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Ringing the changes†

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Abstract

The ancient and continuing art of change ringing, or campanology (how the British ring church bells), is studied from a mathematical viewpoint. An 'extent' on n bells is regarded as a hamiltonian cycle in a Cayley colour graph for the symmetric group S_n, embedded on an appropriate surface. Two methods for variable n (Plain Bob for all n and Grandsire for $n \equiv 3 \pmod 4$) are discussed, and a new method for n odd is introduced. All minimus methods ($n = 4$) and five doubles methods ($n = 5$) are depicted, one of these being the new 'No Call' Doubles.

1. *Introduction*

In the science of change ringing (sometimes called campanology) the central problem is to ring a full extent on n bells. We designate the n bells by the natural numbers $1, 2, \ldots, n$, arranged in descending order of pitch. Bell 1 is called the *treble*, bell n the *tenor*. Each ringing of the n bells, once each in some order, is called a *change*; the special change $1, 2, \ldots, n$ is called *rounds*. An *extent* consists of $n! + 1$ successive changes, satisfying:

(i) the first and last change are both rounds;

(ii) no other change is repeated (so that each possible change other than rounds is rung exactly once);

(iii) from one change to the next, no bell moves more than one position in its order of ringing (positions 1 and n are *not* adjacent, unless $n = 2$);

(iv) no bell rests in the same position for more than two successive changes;

(v) each bell (except perhaps the treble) does the same amount of 'work' (hunting, plain hunting, dodging, etc.);

(vi) The method employed is *palindromic*, or self-reversing (this will be made more precise in Section 2).

Conditions (iv) and (v) are occasionally relaxed in practice, but the other conditions appear to be inviolable.

I venture to guess that rule (i) is for musicality, rule (ii) for thoroughness, rule (iii) for mechanical considerations (dealing with how the bells are hung and rung), rules (iv) and (v) to keep the performance interesting for the ringers, and rule (vi) to ease the memory burden for the ringers.

Typically the number of bells is in the range $3 \leqslant n \leqslant 12$, with names assigned to the extents as in Table 1. The odd-bell names reflect the maximum number of pairs of bells that could be exchanged, in their order of ringing.

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Table 1

n	Name	$n!+1$
3	Singles	7
4	Minimus	25
5	Doubles	121
6	Minor	721
7	Triples	5041
8	Major	40321
9	Caters	362881
10	Royal	3628801
11	Cinques	39916801
12	Maximus	479001601

As an extent of Major takes approximately eighteen hours to ring (the ringers are allowed no visual aids to memory, so this must surely rank as one of the 'major' physical *and* intellectual feats of mankind), extents on more than eight bells are clearly beyond the limits of human endurance; even on eight bells they are extremely rare. Much more tractable, an extent of triples requires under three hours of concentrated ringing. For this reason peals are often attempted, and often succeed; a *peal* consists of at least 5000 and, for $5 \leqslant n \leqslant 7$, exactly 5041 successive changes satisfying rules (i) through (vi) above (but not (ii), if $n < 7$). For $n < 7$, a peal consists of several extents strung together; for $n = 7$ a peal *is* an extent; and, for $n > 7$, a peal is a partial extent. If tower bells are being rung, then each ringer operates one bell (a huge amount of practice is required for the beginner to learn to ring his or her bell even individually, let alone in rounds with the other bells, let alone through all the changes of a peal or an extent); if handbells are employed, then each ringer rings two bells. There is also a bell-ringing machine (designed and constructed by John Carter, a gunsmith and bell-ringer, and currently on loan to the Science Museum in London); it mechanically rings twelve handbells suspended in a cabinet, after being 'programmed' for method.

As generally an even number of bells is provided, for the odd bell methods a 'covering' bell – always the tenor – rings last in every change, in forgiven violation of rules (iv) and (v). It might even be argued that the stability and regularity thus produced is musically pleasing.

Change ringing, as described above, began in England in the middle of the seventeenth century and remains almost exclusively a British art. Today there are over five thousand bell towers in Great Britain, about a half dozen in Canada and a dozen in the United States, and perhaps twice that in Australia. On 27 and 28 July 1963, the first extent of Plain Bob Major was successfully rung on tower bells at the Loughborough Bell Foundry in England. On 27 and 28 December 1977, the same extent was accomplished on handbells in a private home in Farnham, Surrey, England.

Change ringing has been popularized, to an extent, by Dorothy Sayers in her novel *The Nine Tailors* (9) and, more recently, as part of the festivities at the Royal Wedding of Prince Charles and Princess Diana. Only recently have mathematicians begun to study change ringing from their vantage point, although curiously the early composers of peals and extents were doing coset decomposition in symmetric groups a full century before Lagrange. The reader could consult works by Rankin (7), Fletcher (5), Dickin-

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son (4), Price (6), Budden (1), and White (11). For more on the history and practice of change ringing, the reader should seek out Wilson (12), Camp (2), and the weekly publication *The Ringing World* (published in Guildford, Surrey, England).

