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Poincaré theorems for random dynamical systems

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Abstract. In this paper, we establish analytic conjugacy theorems of Poincaré type and Poincaré–Dulac type for analytic random dynamical systems based on their Lyapunov exponents.

1. *Introduction*

The study of reducing an analytic deterministic dynamical system to the simplest possible form may be traced back to Poincaré, Birkhoff, and Siegel. Studying the simplest form, called a normal form, is easier and provides useful information for investigating the original system.

In this paper, we study this problem for analytic random dynamical systems and extend the Poincaré theorems in the deterministic case to random dynamical systems. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\theta^n)_{n\in\mathbb{Z}}$ be a measurable $\mathbb{P}$-measure-preserving dynamical system on Ω. A random dynamical system (or a cocycle) on the space $\mathbb{C}^d$ over the dynamical system θ^n is a measurable map

$$\phi : \mathbb{Z} \times \Omega \times \mathbb{C}^d \to \mathbb{C}^d, \quad (n, \omega, x) \mapsto \phi(n, \omega, x),$$

such that the map $\phi(n, \omega) := \phi(n, \omega, \cdot) : \mathbb{C}^d \to \mathbb{C}^d$ forms a cocycle over θ^n:

$$\phi(0, \omega) = \mathrm{Id} \quad \text{for all } \omega \in \Omega,$$
$$\phi(n + m, \omega) = \phi(n, \theta^m \omega) \circ \phi(m, \omega) \quad \text{for all } m, n \in \mathbb{Z},\ \omega \in \Omega.$$

A typical example is the solution operator for a stochastic differential equation. When $\phi(n, \omega, \cdot)$ is analytic for each n and ω, ϕ is called an analytic random dynamical system. We write the time-one map $\phi(1, \omega, x)$ as $\varphi(\omega, x) := \phi(1, \omega, x)$.

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Then $\varphi(\omega, \cdot)$ is the so-called random diffeomorphism that generates the random dynamical system $\phi(n, \omega, x)$:

$$\phi(n, \omega, \cdot) = \begin{cases} \varphi(\theta^{n-1}\omega, \cdot) \circ \cdots \circ \varphi(\omega, \cdot), & n > 0, \\ I, & n = 0, \\ (\varphi(\theta^n \omega, \cdot))^{-1} \circ \cdots \circ (\varphi(\theta^{-1}\omega, \cdot))^{-1}, & n < 0. \end{cases}$$

We consider an analytic random diffeomorphism $\varphi(\omega, x)$ in $\mathbb{C}^d$ with a fixed point $x = 0$, i.e. $\varphi(\omega, 0) = 0$ for each $\omega \in \Omega$. We assume that $\varphi(\omega, x)$ is locally tempered; that is, there is a tempered ball $U(\omega) = B_{\rho(\omega)}(0) = \{x \mid |x| < \rho(\omega)\}$, where $\rho(\omega)$ is a random variable tempered from below (i.e. $\lim_{n\to\pm\infty}(1/n)\log^- \rho(\theta^n\omega) = 0$, $\mathbb{P}$-almost surely), such that $\sup_{x\in U(\omega)} \|\varphi(\omega, x)\| = C(\omega)$ is tempered from above (i.e. $\lim_{n\to\pm\infty}(1/n)\log^+ C(\theta^n\omega) = 0$, $\mathbb{P}$-almost surely). The size of a tempered ball may decrease as ω varies, but these changes along each orbit $\theta^n\omega$ are at a subexponential rate. The upper bound of φ may grow to infinity as ω varies. However, along each orbit $\theta^n\omega$ it may only increase at a subexponential rate. This non-uniform behavior is one of the intrinsic features of random dynamical systems.

Two local tempered analytic random diffeomorphisms φ and ψ with fixed point $x = 0$ are analytically conjugated if there exists an analytic random diffeomorphism $h(\omega, x)$ defined in a tempered ball $V(\omega)$ with $h(\omega, 0) = 0$ such that

$$h(\theta\omega, \varphi(\omega, x)) = \psi(\omega, h(\omega, x)) \quad \text{for } x \in V(\omega) \text{ a.e.}$$

We write $\varphi(\omega, x)$ as

$$\varphi(\omega, x) = A(\omega)x + F(\omega, x),$$

where $A(\omega) = D\varphi(\omega, 0) \in Gl(d, \mathbb{C})$, $F(\omega, 0) = 0$ and $DF(\omega, 0) = 0$. We assume that the conditions of the Multiplicative Ergodic Theorem hold; that is,

$$\log^+ \|A(\cdot)\| \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \quad \text{and} \quad \log^+ \|A^{-1}(\cdot)\| \in L^1(\Omega, \mathcal{F}, \mathbb{P}).$$

Then, by the celebrated Oseledets' Multiplicative Ergodic Theorem, there exists an invariant set $\tilde{\Omega}$ of full measure such that for each $\omega \in \tilde{\Omega}$, $\Phi(n, \omega)$ (the linear random dynamical system generated by $A(\omega)$) has $p(\omega)$ Lyapunov exponents

$$\lambda_1(\omega) > \cdots > \lambda_{p(\omega)}(\omega).$$

Furthermore, $p(\omega)$, $\lambda_i(\omega)$ are θ^n invariant functions, i.e. $p(\theta\omega) = p(\omega)$ and $\lambda_i(\theta\omega) = \lambda_i(\omega)$. When θ^n is ergodic, $p(\omega)$ and $\lambda_i(\omega)$ are independent of ω.

Our main results may be summarized as follows.

MAIN THEOREM. *Assume that φ is a locally tempered analytic random diffeomorphism in $\mathbb{C}^d$ with a fixed point $x = 0$ and its linearization $A(\omega)$ satisfies the conditions of the Multiplicative Ergodic Theorem.*

(i) Conjugacy in jet class. *If $x = 0$ is stable or unstable fixed point, i.e. all Lyapunov exponents $\lambda_i(\omega)$ for $1 \le i \le p(\omega)$ have the same sign, then any local tempered analytic random diffeomorphism ψ in $\mathbb{C}^d$ with $\varphi - \psi = O(\|x\|^{N(\omega)})$, $N(\omega) > \alpha(\omega)/\beta(\omega)$ is analytically conjugated to φ, where $\alpha(\omega) = \max|\lambda_i(\omega)|$, $\beta(\omega) = \min|\lambda_i(\omega)|$, and $N(\omega)$ is θ-invariant random variable with integer values and is constant when θ is ergodic.*

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(ii) Conjugacy to random polynomial. *If all Lyapunov exponents* $\lambda_i(\omega)$, $1 \le i \le p(\omega)$, *have the same sign, then* φ *is analytically conjugated to a random polynomial of degree not greater than* $\alpha(\omega)/\beta(\omega)$.

