reverse mathematics. Pambuccian (2005) pursues a similar 'reverse geometry', finding a minimal weak axiom system for four results in Euclidean geometry. These are searches for the weakest hypotheses in terms of proof-theoretic strength. They are *not* what Hilbert or Hallett claims for the Desargues property. And as Arana rightly points out 'reverse mathematics' is not an issue of purity. Each of these notions of logical purity, which I have just described, is about equivalence of statements not about proofs; none of them address the key issue: *'Which, if any, proof of a theorem is pure?'*. Nevertheless, these *formal* notions are sometimes capable of detecting the nonexistence of pure proofs.

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PROPOSITION 2.10. *No proof of Desargues proposition from the assumption of three dimensions is strongly logical pure.*

If there were such a proof, every geometry satisfying the Desargues proposition would actually be three-dimensional. This is clearly false; we investigate the subtly different consequence (embeddability) of the Desargues proposition in Section 2.4.

The notion of topical purity builds on an earlier formulation of Arana, 'a proof . . . which draws only on what must be understood or accepted in order to understand that theorem' (Page 38 of Arana, 2008a). Two issues arise. What does it mean to draw on? How can one determine 'must be understood or accepted to understand'. We follow Arana in leaving the second question to individual cases. But there is a more uniform way to understand 'draw on'. We say that a 'concept' is drawn on in a proof when it is given a name. We are going to discuss arguments below which could be formalized as derivations in a basic language. We first observe that introducing relations that are *not* definable in the base language is either a definite sign of impurity or an admission that the formalization of the hypothesis omitted a necessary concept. In the latter case there might be pure argument although the one originally given was not. On the one hand, we argued in Section 2.1 that adding both additional relations and axioms about them often distorts meaning. More strongly if nondefinable relations are added, even with no additional axioms, any claim that this is a pure move, is simply a claim that the original formalization left out crucial concepts. An example shows how this issue focuses on the importance of specifying what is being formalized. I take Presburger arithmetic as an attempt to formalize *addition* on the natural numbers. If one moves from the schema of induction to introducing multiplication by a recursive definition, the topic has vastly changed. $(N, +)$ is decidable; $(N, +, \times)$ is not.

But we argue more strongly that an explicit definition may violate purity concerns. We will discuss what we claim are 'pure' and 'impure' proofs, both invoking explicit definition, of the same fundamental result, explaining the reasons for this diagnosis. And then we argue for the value of each of these proofs.

We seek now a more mathematical formulation of the topical purity introduced in Detlefsen & Arana (2011).[39] Although in earlier sections we avoided the word language because of its multiple meanings in similar situations, we now introduce a specific meaning clarifying one of the three discussed in Section 1.3.

DEFINITION 2.11. *The language for a mathematical topic A is a vocabulary for A with symbols for each of the primitive notions identified by the investigator and axioms for the relations which are sufficient to delimit these concepts.*

Some examples are the language $(+, \cdot, 0, 1)$ for rings (where the ring axioms are specified) and the language $(\vee, \wedge, 0, 1)$ for bounded lattices (with appropriate axioms). The vocabularies differ only in notation; this difference is meaningless without the description of the properties.

Arana and Detlefsen introduced topical purity as concept about the solution of a problem. In accordance with the last couple of pages, we require the context in which this problem arose must be formalized. There is an issue of whether the following definition

[39] Arana suggested the specification was close enough to topicality to not deserve a new name.

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should be construed in the Euclid–Hilbert or Hilbert–Gödel–Tarski frameworks that we laid out in Section §1. For most of the argument it makes no difference. However, Lemma 2.19 and its application in Theorem 2.17 are mathematical theorems invoking the completeness theorem and so require full formalization. Thus officially we see the definition from the second standpoint. I think Arana and Detlefsen saw it more from the Euclid–Hilbert perspective.

DEFINITION 2.12 (Topical purity). *Choose a first-order formalization for the resources which determine the problem in the sense of the first paragraph of this section (Detlefsen–Arana). That is, specify a language including a set of primitive concepts and axioms needed to describe the particular problem and its context A.*

More formally, fix a vocabulary τ and a theory T_0 that implicitly defines the concepts named by the symbols τ. Now a topically pure *proof of ϕ from ψ where ϕ and ψ are τ-sentence is a proof of ψ from ϕ in T_0 that* invokes *only concepts from the context A.*

We will interpret 'invoke' in Definition 2.12 as 'introduce by explicit definition'.

The particular first-order formalization is a real choice. In Pambuccian (2001, 2009) Pambuccian provides examples of distinct interpretations of the basic notions of a proposition which lead to distinct, indeed incompatible, systems; each system can be thought to provide a pure proof for the understanding of the concepts that has been formalized. Pambuccian (2009) studies the Sylvester–Gallai theorem: If the points of a finite set are not all on one line, then there is a line through exactly two of the points. One might conceive of a line in terms of betweenness or as 'the shortest distance between two points'. These provide different contexts; Pambuccian explains three distinct proofs, one using the first concept and two the second. These are based on incompatible axioms systems. He remarks that still another proof holds for planes satisfying a certain Artin–Schreier condition.

A natural question is whether this notion is different from the notion of logical purity described above. To show it differs from strong logical purity, we need only exhibit a proposition which has both topically pure and impure proofs from the same base theory. We will note this in Corollary 2.22.

It is tempting to insist that the analysis of the context and conclusion should elicit all relevant concepts and thus the set of concepts used in the proof should be fixed in the choice of language. Sobocińnski (1955) discusses this criteria on a formal system, primarily in the context of propositional logics. Givant & Tarski (1999) argue that including defined concepts in an axiom system leads to a misleading appearance of simplicity of the axioms. They discuss simplicity in terms of both the 'length' of the axiom system and its complexity in terms of the number of quantifier alternations. While the second has important structural consequences, as we discussed in Section 1.2, either measure of simplicity appears to irrelevant to the notions of purity considered here. Both relate to technicalities of the formalization.

For purity purposes, the 'resources which determine the problem' in the definition of topical purity must seen as a limitation—'which determine the problem and no more'. But this condition is too strong. If it were accepted, the formalizations of Heyting and Hilbert (Hilbert, 1971; Heyting, 1963) would be impure for studying Desargues theorem because they take 'plane' as a primitive concept. And Desargues's theorem is about lines, points, and incidence. But according to the definition of plane in Definition 4.16, these points and lines all lie in the same plane. So 'plane' should be viewed as part of the context and it is not important for purity considerations whether it is taken as a primitive or introduced

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by explicit definition. This is a judgement about this specific case.[40] Such examples are another reason for demanding that base language should reflect both the particular problem and the context.

We reject the criteria of demanding all relevant concepts be fixed among the primitive terms for a number of reasons. It is contrary to the goal of finding 'basic' vocabulary. It is not true to mathematical practice. Mathematical proofs are not carried out as derivations in a fixed formal language. In particular, new concepts are introduced by definition for the purpose of particular proofs. We explore examples of this type of extension in detail in Section 2.4. But a simple example is to consider a formulation of projective geometry (Robinson, 1959) (see the Appendix) that contains only points, lines, and incidence as primitive terms. In order to carry out the proof that Desargues theorem holds in three-dimensional space, one needs the notion of plane. And it is straightforward (Definition 4.16) to introduce the notion of plane as an explicit definition in this system.

If these new definitions are mere abbreviations it seems they should be harmless. Certainly if axioms are added about the new relations, this is no longer harmless (See Example 2.1). In fact, we will argue that, even without additional axioms, explicit definitions can violate purity. That is why we add the requirement that the new definitions remain within context of the original topic. We further illustrate the meaning of this phrase in Section 2.4.

The first use of our characterization of topical purity is to determine cases where there is no topically pure proof of a proposition. Here is an example. We generalize Proposition 2.10 from 'strongly logically pure' to the much broader notion of topical purity.

PROPOSITION 2.13. *There is no topically pure proof of the Desargues proposition in the plane (from PP).*

A proof of this proposition requires three steps. Choose a formalization; then argue that this formalization is appropriate for the context and that there is no topically pure proof in that formalization.

Clearly the Desargues proposition is stated in terms of points, lines, and incidence. And the basic properties of points lines and incidence in a projective plane are given by PP (Definition 2.2), so this is the appropriate formalization. Thus for the projective plane the existence of a topically pure proof would entail that every model of the axioms in Definition 2.2 is a Desarguesian projective plane. Many counterexamples to this assertion have been exhibited in the last 100-odd years. The result extends to the affine plane by Claim 2.6.

This claim is controversial. It is well-known that there are many distinct completions of PP, so it does not exhaust our intuitions about the relations of lines in the projective plane. But it does set the context in the sense of purity issues. We want to argue that certain sufficient conditions clearly add information that is not in the context. It has been argued.[41]

REMARK 2.14. *Counterclaim: Affine Desargues can be proved from planar axioms so there is a pure proof in the plane*

[40] As another justification of considering 'plane' to be part of the context, we could simply ask from the standpoint of three-dimensional geometry whether the Desarguesian proposition can be proved using only resources from the plane.

[41] See Sections 4.1 and 4.7 of Arana & Mancosu (2012)

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To evaluate this claim, we first clarify the mathematical situation. Hilbert (1971) proves two mathematical results:

FACT 2.15. *1. In three-dimensional (affine or projective) geometry, Desargues theorem holds. This depends only on the incidence and order*[42] *axioms.*

2. In two dimensions, the affine Desarguesian theorem can be proved from the incidence axioms, the parallel axiom, and the congruence axiom.