2. *The mathematics of change ringing*

It would be easy to confuse the number of a given bell with the number of the position it occupies in a particular change. To avoid this confusion we regard each change as a function from the set of n positions to the set of n bells, so that domain elements describe positions and range elements describe bells. Thus a change f, recorded as $f(1), f(2), \ldots, f(n)$, would indicate that bell $f(1)$ is to be rung first, bell $f(2)$ second, and so on. Rounds is given by $r(i) = i$, $1 \leqslant i \leqslant n$. In order to satisfy condition (iii) of Section 1, we describe a potential transition from one change to the next by a permutation on the set of n positions which either fixes a given position or interchanges it with an adjacent position. Conversely, each transition satisfying (iii) is of this type. Thus each transition is represented by an involution (that is, by a product of disjoint transpositions) in the symmetric group S_n. Now let $t(n)$ give the number of possible transitions for n bells, and let $F(n)$ give the nth Fibonacci number (where $F(0) = F(1) = 1$); then we have the following well-known result:

THEOREM 2.1. *For* $n \geqslant 2$, $t(n) = F(n) - 1$.

Proof. It is easy to verify that $t(2) = 1$ and that $t(3) = 2$, so we assume the result for $n < k$ and consider $t(k)$, $k \geqslant 4$. Since there are $t(k-1)$ allowable transitions fixing position k, $t(k-2)$ allowable transitions exchanging positions k and $k-1$ and at least one other pair, and the single transposition $(k-1, k)$, we have:

$$\begin{aligned} t(k) &= t(k-1) + t(k-2) + 1 \\ &= F(k-1) - 1 + F(k-2) - 1 + 1 \\ &= F(k) - 1. \end{aligned}$$

Now, if $d_1, d_2, \ldots, d_k$ represent the first k transitions in an extent, then the $(k+1)$st change is the function $rd_1d_2 \ldots d_k$, where – in accordance with the standard notation for function composition – we take products in S_n from right to left. Let Δ be any collection of generating involutions for S_n (if Δ does not generate S_n, then condition (ii) fails) and form the Cayley colour graph $C_\Delta(S_n)$; then we have the immediate but absolutely central result:

THEOREM 2·2. *An 'extent' on n bells, satisfying conditions* (i)–(iii) *and using transition rules from* Δ, *can be rung if and only if* $C_\Delta(S_n)$ *is hamiltonian.*

Since each $d \in \Delta$ has order two, $C_\Delta(S_n)$ has no directed edges by convention (see, for example, White (11)); hence hamiltonian cycles in $C_\Delta(S_n)$ coincide with those in the underlying Cayley graph $G_\Delta(S_n)$. We retain the edge colours of $C_\Delta(S_n)$, however, as they are valuable in constructing the extent itself from the hamiltonian cycle and in checking the palindromic condition (vi). Each method is based upon a *principle*, which usually (but not always) consists of $2n$ changes. The word in S_n describing the $2n-1$ juxtaposed transitions of the principle should be a palindrome:

$$d_1d_2 \ldots d_{n-1}d_nd_{n-1} \ldots d_2d_1.$$

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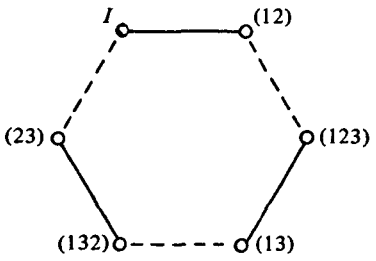


Fig. 1

Table 2

Hamiltonian cycle	Change	Range of change
I	$rI = r$	1, 2, 3
(12)	$r(12)$	2, 1, 3
(123)	$r(123)$	2, 3, 1
(13)	$r(13)$	3, 2, 1
(132)	$r(132)$	3, 1, 2
(23)	$r(23)$	1, 3, 2
I	r	1, 2, 3

These ideas are illustrated in Fig. 1, with $\Delta = \{(12), (23)\}$ for the case $n = 3$. The hamiltonian cycle and the extent it gives rise to shown in Table 2 describe the interior clockwise region boundary of the spherical imbedding of $C_\Delta(S_3)$ depicted. The extent is called 'quick six'; the clockwise boundary of the exterior region produces 'slow six', the only other extent possible on 3 bells. (Not surprisingly, 'slow six' is just 'quick six' in the opposite order.) If we let $a = (12)$ (denoted by solid edges) and $b = (23)$ (denoted by dashed edges), then 'quick six' is completely described by the identity word $(ab)^3$ in S_3 and 'slow six' by $(ba)^3$. In the former case the relevant palindrome is *ababa*; in the latter it is *babab*.