(iii) Poincaré type. *If all Lyapunov exponents* $\lambda_i(\omega)$, $1 \le i \le p(\omega)$, *have the same sign and are non-resonant, i.e.*

$$\lambda_i(\omega) \neq (m, \lambda(\omega)), \quad |m| \ge 2,$$

then φ *is analytically conjugated to its linear part.*

(iv) Poincaré–Dulac type. *If the linear part of* $\varphi(\omega, x)$ *is diagonal and all Lyapunov exponents* $\lambda_i(\omega)$, $1 \le i \le p(\omega)$, *have the same sign, then* φ *is analytically conjugated to a random polynomial which consists of resonant monomials only.*

Remark. The condition that Lyapunov exponents have the same sign corresponds to the Poincaré condition on the eigenvalues for deterministic systems. The non-resonant conditions here are in terms of Lyapunov exponents, which reflects the limit behavior of the system. That the Lyapunov exponents satisfy a non-resonant condition does not imply that the eigenvalues of the linearized system $A(\omega)$ satisfy a non-resonant condition. The linearized system is non-uniform hyperbolic in nature in the sense of Pesin for deterministic systems. The orders of random polynomials in (ii) and (iv) may change as ω varies, but do not change along each orbit $\theta^n\omega$. When the Lyapunov exponents are distinct, the linear part of $\varphi(\omega, x)$ is diagonalizable. When ϕ is a real analytic random diffeomorphism, the restriction of the conjuagacy map to $\mathbb{R}^d$ is a real analytic random diffeomorphism.

The study of reducing a deterministic dynamical system to a linear normal form has a long history. Poincaré [**24**] first studied the reducibility of an analytic differential equation to its linear part. He proved that if the eigenvalues of the linear part belong to the so-called Poincaré domain and satisfy the non-resonant conditions, then the nonlinear equation is analytically conjugated to its linear part. The complementary of the Poincaré domain is the so-called Siegel domain. Siegel [**30**] gave a small divisor condition for the eigenvalues in the Siegel domain under which the nonlinear equation is analytically conjugated to the linear equation. Brjuno [**6**] gave a weaker condition than Siegel's condition and proved that the result also holds. Siegel's theorem was also proved by Arnold [**3**], Moser [**21**], and Zehnder [**35**]. A simple proof of Poincaré's theorem was given by Meyer [**20**]. The Poincaré–Dulac theorem was discussed in [**3**]. A generalized Siegel's theorem for quasiperiodic systems was proved in [**8**].

The study of the reducibility of a deterministic differentiable dynamical system to a linear normal form was initiated by Sternberg. Since then, there has been extensive literature on this problem for finding better conditions for higher regularities. We do not attempt to give an exhaustive list of references. For some of the basic results, we refer to [**4, 5, 7, 10, 11, 22, 27, 28, 31, 32**], and the references therein. A C^0 linearization (a Hartman–Grobman Theorem) associated with the local structural stability may be found in Grobman [**9**], Hartman [**10**], Pugh [**25**] and Palmer [**23**].

Random dynamical systems arise in the modeling of many phenomena in physics, biology, climatology, etc. The random effects, such as noise, are not only introduced

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to compensate for the defects in deterministic models, but are often also rather intrinsic phenomena. The study of random dynamical systems goes back to Ulam and von Neumann [**33**]. It has flourished since the 1980s due to the discovery that stochastic ordinary differential equations generate random dynamical systems. There is extensive literature on this subject, and we refer the reader to [**1, 14, 15, 19, 26**] and the references therein. Here we simply limit ourselves to the conjugacy problem. For random dynamical systems, Wanner [**34**] proved a Hartman–Grobman theorem (see also Arnold [**1**]). A structural stability theorem of deterministic dynamical systems under random perturbations was obtained by Liu [**18**]. In [**2**] (see also [**1**]), Arnold and Xu gave a theorem on formal Linearization of random diffeomorphisms. In [**16**], the present authors established theorems of Sternberg type for random dynamical systems. A theorem of Siegel type for random dynamical systems was presented in [**17**].

In §2, we introduce some basic concepts on random dynamical systems, the Multiplicative Ergodic Theorem and related results. In §3, we establish analytic conjugacy theorems of Poincaré type for analytic random dynamical systems.

2. *Random dynamical systems*

In this section, we review some basic concepts and results on random dynamical systems, which are taken from [**1**]. We first introduce basic concepts on random dynamical systems. Then, we state the multiplicative ergodic theorem and related results for random dynamical systems.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$. $\mathbb{T}$ is endowed with its Borel σ-algebra $\mathcal{B}(\mathbb{T})$.

Definition 2.1. A family $(\theta^t)_{t\in\mathbb{T}}$ of mappings from Ω into itself is called a metric dynamical system if:

(1) $(\omega, t) \to \theta^t\omega$ is $\mathcal{F} \otimes \mathcal{B}(\mathbb{T})$ measurable;

(2) $\theta^0 = \mathrm{id}_\Omega$, the identity on Ω, $\theta^{t+s} = \theta^t \circ \theta^s$ for all $t, s \in \mathbb{T}$;

(3) θ^t preserves the probability measure $\mathbb{P}$.

Definition 2.2. A map

$$\phi : \mathbb{T} \times \Omega \times \mathbb{C}^d \to \mathbb{C}^d, \quad (t, \omega, x) \mapsto \phi(t, \omega, x),$$

is called a random dynamical system (or a cocycle) on the measurable space $(\omega, \mathcal{F})$ over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta^t)$ if:

(1) ϕ is $\mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{C}^d)$-measurable;

(2) the map $\phi(t, \omega) := \phi(t, \omega, \cdot) : \mathbb{C}^d \to \mathbb{C}^d$ forms a cocycle over θ^t:

$$\phi(0, \omega) = \mathrm{Id} \quad \text{for all } \omega \in \Omega,$$
$$\phi(t + s, \omega) = \phi(t, \theta^s\omega) \circ \phi(s, \omega) \quad \text{for all } t, s \in \mathbb{T},\ \omega \in \Omega.$$

In this paper, we consider discrete random dynamical systems. We assume that 0 is a fixed point of $\phi(n, \omega, \cdot)$, i.e. $\phi(n, \omega, 0) = 0$ for any $n \in \mathbb{Z}$ and $\omega \in \Omega$. Then, we rewrite $\phi(n, \omega, x)$ as

$$\phi(n, \omega, x) = \Phi(n, \omega)x + f(n, \omega, x),$$

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where $\Phi(n,\omega) \in Gl(d,\mathbb{R})$, $f(n,\omega,0) = 0$ and $Df(n,\omega,0) = 0$. Here, Df is the derivative of f with respect to x.

We write the time-one map $\phi(1,\omega,x)$ as

$$\phi(\omega,x) := \phi(1,\omega,x) = A(\omega)x + F(\omega,x)$$

where $A(\omega) = \Phi(1,\omega) \in Gl(d,\mathbb{R})$, $F(\omega,x) = f(1,\omega,x)$. Note that $F(\omega,0) = 0$ and $DF(\omega,0) = 0$. Let $\{x_n\}$ be an orbit of $\phi(n,\omega,x)$ with initial point x_0, i.e. $x_n = \phi(n,\omega,x_0)$. Then, $\{x_n\}$ satisfies the equation

$$x_{n+1} = A(\theta^n\omega)x_n + F(\theta^n\omega, x_n).$$

Conversely, a solution of the above equation is also an orbit of $\phi(n,\omega,x_0)$.