We showed in Example 2.1 that adding additional relations and structure can change the interpretations of the basic structure. Here, we note that the problems arise in the specific geometric context. It is true that affine Desargues can be proved from the parallel postulate and congruence axioms (basically side-angle-side) and these are surely planar concepts. But while parallelism is necessary to understand affine Desargues, congruence is not. The proof of Fact 2.15.2 requires extensions of the basic geometric axioms in two distinct ways. First there are additional axioms in the same vocabulary, the parallel postulate. But secondly a new concept of congruence must be introduced; in fact several of them. A priori, one needs relations for segment congruence (4-ary), triangle congruence (6-ary), and angle congruence (perhaps formulated as a four-ary predicate on lines). And a congruence axiom such as SAS must also be posited. The fact that additional axioms are introduced is immediate evidence of impurity. In fact congruence is definitely foreign to the situation as the theorem of Desargues holds in an affine plane over any algebraically closed field. There is no notion of congruence definable in the geometry over such fields (consider the Riemann mapping theorem). To define congruence one must introduce further relations; for example, regard the complexes as a two-dimensional real vector space). Thus, in the spirit of Detlefsen & Arana (2011), there can be no topically pure proof of the Desargues theorem in the plane, even for affine geometry.

The basic point here is that two distinct geometrical contexts are being considered. Metric geometry (or in the Euclidian formulation, geometry with parallels and congruence) is a different subject than projective geometry which encodes only the properties of lines and incidence. But in fact there is no pure proof in the context of affine metric geometry, because the congruence axioms require 'flatness'; Desargues theorem fails in various non-Euclidean geometries. This illustrates an important attribute of the search for pure proofs. It forces the clarification of hypotheses.

The next section explains that although the definition of topical purity allows the introduction of new terms by definition, this introduction is restrained by the informal context A. We will see that without this restriction proofs using manifestly impure notions would meet the requirement for topical purity.

2.4. Purity and the Desargues proposition. As reported by Hallett in (page 227 of Hallett, 2008), Hilbert argues that although the content of the Desargues proposition is manifestly two-dimensional; three-dimensional methods are necessary for its proof. We explore the role of vocabulary versus axioms in understanding this claim. We first want to formalize Hilbert's results on the strength of planar Desargues. Thus the following notation is introduced in the parlance of contemporary model theory. We will further explain the motivation for the introduction of Σ after stating Theorem 2.17.

[42] In fact, the order axioms are a red herring. They are used only to guarantee that the coordinatizing field is ordered. See Bernays Supplement IV in Hilbert (1971).

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NOTATION 2.16. *Fix a vocabulary of points, lines, and the incidence relation.*

1. *PG is the theory of projective geometry*[43] *(asserting the existence of at least three dimensions) and PP is the theory of projective planes as in Definition 2.2.*[44]
2. *Let* Σ *be the collection of sentences* σ *that are satisfied in some projective plane such that* $PG \vdash \sigma$. *That is,* $\sigma \in \Sigma$ *just if* σ *is true in at least one projective plane and in every projective geometry of dimension at least 3.*

Hilbert (page 227 of Hallett, 2008, or page 240 of Hilbert, 2004) conjectured and later proved two results which establish the pivotal role of the Desargues theorem from a geometrical standpoint.

> Is Desargues' Theorem also a sufficient condition for this? i.e. can a system of things (planes) be added in such a way that all Axioms I, II are satisfied, and the system before can be interpreted as a sub-system of the whole system? Then the Desargues Theorem would be the very condition which guarantees that the plane is distinguished in space, and we could say that everything which is provable in space is already provable in the plane from Desargues.

Using Notation 2.16, we formulate the two assertions of this quote in modern terms; the first is 'formalism-free', the second is formal.

THEOREM 2.17 (Hilbert). *1. If* Π *is a Desarguesian projective plane,* Π *can be embedded in three-space.*

2. *If* $\sigma \in \Sigma$ *then* (PP + Desargues) $\vdash \sigma$.

We place the situation in a more general framework. For a formula $\theta(\mathbf{y}, x)$ a tuple $\boldsymbol{a}$ and a sentence ψ, $\psi^{\theta(\boldsymbol{a},x)}$ denotes the relativization of ψ to $\theta(\boldsymbol{a}, x)$. The Hilbert quote above is somewhat ambiguous. What does 'everything' in 'everything which is provable in space' mean? It must mean 'everything about planes'; there is no intent to assert that an arbitrary statement about 3-space (e.g., properties of spherical geometry) is provable from the planar axioms and Desargues. We introduced the set of sentences Σ in Definition 2.16 to formalize this observation. Letting $\theta(\mathbf{y}, x)$ define the plane[45] generated by $\mathbf{y}$, we rephrase the key sentence in formal terms as, ' If $T_2 \models (\forall \mathbf{y})\sigma^{\theta(\mathbf{y},x)}$ then $T_1 \models \sigma$.'

DEFINITION 2.18. *[Interpretation] Let* T_1 *and* T_2 *be two theories in the same vocabulary. We say* T_2 *is* interpretable *in* T_1 *if there is a formula* $\theta(\mathbf{y}, x)$ *such that*

1. *for every* $\psi \in T_2$, $T_1 \models \forall \mathbf{y}\, \psi^{\theta(\mathbf{y},x)}$;
2. *if* $M \models T_2$ *and* $M' \models T_1$ *and* $M \subseteq M'$, *there is an* $\boldsymbol{a} \in M'$ *such that* $M = \theta(M', \boldsymbol{a})$.

---

[43] To be precise, we take the formalization in Robinson (1959) since it has the same vocabulary: points, lines, and incidence for two and multidimensional geometry. The notion of plane is introduced by explicit definition as in Definition 4.16.

[44] Note that the union of the theories PP and PG is inconsistent.

[45] Hilbert's informal use of plane in Hilbert (1971) reinforces our concern about vocabulary. Axiom II,5 refers to a line a lying in the plane ABC. Theorem 5 refers to a line a lying in a plane α (without parameters). Axiom III refers to both points and lines lying in a plane α. This can all be naturally formalized by a 4-ary predicate $P(\mathbf{y}, x)$ which holds if the point x lies on the plane generated by $\mathbf{y} = \langle y_1, y_2, y_3 \rangle$, where for Hilbert this would be an implicit definition of 'generated'. We give an explicit definition in Section 4.

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In our current situation, take T_2 as $PP + Desargues$, T_1 as PG, $\theta(\mathbf{y}, x)$ as the formula: x is on the plane generated by $\mathbf{y} = \langle y_1, y_2, y_3 \rangle$. Note the projective plane M is merely a substructure not an elementary submodel of M'; the structure M satisfies that every two lines intersect. When $\theta(\boldsymbol{a}, x)$ defines M, $M \models \sigma$ if and only if $M' \models \sigma^{\theta(\boldsymbol{a},x)}$.

LEMMA 2.19. *Suppose T_2 is interpretable in T_1. Suppose further that if $M \models T_2$ there is an $M' \models T_1$ with $M \subseteq M'$. If $T_2 \models (\forall \mathbf{y})\sigma^{\theta(\mathbf{y},x)}$ then $T_1 \models \sigma$.*

Proof. Fix $\sigma \in \Sigma$. Let $M \models T_2$, then M extends to a model M' of T_1. By Condition 2) of Definition 2.18, there is an $\boldsymbol{a} \in M'$ such that $M = \theta(M', \boldsymbol{a})$. By Condition 1), for each such $\boldsymbol{a}$, $M' \models \sigma^{\theta(\boldsymbol{a},x)}$; in particular, $M \models \sigma$. Since we have shown every model of T_1 is a model of σ, by the extended completeness theorem $T_1 \vdash \sigma$. □

Proof of 2.17: We discuss 1) at length and prove it (informally) in the Appendix.[46] For part 2, let $\theta(\mathbf{y}, x)$ assert that x is on the plane[47] generated by y_1, y_2, y_3. To apply Lemma 2.19, take T_2 as $PP + Desargues$ and T_1 as PG. Condition 1) of Definition 2.18 holds since every subplane of a three-dimensional space is Desarguesian. Clearly T_2 is interpretable in T_1. Condition 2) holds since planes are definable. Theorem 2.17.1 gives us the second hypothesis of Lemma 2.19. Note that $\sigma \in \Sigma$ (Definition 2.16) implies $PG \vdash \sigma$ and thus $PG \vdash (\forall \mathbf{y})\sigma^{\theta(\mathbf{y},x)}$. By Lemma 2.19, $(PP + Desargues) \vdash \Sigma$. □

Thus if θ is a sentence about projective planes that we show (perhaps in a formalism-free way) to be true in every plane that can be embedded in three-space, then θ can be formally derived from the PP plus the Desargesuian property. The Pappus theorem is an example of a statement concerning projective planes, which is *false* in some planes that can be embedded in three-space.

Hilbert's analysis of the quality of a proof extends beyond topical purity. He wrote, (in unpublished notes of Hilbert that are quoted in Hallett, 2008).

> Nevertheless, drawing on differently constituted means has frequently a *deeper and justified* ground, and this has uncovered beautiful and *fruitful relations*; e.g. the prime number problem and the $\zeta(x)$ function, potential theory and analytic functions, etc. In any case one should never leave such an occurrence of the mutual interaction of different domains unattended.

The role of 'spatial assumptions' is better seen by a more careful examination of Hilbert's proof of Fact 2.15 and Theorem 2.17. He begins (Hilbert, 1971) by noting that the three-dimensional proof of Desargues' theorem (Fact 2.15.1) from the axioms of connection, order, and parallels is well-known. The structure of his proof of each of Fact 2.15.2 and Theorem 2.17 follows that pointed out for the proof of embeddability from Desargues on page 228 of Hallett (2008). A ternary field is a structure with a single ternary operation; roughly $t(a, x, b)$ corresponds to $ax + b$, which satisfies a set of axioms as specified in Dembowski (1977) and Hughes & Piper (1973). But for this correspondence to be literally true the plane coordinatized by the ternary field must satisfy the Desargues property.[48]

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[46] Of course, Hilbert had already proved it; the point of the Appendix is to exhibit a pure proof.