3. *General results*

In (8), Rapaport provided an inductive construction, to establish the following:

THEOREM 3·1. *Let* $\Delta = \{(12), (12)(34)(56)\ldots, (23)(45)(67)\ldots\}$ *for* S_n*; then* $C_\Delta(S_n)$ *is hamiltonian.*

In (11), we required only conditions (i), (ii), and (iii) for an extent, and gave the following two ramifications of the Rapaport construction:

THEOREM 3·2. *An extent on n bells exists, for all n. Moreover, only three transition rules are required.*

THEOREM 3·3. *For n odd an extent exists which also satisfies condition* (iv) *at all but position n, where (at the worst) no bell rests for more than four successive changes.*

In (11) we also displayed an embedding of $C_\Delta(S_4)$, for Rapaport's Δ, in the projective plane. (This is essentially Fig. 2 below.) In this embedding the three left cosets of D_4 (the dihedral group of order 8) in S_4 bound regions which link together nicely to

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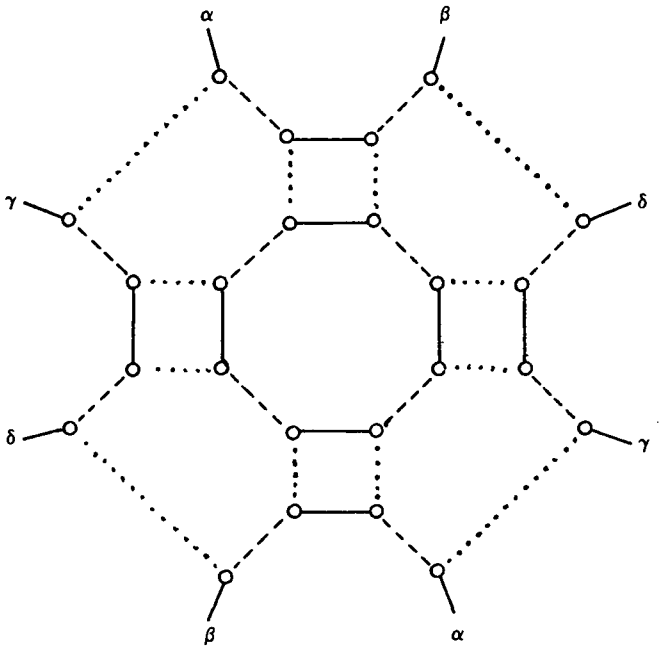


Fig. 2

trace out the hamiltonian cycle giving the extent, in extension of the idea behind Fig. 1. This suggests the efficacy of picturing extents on surfaces, and, as we naturally wish to do this as efficiently as possible, we define the *characteristic of an extent* (on n bells, with transition rules Δ) to be the maximum characteristic of all surface embeddings for $C_\Delta(S_n)$. The *characteristic* of a 2-cell imbedding of a graph G on a surface S is $p-q+r$, where G has p vertices and q edges, and S is decomposed into r regions; this number is constant for given S. As certain transition rules (the 'calls'–so called because the conductor calls them at the appropriate time, while all other transitions are trusted to the memory of the ringers–known as 'bobs' and 'singles') are used infrequently in a particular extent (a single is perhaps employed only *twice* among the $n!$ transitions, for example) we take Δ to include only the other generators for S_n; this has the effect of avoiding unnecessary clutter in $C_\Delta(S_n)$. Thus, in the event that Δ does not generate S_n, a characteristic embedding of an extent may be on a disconnected surface.

For example, the Rapaport $\Delta = \{a, b, c\}$ (where $a = (12)$, $b = (12)(34)(56)\ldots$, and $c = (23)(45)(67)\ldots$) for S_n ($n \geqslant 5$) involves no 'call' generators and the relators $(ab)^2$, $(ac)^6$, and $(bc)^n$ bound regions for an embedding (orientable if and only if $n \equiv 3 \pmod 4$) of $C_\Delta(S_n)$ of characteristic $(n-1)!\,(3-n)/6$, which is thus a lower bound for the characteristic of the extent. Or, the single relator $(abc)^{n-1}$ can be taken to bound regions for an orientable embedding of $C_\Delta(S_n)$ of characteristic $n(n-2)!\,(3-n)/2$, but this is less efficient. Note that Δ does generate S_n, as a and bc are known to do so; see Coxeter and Moser ((3), p. 63). For $n = 4$, the embedding in the projective plane *cannot* be improved (as $C_\Delta(S_4)$ contains a homeomorph of $K_{3,3}$), so that the characteristic of this extent is exactly one.

We now consider $\Delta = \{b, c, d\}$, where $b = (12)(34)(56)\ldots$, $c = (23)(45)(67)\ldots$, and

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$d = (34)(56)(78)\ldots$, for the various *Plain Bob* extents on n bells. Here the *hunting group* generated by b and c is always D_n (as can be seen from its action on a regular $n$gon with vertices labelled, in order, $1, 2, 4, 6, \ldots, 7, 5, 3$), giving a block of $2n$ changes. The generator d is used to link $n-1$ of these blocks together, to give a 'touch' of $2n(n-1)$ changes. Finally, for $n \geqslant 5$, the call generators (the bob $e = (23)(56)(78)\ldots$ and, for $n \geqslant 6$, the single $f = (56)(78)\ldots$) are added to produce the full $n! + 1$ changes of the extent. Here the relators $(bc)^n$, $(bd)^2$, and $(cd)^{n-1}$ bound regions for an always nonorientable embedding of $C_\Delta(S_n)$ of characteristic $-(n-2)!\,(n^2-5n+2)/4$; this is for $n \geqslant 4$. This appears to be the most useful embedding for Plain Bob extents, as the relator $(bc)^n$ exactly portrays the principle embodied in the hunting group, and replicated in the left cosets of that group. Again we note that Δ generates S_n, as bc and bd are known to do so.