The following theorem is Oseledets' Multiplicative Ergodic Theorem (see [**1**]).

THEOREM 2.3. (Multiplicative Ergodic Theorem) *Let Φ be a linear random dynamical system in $\mathbb{C}^d$ with two-side time over the metric dynamical system $(\Omega,\mathcal{F},\mathbb{P},\theta^n)$. Let*

$$\Phi(n,\omega) = \begin{cases} A(\theta^{n-1}\omega)\dots A(\omega), & n > 0, \\ I, & n = 0, \\ A^{-1}(\theta^n\omega)\dots A^{-1}(\theta^{-1}), & n < 0, \end{cases}$$

be generated by $A : \Omega \to Gl(d,\mathbb{R})$ and assume

$$\log^+ \|A(\cdot)\| \in L^1(\Omega,\mathcal{F},\mathbb{P}) \quad \textit{and} \quad \log^+ \|A^{-1}(\cdot)\| \in L^1(\Omega,\mathcal{F},\mathbb{P}). \tag{1}$$

Then there exists an invariant set $\tilde{\Omega}$ of full measure such that for each $\omega \in \tilde{\Omega}$ we have the following.

(i) *The limit $\lim_{n\to\infty}(\Phi(n,\omega)^*\Phi(n,\omega))^{1/2n} =: \Psi(\omega) > 0$ exists.*

(ii) *Let $e^{\lambda_{p(\omega)}(\omega)} < \dots < e^{\lambda_1(\omega)}$ be the different eigenvalues of $\Psi(\omega)$ and let $U_{p(\omega)}(\omega),\dots,U_1(\omega)$ be the corresponding eigenspaces with multiplicities $d_i(\omega) := \dim U_i(\omega)$. Then*

$$\begin{aligned} p(\theta\omega) &= p(\omega) \\ \lambda_i(\theta\omega) &= \lambda_i(\omega) \quad \textit{for all } i \in \{1,\dots,p(\omega)\}, \\ d_i(\theta\omega) &= d_i(\omega) \quad \textit{for all } i \in \{1,\dots,p(\omega)\}, \\ \mathbb{C}^d &= U_1(\omega) \oplus \dots \oplus U_{p(\omega)}. \end{aligned}$$

(iii) *If $(\Omega,\mathcal{F},\mathbb{P},\theta^t)$ is ergodic, then the functions $p(\omega)$, $\lambda_i(\omega)$, and $d_i(\omega)$ are constant on $\tilde{\Omega}$.*

(iv) *For each $\omega \in \tilde{\Omega}$ there exists a splitting*

$$\mathbb{C}^d = E_1(\omega) \oplus \dots \oplus E_{p(\omega)}$$

of $\mathbb{C}^d$ into random subspaces $E_i(\omega)$ with dimension $d^i(\omega)$. If $\Pi_i(\omega) : \mathbb{C}^d \to E_i(\omega)$ denotes the corresponding projection onto $E_i(\omega)$, then

$$A(\omega)\Pi_i(\omega) = \Pi_i(\theta\omega)A(\omega),$$

equivalently

$$A(\omega)E_i(\omega) = E_i(\theta\omega).$$

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(v) *We have*

$$\lim_{n\to\pm\infty}\frac{1}{n}\log\|\Phi(n,\omega)x\| = \lambda_i(\omega) \Longleftrightarrow x \in E_i(\omega)\setminus\{0\}.$$

(vi) *Measurability: the functions* $\omega \to p(\omega) \in \{1,\dots,d\}$, $\omega \to \lambda_i(\omega) \in \mathbb{R}$, $\omega \to d_i(\omega) \in \{1,\dots,d\}$, $\omega \to E_i(\omega)$ *and* $\omega \to \Pi_i(\omega)$ *are measurable.*

Here $\lambda_i(\omega)$ are the so-called Lyapunov exponents and $E_i(\omega)$ are the so-called Oseledets spaces. For the remainder of the paper, we denote $\tilde{\Omega}$ by Ω and assume that these statements are true for all $\omega \in \Omega$.

The next concept is of fundamental importance in the study of random dynamical systems.

Definition 2.4.

(i) A random variable $R : \Omega \to (0,\infty)$ is called tempered with respect to a metric dynamical system θ^t if

$$\lim_{n\to\pm\infty}\frac{1}{n}\log R(\theta^n\omega) = 0 \quad \mathbb{P}\text{-a.s.}$$

(ii) $R : \Omega \to [0,\infty)$ is called tempered from above if

$$\lim_{n\to\pm\infty}\frac{1}{n}\log^+ R(\theta^n\omega) = 0 \quad \mathbb{P}\text{-a.s.}$$

(iii) $R : \Omega \to (0,\infty)$ is called tempered from below if $1/R$ is tempered from above.

LEMMA 2.5. *Let* $f : \Omega \to (0,\infty)$ *be tempered and* $\gamma : \Omega \to (0,\infty)$ *be a* θ*-invariant random variable. Then, there is a tempered random variable* $R(\omega)$ *such that:*

(i) $1/R(\omega) \le f(\omega) \le R(\omega)$*;*

(ii) $e^{-\gamma(\omega)|n|}R(\omega) \le R(\theta^n\omega) \le e^{\gamma(\omega)|n|}R(\omega)$.

This lemma is a consequence of [**1**, Proposition 4.3.3]. In fact, letting for $k \in \mathbb{N}$

$$\tilde{\Omega}_k = \left\{\omega \in \Omega \;\middle|\; \frac{1}{k} \le \gamma(\omega) < \frac{1}{k-1}\right\},$$

then, by [**1**, Proposition 4.3.3], on each $\tilde{\Omega}_k$ there is a $1/k$-slowly varying random variable $R_{1/k}(\omega)$, i.e. $e^{-(1/k)|n|}R_{1/k}(\omega) \le R_{1/k}(\theta^n\omega) \le e^{(1/k)|n|}R_{1/k}(\omega)$, such that

$$\frac{1}{R_{1/k}(\omega)} \le f(\omega) \le R_{1/k}(\omega).$$

Clearly, $e^{-\gamma(\omega)|n|}R_{1/k}(\omega) \le R_{1/k}(\theta^n\omega) \le e^{\gamma(\omega)|n|}R_{1/k}(\omega)$ since $1/k \le \gamma(\omega)$ and $\gamma(\theta\omega) = \gamma(\omega)$. Define a random variable $R : \Omega \to (0,\infty)$ as $R(\omega) = R_{1/k}(\omega)$ for $\omega \in \tilde{\Omega}_k$. Then, $R(\omega)$ satisfies Lemma 2.5.

The next lemma is a consequence of the Multiplicative Ergodic Theorem 2.3 and its proof follows the same lines as Theorem 4.3.4 in [**1**].