[47] Thus θ formalizes Definition 4.16.

[48] In Baldwin (1994) I constructed a non-Desarguesian projective plane which is $\aleph_1$-categorical. In Baldwin (1995), I prove that despite its well-behaved nature from a model-theoretic

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REMARK 2.20. *The structure of the argument:*

1. Any *geometry can be coordinatized by a ternary field.*
2. *If the geometry satisfies*
   (a) *the Desargues proposition or*
   (b) *the parallel postulate and SAS (the congruence axiom in Hilbert's parlance)*

   then the coordinatizing ring is associative (and in fact a skew field[49]*).*
3. *An n-dimensional affine (projective) geometry can be constructed as a set of n $(n + 1)$-tuples from a Skew field and the plane can be embedded in the three-space.*

Hilbert's embedding theorem is an example[50] of what we call a *fully semantic proof*: the entire proof consists of operations on mathematical structures. Geometric statements like Desargues theorem or that two points determine a line or algebraic properties such as the associative law are treated as properties of mathematical objects. These objects might be geometric or algebraic. The proof proceeds from a plane to a 3-space by an explicit definition of binary functions that satisfy the axioms for a skew field. Now the original geometric properties are ignored and a three-space structure is defined on the product of two copies of the skew field.

This proof from Hilbert (1971) introduces a different set of purity concerns. In Hilbert's argument, a field is defined whose elements are equivalence classes of segments. These are not geometric notions and the objects are not in the model but are what model theorist now call 'imaginary elements as in Marker (2002)'.[51] This objection is somewhat reduced by Heyting's proof. Heyting still defines a field, but its elements are points of the given plane. Even if we have the fields as the points on a line, the construction of the three-dimensional model goes far afield from geometry. These new objects do not have 'geometric interpretations'. On the one hand, the objects of the field are viewed as numbers. In Hilbert's formulation, they are equivalence classes of segments (hardly a geometric notion). Even accepting the domain of the field (as points on a single line), the operation of multiplication while explained geometrically is hardly geometrical in a Greek sense. The modern geometry of homogenous quadruples is employed in the construction. This is essentially a metamathematical argument constructing a three-space out of whole cloth and embedding the original plane in it. This seems to be a really new method introduced by Hi1bert.[52] It is very different from Hilbert's geometric construction of counterexamples to Desargues or the geometric arguments for Theorem 4.1 given by Levi (1939) or in the Appendix. At the least it is a precursor of the modern notion of

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standpoint, this plane admits little 'algebraic' structure; in particular the ternary operation can not decomposed into two well-behaved binary operations and no group is interpretable in the structure. I also proved this projective plane is in the definable closure of any line (with no parameters). That is, the plane admits no perspectivities. The task of giving a geometric proof of this result remains open.

49 A skew field or division ring is a structure for the vocabulary $(+, \times, 0, 1)$ which satisfies all the axioms for a field except commutativity of multiplication.

50 Most mathematical proofs are of this sort. Note in particular Theorem 1.7.

51 This is a deliberate decision of Hilbert so as to study the geometry of segments. Already in his 1893–1894 lectures he had established a correspondence between the points on a line and numbers. See pages 68–69 of Hilbert (2004).

52 This was remarked by Hallett (2008).

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the interpretation of one theory in another. Moreover, Hilbert's proof of Desargues in an affine plane (Proposition 2.15.2) with congruence also goes through this metamathematical trick of embedding in three-space and deducing the result from the known proof of Desargues in 3-space. In fact, as pointed out in Arana & Mancosu (2012), Desargues gave a geometric proof of his theorem in the in three-dimensional affine case using the theorem of Menelaus.

Our use of 'metamathematical' in the last paragraph has two senses. Metaphorically, Hilbert is constructing a model and so this is a precursor of model theory. But he has also given a 'formalism-free' proof of Theorem 2.17.2. (That is, with the conclusion expressed as in the quotation before Theorem 4.1.) In the weakest sense this proof is formalism free since there is no formal proof system. But even more, the 'interpretation' makes no reference to the formal vocabulary or any notion of a formal language. But as we noted in proving Theorem 2.19, this formalism-free proof translates to the existence of a formal proof by the extended completeness theorem. (Of course, this translation was not available to Hilbert in 1900.) That is, in the proof of the formal Theorem 2.17.2, we invoked Hilbert's semantic proof of Theorem 2.17.1. Such a translation is a standard consequence of the extended completeness theorem and a routine model-theoretic tool. This does not mean that the semantical proof is tacitly formal; it just expresses the content of the completeness theorem.

Consideration of some of the standard texts in projective geometry of the last half century (Artin, 1957; Dembowski, 1977; Hughes & Piper, 1973) reveals an interesting phenomena. The proof of Desargues proposition is at best barely mentioned.[53] The crux is the understanding of the Desargues proposition in terms of the properties of the group of collineations and in terms of the properties of the coordinatizing ternary ring.

As we noted in Remark 2.20 and Proposition 2.10, the Desargues proposition does not imply there are non-coplanar points. Thus, it is not true that the Desargues proposition implies there is a third dimension. Rather, Hilbert showed, by a *fundamentally non-geometric construction*, one can embed the given plane in three-space. But there is a 'geometric' construction of this embedding, which we present in the Appendix. Thus we have an example where an impure proof provides very significant information. Indeed the very impurity of Hilbert's argument is crucial for the twentieth century development of the theory of plane projective geometry. In fact, this may be the mathematical impact of a proof of impurity. It focusses attention on the proposition in question as an axiom for selecting a new field of study. For example, the fact that there is no pure proof of the Desargues proposition in the plane calls attention to the importance of studying Desarguesian planes. The crucial property, as Hilbert saw, is not the geometric configuration itself but the associated algebraic structure; it was later codified in terms of transitivity properties of the automorphism group (Lenz–Barlotti classification). The new 'algebraic' concepts introduced by Hilbert are all introduced by explicit definition but they stray far from the geometric topic under consideration.

In contrast, I claim that the argument for Theorem 4.1 in the Appendix is topically pure. The crucial point is that Hilbert's argument introduces the notions of coordinatization and field which are foreign to synthetic geometry. In the Appendix, we reinterpret the words, point, line, and plane in terms of certain planar configurations to interpret a 3-space containing π in a Desarguesian plane π but don't introduce significantly new concepts. Since Hilbert's proof is impure, we conclude.

[53] Hartshorne (1967) is an exception.

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FACT 2.21. *The assertion that every Desarguesian plane is embedded in three-space has both topically pure and topically impure proofs from the axioms PP of projective planes.*

COROLLARY 2.22 *The notions of strong logical purity and topical logical purity differ.*

This difference is evident. Topical purity is a property of a particular argument. While, strong logical purity was defined in terms of the existence of a derivation (Definition 2.9) and does not depend on the particular argument advanced for the proposition.

The Desarguesian proposition is a dividing line in the sense of Shelah (e.g., introduction to Shelah, 2009). Its truth implies strong coordinatization properties; its failure implies planarity (in an axiom system that is agnostic on dimension). Specifically, the associativity of the coordinatizing field is used to prove that the relation of tuples from the field: $\mathbf{x} \sim \mathbf{y}$ if there is a 'number' c with $\mathbf{y} = c\mathbf{x}$ by coordinate-wise multiplication (used to introduce homogenous coordinates) is transitive and thus an equivalence relation. Thus we have:

FACT 2.23. *The following are equivalent: A projective plane is*

1. *coordinatized by an* associative *skew field.*
2. *satisfies the Desargues property;*
3. *can be embedded in three-space.*

There is another connection between spatial axioms and associativity. In three-dimensional Euclidean geometry the volume of a cube can be computed. Interpreting XI.32 of Euclid (1956) in modern language yields the formula $V = \ell wh$. (Euclid proves that the volume of a parallelpiped is determined by the area of the base and the height.) The fact that the geometric notion is independent of which side is chosen as the base of the parallelepiped implies the associative law for the coordinatizing field.[54]

Finding the associative field is, in modern terms, an interpretation (Definition 2.18) of the 'field' into the geometry. It proceeds by a sequence of explicit definitions. The proof of the algebraic axioms follows from the geometry. And then the plane is interpreted back into the 3-space over the field. Thus if there is any distinction between algebra and geometry the deduction of 3) from 2) via 1) fails to be a topically pure proof. But this conclusion cannot be established by a characterization of purity such as strong logical purity which concerns the mere existence of proof. In fact, since we know 3) implies 2), the proof that 2) implies 3) is strongly logically pure.[55] The failure of topical purity is seen by consideration of the meaning of concepts introduced in the proof: the introduction of the notion of a skew field which is not a geometric notion is decisive.

This illustrates Tait's maxim: The notion of formal proof was invented to study the existence of proofs, not methods of proof. Or as Burgess (2010) puts it, 'For formal provability to be a good model of informal provability it is not necessary that formal proof should be a good model of informal proof'.

We should not ignore the virtues of a demonstration that there is no pure proof. It shows that additional resources are needed for a particular claim. The Desargues proposition is particulary instructive in showing the value of the searching for the content of those

[54] Serendipitously, this argument was given by Ken Gross in a professional development program for elementary school teachers while I was working on this paper.

[55] We can take the base theory T in Definition 2.9 as PP. The formalization of the embedding is clearer from the argument of the Appendix than from Hilbert.

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additional resources. Hilbert isolated the ability to coordinatize in terms of the Desargues configuration and its connections with the interpretability of division rings. A significant part of twentieth century mathematics, the further development of projective planes, particularly finite projective planes, relied both on the algebraization and on the discovery of the underlying properties of the group of perspectivities of the plane.

Hilbert succeeded in showing a deep connection between algebraic and geometric conceptions by identifying both the algebraic (associativity) and geometric (Desargues proposition) conditions necessary and sufficient for Descartes coordinatization to succeed.