Next consider $\Delta = \{b, c, g\}$, where $b = (12)(34)(56)\ldots$, $c = (23)(45)(67)\ldots$, and $g = (12)(45)(67)\ldots$ for the various *Grandsire* extents on n (odd) bells. Here we see that $\langle a, c\rangle = D_n$ and g is used as a linking permutation. The bob for Grandsire is constructed from Δ, but the single is the new $h = (45)(67)(89)\ldots$. To see that $C_\Delta(S_n)$ is connected (for $n \equiv 3 \pmod 4$), we observe that $gc = (123)$ and that conjugating alternately by b and g (and then taking inverses) produces $\{(12k) | 3 \leqslant k \leqslant n\}$; this is known ((3), p. 66) to generate A_n. But then $\langle A_n, g\rangle = S_n$. We use the relators $(gc)^3$, $(gb)^{n-2}$, and $(cb)^n$ (the latter is used in the principle for Grandsire) to bound regions for an embedding of $C_\Delta(S_n)$ (orientable for $n \equiv 3 \pmod 4$, nonorientable and in two components for $n \equiv 1 \pmod 4$) of characteristic $-(n-1)(n-3)!\,(n^2-5n+3)/3$. For $n \equiv 1 \pmod 4$ the single h is odd, so that $\langle A_n, h\rangle = S_n$.

Finally, we endeavour to improve on the Rapaport generating set, so as to satisfy condition (iv) universally. Her construction is based upon the following lemma:

LEMMA 3·4. *A connected graph regular of degree three has a hamiltonian cycle if there is a set P of polygons and a set Q of squares, each set partitioning the vertex set of the graph and such that no member of P contains every vertex of a member of Q.*

For the Rapaport $\Delta = \{a, b, c\}$, where

$$a = (12), \quad b = (12)(34)(56)\ldots, \quad \text{and} \quad c = (23)(45)(67)\ldots,$$

the set P is composed of $n!/12$ 12-gons $(ac)^6$ (or we could use $(n-1)!/2$ $2n$-gons $(bc)^n$) and the set Q consists of $n!/4$ squares $(ab)^2$. The Rapaport construction of a hamiltonian cycle uses b edges to combine 12-gons from P (or a edges to combine $2n$-gons from P); thus the cycle consists of partial words $(ac)^k b(ca)^h$ (or $(bc)^k a(cb)^h$) and we see that condition (iv) is uniformly satisfied, if n is odd. This strengthens Theorem 3·3.

If we modify Δ to $\Delta' = \{d, b, c\}$ as in Plain Bob (so that $d = (34)(56)(78)\ldots$, instead of (12)), then Lemma 3·4 still applies, to show that $C_\Delta(S_n)$ is hamiltonian (it is clear that Δ' generates S_n, since Δ does). But now the hamiltonian cycle found by Rapaport always involves $\ldots cdc\ldots$, which fixes the bell in position one unduly.

To retain the constant movement of Plain Bob (that is, to avoid the 'stagnation' of Rapaport's $a = (12)$, which fixes $n-2$ bells) but to avoid the dawdling in position one, we now try $\Delta'' = \{b, c, h\}$, where $h = (45)(67)(89)$, and where $n = 2m-1$ is odd. We first check that Δ'' generates $S_n = \langle ch, (bc)^{2m-3}\rangle$ (this argument fails if n is even). Lemma 3·4 then applies, with P determined by (say) $(bc)^n$ and Q by $(ch)^2$. Then we

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obtain a hamiltonian cycle in which b alternates throughout; as b moves every position but the last, which is moved by both c and h, we see immediately that condition (iv) is satisfied. Moreover, there is an efficiency here (over Plain Bob), as only three generators are used; there are no bobs or singles. However, conditions (v) and (vi) have not been checked, and indeed it appears difficult to do so from this inductive construction. The special case $n = 5$ will be considered in detail in Section 5.

The embedding of the 'no call' extent described above is nonorientable, of characteristic $-(n-1)^2(n-4)(n-3)!/4$.

Thus the various extent embeddings we have constructed in this section have had characteristics asymptotic to $n!/k$, where $k \in \{2, 3, 4, 6\}$.