LEMMA 2.6. *Assume that* $A(\omega)$ *satisfies the conditions of the Multiplicative Ergodic Theorem 2.3.*

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(i) *Suppose that all the Lyapunov exponents are negative. Then for each* $0 < \epsilon < 1$ *there exists a tempered function* $K(\omega)$ *such that*

$$\begin{aligned} \|\Phi(n,\omega)\| &\le K(\omega)e^{-((1-\epsilon/2)\beta(\omega))n} \quad \text{for } n \in \mathbb{N}, \\ \|\Phi(-n,\omega)\| &\le K(\omega)e^{((1+\epsilon/2)\alpha(\omega))n} \quad \text{for } n \in \mathbb{N}, \end{aligned} \tag{2}$$

where $\alpha(\omega) = \max|\lambda_i(\omega)|$ *and* $\beta(\omega) = \min|\lambda_i(\omega)|$.

(ii) *Suppose that all the Lyapunov exponents are positive. Then for each* $0 < \epsilon < 1$ *there exists a tempered function* $K(\omega)$ *such that*

$$\begin{aligned} \|\Phi(-n,\omega)\| &\le K(\omega)e^{-((1-\epsilon/2)\beta(\omega))n} \quad \text{for } n \in \mathbb{N}, \\ \|\Phi(n,\omega)\| &\le K(\omega)e^{((1+\epsilon/2)\alpha(\omega))n} \quad \text{for } n \in \mathbb{N}. \end{aligned} \tag{3}$$

As ω varies, $\beta(\omega)$ may be arbitrarily small and $K(\omega)$ may be arbitrarily large. However, along each orbit $\theta^n\omega$, $\beta(\omega)$ is a constant and $K(\omega)$ may increase only at a subexponential rate, which together with condition (2) or (3) imply that the linear system $\Phi(n,\omega)$ is non-uniformly hyperbolic in the sense of Pesin for deterministic dynamical systems.

3. *Analytic conjugacy*

We first give a general theorem (Theorem 3.2) on analytic conjugacy in a jet class. Then we present the implications of this theorem including a theorem of Poincaré type, a theorem of Poincaré–Dulac type and a theorem on normal forms. An example of normal forms for a two-dimensional random dynamical system over an ergodic metric dynamical system is also given.

Before we state our assumption, we introduce a concept on a tempered neighborhood. A ball $U(\omega) = B_{\rho(\omega)}(0) = \{x \mid |x| < \rho(\omega)\}$ is called a tempered ball if $\rho(\omega)$ is a random variable tempered from below.

We assume that the random dynamical system $\phi(n,\omega,x)$ satisfies the following hypothesis.

STANDING HYPOTHESIS.

(i) $\phi(n,\omega,x)$ *is a locally tempered analytic random dynamical system; that is, there is a tempered ball such that* $\sup_{x\in U(\omega)}\|\phi(1,\omega,x)\| = C(\omega)$ *is tempered from above, with fixed point* $x = 0$.

(ii) $A(\omega) = D\phi(1,\omega,0)$ *satisfies condition (1)—the condition of the Multiplicative Ergodic Theorem 2.3.*

Definition 3.1. Two locally tempered analytic random dynamical systems $\phi(n,\omega,x)$ and $\psi(n,\omega,x)$ with a fixed point $x = 0$ are said to be analytically conjugated if there exists an analytic random diffeomorphism $h(\omega,x)$ from a tempered ball $V(\omega)$ to a tempered ball $W(\omega)$ with $h(\omega,0) = 0$ such that

$$h(\theta\omega,\phi(1,\omega,x)) = \psi(1,\omega,h(\omega,x)) \quad \text{for } x \in V(\omega) \text{ a.e.} \tag{4}$$

The conjugacy relationship (4) implies that $h(\theta^n\omega,\phi(n,\omega)x) = \psi(n,\omega)h(\omega,x)$ (h carries an orbit of ϕ to an orbit of ψ) when the orbits stay in the corresponding domains and is a θ-pathwise equation.

The next theorem gives an analytic conjugacy in a jet class.

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THEOREM 3.2. *Assume that the Standing Hypothesis holds and that all Lyapunov exponents $\lambda_i(\omega)$ for $1 \le i \le p(\omega)$ have the same sign. Then any locally tempered analytic random dynamical system ψ with $\phi(1,\omega,x) - \psi(1,\omega,x) = O(|x|^N)$, $N = N(\omega) > \alpha(\omega)/\beta(\omega)$, is analytically conjugated to ϕ, where $\alpha(\omega) = \max_{1\le i\le p(\omega)} |\lambda_i(\omega)|$, $\beta(\omega) = \min_{1\le i\le p(\omega)} |\lambda_i(\omega)|$ and $N(\omega)$ is a θ-invariant random variable with positive integer values.*

Remark. Both $\alpha(\omega)$ and $\beta(\omega)$ are θ-invariant random variables and $N(\omega)$ may be taken as $N(\omega) = [\alpha(\omega)/\beta(\omega)] + 1$. When θ^n is ergodic, $\alpha(\omega)$ and $\beta(\omega)$ are constant, thus N may be any integer larger than $\alpha(\omega)/\beta(\omega)$. When θ^n is not ergodic, the condition $N(\omega) > \alpha(\omega)/\beta(\omega)$ reflexes the non-uniform aspect of system. In the non-ergodic case, since $[\alpha(\omega)/\beta(\omega)]$ is a θ-invariant measurable function with integer values, we may write Ω as a union of countably many disjointed θ-invariant measurable sets

$$\Omega = \bigcup_{i\ge 1} \Omega_i \tag{5}$$

such that on each Ω_i, $[\alpha(\omega)/\beta(\omega)]$ is a constant. Instead of solving (4) for all $\omega \in \Omega$ at once, we solve (4) over each Ω_i for $h_i(\omega, x)$. Patching all h_i together gives the desired $h(\omega, x)$. We point out that equation (4) is θ-pathwise.

We assume from now that all Lyapunov exponents are negative. Otherwise, we consider its time reverse system. The proof of this theorem is based on the following lemmas.

The next lemma states that one may reduce the problem of analytic conjugacy (4) to a problem of solving a linear recurrent functional equation. This idea is based on the so-called homotopy method which was used in [**12, 13**] for finitely smooth normal forms of deterministic dynamical systems. Here we extend it further so that it can be applied to random dynamical systems where the hyperbolicity is non-uniform. Our modified approach also allow us to establish theorems of Sternberg type [**16**].