§3. Distinguishing algebraic and geometric proof. This section is a commentary on the Appendix, arguing that it provides a pure geometric proof of the embeddability theorem while Hilbert's proof is manifestly not pure. For this we need some distinction between algebra and geometry. Algebra deals with numbers (of various sorts); geometry deals with magnitudes. Geometric arguments admit and (as the writing of the Appendix demonstrated) often demand pictures. The distinction is clearly made in the quotation from Newton in Detlefsen & Arana (2011); there should be no arithmetical computations except of 'Quantities truly geometrical'. The essence of coordinatization, fundamental to Hilbert's proof of the embedding theorem, is to reject this notion.

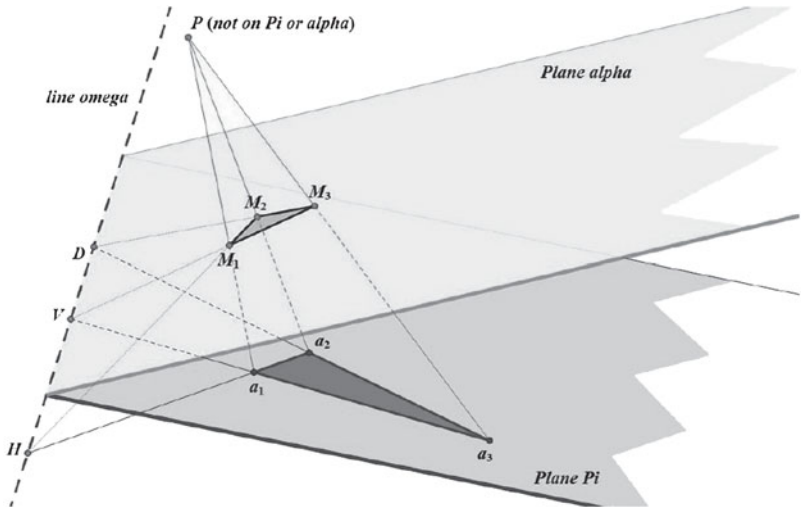
Thus in the Appendix the crucial vocabulary remains points, lines, and planes. There is no introduction of multiplication and addition and no reliance on the development of coordinate geometry. Crucially, however, new 'points' are introduced as certain triples of points and new 'lines' are introduced as sets of these 'points'; explicitly defined equivalence relations play a significant role. This is a more complex argument than the construction of models of non-Desarguesian planes that interpret (pieces) of curves as lines. This level of complexity is implicit in Hilbert.

The number of special cases that appear in the proof below are characteristic of geometric arguments. Algebraic methods (as is clear in the developments of twentieth century algebraic geometry) can clarify the notion of a 'generic configuration'. Thus, the coordinatization of a Desargesuian plane requires a not-quite arbitrary choice of coordinate points. In contrast Claim 4.15 requires a delicate argument to replace 'arbitrary' points by ones that are in general position. A similar situation concerns the relation between Pappian and Desarguian fields. The first is coordinatized by a commutative field, the second by a division ring (which will not be commutative if the plane is not Pappian). Hessenberg (1905) proved this in 1905. But he missed some possible intersections in the Desargues configuration and the proof was completed by Cronheim (1953) in 1953.

Levi (1939) earlier gave a pure geometric proof of the Hilbert embedding theorem. The argument with Howard in the Appendix makes clear the geometric picture that motivates the coding of points in three-space by triples in the plane. Levi does not bring this out and it is unclear if he had the same picture in mind. In particular, he gave the proof in the affine case and then extended to projective planes on general grounds as in Remark 2.4.

§4. Appendix: a geometric proof that Desargues implies embeddability (with William Howard). The diagram below illustrates the three principal ideas of the Appendix.

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The diagram[56] above illustrates Desargues theorem if it is viewed as lying in the plane of the page. Triangles $\langle M_1, M_2, M_3\rangle$ and $\langle a_1, a_2, a_3\rangle$ are centrally perspective by point P and axially perspective by line ω. The Desargues theorem asserts that either perspectivity condition implies the other.

The 3 dimensional proof of the Desargues theorem is seen by considering the diagram to be three-dimensional with triangle $\langle M_1, M_2, M_3\rangle$ in plane α and triangle $\langle a_1, a_2, a_3\rangle$ in plane Π. Assuming central perspectivity each pair of lines through corresponding sides meet in ω, the line of intersection of Π and α.

The interpretation of three-space into a Desarguesian plane Π described below is seen by the projection of M_i to a_i. As a point P varies, it corresponds to the triple $\langle a_1, a_2, a_3\rangle$ obtained by projecting through $\langle M_1, M_2, M_3\rangle$.

THEOREM 4.1. *A Desarguesian projective plane π can be embedded in a three-dimensional geometry.*

Levi's first 'geometric' proof of Theorem 4.1 appeared in Levi (1939). The second author remembers reading a version of the proof but not where.[57] The argument here stems mostly from the second author's reanalysis and differs from Levi by presenting the motivation more clearly and by a quite different application of Desargues. It differs from both Hilbert and Levi in being purely projective with no reference to affine geometry. The crux is that either this argument or Levi's differ from Hilbert's as there is no mention of coordinatization by a skew field. For concreteness, we view projective geometry as axiomatized in Robinson (1959) and recite the exact axioms later.

4.1. Some plane projective geometry. The following key property of a Desarguesian projective plane drives the main work.

DEFINITION 4.2. *Fix a point P and let $\mathcal{S}_P$ denote the set of pairs $\hat{a} = \langle a_1, a_2\rangle$ such that P lies on the line a_1a_2. For any $\ell \in \pi$, and $\hat{a}, \hat{b} \in \mathcal{S}_P$ such that $\hat{a}$ and $\hat{b}$ do not lie on the*

[56] We thank Andreas Mueller for this diagram. He made a series of illustrations in Geometers Sketchpad (some interactive) for the proof in this Appendix. The sketches are available on request.

[57] Victor Pambuccian pointed us to an exact reference, Levi (1939).

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same line through P, define R_ℓ by

$\hat{a} R_\ell \hat{b}$ if and only a_1, a_2 and b_1, b_2 are centrally perspective by a point on ℓ.

LEMMA 4.3. *For any $\ell \in \pi$,*

1. *If $\hat{a}, \hat{b}, \hat{c} \in \mathcal{S}_P$ lie on distinct lines through P then $\hat{a} R_\ell \hat{b}$ and $\hat{c} R_\ell \hat{b}$ imply $\hat{a} R_\ell \hat{c}$*
2. *R_ℓ is an equivalence relation on any set of pairs which determine distinct lines through P.*

Proof. Suppose $\hat{a}, \hat{b}, \hat{c}$ determine distinct lines through P. Suppose $\hat{a} R_\ell \hat{b}$ and $\hat{c} R_\ell \hat{b}$. Note that the result holds trivially if either $a_1 b_1 c_1$ or $a_2 b_2 c_2$ are collinear. So we may assume the triangles $a_1 b_1 c_1$ and $a_2 b_2 c_2$ are proper; by definition of $\mathcal{S}_P$ they are centrally perspective through P. Note that the points of central perspectivity $O_{\hat{a},\hat{b}}$, $O_{\hat{a},\hat{c}}$, $O_{\hat{c},\hat{b}}$ are the intersections of the sides of $a_1 b_1 c_1$ and $a_2 b_2 c_2$. By Desargues' theorem $O_{\hat{a},\hat{b}}$, $O_{\hat{a},\hat{c}}$, $O_{\hat{c},\hat{b}}$ are collinear. By the definition of R_ℓ the first and third are on ℓ so all three are. Applying the first part of the lemma multiple times yields the result. □

4.2. Motivation. The goal here is a direct construction: Given a Desarguesian plane π, we construct a three-dimensional projective space containing π. We will describe incidence in π in English; incidence in the new structure will be given by 'element of'. We use capital Roman letters for the points in a general projective plane, lower-case Roman letters for points in the new space (i.e., 'triangles' in π^3) and list the points on the triangle a as a_1, a_2, a_3 in π. Points in π may be labeled by capital Roman letter or subscripted lower-case Roman letters depending on the context. Planes are labeled by Greek letters and lines by lower case Roman letters.

The basic idea is to reverse the following operation. Fix a plane π in three-space. Then map the three-space onto π by fixing a triangle $M_1 M_2 M_3$ in a plane α not equal to π and mapping each P to the triangle, say $\langle a_1, a_2, a_3 \rangle$ that P projects $M_1 M_2 M_3$ onto in π. A little thought shows there are certain subtleties in this idea. Let ω denote the line where α and π meet and let H denote $M_1 M_2 \wedge \omega$, V denote $M_1 M_3 \wedge \omega$ and D denote $M_2 M_3 \wedge \omega$. Any point not in $\alpha \cup \pi$ will project to a proper triangle in π. Points on α have various special forms which are precisely defined as the sets Ω_i in Definition 4.4. If P is on α but not collinear with one of the edges of $M_1 M_2 M_3$ it projects to three distinct points on ω (in Ω_0). If P is on $M_1 M_2$ but not M_1 or M_2 it projects to $\langle H, H, s \rangle$ for some $s \in \omega - \{H, V, D\}$ (in Ω_1). The cases of M_1, M_2, M_3 are even more special. Each of them projects to a pair from $\{H, V, D\}$ (into Ω_2). A line ℓ in three-space not in α will project to a 'line' of triangles in π which are in perspective from $\ell \wedge \pi$.

4.3. The construction. With a little care for special cases and appropriate use of the Desargues theorem we can reverse the map in Section 4.2 and code the three-space. While, especially in footnotes, we return to the picture of the previous paragraph for motivation, formally, we are now defining a three-space by defining in π a set of points (certain triangles) and the notions of line, incidence, and eventually plane on those points.

