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Euler characteristic formulas for simplicial maps

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This paper is dedicated to Professor Takuo Fukuda on his sixtieth birthday.

Abstract

In this paper, various Euler characteristic formulas for simplicial maps are obtained, which generalize the Izumiya–Marar formula [**14**], the Banchoff triple point formula [**3**] and the formula due to Szücs for maps of surfaces into 3-space [**27**]. Moreover, we obtain new results about the Euler characteristics of the multiple point sets and the images of generic smooth maps and the numbers of their singularities.

1. *Introduction*

Let X be a finite simplicial complex. By definition, the *Euler characteristic* $\chi(X) \in \mathbb{Z}$ of X is given by $\chi(X) = \sum_\sigma (-1)^{\dim \sigma}$, where σ runs through all the simplices of X. For each vertex v of X, we define the *local Euler characteristic* $\chi_v(X) \in \mathbb{Q}$ of X at v by

$$\chi_v(X) = \sum_{\sigma \ni v} \frac{(-1)^{\dim \sigma}}{\dim \sigma + 1},$$

where σ runs through all the simplices of X containing v as a vertex (this definition slightly differs from that given in section 11·2 of [**4**]). Then, since every i-simplex contains $i+1$ vertices, we have that $\chi(X) = \sum_v \chi_v(X)$, where v runs through all the vertices of X. In other words, we can collect the contributions at the vertices so that their sum gives the (global) Euler characteristic.

In this paper, based on such an idea, we give similar formulas for Euler characteristics in various situations. Note that although the (global) Euler characteristic is invariant under subdivision, the local Euler characteristic is *not* invariant under

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subdivision in general. However, in certain important cases, the local Euler characteristic will turn out to be invariant under subdivision.

Let $f: X \to Y$ be a simplicial map between simplicial complexes. Using the above-mentioned idea, we can conveniently define an index $\mathrm{Ind}_f(v)$ for each vertex v in the multiple point set of f so that we get the formula (see Theorem 3·3)

$$\chi(f(X)) = \chi(X) + \sum_v \mathrm{Ind}_f(v). \tag{1·1}$$

Suppose now that $f: X \to Y$ is a simplicial map from a combinatorial n-manifold X into a combinatorial $(n+1)$-manifold Y such that $f(X)$ is two-colourable in its regular neighbourhood $V = V(f(X))$. This means that we can assign colours, say blue or red, to the connected components of $V \setminus f(X)$ so that no two adjacent components have the same colour. Denoting by B the blue region of $V \setminus f(X)$, we define $\chi_B(f) = \chi(\partial V \cap B)$. This gives a topological invariant of f which depends on the embedding of $f(X)$ into Y, while $\chi(f(X))$ depends only on $f(X)$. Then,

$$\chi_B(f) = \chi(X) + \sum_v (\mathrm{Ind}_f(v) + \mathrm{Ind}_B(v)), \tag{1·2}$$

where $\mathrm{Ind}_B(v) \in \mathbb{Q}$ is an index with respect to the blue region B defined conveniently for each vertex v of the multiple point set of f by using the coloration (see Theorem 3·6). We also have a corresponding formula for the red region in terms of $\chi_R(f)$ and $\mathrm{Ind}_R(v)$. One of the most important features of the indices $\mathrm{Ind}_B(v)$ and $\mathrm{Ind}_R(v)$ is that they satisfy nice duality properties at each vertex v (see Proposition 2·10), which shows that they are not artificial ones, but are natural ones. Moreover, in some cases we will see that $\chi_B(f)$ is even (for instance, when $n \equiv 2 \bmod 4$ and Y is orientable, or when n is even and the fundamental class $[f(X)]$ is zero in $H_n(Y)$). In particular, we will obtain a congruence modulo 2 between $\chi(X)$ and the sum of the indices (see Corollary 3·8).

In the second part of this paper, we will apply our formulas in several situations. In Section 4, we consider a smooth map $g: M \to N$ from a closed surface M into a 3-manifold N. If g has only stable singularities, then (1·1) gives the Izumiya–Marar formula [**14**]: $\chi(g(M)) = \chi(M) + T(g) + C(g)/2$ (here $T(g)$ and $C(g)$ denote the numbers of triple points and cross caps of g respectively) and (1·2) gives the congruence of Szücs [**27**]: $\chi(M) \equiv T(g) + BC(g) \bmod 2$ (here $BC(g)$ is the number of 'blue' cross caps of g). Moreover, both results can also be applied when g has singularities of codimension $\leqslant 1$, giving new formulas in terms of the number of singularities of each type. In fact, (1·1) and (1·2) are valid even in the case that g has finitely determined singularities.

In Sections 5–7 we obtain similar formulas for C^0 stable smooth maps $g: M^3 \to N^4$, for simplicial maps $f: X \to Y$ with 1-dimensional double point locus and for immersions with normal crossings $g: M^n \to N^{n+1}$ respectively. In Section 8, we discuss the corresponding formulas in the case where the manifolds involved have non-empty boundaries.

Throughout the paper, all homology and cohomology groups are with $\mathbb{Z}_2$ coefficients unless otherwise indicated. Furthermore, we often abuse the terminology 'simplicial complex' (or 'simplicial map') to indicate the corresponding polyhedron (resp. PL map).

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2. *Euler characteristic formulas for subcomplexes*

In this section, we introduce the notions of local, semi-local and global two-colourability of subcomplexes of combinatorial manifolds and give related Euler characteristic formulas. Let Y be an $(n+1)$-dimensional combinatorial manifold (possibly with boundary) and X a finite n-dimensional subcomplex of Y. Denote by $V(X)$ the regular neighbourhood of X in Y.

Definition 2·1. We say that X is (globally) *two-colourable* in Y if there exist non-empty open subsets B and R of $Y \setminus X$ such that

$$Y \setminus X = B \cup R \quad \text{and} \quad \bar{B} \cap \bar{R} = \partial B = \partial R = X,$$

where $\bar{B}$ and $\bar{R}$ denote the closures of B and R in Y respectively, $\partial B = \bar{B} - B$ and $\partial R = \bar{R} - R$. We call B (or R) the *blue region* (resp. *red region*) and the pair (B, R) is called a *coloration* of X in Y.

We say that X is *semi-locally two-colourable* in Y if it is two-colourable in $V(X)$. Finally, X is said to be *locally two-colourable* in Y if for each simplex σ of X, $\mathrm{Link}\,(b_\sigma, Y'') \cap X$ is two-colourable in $\mathrm{Link}\,(b_\sigma, Y'')$, where b_σ is the barycentre of σ and Y'' is the second barycentric subdivision of Y.

Note that we have the implications:

$$\text{two-colourable} \implies \text{semi-locally two-colourable} \implies \text{locally two-colourable}.$$

Suppose now that the combinatorial manifold Y has no boundary. Then it is easy to see that if X is locally two-colourable in Y, then X must be *homogeneously n-dimensional*; i.e. every simplex of X is contained in some n-dimensional simplex as a face. Furthermore, for each $(n-1)$-simplex, the number of n-simplices which contain it as a face must be even; i.e. X must be *evenly branched* in the terminology of [**18**]. Thus, in particular, X carries a nonzero ($\mathbb{Z}_2$-)*fundamental class* $[X] \in H_n(X)$, which is the homology class represented by the n-cycle consisting of all n-simplices of X. We will denote by $U_X \in H^1(Y)$ the Poincaré dual of the element $i_*[X] \in H^c_n(Y)$, where $i\colon X \to Y$ is the inclusion map and H^c_* denotes the homology with closed support (see [**29**]). Then by using the standard homology theory, we can prove the following lemmas, whose proofs are left to the reader.

Lemma 2·2. *Let X be an n-dimensional subcomplex of an $(n+1)$-dimensional combinatorial manifold Y without boundary. Then X is locally two-colourable if and only if it is homogeneously n-dimensional and evenly branched.*

Lemma 2·3. *Let X and Y be as in Lemma 2·2 and suppose that X is locally two-colourable. Then,*

(1) *X is two-colourable if and only if $U_X = 0$ in $H^1(Y)$.*
(2) *If $H_1(Y) = 0$, then X is always two-colourable.*
(3) *If $i_*[X] = 0$ in $H_n(Y)$, then there exists a coloration (B, R) of X in Y such that $\bar{B} = B \cup X$ is compact.*
(4) *X is semi-locally two-colourable if and only if $i^*U_X = 0$ in $H^1(X)$.*

Definition 2·4. Suppose that X is a finite subcomplex of an $(n+1)$-dimensional combinatorial manifold Y without boundary and that X is semi-locally two-colourable

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in Y. We fix a coloration (B, R) of X in its regular neighbourhood $V(X)$. Note that

$$V(X) = \bigcup_{\sigma} \mathrm{Star}\,(b_\sigma, Y''),$$

where σ runs through all the simplices of X.

Set $X_B = \partial V(X) \cap B$ and $X_R = \partial V(X) \cap R$. Note that both X_B and X_R are non-empty combinatorial n-dimensional submanifolds of Y. Since X is a deformation retract of $V(X)$, we have natural simplicial maps $r_B : X_B \to X$ and $r_R : X_R \to X$. Then, for each simplex σ of X, we set

$$\chi_B(\sigma) = \chi(\mathrm{Link}\,(b_\sigma, Y'') \cap X_B) = \chi(r_B^{-1}(b_\sigma))$$

and

$$\chi_R(\sigma) = \chi(\mathrm{Link}\,(b_\sigma, Y'') \cap X_R) = \chi(r_R^{-1}(b_\sigma)).$$

We can use these numbers to compute the Euler characteristics of X_B and X_R. To see this, we prove an Euler characteristic formula for the source of a simplicial map in terms of the Euler characteristics of the fibres. Such a formula may have been well known; however, we give a brief and elementary proof, since it plays an important role here.

Lemma 2·5. *Let $f : X \to Y$ be a simplicial map between simplicial complexes, where X is finite. Then,*

$$\chi(X) = \sum_{\sigma} (-1)^{\dim \sigma} \chi(f^{-1}(b_\sigma)),$$

where σ runs through all the simplices of Y.