LEMMA 3.3. *Let $\phi_s(n,\omega)x := \phi_s(n,\omega,x)$, $s \in [0,1]$, $n \in \mathbb{Z}$, be a family of locally tempered analytic random dynamical systems in a tempered ball $U(\omega) = B(0,\rho(\omega))$ with fixed point $x = 0$ whose time-one map has the form*

$$\varphi_s(\omega,x) := \phi_s(1,\omega,x) = \phi(1,\omega,x) + sR(\omega,x).$$

If there exists a measurable mapping $r : \Omega \times U(\omega) \times [0,1] \to \mathbb{C}^d$ such that:

(i) *$r(\omega,\cdot,\cdot)$ is analytic and satisfies*

$$\|r(\omega,x,s)\| \le a(\omega)\|x\|^2 \quad \text{for } x \in U(\omega), \tag{6}$$

where $a(\omega)$ is a random variable tempered from above;

(ii) *r satisfies the following equation*

$$\frac{\partial \varphi_s}{\partial x} r(\omega,x,s) - r(\theta\omega, \varphi_s(\omega,x), s) = -R(\omega,x); \tag{7}$$

then ϕ_0 and ϕ_1 are analytically conjugated.

Proof. We first introduce a family of analytic diffeomorphisms defined on $U(\omega) \times [0,1]$

$$\psi_\omega : (x,s) \mapsto (\varphi_s(\omega,x), s)$$

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and a family of vector fields

$$V_\omega = (r(\omega, x, s), 1).$$

We consider ω as a parameter here. Then, using (7), we have

$$D\psi_\omega V_\omega = V_{\theta\omega} \circ \psi_\omega. \tag{8}$$

Denote by g_ω^t the local flow generated by V_ω, which is an analytic random diffeomorphism. Then, from (8), it follows that

$$\psi_\omega \circ g_\omega^t = g_{\theta\omega}^t \circ \psi_\omega. \tag{9}$$

Using (6), we have that g_ω^t is well defined for x in a tempered ball $B(0, \tilde{\rho}(\omega))$, $\tilde{\rho}(\omega) = \rho(\omega)/(1 + a(\omega)\rho(\omega))$ and $t \in [0, 1]$. Write $g_\omega^1(x, 0) = (h(\omega, x), 1)$. Then, we have

$$\begin{aligned}\psi_\omega \circ g_\omega^1(x, 0) &= \psi_\omega(h(\omega, x), 1) = (\varphi_1(\omega, h(\omega, x)), 1)\\ &= (\phi(1, \omega, h(\omega, x)) + R(\omega, h(\omega, x)), 1)\end{aligned}$$

and

$$g_{\theta\omega}^1 \circ \psi_\omega(x, 0) = g_{\theta\omega}^1(\varphi_0(\omega, x), 0) = g_{\theta\omega}^1(\phi(\omega, x), 0) = (h(\theta\omega, \phi(1, \omega, x)), 1),$$

which together with (9) yields

$$h(\theta\omega, \phi(1, \omega, x)) = \phi(1, \omega, h(\omega, x)) + R(\omega, h(\omega, x)).$$

This completes the proof of this lemma. □

For simplicity, we set $(f_* R)(x) := (Df\ R) \circ f^{-1}(x)$ for an analytic diffeomorphism $f : \mathbb{C}^d \to \mathbb{C}^d$ and a vector field R in $\mathbb{C}^d$. Note that $f_*(g_* R) = (f \circ g)_* R$. Then, we have a formal solution to equation (7).

LEMMA 3.4. *The function*

$$r(\omega, x, s) = -\sum_{m=1}^{\infty}((\phi_s(m, \omega)^{-1})_* R(\theta^{m-1}\omega, \varphi_s^{-1}(\theta^{m-1}\omega, \cdot)))(x) \tag{10}$$

is a formal solution of (7).

Proof. The proof of this lemma is quite straightforward. Let $\tilde{R}(\omega, x, s) = R(\omega, \varphi_s^{-1}(\omega, x))$. Then, equation (7) becomes

$$((\varphi_s)_* r(\omega, \cdot, s))(x) = r(\theta\omega, x, s) - \tilde{R}(\omega, x, s). \tag{11}$$

Note that $\varphi_s(\omega, \cdot) = \phi_s(1, \omega)$. Then, we have

$$\begin{aligned}(\varphi_s)_* r &= -\sum_{m=1}^{\infty}((\phi_s(1, \omega)\phi_s(m, \omega)^{-1})_* \tilde{R}(\theta^{m-1}\omega, \cdot, s))(x)\\ &= -\sum_{m=1}^{\infty}((\phi_s(m-1, \theta\omega)^{-1})_* \tilde{R}(\theta^{m-1}\omega, \cdot, s))(x)\\ &= -\tilde{R}(\omega, x, s) - \sum_{m=1}^{\infty}((\phi_s(m, \theta\omega)^{-1})_* \tilde{R}(\theta^m\omega, \cdot, s))(x)\\ &= -\tilde{R}(\omega, x, s) + r(\theta\omega, x, s).\end{aligned}$$

□

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Let $R(\omega, x) = \phi(1, \omega, x) - \psi(1, \omega, x)$ and $\phi_s(t, \omega, x)$ be a family of random dynamical systems whose time-one maps have the form

$$\phi_s(1, \omega, x) = \phi(1, \omega, x) + sR(\omega, x), \quad s \in D_2 := \{z \in \mathbb{C} : |z| < 2\}.$$

Assume that both $\phi_s(1, \omega, x)$ and $\phi_s(-1, \omega, x)$ are well defined in a tempered ball $B_{\rho(\omega)} = B(0, \rho(\omega))$.

The next lemma states there is a tempered ball on which all positive orbits are well defined, each orbit decays exponentially and non-uniformly, and the derivative of the inverse of ϕ_s is bounded by an exponential function.

LEMMA 3.5. *For each $0 < \epsilon < 1$, there exists a random variable $d(\omega)$ tempered from above and a tempered ball $B_{b(\omega)}$ such that:*

(i) $\phi_s(n, \omega, \cdot)B_{b(\omega)} \subset B_{b(\theta^n\omega)}$ *for all* $n \in \mathbb{N}, s \in D_2$;

(ii) $|\phi_s(n, \omega, x)| \le d(\omega)e^{-(1-\epsilon)\beta(\omega)n}|x|$ *for all* $n \in \mathbb{N}$, $x \in B_{b(\omega)}, s \in D_2$;

(iii) $\|D_x\phi_s^{-1}(n, \omega, x)\| \le d(\omega)e^{(1+\epsilon)\alpha(\omega)n}$ *for all* $n \in \mathbb{N}, x \in \phi_s(n, \omega, \cdot)B_{b(\omega)}, s \in D_2$.