DEFINITION 4.4 (The Model: I). *Fix a line ω in the Desarguesian plane π and distinct points H, V, D, called the coordinate points, on ω. $\mathcal{S}$ is a collection of ordered triples from π.*

1. *The set of regular points $\mathcal{S}_r$ in the new three space consists of the proper triangles $a = a_1 a_2 a_3$ in π such that H is on $a_1 a_2$, D is on $a_2 a_3$, and V is on $a_1 a_3$. We call them regular triangles.*

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$\mathcal{S}$ also contains several exceptional cases.

(a) degenerate triangles: $a_1 = a_2 = a_3$. That is, $a \in \pi$.

(b) 'triangles' on ω:[58]

i. Ω_0 denotes the set of special triangles, ordered triples $\langle a_1, a_2, a_3\rangle$, of distinct points on ω.[59]

ii. Ω_1 is the set of very special triangles, triples of points from ω of the form $\langle H, H, x\rangle, \langle V, x, V\rangle, \langle x, D, D\rangle$ with $x \in \pi - \{H, V, D\}$.[60]

iii. Ω_2 denotes the three 'triples' $\langle *, H, V\rangle, \langle H, *, D\rangle, \langle V, D, *\rangle$ where $*$ indicates that one coordinate is undefined.[61]

$\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2 \cup \omega$ is called the plane at infinity.

2. *A* regular line *m in the three-space is a set of $a \in \mathcal{S}$ such that for some fixed $0_m \in \pi$ any $a, b \in m$ are centrally perspective*[62] *with center O_m. Note that O_m is a degenerate point in $\mathcal{S}$ of type 1a) and $O_m \in m$.*
$\mathcal{L}$ denotes the set of all lines including those entirely contained in Ω and $\mathcal{L}_r$ denotes the set of regular lines (those which contain a regular triangle).
The lines on Ω are described in Definition 4.26.

We will say a special triangle b is centrally perspective with a regular triangle a if the lines $a_i b_i$ (for $i \leq 3$) meet in a point. We write $O_{a,b}$ for the point on π of central perspectivity of a and b. We say two or more points in $\mathcal{S}$ are *regularly collinear* if there is a regular line through them. Suppose $a, b, c \in \mathcal{S}$ are three points that are not regularly collinear and that a is a regular triangle. Then ab and ac are two distinct regular lines.

We begin by observing a natural classification of regular lines in $\mathcal{S}$.

REMARK 4.5 (Classifying regular lines). *Each regular line $r \in \mathcal{L}_r$ (i.e., not on Ω) is given by a triple (r_1, r_2, r_3) of lines in π. (In case 3a below one of the r_i is not defined; all triangles on the line share a point.) r_i goes through the x_i coordinate of the triangles in r. The intersection of the r_i is O_r, the vertex.*

type 1 *None of the coordinates H, V, D is on any r_i; the r_i are distinct.*
type 2 *Two of the r_i agree and go through a coordinate point, H, V, or D.*[63]

		$r_1 \wedge \omega$	$r_2 \wedge \omega$	$r_3 \wedge \omega$
2a)	$r_1 = r_2$	H	H	$\omega - \{V, D\}$
2b)	$r_1 = r_3$	V	$\omega - \{H, D\}$	V
2c)	$r_2 = r_3$	$\omega - \{H, V\}$	D	D

58 These points corresponding to the images of elements of the plane α.

59 These points are in central perspective with a regular triangle and are discussed in Remark 4.6.

60 These are also in perspective with a regular triangle; see Remark 4.5.

61 In our prototype, $M_1 \mapsto \langle *, H, V\rangle$, $M_2 \mapsto \langle H, *, D\rangle$, $M_3 \mapsto \langle V, D, *\rangle$. These points lie on the type 3 lines in Remark 4.5.

62 More precisely, $a_1a_2a_3$ and $b_1b_2b_3$ are centrally perspective at $a_1b_1 \wedge a_2b_2 \wedge a_3b_3 = O_m$. There are three special cases, for example, when a_1a_2 and b_1b_2 are the same line through H in π. In this case, letting s denote $a_3b_3 \wedge \omega$, the very special point $\langle H, H, s\rangle$ is on the line.

63 If one thinks of the affine part of the plane the lines in this class are horizontal, vertical, and diagonal in that order. Note the special case when, for example, in 2a), the intersection point is $\langle H, H, H\rangle$; this is the image of a triangle whose plane contains M_1M_2. Lines of this form contain no special triangles.

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type 3 *One of the r_i is undefined and so has no intersection with ω. O_r is a vertex of every regular triangle in r.*

	$r_1 \wedge \omega$	$r_2 \wedge \omega$	$r_3 \wedge \omega$
3*a*)	*undefined*	H	V
3*b*)	H	*undefined*	D
3*c*)	V	D	*undefined*

REMARK 4.6 (Special triangles). *Specifying two points on a regular triangle b determines the third and the labeling is determined by, for example, b_1 is the vertex that is the intersection of lines through Hb_2 and Vb_3. We establish the analogous claim for special triangles in Lemma 4.7.*

LEMMA 4.7. *Let s_1, s_2 be two points in $\omega - \{H, V, D\}$. Then there is a unique point s_3 on ω so that $s = s_1 s_2 s_3$ is in central perspectivity with a regular triangle $a_1 a_2 a_3$. Thus, $s \in \Omega_0$.*

Proof. Choose any regular triangle $a_1 a_2 a_3$. Let P on π be the intersection of $s_1 a_1$ and $s_2 a_2$. Since the s_i are not in $\{H, V, D\}$, P is not on a line extending a side of $a_1 a_2 a_3$. Choose s_3 as Pa_3 intersect ω. We show that s_3 does not depend on the choice of $a_1 a_2 a_3$. Let $b_1 b_2 b_3$ be another regular triangle and by the same procedure choose a point of perspectivity Q and an $s_3' = Qb_3 \wedge \omega$ so that $s_1 s_2 s_3'$ and $b_1 b_2 b_3$ are in perspective from Q. By the converse of Desargues, the regular triangles $a_1 a_2 a_3$ and $b_1 b_2 b_3$ are centrally perspective through some point R. Further,

1. $a_1 b_1 s_1$ and $a_2 b_2 s_2$ are centrally perspective by H and
2. $a_2 b_2 s_2$ and $a_3 b_3 s_3$ are centrally perspective by $a_2 a_3 \wedge \omega = D$.

Applying Desargues, 1) yields $a_1 b_1 \wedge a_2 b_2 = R$, $b_1 s_1 \wedge b_2 s_2 = Q$, and $a_1 s_1 \wedge a_2 s_2 = P$ are collinear. And 2) implies $a_2 b_2 \wedge a_3 b_3 = R$, $a_2 s_2 \wedge a_3 s_3 = P$ are collinear with $b_2 s_2 \wedge b_3 s_3$. That is $b_2 s_2 \wedge b_3 s_3$ lies on PR and so must equal Q. But then $Qb_3 \wedge \omega = b_3 s_3 \wedge \omega$ and $s_3 = s_3'$. □

We make two crucial observations.

FACT 4.8. *If a and b are in $\mathcal{S}$ and at most one of them is on Ω, $O_{a,b}$ is well-defined. Every regular line meets Ω.*

Proof. The first sentence is clear from Lemma 4.7 when $a \in \mathcal{S}_r$ and $b \in \Omega_0$ has two coordinates from $\omega - \{H, V, D\}$. But even in case $b \in \Omega_1 \cup \Omega_2$, there are two lines $a_i b_i$ and $a_j b_j$ (for some i, j among 1, 2, 3) which are distinct; their intersection is $O_{a,b}$. If $a \in \pi$ then for any b, $O_{a,b} = a$. The second sentence is easy to check. □

We now give an external definition using the tools of our construction of the plane generated by three points, where one is regular. We will show that in $(\mathcal{S}, \mathcal{L}, \epsilon)$ it describes the same notion as the intrinsic Definition 4.16.

DEFINITION 4.9 (The Model: II). *If $a, b, c \in \mathcal{S}$ are three points that are not regularly collinear and a is a regular triangle, the* plane $\alpha = \langle a, b, c \rangle$ generated by *abc is the union of all lines $\ell_x \in \mathcal{S}$ that pass through a and some point x on the line $\ell_\alpha = O_{a,b} O_{a,c}$ (called the trace of ℓ_α) in π. Formally*

$$\alpha = \bigcup_{x \in \ell_\alpha} \ell_x.$$

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REMARK 4.10. *Since the* 7 *element Fano plane is the only projective plane with exactly three points on a line and it is coordinatized by the* 2 *element field, we can assume that each line in* π *has at least* 4 *elements. Consequently each regular line in* $\mathcal{S}$ *contains at least two regular points.*

We want to apply Desargues's theorem to study planes containing regular points. The difficulty is that applications of Desargues theorem to establish the transitivity of the relations R_ℓ from Definition 4.2 require that certain triangles not have sides on a common line. From now though Lemma 4.15 we see how to adjust situations to avoid this problem.

DEFINITION 4.11. *By a* direction *of a regular triangle* b *we mean any of the lines that extend a side of* b *to one of* $\{H, V, D\}$ *(For example, the line* b_1b_2H*; to specify the direction we write* X*-direction for some* $X \in \{H, V, D\}$*.).*

DEFINITION 4.12. *Suppose* $a, b, c \in \mathcal{S}$*. We say a pair of regular triangles* a, b *are* aligned *if they share a direction.*

No regular triangle is aligned with a special triangle. Any pair of elements of Ω *is* aligned.

We say three triangles a, b, c *are* askew *if there is a direction in which no pair of them is aligned.*

We will need some stronger variants on Lemma 4.3. First we extend the relation R_ℓ from pairs to triangles.