Proof. Consider the second barycentric subdivision Y'' of Y. For each simplex σ of Y, set $N_\sigma = \mathrm{Star}\,(b_\sigma, Y'')$, which is the union of the simplices of Y'' containing b_σ. Then we see easily that $f^{-1}(b_\sigma)$ is a deformation retract of $f^{-1}(N_\sigma)$. In particular, we have $\chi(f^{-1}(N_\sigma)) = \chi(f^{-1}(b_\sigma))$.

For a non-negative integer k, let X_k be the union of all $f^{-1}(N_\sigma)$, where σ runs through all the simplices of Y of dimension $\leqslant k$. Let us show that

$$\chi(X_k) = \sum_{\dim \sigma \leqslant k} (-1)^{\dim \sigma} \chi(f^{-1}(b_\sigma)) \tag{2·1}$$

by induction on k. For $k = 0$, this is obvious by the above observation. Suppose that (2·1) holds for k. Let W_{k+1} be the union of all $f^{-1}(N_\sigma)$, where σ runs through all the $(k+1)$-simplices of Y. Since $X_{k+1} = X_k \cup W_{k+1}$, we have

$$\chi(X_{k+1}) = \chi(X_k) + \chi(W_{k+1}) - \chi(X_k \cap W_{k+1}).$$

It is easy to see that, for each $(k+1)$-simplex σ of Y, $F_\sigma = f^{-1}(\mathrm{Link}\,(b_\sigma, Y'') \cap \sigma)$ is a deformation retract of $X_k \cap f^{-1}(N_\sigma)$. Since F_σ is homeomorphic to $f^{-1}(b_\sigma) \times S^k$, we have $\chi(X_k \cap f^{-1}(N_\sigma)) = \chi(F_\sigma) = \chi(f^{-1}(b_\sigma))\chi(S^k)$ and hence

$$\chi(W_{k+1}) - \chi(X_k \cap W_{k+1}) = (-1)^{k+1} \sum_{\dim \sigma = k+1} \chi(f^{-1}(b_\sigma)).$$

By our induction hypothesis, we obtain (2·1) for $k+1$. Thus (2·1) holds for all k and the result follows immediately. □

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Now, applying the above lemma to the simplicial maps $r_B\colon X_B \to X$ and $r_R\colon X_R \to X$, we obtain the following.

PROPOSITION 2·6. *Let X be a semi-locally two-colourable subcomplex of Y. Then,*

$$\chi(X_B) = \sum_{\sigma} (-1)^{\dim \sigma} \chi_B(\sigma) \quad \text{and} \quad \chi(X_R) = \sum_{\sigma} (-1)^{\dim \sigma} \chi_R(\sigma),$$

where σ runs through all the simplices of X.

Now, we introduce the (local) index of a vertex of X with respect to the coloration (B, R). Then we will get the main result of this section, which states that the Euler characteristic of X_B (or X_R) can be computed as the Euler characteristic of X plus the sum of the local contributions at the vertices.

Definition 2·7. Suppose that we are in the situation of Definition 2·4. If v is a vertex of X, then we define the *blue index* $\mathrm{Ind}_B(v) \in \mathbb{Q}$ and the *red index* $\mathrm{Ind}_R(v) \in \mathbb{Q}$ of v with respect to the coloration (B, R) by

$$\mathrm{Ind}_B(v) = \sum_{\sigma \ni v} \frac{(-1)^{\dim \sigma}}{\dim \sigma + 1} (\chi_B(\sigma) - 1) \quad \text{and} \quad \mathrm{Ind}_R(v) = \sum_{\sigma \ni v} \frac{(-1)^{\dim \sigma}}{\dim \sigma + 1} (\chi_R(\sigma) - 1)$$

respectively, where σ runs through all the simplices of X containing v as a vertex.

THEOREM 2·8. *Let X be a semi-locally two-colourable subcomplex of Y. Then,*

$$\chi(X_B) = \chi(X) + \sum_{v \in X^{(0)}} \mathrm{Ind}_B(v) \quad \text{and} \quad \chi(X_R) = \chi(X) + \sum_{v \in X^{(0)}} \mathrm{Ind}_R(v),$$

where $X^{(0)}$ denotes the set of vertices of X.

Proof. By Proposition 2·6, we have

$$\chi(X_B) - \chi(X) = \sum_{\sigma} (-1)^{\dim \sigma} (\chi_B(\sigma) - 1).$$

Note that each simplex σ contains exactly $\dim \sigma + 1$ vertices. Therefore,

$$\chi(X_B) - \chi(X) = \sum_{v \in X^{(0)}} \sum_{\sigma \ni v} \frac{(-1)^{\dim \sigma}}{\dim \sigma + 1} (\chi_B(\sigma) - 1),$$

which gives the desired result for $\chi(X_B)$. For $\chi(X_R)$ the proof is the same.

In the rest of this section, we will show that the blue and the red indices have nice duality properties. Before that, we need to prove the following lemma.

LEMMA 2·9. *For every finite simplicial complex W and every vertex v of W, we have*

$$\sum_{\sigma \ni v} \frac{(-1)^{\dim \sigma}}{\dim \sigma + 1} \chi(\mathrm{Link}\,(b_\sigma, W'')) = 0,$$

where σ runs through all the simplices of W containing v as a vertex.

Proof. For each fixed simplex τ containing v as a vertex and with $\dim \tau = m$, we have

$$\sum_{v \in \sigma \preceq \tau} \frac{(-1)^{\dim \sigma}}{\dim \sigma + 1} = \sum_{i=0}^{m} \frac{(-1)^i}{i+1} \binom{m}{i} = \frac{1}{m+1} = \frac{1}{\dim \tau + 1}.$$

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On the other hand, it is not difficult to show that

$$\chi(\mathrm{Link}\,(b_\sigma, W'')) = 1 + \sum_{\tau \succeq \sigma} (-1)^{\dim \tau + 1},$$

where τ runs through all the simplices of W containing σ as a face (including σ itself). Therefore,

$$\begin{aligned}\sum_{\sigma \ni v} \frac{(-1)^{\dim \sigma}}{\dim \sigma + 1} \chi(\mathrm{Link}\,(b_\sigma, W'')) &= \sum_{\sigma \ni v} \frac{(-1)^{\dim \sigma}}{\dim \sigma + 1} \left(1 + \sum_{\tau \succeq \sigma} (-1)^{\dim \tau + 1}\right) \\ &= \sum_{\sigma \ni v} \frac{(-1)^{\dim \sigma}}{\dim \sigma + 1} - \sum_{\tau \ni v} \left((-1)^{\dim \tau} \sum_{v \in \sigma \preceq \tau} \frac{(-1)^{\dim \sigma}}{\dim \sigma + 1}\right) \\ &= \sum_{\sigma \ni v} \frac{(-1)^{\dim \sigma}}{\dim \sigma + 1} - \sum_{\tau \ni v} \frac{(-1)^{\dim \tau}}{\dim \tau + 1} = 0.\end{aligned}$$

Proposition 2·10. *Let X be a semi-locally two-colourable subcomplex of Y and let v be a vertex of X. Then,*

(1) $\mathrm{Ind}_B(v) + \mathrm{Ind}_R(v) = 0$, *if n is even.*
(2) $\mathrm{Ind}_B(v) = \mathrm{Ind}_R(v) = -\chi_v(X)$, *if n is odd, where $\chi_v(X)$ denotes the local Euler characteristic of X at v introduced in Section* 1.

Proof. By the definition of $\mathrm{Ind}_B(v)$ and $\mathrm{Ind}_R(v)$, we have

$$\begin{aligned}\mathrm{Ind}_B(v) + \mathrm{Ind}_R(v) &= \sum_{\sigma \ni v} \frac{(-1)^{\dim \sigma}}{\dim \sigma + 1} (\chi_B(\sigma) - 1) + \sum_{\sigma \ni v} \frac{(-1)^{\dim \sigma}}{\dim \sigma + 1} (\chi_R(\sigma) - 1) \\ &= \sum_{\sigma \ni v} \frac{(-1)^{\dim \sigma}}{\dim \sigma + 1} (\chi_B(\sigma) + \chi_R(\sigma) - 2),\end{aligned}$$

where σ runs through all the simplices of X containing v as a vertex. It is not difficult to show, by using the definitions of $\chi_B(\sigma)$ and $\chi_R(\sigma)$, that

$$\begin{cases} \chi_B(\sigma) + \chi_R(\sigma) + \chi(\mathrm{Link}\,(b_\sigma, X'')) = 2, & \text{if } n \text{ is even,} \\ \chi_B(\sigma) + \chi_R(\sigma) - \chi(\mathrm{Link}\,(b_\sigma, X'')) = 0, & \text{if } n \text{ is odd,} \end{cases}$$

for all σ. Then the above lemma implies that

$$\mathrm{Ind}_B(v) + \mathrm{Ind}_R(v) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ -2\chi_v(X), & \text{if } n \text{ is odd.} \end{cases}$$

Suppose now that n is odd. It is easy to see from the definition of $\chi_B(\sigma)$ that

$$\chi(\mathrm{Link}\,(b_\sigma, \bar{B}'')) = -\chi_B(\sigma) + \chi(\mathrm{Link}\,(b_\sigma, X'')).$$

In particular, it follows from the above lemma that

$$\sum_{v \in \sigma \subset X} \frac{(-1)^{\dim \sigma}}{\dim \sigma + 1} \chi(\mathrm{Link}\,(b_\sigma, \bar{B}'')) = -\sum_{v \in \sigma \subset X} \frac{(-1)^{\dim \sigma}}{\dim \sigma + 1} \chi_B(\sigma).$$

Since n is odd, we see that in the left-hand side we get the same sum even if we consider only the simplices σ of $\bar{B}$ (instead of X) containing v as a vertex. Thus, by

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applying again the above lemma, we see that it is equal to zero and we get

$$\mathrm{Ind}_B(v) = \sum_{v\in\sigma\subset X} \frac{(-1)^{\dim\sigma}}{\dim\sigma+1}(\chi_B(\sigma)-1) = -\chi_v(X).$$

Hence, we also have that $\mathrm{Ind}_R(v) = -\chi_v(X)$ and this completes the proof.