Proof. By Lemma 2.6, there exists a tempered function $K(\omega)$ such that

$$\begin{aligned} \|\Phi(n, \omega)\| &\le K(\omega)e^{-((1-\epsilon/2)\beta(\omega))n} \quad \text{for } n \in \mathbb{N}, \\ \|\Phi(-n, \omega)\| &\le K(\omega)e^{((1+\epsilon/2)\alpha(\omega))n} \quad \text{for } n \in \mathbb{N}. \end{aligned} \tag{12}$$

We write $\phi_s(1, \omega, x)$

$$\phi_s(1, \omega, x) = \Phi(1, \omega)x + F(\omega, x, s),$$

where $\Phi(1, \omega) = D\phi(1, \omega, 0)$ is the linear part of $\phi_s(1, \omega, x)$. Let

$$B_2(\omega) = \sup_{|x| \le \rho(\omega)/2, s \in D_2} \|D_x^2 F\|.$$

Then $\|D_x F\| \le B_2(\omega)|x|$ for $|x| \le \rho(\omega)/2$. Denote

$$c(\omega) = \min\left\{\frac{\rho(\omega)}{2}, \frac{\epsilon\beta(\omega)e^{-\beta(\omega)}}{2K(\theta\omega)B_2(\omega)}\right\}.$$

Then $c(\omega)$ is a random variable tempered from below and

$$B_1(\omega) := \sup_{|x| < c(\omega),\, s \in D_2} \|D_x F(\omega, x)\| < \frac{\epsilon\beta(\omega)e^{-\beta(\omega)}}{2K(\theta\omega)}. \tag{13}$$

Denote by $x_n = \phi_s(n, \omega, x)$ with $x_0 = x$. Since x_n satisfies the constant of variation formula

$$x_n = \Phi(n, \omega)x + \sum_{i=0}^{n-1} \Phi(n - 1 - i, \theta^{i+1}\omega)F(\theta^i\omega, x_i, s). \tag{14}$$

Using (12), if $|x_i| < c(\theta^i\omega)$, $0 \le i \le n - 1$, we have

$$|x_n| \le e^{-(1-\epsilon/2)\beta(\omega)n}K(\omega)|x_0| + \sum_{i=0}^{n-1} e^{-(1-\epsilon/2)\beta(\omega)(n-1-i)}K(\theta^{1+i}\omega)B_1(\theta^i\omega)|x_i|,$$

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which together with (13) implies that

$$e^{(1-\epsilon/2)\beta(\omega)n}|x_n| \le K(\omega)|x_0| + \frac{\epsilon\beta(\omega)}{2}\sum_{i=0}^{n-1} e^{(1-\epsilon/2)\beta(\omega)i}|x_i|.$$

It follows from the Discrete Time Gronwall–Bellman Lemma [**1**, p. 327] that

$$|x_n| \le K(\omega)|x_0|e^{-(1-\epsilon/2)\beta(\omega)n}\left(1+\frac{\epsilon\beta(\omega)}{2}\right)^n \le K(\omega)|x_0|e^{-(1-\epsilon)\beta(\omega)n}.$$

Lemma 2.5 implies that we may choose a random variable $b(\omega)$ tempered from below:

$$b(\omega) \le \frac{1}{1+K(\omega)}e^{(1-\epsilon)\beta(\omega)n}c(\theta^n\omega) \quad \text{for all } n \in \mathbb{N}.$$

Then the conclusions (i) and (ii) are proved with any $d(\omega) \ge K(\omega)$. We will fix $d(\omega)$ later.

Next we show (iii). We consider the time-minus-one map of ϕ_s:

$$\phi_s(-1,\omega,x) = \Phi(-1,\omega)x + \tilde{F}(\omega,x,s).$$

Denote by $\mathcal{U}_n(\omega) = \phi_s(n,\theta^{-n}\omega,\cdot)B_{b(\theta^{-n}\omega)}$, Then by (i), $\mathcal{U}_n(\omega) \subset B_{\rho(\omega)}$. Thus, for each $x \in \mathcal{U}_n(\omega)$, $\phi_s(-n,\omega,x)$ is well defined. Let

$$\tilde{B}_1(\omega) = \sup_{|x|<\rho(\omega),s\in D_2} \|D_x\tilde{F}\|.$$

Since $D_x\tilde{F}(\omega,0,s) = 0$, we choose $\rho(\omega)$ so small that

$$\tilde{B}_1(\omega)K(\theta^{-1}\omega) < \frac{\epsilon\alpha(\omega)}{3}. \tag{15}$$

Since x_{-n} satisfies the constant of variation formula

$$x_{-n} = \Phi(-n,\omega)x + \sum_{i=0}^{n-1}\Phi(-n+i+1,\theta^{-i-1}\omega)\tilde{F}(\theta^{-i}\omega,x_{-i},s), \tag{16}$$

differentiating x_{-n} in x and using (12), we have

$$|D_x(x_{-n})| \le e^{(1+\epsilon/2)\alpha(\omega)n}K(\omega) + \sum_{i=0}^{n-1} e^{(1+\epsilon/2)\alpha(\omega)(n-1-i)}K(\theta^{-1-i}\omega)\tilde{B}_1(\theta^{-i}\omega)|D_x(x_{-i})|.$$

Then, using (15), we obtain

$$e^{-(1+\epsilon/2)\alpha(\omega)n}|D_x(x_{-n})| \le K(\omega) + \frac{\epsilon\alpha(\omega)}{3}\sum_{i=0}^{n-1} e^{-(1+\epsilon/2)\alpha(\omega)i}|D_x(x_{-i})|.$$

It follows from the Discrete Time Gronwall–Bellman Lemma [**1**, p. 327] that

$$|D(x_{-n})_x| \le K(\omega)e^{(1+\epsilon/2)\alpha(\omega)n}\left(1+\frac{\epsilon\alpha(\omega)}{3}\right)^n < K(\omega)e^{(1+5\epsilon/6)\alpha(\omega)n},$$

which, by cocycle properties, implies that

$$\|D_x\phi_s^{-1}(n,\omega,x)\| = \|D_x\phi_s(-n,\theta^n\omega,x)\| \le K(\theta^n\omega)e^{(1+5\epsilon/6)\alpha(\omega)n} \le d(\omega)e^{(1+\epsilon)\alpha(\omega)n},$$

for some tempered function $d(\omega)$. Here, Lemma 2.5 is used to have the random variable $d(\omega)$ such that $K(\omega) \le d(\omega)$ and $d(\theta^n\omega) \le d(\omega)e^{(\epsilon/6)\alpha(\omega)n}$. This completes the proof. □

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Let

$$R_1(\omega, x, s) = R(\omega, \phi_s^{-1}(1, \omega, x)).$$

We may assume that R_1 is well defined in a tempered ball $|x| < 2c(\omega)$. For a θ-invariant function $N = N(\omega) \geq 2$ with integer values, we set

$$C(\omega) = \sup_{|x|<c(\omega), s\in D_2} \|D_x^N R_1\|.$$

Since $N(\omega)$ is a θ-invariant function with integer values, Ω may be decomposed into a union of countably many disjointed θ-invariant measurable sets $\tilde{\Omega}_i$ on which $N(\omega)$ is a constant and $C(\omega)$ is tempered from above. Clearly, $C(\omega)$ is tempered from above in Ω. Using the Taylor expansion, we obtain

$$|R_1(\omega, x, s)| \leq C(\omega)|x|^N \quad \text{for } |x| \leq c(\omega). \tag{17}$$

The next lemma states that the formal solution defined in Lemma 3.4 with $R = R_1$ actually converges to an analytic function.