DEFINITION 4.13. *For any* $\ell \in \pi$*, and* $a, b \in \mathcal{S}$ *such that* $O_{a,b}$ *exists define* R_ℓ *by* $aR_\ell b$ *if and only* $O_{a,b} \in \ell$*.*

LEMMA 4.14. *Suppose* $a, b, c \in S$*, and at most one is in* Ω*, then* R_ℓ *is transitive on* $\{a, b, c\}$*.* R_ℓ *is transitive on any finite set of elements in* $\mathcal{S}$ *with at most one on* Ω *and the rest in* S_r*.*

Proof. Suppose a, b, c are all aligned in a direction, say D for simplicity. Then $a_1, a_3, b_1, b_3, c_1, c_3$ lie on a line ℓ and each of $O_{a,b}$, $O_{a,c}$, $O_{b,c}$ must lie on that line. Assuming this case does not occur, we prove.

CLAIM 4.15. *We can choose* d *such that*

1. *for two of the triangles (say* a, c*)* $O_{a,d} = O_{c,d} = O_{a,c}$*;*
2. a, d, b *and* b, c, d *are askew.*

Proof of Claim: If one of the points is in Ω, a, b, c are automatically askew. So we consider the case where all are regular. There are two cases.

First suppose that two of the triangles are aligned in two directions. Fix these as b, c and suppose they both D-aligned and V-aligned. (The choice of names of the triangles in the statement of Claim 4.15 depend on the choice made here.) Then $c_3 = b_3$ and since the three are not X-aligned for any X, if there is any alignment with a, it is along H. Suppose a and c are H-aligned. Choose d_3 on c_3a_3 not equal to either of them. Then let $d_1 = d_3D \wedge a_1c_1$ and $d_2 = d_3V \wedge a_2c_2$. Now neither d, b, a nor d, b, c is D-aligned.

The second case is when no pair is aligned in two directions but perhaps each pair is aligned. Suppose for example, that a_1, a_2 and b_1, b_2 are collinear. By our case analysis, we may suppose that a, c are not both D and V aligned. For concreteness, say they are D aligned. And we may also assume that $a_2 \neq c_2$ as equality would imply that a and c are H-aligned. If they were, c_1 would be on both a_1a_3D and a_1b_1H so $a_1 = c_1$ and we

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would have all three H-aligned contrary to the first line of the proof of Lemma 4.14. Thus, $a_3 \neq c_3$ and $a_2 \neq c_2$. Choose regular d_1 on $a_1c_1O_{a,c}$ but not among a_1, c_1 or $O_{a,c}$. Now let d_2 be $Hd_1 \wedge a_2c_2$ and d_3 be $Dd_2 \wedge a_3c_3$. Both of these points are defined

The construction guarantees that c, b, d are not H-aligned and b, a, d are not aligned in the D (i.e., a_2a_3) direction. □

This implies that R_ℓ is transitive on any triple of elements in $\mathcal{S}$ with at most one on Ω and the rest in S_r. For $a, b, c \in S_r$, the result follows from Lemma 4.3 applied to the two triples in 2) along the non-aligned direction. That is, suppose $aR_\ell c$ and $bR_\ell c$. We have $O_{c,d} = O_{a,d}$ and $O_{b,c}$ are both on ℓ so applying Lemma 4.3 in the H-direction to c, b, d $O_{b,d} \in \ell$, that is, $bR_\ell d$. We have $bR_\ell d$ and by construction $aR_\ell d$, so applying the Lemma 4.3 in the D-direction on a, b, d, we deduce $aR_\ell b$. If one of the points is on Ω, the result is easier as we don't have to do so much adjustment to apply Lemma 4.3.

For the second sentence, suppose we are given a set A of n regular triangles $a_1, \ldots a_n \in S_r$ and $a_{n+1} \in \Omega$. It suffices to show R_ℓ is transitive on each three element subset of A. We can establish this by applying part 1) independently to each such triple. □

Robinson (1959), whose axiomatization we are using, unlike Hilbert, does not take *plane* as a primitive but makes the following intrinsic definition.

DEFINITION 4.16. *The plane generated by a, b, c is the collection of all points on lines that contain a and intersect bc.*

We show in Lemma 4.21 that our external notion of plane, Definition 4.9, coincides with the intrinsic definition in the language of the geometry that we give now. For this, we need the consequences of Desargues we have just developed and work extensively with the extrinsic definition until proving the equivalence in Lemma 4.21.

LEMMA 4.17. *Suppose a, b, c are distinct points in $\mathcal{S}$ and a is a regular triangle. If d is a regular triangle on the* plane *α generated by abc (ℓ_α and a) in the sense of Definition 4.9, then α is also generated by dbc (by ℓ_α and d) in the same sense.*

Proof. Suppose the trace of α on π is $\ell = \ell_\alpha$. If $b, c \in \Omega$, each of the triples a, b, d and a, c, d is askew. In particular there is a direction X such that the three lines through X extending sides of a, b, d are distinct. If at most one point is in Ω we apply Lemma 4.14 to guarantee the triangles are askew. For any of these points u in $\mathcal{S}$, let $\hat{u}$ denote the restriction to the relevant pair in $\mathcal{S}_X$. We have $\hat{d}R_\ell\hat{a}$ and $\hat{b}R_\ell\hat{a}$. Hence we can apply the Lemma 4.3.1 for $\mathcal{S}_X$, to conclude $\hat{d}R_\ell\hat{b}$. Similarly, $\hat{d}R_\ell\hat{a}$ and $\hat{c}R_\ell\hat{a}$ imply by Lemma 4.3.1, $\hat{d}R_\ell\hat{c}$. So $O_{d,b}O_{d,c} = \ell_\alpha$ so $\alpha = \langle d, b, c\rangle$ and α is generated by ℓ_α and d. □

LEMMA 4.18. *If $\alpha = \langle a, b, c\rangle$ is generated by abc, where a is regular, then there is a $k \in \{1, 2, 3\}$ such that*

1. *For every regular point $d = \langle d_1, d_2, d_3\rangle$ in α, d_k is not on ℓ_α.*
2. *$s_k \wedge \omega$ is defined for the line through bc, which is given by $s = \langle s_1, s_2, s_3\rangle$.*
3. *For every regular line $s = \langle s_1, s_2, s_3\rangle$ in α, s_k is defined.*

Proof. Note first that some coordinate a_i of a, must not be on ℓ_α since a is proper triangle in π. We will show that in this case no regular triangle d in α has d_i on α. For simplicity, suppose $i = 1$. Suppose for contradiction some $d \in \alpha$ has d_1 on ℓ_α. Then $d_1 = O_{a,d}$ so d_1d_2H, $d_1a_2d_2$ and thus $d_1d_2a_2H$ are collinear in π. But then since a_1a_2H is a line, a_1 is also on this line. Similarly $d_1d_3a_3a_1V$ is also a line. But they meet in both a_1 and d_1. So $a_1 = d_1$ contrary to a_1 not on ℓ_α.

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For the second claim, continue with the case $k = 1$. Suppose the line $s = \langle s_1, s_2, s_3\rangle \in \alpha$ is through b, c, but s_1 is not defined. Then $b_1 = c_1$. But $O_{a,b}$ lies on a_1b_1 and $O_{a,c}$ lies on a_1c_1, so $b_1 = c_1$ is on ℓ_α contradicting part 1.

By Lemma 4.17, we can choose any d on s and assume α is generated by d and bc, yielding iii) from ii). □

We need a bit more information about the composition of lines.

LEMMA 4.19. *Let $r = \langle r_1, r_2, r_3\rangle$ be a line with $r_k \wedge \omega$ defined, say $k = 1$. For any point b_1 on π, there is a unique triple $\langle b_1, b_2, b_3\rangle$ on r.*

Proof. By the inventory of lines, Remark 4.5, we can choose a k, (say 2 for concreteness) so that $r_2 \neq r_1$. Consider first the case that r is of type 1 or type 2. Then, again by Remark 4.5, H is not on r_1 nor r_2. Let $b_2 = b_1H \wedge r_2$. If b_1 is not on ω then neither is b_2 and $b_3 = r_3 \wedge b_2D$. Then H is not on either r_1 or r_2. If b_1 is on ω then so is b_2. But $r_2 \wedge Hb_1 = b_2$ so neither b_1 nor b_2 is H. By Lemma 4.6, we can choose b_3 as required. If r is of type 3, we are in case Remark 4.5.3c and $\langle b_1, b_2, b_3\rangle = \langle V, D, *\rangle$. □

LEMMA 4.20. *If a, b, c are regular, any two regular lines in the plane α generated by abc in sense of Definition 4.9 intersect.*

Proof. Let r and $s = \langle s_1, s_2, s_3\rangle$ be two such lines. Lemma 4.17 allows us to assume a is on r and so r has the form $y'a$ for some y' on ℓ_α. We need to show that $y'a \wedge s$ is nonempty. By Lemma 4.18.2, there is a k such that s_k is defined. For simplicity, assume $k = 1$. Let $d_1 = y'a_1 \wedge s_1$. By Lemma 4.19 there is a unique triple $d = \langle d_1, d_2, d_3\rangle$ on s. We claim d is also on $y'a$. By the definition of α, O_{da} lies on ℓ_α. But d_1 is on a_1y' so $O_{da} = y'$. Thus $d = y'a \wedge bc$ as required. □

LEMMA 4.21. *If a, b, c are regular, the plane α generated by abc in sense of Definition 4.9 is the same as that given by Definition 4.16:*

$$\alpha = \bigcup_{x \in bc} \ell_{xa}$$

where $\ell_{xa} \in \mathcal{L}$ is through the regular triangles x and a.

Proof. Let d be a regular triangle on bc. We must show $O_{a,d}$ lies on $\ell_\alpha = O_{a,c}O_{a,b} \in \pi$. By Lemma 4.14, $O_{b,c}$ is on ℓ_α. Since d is on bc, $O_{b,d} = O_{b,c} \in \ell_\alpha$. By Lemma 4.14 again, $O_{a,d}$ is on ℓ_α as required.