Remark 2·11. Summing up the duality equations of the above proposition for all $v \in X^{(0)}$, we obtain

$$\begin{cases} \chi(X_B)+\chi(X_R)-2\chi(X)=0, & \text{if } n \text{ is even,} \\ \chi(X_B)=\chi(X_R)=0, & \text{if } n \text{ is odd.} \end{cases}$$

These equations themselves can easily be proved by using Poincaré duality together with the fact that $V(X)$ is a compact $(n+1)$-dimensional manifold with $\partial V(X) = X_B \cup X_R$. In other words, the above proposition shows that such relations among the global Euler characteristics also hold locally at each vertex of X.

Note also that when n is odd, the above proposition shows that the indices $\mathrm{Ind}_B(v)$ and $\mathrm{Ind}_R(v)$ depend only on X and not on a particular embedding of X into Y.

3. *Euler characteristic formulas for simplicial maps*

Now we consider a simplicial map $f: X \to Y$ between simplicial complexes, where X is finite. We start by giving an Euler characteristic formula for $f(X)$.

Definition 3·1. Let $D(f)$ denote the closure of the set $\{x \in X: f^{-1}(f(x)) \neq \{x\}\}$ in X and set $M(f) = f(D(f))$. It is not difficult to see that $D(f)$ and $M(f)$ are subcomplexes of X and Y respectively. We call $D(f)$ the *self-intersection set* of f and $M(f)$ the *multiple point set* of f.

For a vertex v of $M(f)$, define the *index* of v with respect to f by

$$\mathrm{Ind}_f(v) = \chi_v(M(f)) - \sum_{w\in(f^{-1}(v))^{(0)}} \chi_w(D(f)) \in \mathbb{Q},$$

where $(f^{-1}(v))^{(0)}$ denotes the set of vertices of $f^{-1}(v)$. Note that the index depends only on the surjective map $f|D(f): D(f) \to M(f)$.

Remark 3·2. Geloneze Neto [**8**] has considered the case where X is a combinatorial surface, Y is a combinatorial 3-dimensional manifold and f is non-degenerate (see [**26**] or Definition 3·9). He has defined a similar index $I_{w,v} \in \mathbb{Q}$ for each pair (w, v) of vertices such that $f(w) = v$ with $w \in X$ and $v \in Y$. In fact, in his situation, our index $\mathrm{Ind}_f(v)$ coincides with $\sum_{w\in(f^{-1}(v))^{(0)}} I_{w,v}$ for each vertex v of $M(f)$. The following result generalizes theorem 12 of [**8**].

Theorem 3·3. *Let $f: X \to Y$ be a simplicial map between simplicial complexes, where X is finite. Then,*

$$\chi(f(X)) = \chi(X) + \sum_{v\in M(f)^{(0)}} \mathrm{Ind}_f(v),$$

where $M(f)^{(0)}$ denotes the set of vertices of $M(f)$.

Proof. Let us define $\mathrm{Ind}_f(v) = 0$ for each vertex v of $f(X) \setminus M(f)$. Then, it is obvious that

$$\mathrm{Ind}_f(v) = \chi_v(f(X)) - \sum_{w\in(f^{-1}(v))^{(0)}} \chi_w(X)$$

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for each vertex v of $f(X)$. This gives that

$$\sum_{v\in M(f)^{(0)}} \mathrm{Ind}_f(v) = \sum_{v\in (f(X))^{(0)}} \chi_v(f(X)) - \sum_{w\in X^{(0)}} \chi_w(X) = \chi(f(X)) - \chi(X),$$

where $(f(X))^{(0)}$ denotes the set of vertices of $f(X)$.

Now we will use the results of Section 2 to obtain Euler characteristic formulas for semi-locally two-colourable maps. Suppose that X and Y are combinatorial manifolds without boundary with X compact and $\dim Y = n+1$, and that $f\colon X \to Y$ is a simplicial map. Note that X may not necessarily be of dimension n.

Definition 3·4. We say that $f\colon X \to Y$ is globally (resp. semi-locally or locally) *two-colourable* if so is $f(X)$ in Y in the sense of Definition 2·1.

Suppose that f is semi-locally two-colourable and let (B, R) be a coloration of $f(X)$ in its regular neighbourhood $V(f(X))$. For each vertex v of $f(X)$, we can consider the indices $\mathrm{Ind}_B(v)$ and $\mathrm{Ind}_R(v)$. Moreover, we set $\chi_B(f) = \chi(f(X)_B)$ and $\chi_R(f) = \chi(f(X)_R)$ (see Definitions 2·4 and 2·7).

Remark 3·5. Note that the blue and the red indices can be computed by using only those simplices lying on $M(f)$. That is, we have that

$$\mathrm{Ind}_B(v) = \sum_{M(f)\supset\sigma\ni v} \frac{(-1)^{\dim\sigma}}{\dim\sigma+1}(\chi_B(\sigma)-1),$$

where σ runs through all the simplices of $M(f)$ containing v as a vertex. The same is true for the red index. In particular, we also get that if a vertex v of $f(X)$ is not in $M(f)$, then $\mathrm{Ind}_B(v) = \mathrm{Ind}_R(v) = 0$.

In fact, it suffices to show that $\chi_B(\sigma) = \chi_R(\sigma) = 1$ for each simplex σ of $f(X)$ not contained in $M(f)$. If such a σ exists, then $f(X) \neq M(f)$ and hence X must be a combinatorial manifold of dimension n. Thus $\mathrm{Link}\,(b_\sigma, Y'') \bigcap f(X)$ is a combinatorial $(n-1)$-sphere embedded in the combinatorial n-sphere $\mathrm{Link}\,(b_\sigma, Y'')$. Then by Alexander duality, we see easily that

$$\chi_B(\sigma) = \chi(\mathrm{Link}\,(b_\sigma, Y'') \bigcap f(X)_B) = \chi_R(\sigma) = \chi(\mathrm{Link}\,(b_\sigma, Y'') \bigcap f(X)_R) = 1.$$

Therefore, as a consequence of Theorems 2·8 and 3·3 and the above remark, we arrive at the main theorem of this section.

Theorem 3·6. *Let $f\colon X \to Y$ be a semi-locally two-colourable simplicial map between combinatorial manifolds with X compact and (B, R) a coloration of $f(X)$ in its regular neighbourhood. Then,*

$$\chi_B(f) = \chi(X) + \sum_{v\in M(f)^{(0)}} (\mathrm{Ind}_f(v) + \mathrm{Ind}_B(v))$$

and

$$\chi_R(f) = \chi(X) + \sum_{v\in M(f)^{(0)}} (\mathrm{Ind}_f(v) + \mathrm{Ind}_R(v)).$$

When $n = \dim f(X) = \dim Y - 1$ is odd, we have $\chi_B(f) = \chi_R(f) = 0$, since $f(X)_B$ and $f(X)_R$ are closed n-dimensional manifolds. Furthermore, if $\dim X$ is also odd, then we have $\chi(X) = 0$. Thus, we obtain

$$\sum_{v\in M(f)^{(0)}} (\mathrm{Ind}_f(v) + \mathrm{Ind}_B(v)) = \sum_{v\in M(f)^{(0)}} (\mathrm{Ind}_f(v) + \mathrm{Ind}_R(v)) = 0.$$

In fact, such equalities also hold locally at each vertex v as follows.

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PROPOSITION 3·7. *Let $f: X \to Y$ be a semi-locally two-colourable simplicial map as in Theorem* 3·6 *and suppose that both* $\dim X$ *and* $n = \dim f(X)$ *are odd. Then, for each vertex v of $f(X)$, we have*

$$\mathrm{Ind}_f(v) = -\mathrm{Ind}_B(v) = -\mathrm{Ind}_R(v) = \chi_v(f(X)).$$

Proof. By Proposition 2·10, we have only to show that $\mathrm{Ind}_f(v) = \chi_v(f(X))$. Using Definition 3·1, we see easily that

$$\mathrm{Ind}_f(v) = \chi_v(f(X)) - \sum_{w \in (f^{-1}(v))^{(0)}} \chi_w(X).$$

Thus we have only to show that if X is an odd dimensional combinatorial manifold without boundary, then the local Euler characteristic $\chi_w(X)$ vanishes for each vertex w of X.

In fact, consider the natural embedding $X = X \times \{0\} \to X \times [-1, 1]$. Obviously, it is semi-locally two-colourable and we see easily that $\mathrm{Ind}_B(w) = \mathrm{Ind}_R(w) = 0$ for each vertex w of X. Thus by Proposition 2·10, we have $\chi_w(X) = 0$ for each vertex w of X.

Now, we analyse the case where n is even. Recall that in this case we have that $\mathrm{Ind}_B(v) = -\mathrm{Ind}_R(v)$ by Proposition 2·10. Moreover, in some cases we can ensure that $\chi_B(f)$ and $\chi_R(f)$ are even and then Theorem 3·6 gives a congruence modulo 2 as follows.

COROLLARY 3·8. *Suppose that $f: X \to Y$ is a simplicial map with $n = \dim f(X)$ being even such that either*

(1) *the map f is semi-locally two-colourable, Y is orientable and $n \equiv 2 \bmod 4$, or*
(2) *the map f is locally two-colourable and the fundamental homology class $[f(X)]$ vanishes in $H_n(Y)$.*

Then,

$$\chi(X) \equiv \sum_{v \in M(f)^{(0)}} (\mathrm{Ind}_f(v) + \mathrm{Ind}_B(v)) \equiv \sum_{v \in M(f)^{(0)}} (\mathrm{Ind}_f(v) + \mathrm{Ind}_R(v)) \bmod 2.$$

Proof. When (1) holds, $f(X)_B$ and $f(X)_R$ are orientable of dimension n and hence have even Euler characteristics. When (2) holds, f is globally two-colourable by Lemma 2·3. Furthermore, $(i_B)_*[f(X)_B] = (i_R)_*[f(X)_R] = [f(X)] = 0$ in $H_n(Y)$, where $i_B: f(X)_B \to Y$ and $i_R: f(X)_R \to Y$ are the inclusion maps. Thus $f(X)_B$ and $f(X)_R$ bound compact $(n+1)$-dimensional manifolds embedded in Y and hence have even Euler characteristics. Then, in both cases, the result follows from Theorem 3·6.