4. *Minimus*

The eleven minimus extents given in (10) are algebraically described and graphically depicted in this section. We see from Theorem 2·1 that there are four allowable transition rules for four bells; they are:

$a = (12)(34)$ ————

$b = (23)$ – – – – – – –

$c = (34)$ · · · · · · · · ·

$d = (12)$ 

Each extent is described, in Table 3, by the identity word in S_4 giving the hamiltonian cycle in the Cayley colour graph. Since all Cayley colour graphs are vertex-transitive, the initial vertex of the hamiltonian cycle can be arbitrarily selected; the remainder of the cycle is then uniquely determined.

Table 3

Minimus extent	Algebraic description
Plain Bob	$((ab)^3ac)^3$
Reverse Bob	$(abad(ab)^2)^3$
Double Bob	$(abadabac)^3$
Canterbury	$(abcdcbab)^3$
Reverse Canterbury	$(db(ab)^2dc)^3$
Double Canterbury	$(dbcdcbcd)^3$
Single Court	$(db(ab)^2db)^3$
Reverse Court	$(ab(cb)^2ab)^3$
Double Court	$(db(cb)^2db)^3$
St Nicholas	$(dbadabdc)^3$
Reverse St Nicholas	$(abcdcbac)^3$

Obviously, each word uses twenty-four symbols, but it is worth noting that only Plain Bob requires as few as four letters to be written down in its algebraic description. We observe that none of the minimus extents satisfies all six of the conditions given in section 1: all fail (v), and only the first three listed satisfy (iv).

Now follow the graphical pictures of the corresponding Cayley graphs. Fig. 2 serves directly for Plain Bob Minimus and Reverse Court Minimus; it also applies to Reverse Bob and Single Court, with d replacing c. These four extents all have characteristic one.

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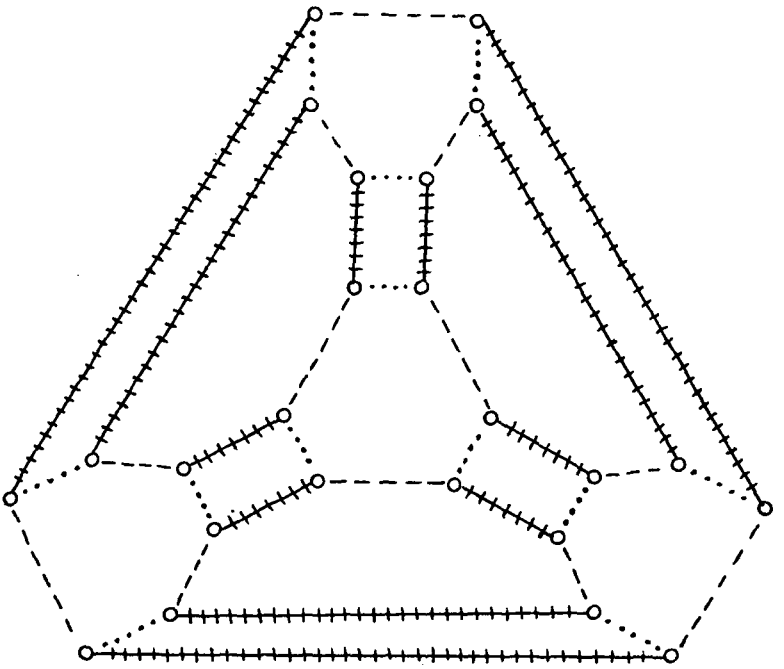


Fig. 3

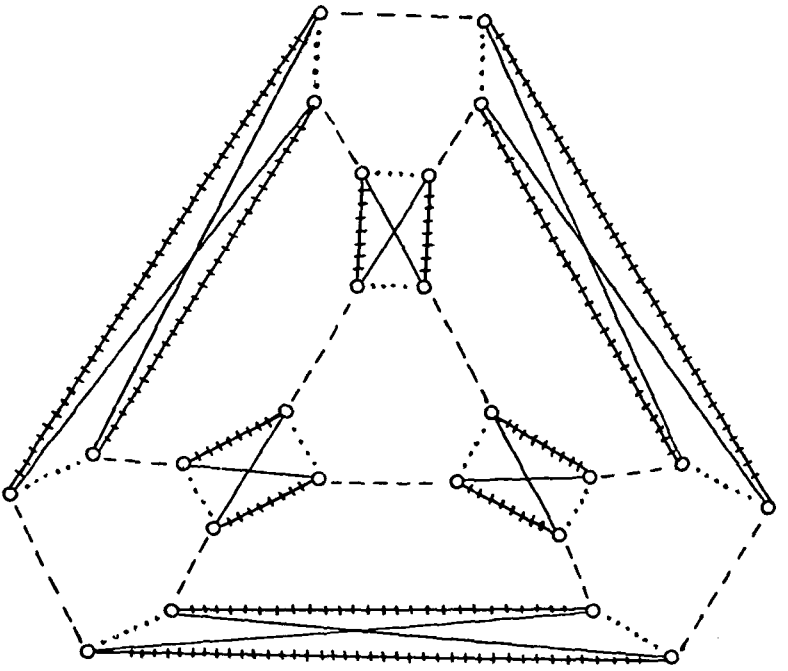


Fig. 4

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Fig. 3 pictures Double Court and Double Canterbury; these two extents have characteristic two. (They are the only *planar* extents.) Fig. 4 depicts Reverse St Nicholas, St Nicholas, Reverse Canterbury, Canterbury, and Double Bob; these are *immersions* and not embeddings, as the redundant generator a is allowed to diagonalize six quadrilaterals. In the language of topological graph theory, this immersion shows that, for $\Delta = \{a, b, c, d\}$, the crossing number of $C_\Delta(S_4)$ is at most six.