For the converse, let $y \in \alpha$. Then y is on a line $y'ay$ with y' on ℓ_α and we need to show that $y'a \wedge bc$ is nonempty. Apply Lemma 4.20 with $r = y'a$ and $s = bc$. □

4.4. The plane at infinity. The points of Ω are a very special plane in $\mathcal{S}$. In this subsection, we study the lines on this plane and conclude with Lemma 4.28 verifying that Ω is in fact a plane in the sense of Definition 4.16.

REMARK 4.22. *Note that if b, c are regular triangles, the point $O_{b,c} \in \pi$ is on every plane containing b, c. Moreover if b, c are in each of two planes α and β the line through b, c is the intersection of α and β. As, if there were some regular d in the intersection but not on that line, Lemma 4.17 and the argument in the first paragraph of Lemma 4.23 shows $\alpha = \beta$.*

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We need to describe the line through two points on Ω: some effort is needed to define these lines. The key is to establish Remark 4.22 for special triangles.

LEMMA 4.23. *Suppose $b, c \in \Omega$. There is a point t on ω such that t is in every plane α that contains a regular point of $\mathcal{S}$ and b, c.*
If α is not contained in Ω, $\{t\} = \alpha \wedge \omega$.

Proof. Consider two distinct planes α and β which both contain b, c. Say $\alpha = \langle a_1, b, c\rangle$ and $\beta = \langle a_2, b, c\rangle$. Then $\ell_\alpha \wedge \ell_\beta = t$ is a point (degenerate triangle) on π. If t is not on ω, choose a regular triangle t' on tb in $\mathcal{S}$. By two applications of Lemma 4.17 (replacing either a_1 or a_2 by t'), $\alpha = \langle a_1, b, c\rangle = \langle t', b, c\rangle = \langle a_2, b, c\rangle = \beta$. So any plane γ containing b, c meets α in $\ell_\alpha \wedge \ell_\gamma = \{t\}$.
If α is not contained in Ω, we can generate α as $\langle a, b, c\rangle$ for any regular $a \in \alpha$. But then neither O_{ab} nor O_{ac} is on ω so $\omega \cap \alpha$ must be singleton (otherwise α would contain all of π). □ □

Since $t = O_{at} \in \ell_\alpha = O_{ab}O_{ac}$, we have:

COROLLARY 4.24 *Suppose $b, c \in \Omega$ and α is a plane containing b, c. Then for any regular point a on α, $\alpha = \langle a, b, c\rangle = \langle a, c, t\rangle = \langle a, b, t\rangle$ where $t = \ell_\alpha \wedge \omega$.*

Now we show that the intersection of a plane with Ω is determined by any pair of points in that intersection.

LEMMA 4.25. *Suppose α, β are planes that are not contained in Ω and $b, c \in \Omega$, $b, c \in \alpha \cap \beta$. If $d \in \Omega \cap \alpha$, then $d \in \Omega \cap \beta$.*

Proof. By Lemma 4.23, α and β intersect in a point $t \in \omega$. Choose any regular point $a \in \beta$, so $\beta = \langle a, b, c\rangle$. By Lemma 4.23, $t \in \beta$ since $b, c \in \alpha \wedge \beta$. Applying Corollary 4.24, $\beta = \langle a, b, c\rangle = \langle a, c, t\rangle$.
Let γ be the plane generated by a, d, c. Then $t \in \gamma$ since α and γ are planes containing a and d. Again by Corollary 4.24, $\gamma = \langle a, d, c\rangle = \langle a, t, c\rangle = \beta$. So $\gamma = \beta$ and $d \in \beta$ as required. □

Lemma 4.25 justifies the following Definition 4.26 of a line through two points in Ω.

DEFINITION 4.26. *For $a, b \in \Omega$ the line through a, b is the intersection of some (any) regular plane that contains a and b with Ω. Note this line includes the point t from Lemma 4.23.*

LEMMA 4.27. *Any two regular planes in $\mathcal{S}$ intersect in a line.*

Proof. Let α and β be two regular planes. If there are two points b, c in their intersection the intersection is the line through b, c by Remark 4.22. In particular we are finished if $\ell_\alpha = \ell_\beta$.
So we must show two regular planes α, β intersect in at least two points. One point is $e = \ell_\alpha \wedge \ell_\beta$. To find a second, let r be a regular line in β. We will prove that r intersects α is a point different from e. For this, choose $a \in \alpha$ but not on ℓ_α. Let γ be the plane generated by r and a. Then γ contains both r and the line $s = ae$. By Lemma 4.20, r and s meet in a point, which since s lies on α, must be in α. Thus, $r \wedge s$ is our second point. □

LEMMA 4.28. *Ω is a plane.*

Proof. Let a, b, c lie on Ω. Let α, β, γ three planes, each containing a regular point and containing ab, bc, ac respectively. For any $d \in \Omega$, Definition 4.16 shows it suffices to that

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$ad \wedge \boldsymbol{c}\neg\emptyset$. Let δ be a regular plane which intersects Ω in the line ad. By Lemma 4.27 or Definition 4.26 δ and β intersect in a line m. Then m and bc are both in β. So by Lemma 4.21 and Lemma 4.20, $m \wedge bc$ is the required point of intersection. □

4.5. Verifying the axioms. For concreteness, we specify the axioms as formulated in Robinson (1959). The term plane is not primitive.

DEFINITION 4.29 (Basic axioms). **Axiom 1** *There are at least two distinct points.*

Axiom 2 *Two distinct points A and B determine one and only one line on both A and B.*

Axiom 3 *If A and B are distinct points, there is at least one point distinct from A and B on the line M_1M_2.*

Axiom 4 *If A and B are distinct points, there is at least one point not on the line AB.*

First we verify the basic axioms.

LEMMA 4.30. $(\mathcal{S}, \mathcal{L}, \epsilon)$ *satisfies the basic axioms.*

Proof.

Axiom 1 Any two distinct points a_1, a_2 in $\pi - \omega$ determine a unique regular triangle $a_1a_2a_3 \in \mathcal{S}$ by drawing the required lines through V and D.

Axiom 2 Clearly the converse of Desargues implies any two triangles in $\mathcal{S}$ with at most one on Ω are centrally perspective by some point. The lines of perspectivity have a unique intersection so there is a single point of perspectivity which indexes the line. The axiom holds for pairs of points in Ω using Definition 4.26 and Lemma 4.25.

Axiom 3 Given two regular triangles a, b which are in perspective from a point O and so are on a line in $\mathcal{S}$, choose any c_1 on $\ell_1 = 0a_1b_1$ and let c_2 be the intersection of c_1K and $\ell_2 = 0a_2b_2$; now the converse of Desargues theorem shows, since the triangle $c_1c_2c_3$ generated from c_1c_2 by the procedure in the proof of Axiom 2 is axially perspective with $a_1a_2a_3$, they are centrally perspective. So O is on c_1a_1.

Axiom 4 Modify the proof for Axiom 3. Given two regular triangles a, b which are in perspective from a point O, choose $P \neq O$ and draw ℓ_1, ℓ_2, ℓ_3 through O and each of the a_i. Now repeat the rest of the argument for Axiom 3, using the new choices for ℓ_1, ℓ_2 to find c not on the line $ab \in \mathcal{S}$. □

Two more axioms in Robinson (1959) relate to the three-dimensionality. For convenience in the proofs, we label the points in Pasch's axiom with lower-case letters.

DEFINITION 4.31. *(Pasch's axiom:)*

Axiom 5 *Suppose A, B, C are three noncollinear points, D is a distinct point on BC and E is a distinct point on CA. Then there is a point F on the intersection of AB and DE.*

LEMMA 4.32. $(\mathcal{S}, \mathcal{L}, \epsilon)$ *satisfies Pasch's axiom.*

Proof. Fix a, b, c as in Pasch's axiom and suppose they generate the plane α. If these three points are in Ω, then $\alpha = \Omega$ and we must show two lines r, s in Ω intersect. For some regular planes α and β, $\alpha \wedge \Omega = r$ and $\beta \wedge \Omega = s$. By Lemma 4.35 $\alpha \wedge \beta$ intersect in a line q and $q \wedge \Omega$ is the required intersection of r and s.

So we may assume a is regular. And in fact, if b or c is in Ω by replacing b by b' a regular point on ab (or c' a regular point on ac), we generate the same plane since $\ell_\alpha = O_{a,b}O_{a,c} = O_{a,b'}O_{a,c'}$. But then by Lemma 4.21, the plane α is also the plane generated by a, b, c in sense of Robinson. By Lemma 4.20, we have Pasch's axiom except

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if d and e are in Ω. But consider the regular line ab. Note by Fact 4.8 that every regular line intersects Ω and we finish by Definition 4.26. □

To specify the geometry is three-dimensional, De Robinson adds:

DEFINITION 4.33. *(3-space axioms)*

Axiom 6 *If A, B, C are three noncollinear points, there is at least one point D not on the plane ABC.*

Axiom 7 *Any two distinct planes have a line in common.*

REMARK 4.34. *Axiom 6, the assertion that there is a point off each plane, is easily checked by considering the cases by which a plane is generated.*

LEMMA 4.35. *Axiom 7: Any two planes in $\mathcal{S}$ intersect in a line.*

Proof. Let α and β be arbitrary planes. For regular planes this is Lemma 4.27. If there are two points b, c in their intersection the intersection is the line through b, c by Remark 4.22 and Lemma 4.25 (in the case one of the planes is Ω). For the $\beta = \Omega$ case, choose any a on α and two distinct lines through a. By Fact 4.8 they intersect in (necessarily distinct) points. □

BIBLIOGRAPHY

Arana, A. (2008a). Logical and semantic purity. *Protosociology*, **25**, 36–48.