Definition 3·9. Let $f: X \to Y$ be a simplicial map between combinatorial manifolds without boundary, where X is compact. We say that f is *non-degenerate* if f preserves the dimension of each simplex of X (see [**26**]). Furthermore, we say that f is *strongly non-degenerate* if it is non-degenerate and we have $f^{-1}(f(\mathrm{Int}\,\sigma)) = \mathrm{Int}\,\sigma$ for each n-simplex σ of X, where $\dim X = n$. That is, f is strongly non-degenerate if it is non-degenerate and $D(f)$ and $M(f)$ have dimension $< \dim X$. It is easy to see that if $\dim Y = n+1$, then a strongly non-degenerate map is always locally two-colourable (see Lemma 2·2) and thus, under some homological conditions, we can ensure that it is semilocally two-colourable. Furthermore, if f is strongly non-degenerate, then we have $[f(X)] = f_*[X] \in H_n(Y)$ in the above corollary.

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4. *Singular surfaces in 3-manifolds*

Let us consider a strongly non-degenerate simplicial map $f: M \to N$ from a closed combinatorial surface M into a combinatorial 3-manifold N without boundary. For a vertex $v \in M(f)$, $\gamma_v = \mathrm{Link}\,(v, N'') \bigcap f(M)$ is a family of PL immersed closed curves with finitely many crossings in the 2-sphere $\mathrm{Link}\,(v, N'')$. For each point $x \in N$, we denote by μ_x the number of points in $f^{-1}(x)$ and we set

$$d_v = \sum_{w \in C_v} (\mu_w - 1),$$

where $C_v \subset \gamma_v$ is the set of crossings of γ_v. Then, by Definition 3·1,

$$\mathrm{Ind}_f(v) = 1 - \mu_v + \frac{d_v}{2} \tag{4·1}$$

for each $v \in M(f)^{(0)}$.

Now suppose that the strongly non-degenerate simplicial map $f: M \to N$ is semi-locally two-colourable. We fix a coloration (B, R) of $f(M)$ in its regular neighbourhood in N. Then, by Remark 3·5,

$$\mathrm{Ind}_B(v) = \chi(\mathrm{Link}\,(v, N'') \bigcap f(M)_B) - 1 - \frac{d_v}{2}$$

and

$$\mathrm{Ind}_R(v) = \chi(\mathrm{Link}\,(v, N'') \bigcap f(M)_R) - 1 - \frac{d_v}{2}$$

for each $v \in M(f)^{(0)}$.

Note that $\mathrm{Ind}_f(v)$, $\mathrm{Ind}_B(v)$ and $\mathrm{Ind}_R(v)$ are invariant under subdivision, since $D(f)$ and $M(f)$ are 1-dimensional. Combining the above observations with Theorem 3·3 or Theorem 3·6, we obtain Euler characteristic formulas for $f(M)$, $f(M)_B$ or $f(M)_R$.

Remark 4·1. The formula corresponding to Theorem 3·3 for nondegenerate simplicial maps of surfaces into combinatorial 3-dimensional manifolds has already been obtained by Geloneze Neto [**8**, theorem 12]. Taguchi [**28**] has also obtained a similar formula in a slightly different situation. In fact, Taguchi has considered smooth maps of smooth compact surfaces into smooth n-dimensional manifolds with $n \geqslant 3$ which are locally cone-like in his terminology. In this case, such a map is always triangulable and the triangulation is strongly non-degenerate. Then his result coincides with ours.

As an application of our results, let us consider the following class of smooth maps of surfaces into 3-dimensional manifolds.

Definition 4·2. Let M and N be smooth manifolds without boundary of dimensions 2 and 3 respectively. A smooth map $f: M \to N$ is said to be *globally finitely determined* if for each $p \in N$, $S_p = f^{-1}(p)$ consists of finitely many points and the multi-germ $f: (M, S_p) \to (N, p)$ is finitely determined in the usual sense (see, e.g. [**31**]).

In the following, for a topological space X, $\mathrm{Cone}\,(X)$ denotes the cone over X; i.e., $\mathrm{Cone}\,(X) = X \times [0,1]/X \times \{0\}$, where the point in $\mathrm{Cone}\,(X)$ corresponding to $X \times \{0\}$ is called the *vertex* of $\mathrm{Cone}\,(X)$. Furthermore, for a continuous map $f: X \to Y$, $\mathrm{Cone}\,(f): \mathrm{Cone}\,(X) \to \mathrm{Cone}\,(Y)$ denotes the continuous map defined by $f(x,t) = (f(x), t)$ $((x,t) \in X \times [0,1]/X \times \{0\})$. The following lemma follows from [**7**].

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LEMMA 4·3. *Let $f: M \to N$ be a proper smooth map. If f is globally finitely determined, then for each $p \in f(M)$, there exists an immersion with normal crossings $\gamma_p: S^1_1 \cup \cdots \cup S^1_{\mu_p} \to S^2$ such that the multi-germ $f: (M, S_p) \to (N, p)$ is C^0 right-left equivalent to the multi-germ*

$$\tilde{\gamma}_p: (\mathrm{Cone}\,(S^1_1) \cup \cdots \cup \mathrm{Cone}\,(S^1_{\mu_p}), V_{\mu_p}) \to (\mathrm{Cone}\,(S^2), V),$$

where μ_p is the number of points in S_p, V_{μ_p} is the set consisting of the vertices of $\mathrm{Cone}\,(S^1_1), \ldots, \mathrm{Cone}\,(S^1_{\mu_p})$, *$V$ is the vertex of* $\mathrm{Cone}\,(S^2)$ *and* $\tilde{\gamma}_p = \mathrm{Cone}\,(\gamma_p|S^1_1) \cup \cdots \cup \mathrm{Cone}\,(\gamma_p|S^1_{\mu_p})$.

Proof. Since everything is local, we may assume that $M = \mathbb{R}^2$ and $N = \mathbb{R}^3$. For simplicity, let us consider the case where $\mu_p = 1$ and consider the mono-germ $f: (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$. Since f is finitely determined, we may assume that the r-jet $W = j^r f(0) \in J^r(2, 3)$ determines f for some $r \geqslant 1$, where $J^r(2, 3)$ is the set of r-jets of smooth map germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$. Then by [**7**, theorem 3], there exists an integer $s \geqslant r$ and a closed semi-algebraic subset $\Sigma_W \subset (\pi^s_r)^{-1}(W)$ having positive codimension such that every map germ $g: (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ with $j^s g(0) \in (\pi^s_r)^{-1}(W) \setminus \Sigma_W$ is equivalent to the cone over a C^0 stable map of the circle into S^2, where $\pi^s_r: J^s(2, 3) \to J^r(2, 3)$ is the natural projection. Adding higher degree terms to f appropriately, we may assume that $j^s f(0) \in (\pi^s_r)^{-1}(W) \setminus \Sigma_W$, since f is finitely determined. Then the result follows, since a smooth map of the circle into S^2 is C^0 stable if and only if it is an immersion with normal crossings.

Remark 4·4. Although Fukuda [**7**] considers only the mono-germ case, the same arguments and results are valid also for the multi-germ case. Note that we can define globally finitely determined smooth maps $f: M \to N$ for manifolds M and N with $\dim M \leqslant \dim N$ in general. In this case also, a result similar to Lemma 4·3 holds. Note that if a proper smooth map $f: M \to N$ is C^0 stable, then it is globally finitely determined.

Remark 4·5. Examining Fukuda's argument [**7**] more carefully, we see that globally finitely determined proper maps are Thom stratifiable in the sense of [**24**]. Then by Shiota's result [**24**], there always exists a triangulation for such a map. Furthermore, such a triangulation is always strongly non-degenerate.

If $f: M \to N$ is a globally finitely determined proper map from a surface into a 3-manifold, then f is an immersion with transverse double points (which appear along a curve) except at isolated points. These isolated singularities can be classified according to their codimensions (intuitively, the codimension is equal to the minimum number of parameters so that we can locally embed the map into a generic family of maps). For instance, it is well known that the singularities of codimension $\leqslant 1$ are the following (see, for example [**10**, section 2·5]):

(i) Codimension 0 (or stable) singularities:
  (a) cross cap;
  (b) transverse triple point.

(ii) Codimension 1 singularities:
  (a) type **B**, degenerate cross cap of elliptic type;
  (b) type **K**, degenerate cross cap of hyperbolic type;
  (c) type **E**, double tangency of elliptic type;

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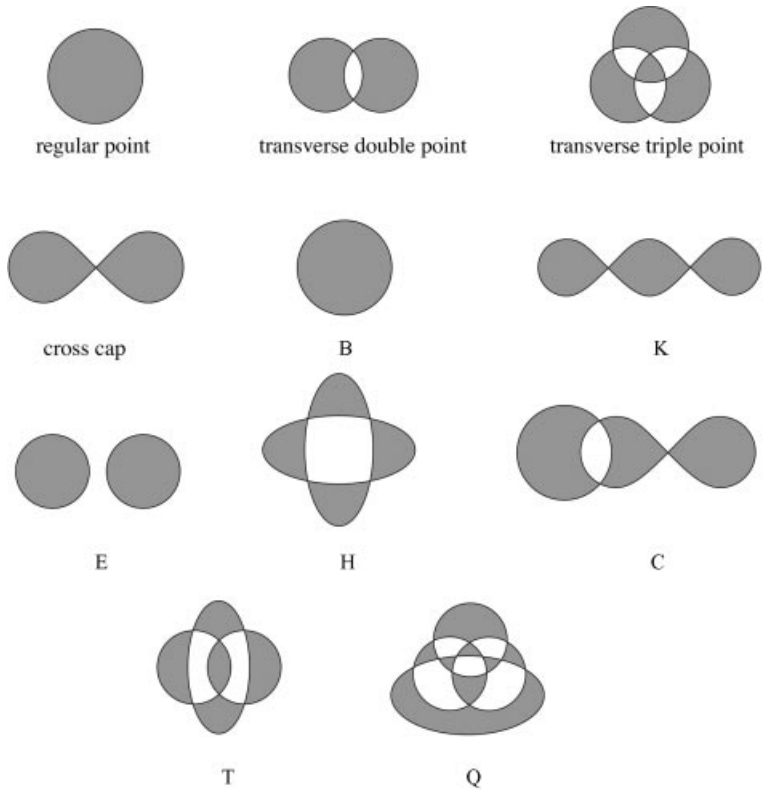


Fig. 1.