LEMMA 3.6. *If $N(\omega) > \alpha(\omega)/\beta(\omega)$, the functions $r(\omega, \cdot, \cdot)$ defined by*

$$r(\omega, x, s) = -\sum_{n=1}^{\infty} (\phi_s^{-1}(n, \omega, x)_* R_1(\theta^{n-1}\omega, \cdot, s))(x), \quad (x, s) \in B_{b(\omega)} \times D_2, \tag{18}$$

is analytic and satisfy (6).

Proof. It is enough to show that the series (18) converges uniformly in x for each fixed $\omega \in \Omega$. Fix $\epsilon > 0$ so small that $(1-\epsilon)\beta(\omega)N > (1+2\epsilon)\alpha(\omega)$. Then, from Lemma 3.5 and (17) it follows that for $|x| < b(\omega)$

$$\begin{aligned}
|r(\omega, x, s)| &\leq \sum_{n=1}^{\infty} |(\phi_s^{-1}(n, \omega)_* R_1(\theta^{n-1}\omega, \cdot, s))(x)| \\
&\leq \sum_{n=1}^{\infty} d(\omega) e^{(1+\epsilon)\alpha(\omega)n} C(\theta^{n-1}\omega) |\phi_s(n, \omega)|^N \\
&\leq \sum_{n=1}^{\infty} d(\omega)^{N+1} e^{(1+\epsilon)\alpha(\omega)n} C(\theta^{n-1}\omega) e^{-(1-\epsilon)\beta(\omega)nN} |x|^N \\
&\leq \sum_{n=1}^{\infty} d(\omega)^{N+1} C(\theta^{n-1}\omega) e^{-[(1-\epsilon)\beta(\omega)N-(1+\epsilon)\alpha(\omega)]n} |x|^N \\
&\leq \sum_{n=1}^{\infty} d(\omega)^{N+1} \tilde{C}(\omega) e^{-[(1-\epsilon)\beta(\omega)N-(1+2\epsilon)\alpha(\omega)]n} |x|^N,
\end{aligned} \tag{19}$$

where Lemma 2.5 is used to have a random variable $\tilde{C}(\omega)$ from above such that $C(\omega) \leq \tilde{C}(\omega)$ and $\tilde{C}(\theta^{n-1}\omega) \leq e^{\epsilon\alpha(\omega)n}\tilde{C}(\omega)$. Since $(1-\epsilon)\beta(\omega)N > (1+2\epsilon)\alpha(\omega)$, the last series above converges uniformly for $|x| < b(\omega)$ and is bounded by

$$d(\omega)^{N+1}\tilde{C}(\omega)/(1 - e^{-[(1-\epsilon)\beta(\omega)N-(1+2\epsilon)\alpha(\omega)]})|x|^N \leq a(\omega)|x|^N,$$

where

$$a(\omega) = \frac{d(\omega)^{N+1}\tilde{C}(\omega)}{1 - e^{-[(1-\epsilon)\beta(\omega)N-(1+2\epsilon)\alpha(\omega)]}}$$

is a tempered function from above. This completes the proof of the lemma. □

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Proof of Theorem 3.2. Let $R(\omega, x) = \phi(1, \omega, x) - \psi(1, \omega, x)$ and $\phi_s(t, \omega, x)$ be a family of random dynamical systems whose time-one maps have the form

$$\phi_s(1, \omega, x) = \phi(1, \omega, x) + sR(\omega, x), \quad s \in D_2.$$

Since $R = O(|x|^N)$, we may choose a common tempered ball $B_{\rho(\omega)} = B(0, \rho(\omega))$ on which $\phi_s(1, \omega, x)$ and $\phi_s(-1, \omega, x)$ are defined and (15) holds. Summarizing Lemmas 3.3, 3.4 and 3.6 gives Theorem 3.2. □

As a corollary of Theorem 3.2, we have the following.

THEOREM 3.7. *Assume that the Standing Hypothesis holds and that all Lyapunov exponents $\lambda_i(\omega)$, $1 \le i \le p(\omega)$, have the same sign, then ϕ is analytically conjugated to a random polynomial of degree not greater than* $\max_{1\le i\le p(\omega)} |\lambda_i(\omega)| / \min_{1\le i\le p(\omega)} |\lambda_i(\omega)|$.

The next theorem is an extension of Poincaré theorem for deterministic systems.

THEOREM 3.8. (Poincare type) *Assume that the Standing Hypothesis holds and that all Lyapunov exponents $\lambda_i(\omega)$, $1 \le i \le p(\omega)$, have the same sign and are non-resonant, i.e.*

$$\lambda_i(\omega) \ne (m, \lambda(\omega)), \quad |m| \ge 2, \tag{20}$$

then ϕ is analytically conjugated to its linear part.

Proof. Let $N(\omega) = [\max |\lambda_i(\omega)| / \min |\lambda_i(\omega)|] + 1$. We decompose Ω as in (5). Then $N(\omega)$ is a constant on each θ-invariant set Ω_i. Thus, restricting θ to Ω_i and using in [**1**, Corollary 8.2.10] we obtain a unique near-identity polynomial transformation $H_{i,N}(\omega, x) = x + \sum_{n=2}^{N} h_{i,n}(\omega, x)$ over Ω_i with tempered coefficients such that ϕ is transformed by $H_{i,N}$ to ψ, whose N-jet in the Taylor expansion of order N is linear, $\psi_i(\omega, x) = A(\omega)x + O(|x|^N)$. Since Ω is the union of countably many disjoint θ-invariant measurable sets Ω_i, patching $H_{i,N}$ and ψ_i together, respectively, we obtain a near-identity random polynomial $H(\omega, x)$ and an analytic random diffeomorphism ψ with the form $\psi(\omega, x) = A(\omega)x + O(|x|^N)$ such that ϕ is analytically conjugated to ψ through $H(\omega, x)$. By Theorem 3.2, ψ is analytically conjugated to its linear part in some tempered ball. This completes the proof. □

Finally, we give a theory of normal forms for locally tempered analytic random diffeomorphisms with diagonal linear part. We follow Arnold [**1**]. Let ϕ be a locally tempered analytic random diffeomorphism in $\mathbb{C}^d$ with fixed point $0, \phi(\omega, 0) = 0$. The idea of normal form theory is to find a random coordinate transformation h such that the random conjugacy $\psi(\omega) = h^{-1}(\theta\omega) \circ \phi(\omega) \circ h(\omega)$, equivalently

$$h(\theta\omega, \psi(\omega, x)) = \phi(\omega, h(\omega, x)), \tag{21}$$

has the simplest possible form. Theorem 3.8 states that if all Lyapunov exponents of the linearized random dynamical system have the same sign and are non-resonant, then ϕ is analytically conjugated to its linear part $A(\omega) = D\phi(\omega, 0)$, thus the normal form is a linear form. We now consider the resonant case. We assume that there is a random basis $\mathcal{S}(\omega) = (u_1(\omega), \ldots, u_d(\omega))$ of $\mathbb{C}^d$ so that in the $\mathcal{S}(\omega)$, $\mathcal{S}(\theta\omega)$ bases the matrix representation $A(\omega)$ of $D\phi(\omega, 0)$ is diagonal (this is the case when the Lyapunov exponents

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of $A(\omega)$ are distinct, see [**1**]) and the change from the standard basis to the random basis $F(\omega)$ is a Lyapunov cohomology. We denote by $H_{n,d}$ the vector space of homogeneous polynomials of degree n in d variables with values in $\mathbb{C}^d$ and let

$$\mathcal{H}_{n,d} := \{f : \Omega \to H_{n,d} \text{ measurable}\}$$

be the vector space of random homogeneous polynomials of degree $n \geq 2$ in d variables with values in $\mathbb{C}^d$.