Arana, A. (2008b). On formally measuring and eliminating extraneous notions in proofs. *Philosophia Mathematica*, **3**, 1–19.

Arana, A. (2011). Purity in arithmetic: Some formal and informal issues. To appear.

Arana, A., & Mancosu, P. (2012). On the relationship between plane and solid geometry. *Review of Symbolic Logic*, **5**, 294–353.

Artin, E. (1957). *Geometric Algebra*. New York: Interscience.

Baldwin, J. T. (1994). An almost strongly minimal non-Desarguesian projective plane. *Transactions of the American Mathematical Society*, **342**, 695–711.

Baldwin, J. T. (1995). Some projective planes of Lenz Barlotti class I. *Proceedings of the American Mathematical Society*, **123**, 251–256.

Baldwin, J. T. (2009). *Categoricity*. Number 51 in University Lecture Notes. Providence, RI: American Mathematical Society. Available from: http://www.math.uic.edu/~jbaldwin.

Barwise, J., & Feferman, S. editors. (1985). *Model-Theoretic Logics*. Heidelberg, Germany: Springer-Verlag.

Burgess, J. P. (2010). Putting structuralism in its place. To appear.

Cronheim, A. (1953). A proof of Hessenberg's theorem. *Proceedings of the American Mathematical Society*, **4**, 219–221.

Dembowski, P. (1977). *Finite Geometries*. Springer-Verlag.

Detlefsen, M., & Arana, A. (2011). Purity of methods. *Philosophers Imprint*, **11**, 1–20.

Educational Development Corporation. (2009). *CME Algebra I*. Boston, MA: Pearson. EDC: Educational Development Corporation, Newton, MA.

Ehrlich, P. (2001). Number systems with simplicity hierarchies: A generalization of conways theory of surreal numbers. *The Journal of Symbolic Logic*, **66**, 1231–1258.

Eklof, P. (1976). Whitehead's problem is undecidable. *American Mathematical Monthly*, **83**, 775–788.

https://doi.org/10.1017/S1755020312000263 Published online by Cambridge University Press

Euclid. (1956). *Euclid's Elements*. New York: Dover. In three volumes, translated by T.L. Heath; first edition 1908.
Feferman, S. (2008). Tarski's conceptual analysis of semantical notions. In Patterson, D., editor. *New Essays on Tarski and Philosophy*. Oxford, UK: Oxford University Press.
Givant, S., & Tarski, A. (1999). Tarski's system of geometry. *Bulletin of Symbolic Logic*, **5**, 175–214.
Gödel, K. (1946). Remarks before the Princeton bicentennial conference of problems in mathematics. In Feferman, S., et. al., editors. *Kurt Gödel: Collected Works*, Vol. 1. New York: Oxford University Press, 1990. 1929 PhD. thesis reprinted.
Grätzer, G. (1968). *Universal Algebra*. Princeton, NJ: Van Nostrand.
Hallett, M. (2008). Reflections on the purity of method in Hilbert's Grundlagen der Geometrie. In Mancosu, P., editor. *The Philosophy of Mathematical Practice*. Oxford, UK: Oxford University Press, pp. 198–256.
Hartshorne, R. (1967). *Geometry: Foundations of Projective Geometry*. New York: W.A. Benjamin.
Henderson, D., & Taimina, D. (2005). How to use history to clarify common confusions in geometry. In Shell-Gellasch, A., and Jardine, D., editors. *From Calculus to Computers*, Vol. 68 of *MAA Math Notes*. Providence, RI: Mathematical Association of America, pp. 57–74.
Hessenberg, G. (1905). Beweis des desarguesschen satzes aus dem pascalschen. *Mathematische Annalen*, **61**, 161–172.
Heyting, A. (1963). *Axiomatic Projective Geometry*. New York: John Wiley & Sons.
Hilbert, D. (1971). *Foundations of Geometry*. Lasalles, IL: Open Court Publishers. Original German publication 1899: translation from tenth edition, Bernays 1968.
Hilbert, D. (2004). In Hilbert and Mayer, editors. *David Hilbert's Lectures on the Foundations of Geometry 1891-1902*. Heidelberg, Germany: Springer, pp. xviii + 665.
Hodges, W. (1987). What is a structure theory? *Bulletin of the London Mathematics Society*, **19**, 209–237.
Hrushovski, E., & Zilber, B. (1993). Zariski geometries. *Bulletin of the American Mathematical Society*, **28**, 315–324.
Hughes, D. R., & Piper, F. C. (1973). *Projective Planes*. Heidelberg, Germany: Springer-Verlag.
Jech, T. (1978). *Set Theory*, Vol. 79 of *Pure and Applied Mathematics*. New York: Academic Press.
Keisler, H. J. (1960). Theory of models with generalized atomic formulas. *The Journal of Symbolic Logic*, **25**, 1–25.
Kennedy, J. (To appear). On formalism freeness.
Kirby, J. (2008). *Abstract Elementary Categories*. Available from: http://www.uea.ac.uk/∼ccf09tku/pdf/aecats.pdf.
Kirby, J. (2010). On quasiminimal excellent classes. *Journal of Symbolic Logic*, **75**, 551–564. Available from: http://arxiv.org/PS_cache/arxiv/pdf/0707/0707.4496v3.pdf.
Lang, S. (1964). *Algebraic Geometry*. New York: Interscience.
Levi, F. W. (1939). On a fundamental theorem of geometry. *Journal of the Indian Mathematical Society*, **3–4**, 82–92.
Lieberman, M. (2011). Category-theoretic aspects of abstract elementary classes. *Annals of Pure and Applied Logic*, **162**(11), 903–915.
Manders, K. (2008). Diagram-based geometric practice. In Mancosu, P., editor. *The Philosophy of Mathematical Practice*. Oxford, UK: Oxford University Press, pp. 65–79.
Marker, D. (2002). *Model Theory: An introduction*. Heidelberg, Germany: Springer-Verlag.

https://doi.org/10.1017/S1755020312000263 Published online by Cambridge University Press

Morley, M. (1965a). Categoricity in power. *Transactions of the American Mathematical Society*, **114**, 514–538.

Morley, M. (1965b). Omitting classes of elements. In Addison, Henkin, & Tarski, editors. *The Theory of Models*. Amsterdam: North-Holland, pp. 265–273.

Pambuccian, V. (2001). A methodologically pure proof of a convex geometry problem. *Beiträge zur Algebra und Geometrie, Contributions to Algebra and Geometry*, **42**, 40–406.

Pambuccian, V. (2005). Euclidean geometry problems rephrased in terms of midpoints and point-reflections. *Elemente der Mathematik*, **60**, 19–24.

Pambuccian, V. (2009). A reverse analysis of the Sylvester-Gallai theorem. *Notre Dame Journal of Formal Logic*, **50**, 245–259.

Pierce, D. (2011). *Numbers*. Available from: http://metu.edu.tr/∼dpierce/Mathematics/Numbers.

Robinson, G. d. B. (1959). *Foundations of Geometry*. Toronto, Canada: Toronto Press.

Shelah, S. (1983). Classification theory for nonelementary classes. I. the number of uncountable models of $\psi \in L_{\omega_1\omega}$ part A. *Israel Journal of Mathematics*, **46**(3), 212–240. paper 87a.

Shelah, S. (2009). *Classification Theory for Abstract Elementary Classes*. Studies in Logic. College Publications. Available from: http://www.collegepublications.co.uk, 2009. Binds together papers 88r, 600, 705, 734 with introduction E53.

Shoenfield, J. (1967). *Mathematical Logic*. Addison-Wesley.

Smith, J. T. (2010). Definitions and non-definability in geometry. *American Mathematical Monthly*, **117**, 475–489.

Smullyan, R. M. (1961). *Theory of Formal Systems*. Princeton, NJ: Princeton University Press.

Sobocińnski, B. (1955). On well-constructed axiom systems. *Polish Society of Arts and Sciences Abroad*, 1–12.

Spivak, M. (1980). *Calculus*. Houston, TX: Publish or Perish Press.

Suppes, P. (1956). *Introduction to Logic*. New York: Van Nostrand.

Tarski, A. (1931). Sur les ensemble définissable de nombres réels I. *Fundamenta Mathematica*, **17**, 210–239.

Tarski, A. (1952). Some notions and methods on the borderline of algebra and metamathematics. In *Proceedings of the International Congress of Mathematicians, Cambridge, Massachusetts, U.S.A., August 30-September 6,1950*, Vol. 1. Providence, RI: American Mathematical Society, pp. 705–720.

Tarski, A. (1954). Contributions to the theory of models, I and II. *Indagationes Mathematicae*, **16**, 572–582.

Urbanik, R., & Hämäri, K. S. (2012). Busting a Myth about Leśniewski and Definitions. *History and Philosophy of Logic*, **33**, 159–189.

van den Dries, L. (1986). A generalization of the Tarski-Seidenberg theorem, and some nondefinability results. *Bulletin Of The American Mathematical Society*, **15**, 189–193.

van den Dries, L. (1999). *Tame Topology and O-Minimal Structures*. London Mathematical Society Lecture Note Series, 248.

Vaught, R. L. (1986). Alfred Tarski's work in model theory. *Journal of Symbolic Logic*, **51**, 869–882.

Vis, T. (Unpublished notes). *Desargues Theorem*. Unpublished notes. Available from: http://math.ucdenver.edu/∼tvis/Teaching/4220spring09/Notes/Desargues.pdf.

https://doi.org/10.1017/S1755020312000263 Published online by Cambridge University Press

Zilber, B. I. (2005). A categoricity theorem for quasiminimal excellent classes. In *Logic and its Applications*, Contemporary Mathematics. Providence, RI: AMS, pp. 297–306.
Zilber, B. I. (2010). *Zariski Geometries, Geometry from the Logician's Point of View*. Cambridge, UK: London Mathematical Society; Cambridge University Press.

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