(d) type **H**, double tangency of hyperbolic type;
(e) type **C**, double point, with a cross cap at one of the points;
(f) type **T**, nontransverse triple point;
(g) type **Q**, quadruple point.

The corresponding curves γ_p as in Lemma 4·3 are depicted in Fig. 1, where the shadowed regions correspond to blue regions for a local coloration. Note that, for a point $p \in M(f)$, d_p coincides with the number of crossing points of γ_p.

Then, we have the corresponding indices as in Table 1. Here, for those singular points for which the blue and the red indices are different from each other, we say that such a singular point is *blue* if the local coloration is as depicted in Fig. 1 and *red* otherwise.

Thus we obtain the following.

Theorem 4·6. *Let $f: M \to N$ be a smooth map from a smooth closed surface M into a smooth 3-dimensional manifold N which has, at worst, codimension one singularities.*

(1) *Then,*

$$\chi(f(M)) = \chi(M) + T(f) + \frac{C(f)}{2} + K(f) - E(f) + H(f) + 3Q(f),$$

where $T(f)$ is the number of triple points of f (including the singularities of type **T**), *$C(f)$ is the number of cross caps of f (including the singularities of type* **C**) *and $K(f), E(f), H(f)$ and $Q(f)$ are the numbers of singularities of types* **K**, **E**, **H** *and* **Q** *of f respectively.*

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Table 1. *Indices for singularities at most of codimension 1*

p	μ_p	d_p	$\chi(S_{p,\varepsilon} \cap B)$	$\mathrm{Ind}_f(p)$	$\mathrm{Ind}_B(p)$	$\mathrm{Ind}_R(p)$
Regular point	1	0	1	0	0	0
Transverse double point	2	2	2	0	0	0
Transverse triple point	3	6	4	1	0	0
Blue cross cap	1	1	2	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
Red cross cap	1	1	2	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
B	1	0	1	0	0	0
K (blue)	1	2	3	1	1	-1
K (red)	1	2	3	1	-1	1
E (blue)	2	0	2	-1	1	-1
E (red)	2	0	2	-1	-1	1
H (blue)	2	4	4	1	1	-1
H (red)	2	4	4	1	-1	1
C (blue)	2	3	3	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
C (red)	2	3	3	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
T (blue)	3	6	5	1	1	-1
T (red)	3	6	5	1	-1	1
Q (blue)	4	12	8	3	1	-1
Q (red)	4	12	8	3	-1	1

(2) *Suppose either that f is semi-locally two-colourable and N is orientable, or that $f_*[M] = 0$ in $H_2(N)$, where $[M] \in H_2(M)$ denotes the fundamental class of M. Then,*

$$\chi(M) \equiv TT(f) + BC(f) \pmod 2,$$

where $TT(f)$ is the number of transverse triple points of f and $BC(f)$ is the number of blue cross caps of f (including the singularities of type **C**). *Here a cross cap p is blue if* $\mathrm{Ind}_B(p) = \frac{1}{2}$.

Note that Theorem 4·6(1) has already been obtained in [**10**, theorem 3·6]. Note also that Theorem 4·6(1) generalizes Izumiya–Marar's formula [**12**–**14**] and that (2) generalizes Banchoff's result [**3**] and also Szücs' result [**27**].

Remark 4·7. One can prove Theorem 3·3 combined with (4·1) for globally finitely determined smooth maps $f: M \to N$ of closed surfaces into 3-dimensional manifolds by using vector field arguments as follows. First we stratify $f(M)$ as an abstract stratified set in the following manner. As two dimensional strata, we take the connected components of the set of regular points. As 1-dimensional strata, we take the connected components of the set of transverse double points. The rest constitutes a set of isolated points, which are the 0-dimensional strata. Then we can construct a totally radial vector field $\mathbf{v}$ on this stratified set $f(M)$ in the sense of [**25**] as follows. On the 0-dimensional strata, we put $\mathbf{v} = 0$. On each 1-dimensional stratum which is homeomorphic to the circle, we define $\mathbf{v}$ to be an arbitrary nowhere zero tangent vector field. On each 1-dimensional stratum which is homeomorphic to the real line $\mathbb{R}$, we construct the vector field $\mathbf{v}$ so that it is equivalent to the vector field

$$-\frac{x}{1 + e^x + e^{-x}} \left(\frac{\partial}{\partial x} \right)$$

on $\mathbb{R}$, where x denotes the coordinate of $\mathbb{R}$. Then on each 2-dimensional stratum, we

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consider an arbitrary tangent vector field with isolated singularities such that the vector field $\mathbf{v}$ is totally radial in the sense of Simon [**25**, définition 3] (see also [**15**]). Then by [**25**, section 2·2, théorème 2], we have

$$\chi(f(M)) = n_0 - n_1 + \sum_{\mathbf{v}(x)=0} \mathrm{Ind}\,(\mathbf{v}|X, x),$$

where n_0 denotes the number of 0-dimensional strata, n_1 the number of 1-dimensional strata homeomorphic to $\mathbb{R}$ and X runs through all the 2-dimensional strata. Furthermore, by considering the stratification and the radial vector field on M naturally induced by f, we obtain

$$\chi(M) = \sum_p \mu_p - 2n_1 + \sum_{\mathbf{v}(x)=0} \mathrm{Ind}\,(\mathbf{v}|X, x),$$

where p runs through all the 0-dimensional strata. On the other hand, by examining the graph structure of $M(f)$, we see that

$$\sum_p d_p = 2n_1.$$

Thus we have

$$\chi(f(M)) - \chi(M) = \sum_p \left(1 - \mu_p + \frac{d_p}{2}\right),$$

which is nothing but Theorem 3·3 for globally finitely determined maps (see (4·1)).

Problem 4·8. Applying our results to the tangent developable of a generic space curve, can one obtain a new formula about the numbers of singularities of the curve? For related results see [**19**, **21**]. Note that Ishikawa [**11**] has obtained a topological classification of the tangent developable (mono-)germs of a large class of space curve germs. However, in order to apply our results, we also need a classification of tangent developables as multi-germs. We can also consider a similar problem for the dual of a generic space curve, which concerns the tangent planes to the space curve.

Remark 4·9. Let $f\colon M \to N$ be a globally finitely determined smooth proper map of an n-dimensional closed manifold into a $(2n-1)$-dimensional manifold N $(n \geqslant 2)$. Then, for each $p \in f(M)$, the multi-germ $f\colon (M, S_p) \to (N, p)$ is equivalent to the cone over an immersion with normal crossings of a disjoint union of homotopy $(n-1)$-spheres into S^{2n-2} in a sense similar to Lemma 4·3 (see Remark 4·4 and [**7**]). Let μ_p denote the number of homotopy $(n-1)$-spheres and d_p the number of crossings of the immersion. Then, with respect to an arbitrary triangulation of f (see Remark 4·5), we have

$$\mathrm{Ind}_f(p) = 1 - \mu_p + \frac{d_p}{2}.$$

For example, if p is an ordinary cross cap, then its index is equal to $\frac{1}{2}$. Then, using Theorem 3·3, we obtain an Euler characteristic formula for the image of such a globally finitely determined map f, which generalizes the result of Izumiya and Marar [**13**, theorem 1].

In the last part of this section, we give an estimate for the number of connected components of the complement $N \setminus f(M)$ of the image of a strongly non-

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degenerate simplicial map $f\colon M \to N$ from a closed combinatorial surface M into a 3-dimensional combinatorial manifold N without boundary. For each point $p \in M(f)$, we see that

$$\sum_{q \in f^{-1}(p)} \chi_q(D(f)) = \mu_p - d_p \quad \text{and} \quad \chi_p(M(f)) = 1 - \frac{d_p}{2}.$$

This gives

$$\chi(D(f)) = \sum_p (\mu_p - d_p) \quad \text{and} \quad \chi(M(f)) = \sum_p \left(1 - \frac{d_p}{2}\right),$$

where p runs through all the vertices of $M(f)$.

As has been observed in [**20**, section 4], if $H_1(N) = 0$, then we get the following estimates:

$$2 + \beta_1(D(f)) - \beta_1(M(f)) - \beta_1(M) \leqslant \beta_0(N - f(M)) \leqslant 2 + \beta_1(D(f)),$$

where β_i denotes the ith Betti number in $\mathbb{Z}_2$-coefficients. In particular, β_0 denotes the number of connected components. Since $\beta_1 = \beta_0 - \chi$ for 1-dimensional simplicial complexes, we obtain the following.

THEOREM 4·10. *Let $f\colon M \to N$ be a strongly non-degenerate simplicial map of a closed combinatorial surface M into a 3-dimensional combinatorial manifold N without boundary. Assume that $H_1(N) = 0$. Then,*

$$\beta_0(N - f(M)) \leqslant 2 + \beta_0(D(f)) + \sum_{p \in M(f)^{(0)}} (d_p - \mu_p)$$

and

$$\beta_0(N - f(M)) \geqslant 2 + \beta_0(D(f)) - \beta_0(M(f)) - \beta_1(M) + \sum_{p \in M(f)^{(0)}} \mathrm{Ind}_f(p).$$

For example, using the same notation as in Theorem 4·6, we have the following generalization of [**20**, theorem 4·2].

COROLLARY 4·11. *Let $f\colon M \to N$ be a smooth map of a smooth closed surface M into a smooth 3-dimensional manifold N which has, at worst, codimension 1 singularities. Then,*

$$\beta_0(N - f(M)) \leqslant 2 + \beta_0(D(f)) + 3T(f) + K(f) - 2E(f) + 2H(f) + \mathbf{C}(f) + 8Q(f)$$

and

$$\begin{aligned}\beta_0(N - f(M)) \geqslant 2 + \beta_0(D(f)) - \beta_0(M(f)) - \beta_1(M)\\ + T(f) + \frac{C(f)}{2} + K(f) - E(f) + H(f) + 3Q(f),\end{aligned}$$

where $\mathbf{C}(f)$ denotes the number of singularities of type $\mathbf{C}$ of f.