5. *Doubles*

There are sixteen doubles extents given in (10); there are undoubtedly many more. The seven possible transitions (see Theorem 2·1) are:

$a = (12)(34)$ ——————

$b = (23)(45)$ ||||||||||||||||

$c = (34)$ · · · · · · · ·

$d = (23)$ + + + +

$e = (12)$

$f = (12)(45)$ ›››››››››

$g = (45)$ ——————

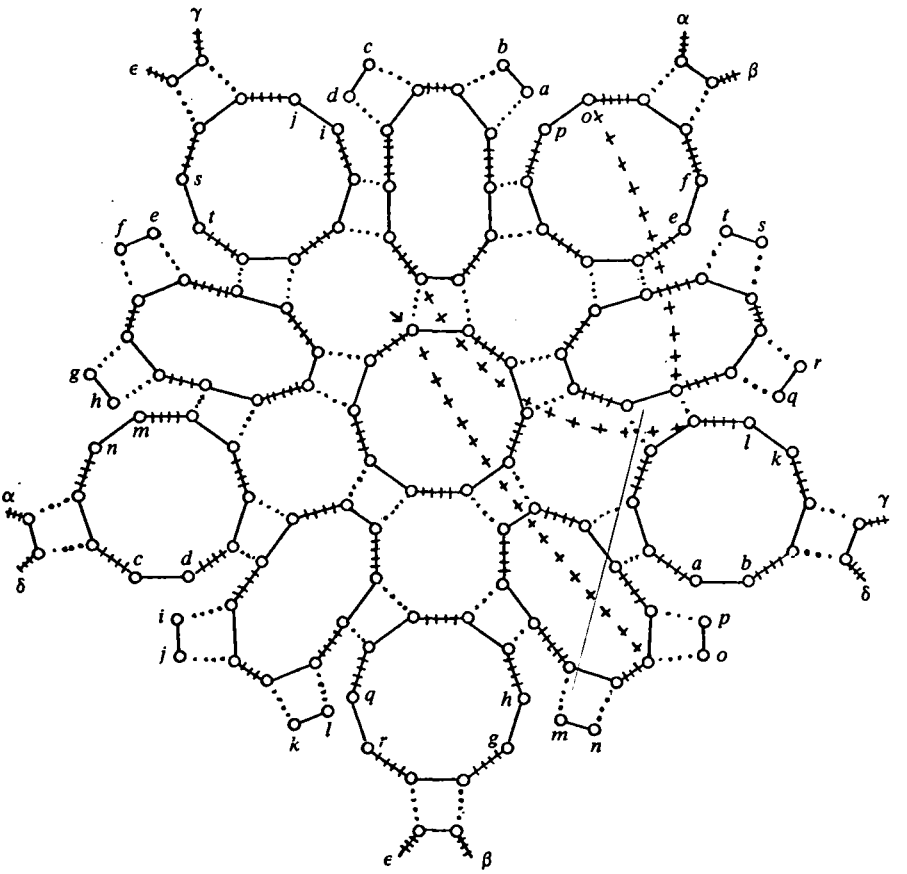


Fig. 5

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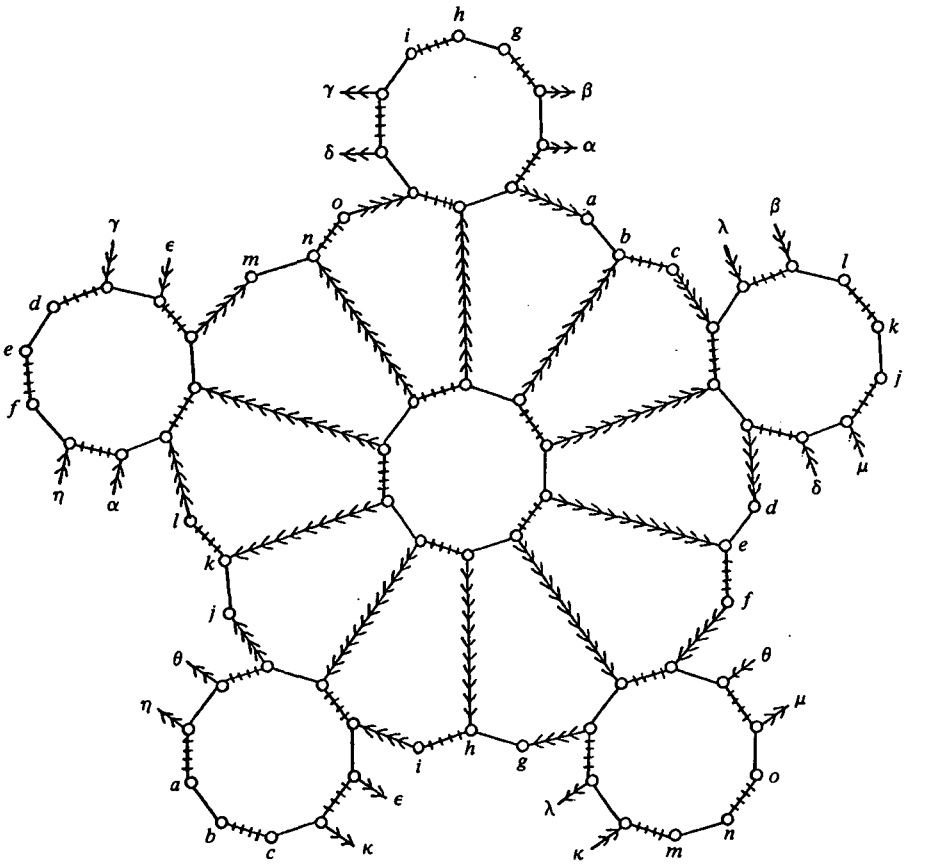


Fig. 6

We shall not need to represent e for the extents discussed in this section, which are:

$$\text{Plain Bob} = (((ab)^4 ac)^3 (ab)^4 ad)^3$$
$$\text{Grandsire} = ((fb(ab)^4 fb(ab)^3 fb)^2 fb(ab)^4 fb(ab)^3 fg)^2$$
$$\text{Stedman} = (fba(f(bf)^2 ab(fb)^2 a)^4 f(bf)^2 abfd)^2$$
$$\text{No Call} = ((ag)^3 (ab)^3 ag(ab)^2 (ag)^2 ab)^5.$$

Fig. 5 shows $C_\Delta(S_5)$ for Plain Bob Doubles ($\Delta = \{a, b, c\}$) embedded in the nonorientable surface N_5, of characteristic -3. Indeed Plain Bob Doubles *has* characteristic -3, as the graph which results when each decagon is contracted to a vertex is $K_{6,6}$ less a 1-factor, for which the resulting quadrilateral embedding is necessarily optimal, and contraction cannot decrease characteristic. If we start the hamiltonian cycle at the distinguished vertex in Fig. 5 and call the bob d as indicated by the superimposed edges, then the 121 rows of the extent can be readily obtained.

Fig. 6 shows $C_\Delta(A_5)$, $\Delta = \{a, b, f\}$ embedded in the nonorientable surface N_6, of characteristic -4. The first 59 symbols of the identity word for Grandsire Doubles trace all but one edge of a hamiltonian cycle for $C_\Delta(A_5)$ on N_6. The single g then shunts us across to a *second copy* of the same $C_\Delta(A_5)$ on another, disjoint, surface N_6. (Thus $C_\Delta(S_5)$ has two components, one for each coset of A_5.) The same hamiltonian path is

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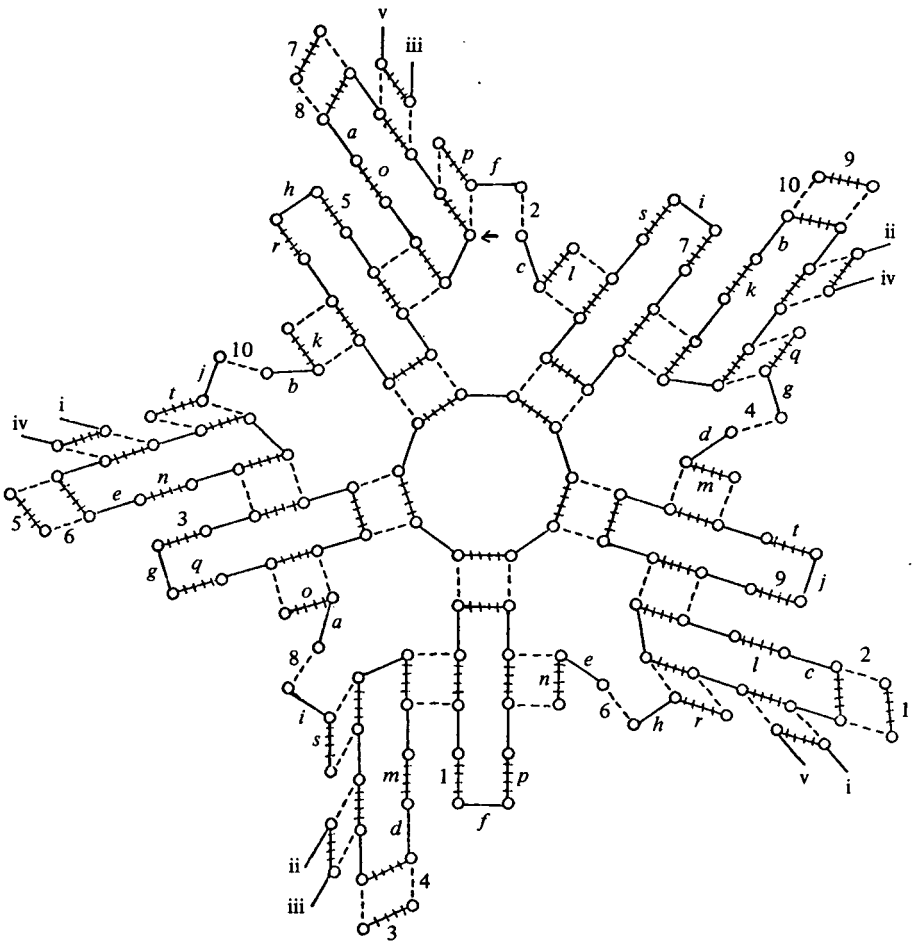