Let $A(\omega) = \operatorname{diag}(a_1(\omega), \ldots, a_d(\omega))$ and $\lambda(\omega) = (\lambda_1(\omega), \ldots, \lambda_d(\omega))$ be the Lyapunov exponents of the linear cocycle generated by $A(\omega)$.

Definition 3.9. The vector-valued monomial $x^m u_i(\omega)$ is said to be resonant if $\lambda_i(\omega) = (m, \lambda(\omega))$, $|m| \geq 2$.

We now expand the three mappings ϕ, h and ψ in equation (21) as formal Taylor series at 0:

$$\phi(\omega, x) \sim A(\omega)x + \sum_{n=2}^{\infty} \phi_n(\omega, x),$$

$$\psi(\omega, x) \sim A(\omega)x + \sum_{n=2}^{\infty} \psi_n(\omega, x),$$

and

$$h(\omega, x) \sim x + \sum_{n=2}^{\infty} h_n(\omega, x),$$

where $\phi_n, \psi_n, h_n \in \mathcal{H}_{n,d}$. Inserting these expansions into equation (21), equating coefficients yields the following result. The nth-order terms of the formal power series expansion of ψ are successively given by the nth-order terms of ϕ and h and lower-order terms. More specifically, $\psi_1(\omega) = \phi_1(\omega) = A(\omega)$ and

$$A(\omega)h_n(\omega, x) - h_n(\theta\omega, A(\omega)x) = \psi_n - \gamma_n, \quad n \geq 2. \tag{22}$$

Here

$$\gamma_n(\omega) = T_n((\phi(\omega) - A(\omega)) \circ S_{n-1}h(\omega) + S_{n-1}h(\theta\omega) \circ S_{n-1}\psi(\omega)),$$

where $T_n(f)$ denotes the terms of order n in the formal Taylor expansion of f and $S_N(f) := \sum_{n=1}^{N} T_n(f)$ is the N-jet of f.

Let $h_m^i(\omega)$, $\psi_m^i(\omega)$ and $\gamma_m^i(\omega)$ denote the coefficients of $x^m u_i(\omega)$ of functions h_n, ψ_n and γ_n, respectively. For $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d$, let $a(\omega)^m = a_1(\omega)^{m_1} \ldots a_d(\omega)^{m_d}$. Then the homological equation (22) gives

$$a_i(\omega)h_m^i(\omega) - a(\omega)^m h_m^i(\theta\omega) = \psi_m^i(\omega) - \gamma_m^i(\omega),$$

which has a tempered solution for $\psi_m^i(\omega) = 0$:

$$h_m^i(\omega) = \begin{cases} -a_i(\omega)^{-1}\gamma_m^i(\omega) - a_i(\omega)^{-1} \displaystyle\sum_{k=1}^{\infty} \prod_{j=1}^{k} (a(\theta^{j-1}\omega)^m a_i(\theta^j\omega)^{-1})\gamma_m^i(\theta^k\omega) \\ \qquad \text{if } \lambda_i(\omega) > (m, \lambda(\omega)), \\ \displaystyle\sum_{k=0}^{\infty} \left(\prod_{j=0}^{k} a_i(\theta^{-j}\omega) a(\theta^{-j-1}\omega)^{-m} \right) \gamma_m^i(\theta^{-k-1}\omega) \\ \qquad \text{if } \lambda_i(\omega) < (m, \lambda(\omega)) \end{cases}$$

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and has a solution $h^i_m(\omega) = 0$ for $\psi^i_m(\omega) = \gamma^i_m(\omega)$ if $\lambda_i(\omega) = (m, \lambda(\omega))$. Summarizing the above discussion together with Theorem 3.2 gives the following.

THEOREM 3.10. (Poincaré–Dulac type) *Assume that the Standing Hypothesis holds and that all Lyapunov exponents $\lambda_i(\omega), 1 \le i \le p(\omega)$, have the same sign. Then ϕ is analytically conjugated to a random polynomial mapping of degree N which consists of resonant monomials only.*

Example. Let ϕ be a locally tempered analytic random diffeomorphism in $\mathbb{C}^2$ with a fixed point 0 over an ergodic metric dynamical system $\{\theta^n\omega\}_{n\in\mathbb{Z}}$. Let λ_1, λ_2 with $|\lambda_1| \ge |\lambda_2|$ be the Lyapunov exponents of ϕ. We assume that λ_1 and λ_2 have the same sign. Then we have the following.

(i) $\lambda_1/\lambda_2 \notin \mathbb{N} \setminus \{1\}$. ϕ is analytically conjugated to its linear part.

(ii) $\lambda_1/\lambda_2 = n \in \mathbb{N} \setminus \{1\}$. ϕ is analytically conjugated to

$$\psi(\omega, x) = \begin{pmatrix} a_1(\omega)x_1 + c(\omega)x_2^n \\ a_2(\omega)x_2 \end{pmatrix}.$$

Remark. The theorem of Poincaré type holds for continuous time random dynamical systems. Consider an analytic continuous time random dynamical system $\phi(t, \omega, x)$ with a fixed point $x = 0$, $\phi(t, \omega, 0)$=0. Assume that $D\phi(t, \omega, 0)$ satisfies the conditions of the Multiplicative Ergodic Theorem and their Lyapunov exponents have the same sign and are non-resonant. Then by Theorem 3.8, the random diffeomorphism $\phi(1, \omega, \cdot)$ is analytically conjugate to its linear part $D\phi(1, \omega, 0)$ through an analytical random diffeomorphism $h(\omega, x)$. Furthermore, one can show that $h(\theta^t\omega, x)$ is continuous in t. Using the standard transformation modified from the deterministic case

$$H(\omega, x) = \int_0^1 D\phi(s, \theta^s\omega, 0)h(\theta^s, \phi(s, \omega, x))\, ds,$$

we have

$$H(\theta^t\omega, \phi(t, \omega, x)) = D\phi(t, \omega, 0)H(\omega, x).$$

If the continuous time random dynamical system $\phi(t, \omega, x)$ is generated by a random differential equation, then $H(\theta^t\omega, x)$ is absolutely continuous since the random dynamical system is absolutely continuous in t. Thus, H transforms the random differential equation to its linear part. Hence, the Poincaré linearization theorem holds for random differential equations.

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