Remark 4·12. In the second inequality of Corollary 4·11, we always have

$$\beta_0(D(f)) - \beta_0(M(f)) - E(f) \geqslant 0.$$

In particular,

$$\beta_0(N - f(M)) \geqslant 2 - \beta_1(M) + T(f) + \frac{C(f)}{2} + K(f) + H(f) + 3Q(f).$$

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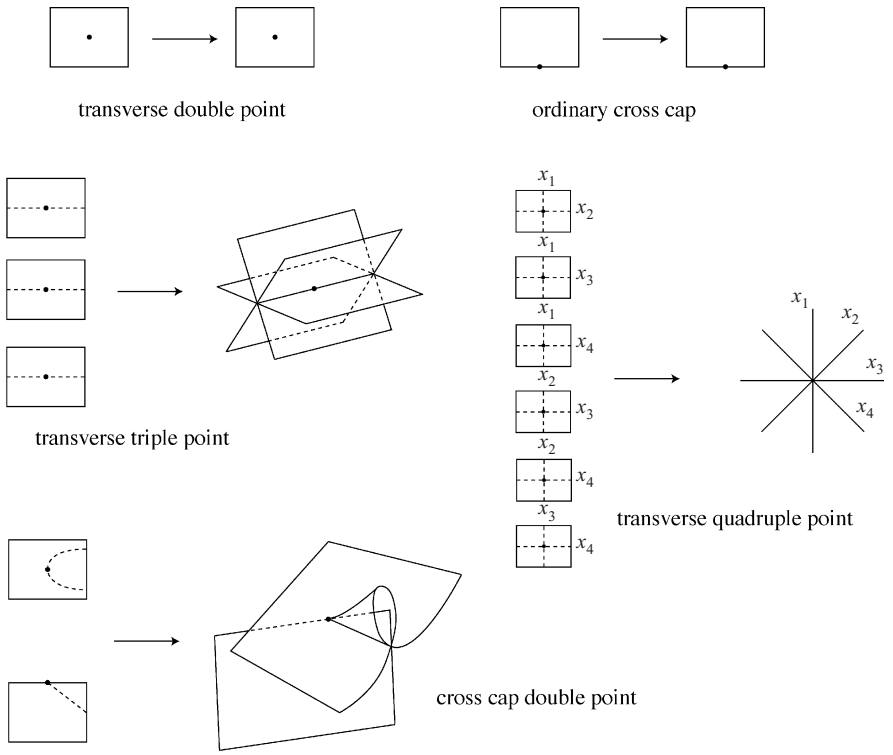


Fig. 2.

5. C^0 *stable maps from* 3*-manifolds into* 4*-manifolds*

Let M and N be smooth manifolds of dimensions 3 and 4 respectively, where M is closed. Consider a C^0 stable smooth map $f: M \to N$. Then f presents the following types of singularities: transverse double points, transverse triple points, ordinary cross caps, transverse quadruple points and transverse intersections of the set of cross caps and the regular points. We call the last type of a singularity a *cross cap double point*. Note that the set of transverse double points is of dimension 2, that those of transverse triple points and of ordinary cross caps are of dimension 1 and that those of transverse quadruple points and of cross cap double points are of dimension 0. Define $\Delta_2(f)$ to be the quotient of the closure of

$$\{(x_1, x_2) \in M \times M : f(x_1) = f(x_2), x_1 \neq x_2\}$$

by the involution $t: M \times M \to M \times M$ defined by $t(x_1, x_2) = (x_2, x_1)$. Note that $\Delta_2(f)$ is a compact 2-dimensional manifold with boundary corresponding to the cross caps. Furthermore, the natural map $\varphi: \Delta_2(f) \to N$ induced by f is an immersion except at finitely many points corresponding to the cross cap double points and $\varphi(\Delta_2(f)) = M(f)$. The local behaviour of the map φ at each point of $M(f)$ is depicted in Fig. 2, where the figures for the image represent the corresponding *abstract* polyhedra and are *not embedded* in 4-space, and in the figure for a transverse quadruple point, the images of two planes intersect only along the corresponding segments. Similarly, the local behaviour of the map $f|D(f): D(f) \to M(f)$ at each point of $M(f)$ is depicted in Fig. 3.

Let us fix a triangulation of $f: M \to N$ and we use the same notation for it (see

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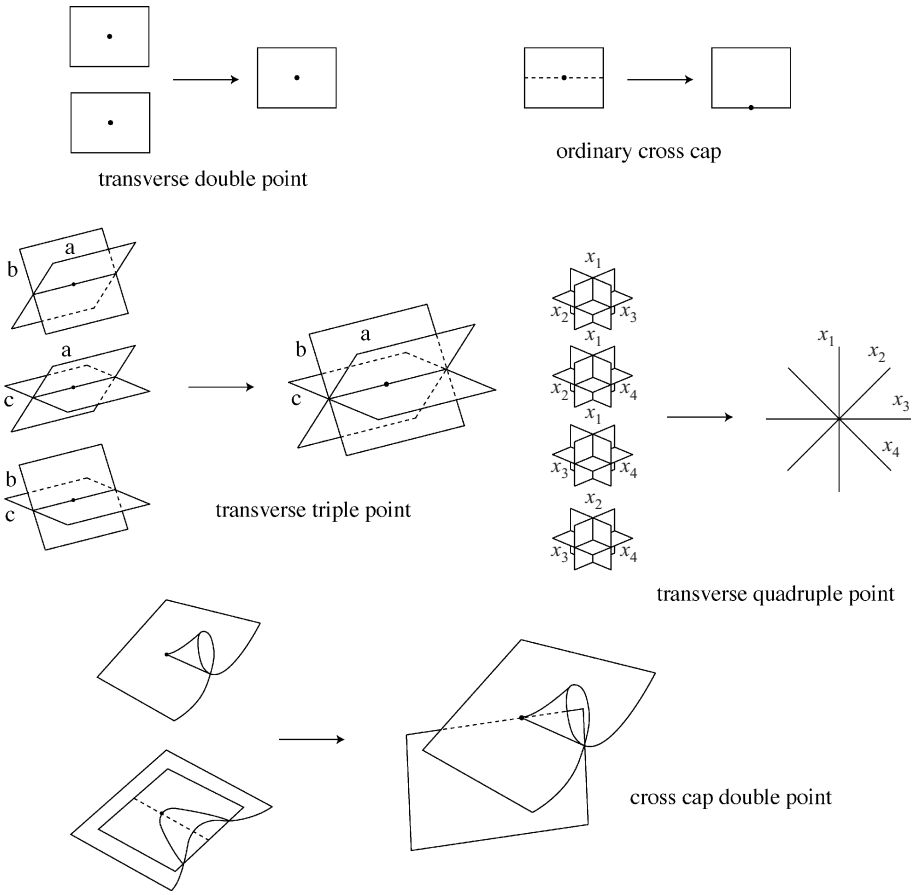


Fig. 3.

[**24**] or [**30**]). If $v \in M(f)$ is a transverse double point, then $\varphi^{-1}(v)$ consists of a single point, say w, and φ is locally an embedding at w. Thus,

$$\chi_v(M(f)) = \chi_w(\Delta_2(f)) \quad \text{and} \quad \sum_{v' \in f^{-1}(v)} \chi_{v'}(D(f)) = 2\chi_w(\Delta_2(f)),$$

which implies that $\mathrm{Ind}_f(v) = -\chi_w(\Delta_2(f))$.

If $v \in M(f)$ is a transverse triple point, then $\varphi^{-1}(v)$ consists of three points, say w_1, w_2 and w_3. Denoting by T the closure of the set of triple points of f in N, we see that

$$\chi_v(M(f)) = \sum_{i=1}^{3} \chi_{w_i}(\Delta_2(f)) - 2\chi_v(T)$$

and

$$\begin{aligned}\sum_{v' \in f^{-1}(v)} \chi_{v'}(D(f)) &= (\chi_{w_1}(\Delta_2(f)) + \chi_{w_2}(\Delta_2(f)) - \chi_v(T)) \\ &\quad + (\chi_{w_2}(\Delta_2(f)) + \chi_{w_3}(\Delta_2(f)) - \chi_v(T)) \\ &\quad + (\chi_{w_3}(\Delta_2(f)) + \chi_{w_1}(\Delta_2(f)) - \chi_v(T)) \\ &= 2\sum_{i=1}^{3} \chi_{w_i}(\Delta_2(f)) - 3\chi_v(T).\end{aligned}$$

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In particular, since $\chi_v(T) = 0$,

$$\mathrm{Ind}_f(v) = -\sum_{w\in\varphi^{-1}(v)} \chi_w(\Delta_2(f)) + \chi_v(T) = -\sum_{w\in\varphi^{-1}(v)} \chi_w(\Delta_2(f)).$$

If $v \in M(f)$ is an ordinary cross cap, then $\varphi^{-1}(v)$ consists of a single point, say w, and φ is locally an embedding at w. Denoting by C the closure of the set of cross caps of f in N, we get

$$\chi_v(M(f)) = \chi_w(\Delta_2(f)) \quad \text{and} \quad \chi_{v'}(D(f)) = 2\chi_w(\Delta_2(f)) - \chi_v(C),$$

where $v' = f^{-1}(v)$. This gives that $\mathrm{Ind}_f(v) = -\chi_w(\Delta_2(f)) + \chi_v(C) = -\chi_w(\Delta_2(f))$, since again $\chi_v(C) = 0$.

If $v \in M(f)$ is a transverse quadruple point, then $\varphi^{-1}(v)$ consists of six points, say $w_1, w_2, \ldots, w_6$. In this case,

$$\chi_v(M(f)) = \sum_{i=1}^{6} \chi_{w_i}(\Delta_2(f)) - 2\chi_v(T) - 3$$

and

$$\sum_{v'\in f^{-1}(v)} \chi_{v'}(D(f)) = 2\sum_{i=1}^{6} \chi_{w_i}(\Delta_2(f)) - 4\left(1 - \tfrac{6}{2}\right) - 4.$$

Therefore,

$$\mathrm{Ind}_f(v) = -\sum_{w\in\varphi^{-1}(v)} \chi_w(\Delta_2(f)) - 2\chi_v(T) - 7 = -\sum_{w\in\varphi^{-1}(v)} \chi_w(\Delta_2(f)) - 1,$$

since $\chi_v(T) = 1 - \frac{8}{2} = -3$.