Fig. 7

repeated on this second surface, and the single g returns us to the initial vertex on the first surface. The result is a hamiltonian cycle in $C_{\Delta \cup \{g\}}(S_5)$, depicted on a topological space consisting of a disconnected surface whose two homeomorphic components are joined by a pair of line segments.

Fig. 6 also serves for Stedman Doubles, in the same manner, if we use d for the single instead of g. Thus both these extents have characteristic at least minus 8.

We note that the same surface (two copies of N_6) can be used for

$$\text{St Simon Doubles} = ((ab(af)^2abac)^3 ab(af)^2abad)^3,$$

except that c is used nine times (of 120) as part of Δ, and d is used as a bob three times; the hamiltonian cycle, due to the action of c, does not respect the cosets of A_5 – that is, it transfers frequently from one N_6 to the other.

Finally, Fig. 7 shows $C_\Delta(S_5)$, $\Delta = \{a, b, g\}$, embedded in the nonorientable surface N_{10}, also of characteristic -8. The identity word $((ag)^3(ab)^3ag(ab)^2(ag)^2ab)^5$ for 'No Call' Doubles (so named by the author, as no bobs or singles are required) traces out a hamiltonian cycle in $C_\Delta(S_5)$. (In fact, the cycle was found in this figure *first*, as suggested

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Table 4.

'No Call' Doubles				
12345	45123	23451	51234	34512
21435	54213	32541	15324	43152
21453	54231	32514	15342	43125
12543	45321	23154	51432	34215
12534	45312	23145	51423	34251
21354	54132	32415	15243	43521
21345	54123	32451	15234	43512
12435	45213	23541	51324	34152
14253	42531	25314	53142	31425
41523	24351	52134	35412	13245
45132	23415	51243	34521	12354
54312	32145	15423	43251	21534
53421	31254	14532	42315	25143
35241	13524	41352	24135	52413
35214	13542	41325	24153	52431
53124	31452	14235	42513	25341
51342	34125	12453	45231	23514
15432	43215	21543	54321	32154
14523	42351	25134	53412	31245
41253	24531	52314	35142	13425
41235	24513	52341	35124	13452
14325	42153	25431	53214	31542
14352	42135	25413	53241	31524
41532	24315	52143	35421	13254
				12345

by the fivefold symmetry, and then the word was written down from the picture; start at the indicated vertex, for example.) The corresponding 121 rows of this extent are given in Table 4.

From the table it is a simple matter to check not only that conditions (i) through (iv) are satisfied, but that condition (v) holds as well: each bell does the same amount of work as every other. In particular, the path of bell 1 in column i is exactly mimicked by that of bell 4 in column $i+1$, bell 2 in column $i+2$, bell 5 in column $i+3$, and bell 3 in column $i+4$, $1 \leqslant i \leqslant 5$ (arithmetic in Z_5, writing '5' for '0'). Unfortunately, condition (vi) just fails, as *agagagababababagababagaga* is not palindromic. There are four pairs of symbols (such as the third '*g*' and the second '*b*') that could be switched to obtain a palindrome, but sadly each such switch destroys the much more crucial hamiltonian property. Perhaps the efficiency of 'No Call'–in that only three transition rules are required (no other doubles extent uses this few)–could argue its merit, in spite of the failure of the final condition.†

† John Elliott, in his article 'Doubles Principles' (*The Ringing World*, 8 May, 15 May and 24 July 1981), generated by computer eight doubles extents not requiring bobs or singles. Each of these is palindromic, and in fact satisfies all conditions (i)–(vi); but it appears that none employs fewer than five generators. In the 12 June 1981 issue of *The Ringing World*, a writer identified only as S. J. P. commented that the Central Council of Church Bell Ringers relaxes rule (iv) by allowing a bell to rest in the same position for up to four successive changes and abolishes rule (vi) – removed 'some years ago' – altogether! This latter, the writer continues, allows 'any old rubbish to qualify as a method'.

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