Finally, if $v \in M(f)$ is a cross cap double point, then $\varphi^{-1}(v)$ consists of two points, say w_1 and w_2. Now,

$$\chi_v(M(f)) = \sum_{i=1}^{2} \chi_{w_i}(\Delta_2(f)) - 2\chi_v(T) + 1$$

and

$$\sum_{v'\in f^{-1}(v)} \chi_{v'}(D(f)) = 2\sum_{i=1}^{2} \chi_{w_i}(\Delta_2(f)) - \chi_v(T) + 1.$$

This gives that

$$\mathrm{Ind}_f(v) = -\sum_{w\in\varphi^{-1}(v)} \chi_w(\Delta_2(f)) - \chi_v(T) = -\sum_{w\in\varphi^{-1}(v)} \chi_w(\Delta_2(f)) - \tfrac{1}{2},$$

since $\chi_v(T) = 1 - \frac{1}{2} = \frac{1}{2}$.

Combining the above calculations with Theorem 3·3, we obtain the following result which coincides with that of Houston [**10**, theorem 3·5]. This is because $\chi(D^2(f)^\sigma) = \chi(D^3(f)) = 0$ and $\chi(D^2(f))/2 = \chi(\Delta_2(f))$ in his terminologies.

Theorem 5·1. *Let $f\colon M \to N$ be a C^0 stable smooth map of a smooth closed 3-dimensional manifold M into a smooth 4-dimensional manifold N. Then we have*

$$\chi(f(M)) = -\chi(\Delta_2(f)) - Q(f) - \frac{CD(f)}{2},$$

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where $Q(f)$ and $CD(f)$ denote the numbers of transverse quadruple points and the cross cap double points of f respectively.

6. *Maps with 1-dimensional double point locus*

Using our result, we can also recover the following special case of a result of Houston [**10**, theorem 3·2]. In fact, he has obtained Euler characteristic formulas for the images of a more general class of maps in terms of the alternating homology and the multiple point sets.

PROPOSITION 6·1. *Let $f: X \to Y$ be a non-degenerate simplicial map between simplicial complexes, where X is finite. Suppose that $D(f)$ is at most 1-dimensional, that f has finitely many triple points and that f has no quadruple points. Then we have*

$$\chi(f(X)) = \chi(X) + \tfrac{1}{2}\left(\chi(\tilde{\Delta}_2(f)^t) - \chi(\tilde{\Delta}_2(f))\right) + T(f),$$

where $\tilde{\Delta}_2(f)$ is the closure of the set

$$\{(x_1, x_2) \in X \times X : f(x_1) = f(x_2), x_1 \neq x_2\}$$

in $X \times X$, $\tilde{\Delta}_2(f)^t$ is the intersection of $\tilde{\Delta}_2(f)$ with the diagonal of $X \times X$, and $T(f)$ denotes the number of triple points of f in $f(X)$.

Proof. It follows from Theorem 3·3 and our hypothesis, that

$$\begin{aligned}\chi(f(X)) - \chi(X) &= \chi(M(f)) - \chi(D(f)) \\ &= c_0(M(f)) - c_1(M(f)) - c_0(D(f)) + c_1(D(f)),\end{aligned}$$

where c_i denotes the number of i-simplices. Since f has finitely many triple points and $D(f)$ is 1-dimensional, we see easily that $c_1(\tilde{\Delta}_2(f)) = c_1(D(f)) = 2c_1(M(f))$. Hence,

$$\chi(f(X)) - \chi(X) = c_0(M(f)) - c_0(D(f)) + \tfrac{1}{2}c_1(\tilde{\Delta}_2(f)).$$

On the other hand, by our hypothesis, we see that $\tilde{\Delta}_2(f)^t$ consists of finitely many points. Thus,

$$\tfrac{1}{2}\left(\chi(\tilde{\Delta}_2(f)^t) - \chi(\tilde{\Delta}_2(f))\right) + T(f) = \tfrac{1}{2}c_0(\tilde{\Delta}_2(f)^t) - \tfrac{1}{2}c_0(\tilde{\Delta}_2(f)) + \tfrac{1}{2}c_1(\tilde{\Delta}_2(f)) + T(f).$$

Therefore, we have only to show that

$$c_0(M(f)) - c_0(D(f)) = \tfrac{1}{2}\left(c_0(\tilde{\Delta}_2(f)^t) - c_0(\tilde{\Delta}_2(f))\right) + T(f).$$

Since f has no quadruple points,

$$\begin{aligned}&\tfrac{1}{2}\left(c_0(\tilde{\Delta}_2(f)^t) - c_0(\tilde{\Delta}_2(f))\right) + T(f) \\ &\quad = -\tfrac{1}{2}\sharp\{(x_1, x_2) \in D(f)^{(0)} \times D(f)^{(0)} : f(x_1) = f(x_2), x_1 \neq x_2\} + T(f) \\ &\quad = -\sharp\{y \in M(f)^{(0)} : \sharp f^{-1}(y) = 2\} - 3\sharp\{y \in M(f)^{(0)} : \sharp f^{-1}(y) = 3\} + T(f) \\ &\quad = -\sharp\{y \in M(f)^{(0)} : \sharp f^{-1}(y) = 2\} - 2\sharp\{y \in M(f)^{(0)} : \sharp f^{-1}(y) = 3\} \\ &\quad = c_0(M(f)) - c_0(D(f)),\end{aligned}$$

where $\sharp$ denotes the number of elements in the set. This completes the proof.

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Remark 6·2. In the above situation, we have

$$\tfrac{1}{2}\big(\chi(\tilde{\Delta}_2(f)^t) - \chi(\tilde{\Delta}_2(f))\big) = -\chi(\Delta_2(f)) + \chi(\tilde{\Delta}_2(f)^t),$$

where $\Delta_2(f)$ is the quotient space of $\tilde{\Delta}_2(f)$ by the involution $t\colon X \times X \to X \times X$ defined by $t(x_1, x_2) = (x_2, x_1)$. Combining this with Proposition 6·1, we can prove Proposition 7·1 in Section 7 for smooth immersions with normal crossings from smooth surfaces into smooth 3-dimensional manifolds.

7. *Immersions with normal crossings*

Suppose that $f\colon M \to N$ is a smooth immersion with normal crossings from a closed n-dimensional manifold M into an $(n+1)$-dimensional manifold N. For $i = 1, 2, \ldots, n+1$, let us define $\Delta_i(f)$ to be the quotient of the closure of

$$\{(x_1, \ldots, x_i) \in M \times \cdots \times M\colon f(x_1) = \cdots = f(x_i), x_k \neq x_l\ (k \neq l)\}$$

by the natural action of the permutation group on i letters. Note that $\Delta_i(f)$ is a smooth closed manifold of dimension $n + 1 - i$. Furthermore, f induces a natural map $\varphi_i\colon \Delta_i(f) \to N$, which is a smooth immersion and whose image coincides with the closure of the set of i-tuple points of f in N (see also [**2**]).

Proposition 7·1. *Let $f\colon M \to N$ be a smooth immersion with normal crossings from a closed n-dimensional manifold M into an $(n+1)$-dimensional manifold N. Then,*

$$\chi(f(M)) = \sum_{i=1}^{n+1} (-1)^{i-1} \chi(\Delta_i(f)).$$

Proof. For i with $1 \leqslant i \leqslant n$, set $M(f)_i = \varphi_i(\Delta_i(f)) - \varphi_{i+1}(\Delta_{i+1}(f))$ and set $M(f)_{n+1} = \varphi_{n+1}(\Delta_{n+1}(f))$. Let us fix a triangulation of f and we use the same notation for it. For a vertex $v \in f(M)$, $1 \leqslant i \leqslant n+1$ and $0 \leqslant j \leqslant n+1-i$, let $\alpha_{i,v}^j$ denote the number of j-simplices of $f(M)$ which contain v as a vertex and whose interiors lie in $M(f)_i$ and set

$$\beta_{i,v} = \sum_{j=0}^{n+1-i} \frac{(-1)^j}{j+1} \alpha_{i,v}^j.$$

In this way, if v is a vertex of $\varphi_i(\Delta_i(f))$,

$$\sum_{w \in \varphi_i^{-1}(v)} \chi_w(\Delta_i(f)) = \sum_{r=i}^{n+1} \binom{r}{i} \beta_{r,v}.$$

This gives that for each vertex v of $M(f)_i$,

$$\begin{aligned}
\sum_{j=1}^{i} \left((-1)^{j-1} \sum_{w \in \varphi_j^{-1}(v)} \chi_w(\Delta_j(f)) \right) &= \sum_{j=1}^{i} \left((-1)^{j-1} \sum_{r=j}^{i} \binom{r}{j} \beta_{r,v} \right) \\
&= \sum_{r=1}^{i} \left(\sum_{j=1}^{r} (-1)^{j-1} \binom{r}{j} \right) \beta_{r,v} \\
&= \sum_{r=1}^{i} \beta_{r,v} = \chi_v(f(M)).
\end{aligned}$$

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Thus,

$$\begin{aligned}\chi(f(M)) &= \sum_{v\in(f(M))^{(0)}} \chi_v(f(M)) = \sum_{i=1}^{n+1} \sum_{v\in(M(f)_i)^{(0)}} \chi_v(f(M)) \\ &= \sum_{i=1}^{n+1} \sum_{v\in(M(f)_i)^{(0)}} \sum_{j=1}^{i} \left((-1)^{j-1} \sum_{w\in\varphi_j^{-1}(v)} \chi_w(\Delta_j(f)) \right) \\ &= \sum_{j=1}^{n+1} \left((-1)^{j-1} \sum_{i=j}^{n+1} \sum_{v\in(M(f)_i)^{(0)}} \sum_{w\in\varphi_j^{-1}(v)} \chi_w(\Delta_j(f)) \right) \\ &= \sum_{j=1}^{n+1} \left((-1)^{j-1} \sum_{w\in\Delta_j(f)^{(0)}} \chi_w(\Delta_j(f)) \right) = \sum_{j=1}^{n+1} (-1)^{j-1} \chi(\Delta_j(f)).\end{aligned}$$

This completes the proof.

Suppose that $f\colon M \to N$ is semi-locally two-colourable and we fix a coloration (B, R) of $f(M)$ in its regular neighborhood $V(f(M))$. Since f is an immersion with normal crossings, we see easily that $\mathrm{Ind}_B(v) = \mathrm{Ind}_R(v)$ for every vertex v of $f(M)$. Thus by Theorem 3·6 we have $\chi_B(f) = \chi_R(f)$. On the other hand, if n is even, it follows that $2\chi(f(M)) = \chi_B(f) + \chi_R(f)$ as has been observed in Remark 2·11. Thus we obtain the following result.

Proposition 7·2. *Let $f\colon M \to N$ be a smooth immersion with normal crossings from a smooth closed n-dimensional manifold M into a smooth $(n+1)$-dimensional manifold N. If n is even and f is semi-locally two-colourable, then*

$$\chi_B(f) = \chi_R(f) = \chi(f(M)) = \sum_{i:\mathrm{odd}} \chi(\Delta_i(f)).$$

As a corollary, we obtain the following refinement of a result of [**2**].

Corollary 7·3. *Let $f\colon M \to N$ be a smooth immersion with normal crossings from a smooth closed n-dimensional manifold M into a smooth $(n+1)$-dimensional manifold N with n even. Suppose either that f is semi-locally two-colourable, N is orientable and $n \equiv 2 \pmod 4$, or that $f_*[M] = 0$ in $H_n(N)$. Then,*

$$\chi(f(M)) \equiv \sum_{i:\mathrm{odd}} \chi(\Delta_i(f)) \equiv 0 \pmod 2.$$

When $N = \mathbb{R}^{n+1}$ and n is even, by a result of Herbert [**9**] (see also [**5**, 1·8 remark (d)] and [**23**, proposition 3·1]), we always have $\chi(M) = \chi(\Delta_1(f)) \equiv \chi(\Delta_{n+1}(f)) \pmod 2$.

Corollary 7·4. *Let $f\colon M \to \mathbb{R}^{n+1}$ be a smooth immersion with normal crossings of a smooth closed n-dimensional manifold M with n even. Then,*

$$\chi(f(M)) \equiv \sum_{3\leqslant i\leqslant n-1, i:\mathrm{odd}} \chi(\Delta_i(f)) \equiv 0 \pmod 2.$$

For example, in the above situation, if $n = 4$, then $\chi(\Delta_3(f)) \equiv 0 \pmod 2$, where $\Delta_3(f)$ is a closed surface.

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When n is odd, we have

$$\chi(f(M)) = -\sum_{i:\text{even}} \chi(\Delta_i(f)).$$

In particular, for $n = 3$, it follows that $\chi(f(M)) = -\chi(\Delta_2(f)) - Q(f)$, which is also a consequence of Theorem 5·1. By results of Freedman [**6**] and Koschorke [**16**] (see also [**1**]), we have $\chi(\Delta_2(f)) \equiv Q(f) \pmod 2$, if $N = \mathbb{R}^4$. Hence we see that $\chi(f(M))$ is even, provided that $n = 3$ and $N = \mathbb{R}^4$. When $n = 1$, the figure 8 immersion of the circle into $\mathbb{R}^2$ shows that $\chi(f(M))$ is not necessarily an even number. We do not know if $\chi(f(M))$ is always even for each $n \neq 1$ when $N = \mathbb{R}^{n+1}$ (see also [**2**]).

8. *Relative version*

In this last section, we will dedicate some words in order to show how to update our techniques to the case of manifolds with boundary.

First note that Theorem 3·3 holds for any simplicial map $f: X \to Y$ between general simplicial complexes. In particular, it can be applied to the case where X and/or Y are manifolds with boundary. Then let us explain how to generalize the formulas for semi-locally two-colourable maps to such cases.

Let X be a homogeneously n-dimensional finite simplicial complex. We denote by ∂X the subcomplex of X consisting of the $(n-1)$-simplices σ of X with the number of n-simplices of X containing σ as a face equal to an *odd* integer, together with their faces. Note that ∂X is homogeneously $(n-1)$-dimensional. As is easily seen, X carries a non-zero *fundamental class* $[X, \partial X] \in H_n(X, \partial X)$, which is represented by the n-cycle consisting of all the n-simplices of X. Furthermore, the image of $[X, \partial X]$ by the boundary homomorphism $\partial: H_n(X, \partial X) \to H_{n-1}(\partial X)$ coincides with the fundamental class of ∂X. In particular, ∂X is evenly branched. Let Y be an $(n+1)$-dimensional combinatorial manifold with boundary ∂Y (possibly empty) and X a finite subcomplex of Y. Then it is not difficult to show that X is locally two-colourable in the sense of Definition 2·1 if and only if X is homogeneously n-dimensional, $\partial X \subset \partial Y$ and $X \bigcap \partial Y$ has dimension $< n$ (compare this with Lemma 2·2). We can also prove results similar to Lemma 2·3 in this case as well. Now suppose that X is semi-locally two-colourable. We fix a coloration (B, R) of X in its regular neighbourhood $V(X)$. We set $X_B = V(X) \bigcap \overline{(Y - V(X))} \bigcap B$ and $X_R = V(X) \bigcap \overline{(Y - V(X))} \bigcap R$, which are nonempty n-dimensional combinatorial submanifolds of Y with boundary. Furthermore, we define $\chi_B(\sigma)$ and $\chi_R(\sigma)$ for each simplex σ of X and the blue and the red indices of each vertex of X as in Definitions 2·4 and 2·7 respectively. Under such definitions, Theorem 2·8 holds in our situation as well.

Now let $f: X \to Y$ be a simplicial map between combinatorial manifolds (possibly with boundary) with X compact. As in Definition 3·4, we can define the notion of global (semi-local or local) two-colorability of f. Suppose now that f is semi-locally two-colourable and let us fix a coloration (B, R) of $f(X)$ in its regular neighbourhood $V(f(X))$. We denote by $\chi_B(f)$ and $\chi_R(f)$ the Euler characteristics of $f(X)_B$ and $f(X)_R$ respectively. Then Theorem 3·6 holds in our situation as well.

Note, however, that the duality properties for blue and red indices as in Proposition 2·10 must be modified appropriately at vertices on the boundaries. When the self-intersection set $D(f)$ and the multiple point set $M(f)$ are 1-dimensional, the blue

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and the red indices simplify to

$$\mathrm{Ind}_B(v) = \chi_B(v) - 1 - \frac{1}{2}\sum_{\sigma \ni v}(\chi_B(\sigma) - 1)$$

and

$$\mathrm{Ind}_R(v) = \chi_R(v) - 1 - \frac{1}{2}\sum_{\sigma \ni v}(\chi_R(\sigma) - 1)$$

respectively, where σ runs through all the 1-simplices of $M(f)$ containing v as a vertex. In particular, the blue and the red indices are invariant under subdivision.

For example, let us consider a generic neat smooth map $g\colon M \to N$ from a smooth compact surface M with boundary into a smooth 3-dimensional manifold N with boundary as considered in [**22**] (recall that a map $g\colon M \to N$ is *neat* if $g^{-1}(\partial N) = \partial M$). Then g is a C^0 stable map on the interior. On the boundary g is regular except at some isolated points, where we find transverse boundary double points.

Let $f\colon X \to Y$ be a triangulation of g. Then, for a boundary double point v, we have

$$\mathrm{Ind}_f(v) = (1-2) - \tfrac{1}{2}(1-(1+1)) = -\tfrac{1}{2}.$$

Moreover, if f is semi-locally two-colourable, then we have

$$\mathrm{Ind}_B(v) = \mathrm{Ind}_R(v) = (2-1) - \tfrac{1}{2}(2-1) = \tfrac{1}{2}$$

for a boundary double point v. Thus we obtain the following Euler characteristic formulas.

Corollary 8·1. *Let $g\colon M \to N$ be a generic neat smooth map from a smooth compact surface M with boundary into a smooth 3-dimensional manifold N with boundary.*

(1) *Let $T(g), C(g)$ and $D(g)$ denote the numbers of triple points, cross caps and boundary double points of g respectively. Then,*

$$\chi(g(M)) = \chi(M) + T(g) + \frac{C(g)}{2} - \frac{D(g)}{2}.$$

(2) *Suppose that g is semi-locally two-colourable and denote by $BC(g)$ and $RC(g)$ the numbers of blue and red cross caps of g respectively. Then,*

$$\chi_B(g) = \chi(M) + T(g) + BC(g) \quad \text{and} \quad \chi_R(g) = \chi(M) + T(g) + RC(g).$$

(3) *Suppose that g is semi-locally two-colourable and N is orientable. We denote by $\beta_0(\partial g(M)_B)$ the number of boundary components of $g(M)_B$. Then,*

$$\chi(M) \equiv T(g) + BC(g) + \beta_0(\partial g(M)_B) \bmod 2.$$

(4) *Suppose that $g_*[M, \partial M] = 0$ in $H_2(N, \partial N)$ and choose the coloration (B, R) of $g(M)$ in N so that $\bar{B}$ is compact (see Lemma 2·3(3)). Then,*

$$\chi(M) \equiv T(g) + BC(g) + D(g) + \chi(\bar{B} \cap \partial N) \bmod 2.$$

Proof. Parts (1) and (2) follow directly from Theorems 3·3 and 3·6 respectively. As to part (3), by our hypothesis, we see that $g(M)_B$ is orientable. Thus $\chi_B(g) = \chi(g(M)_B) \equiv \beta_0(\partial g(M)_B) \bmod 2$. Then the result follows from part (2). As to part (4), $W = \overline{B - V(g(M))}$ is a compact 3-dimensional manifold with boundary and hence

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we have

$$\chi(g(M)_B) + \chi(\bar{B} \bigcap \partial N) + D(g) = \chi(\partial W) \equiv 0 \mod 2.$$

Thus the result follows from part (2) and Definition 3·9.

Note that part (1) of the above corollary has essentially been obtained in Theorem 2 of [**17**]. On the other hand, the part (3) is nothing but theorem 4·2 of [**22**], since $D(g) + \chi(\bar{B} \bigcap \partial N)$ coincides with the topological Euler characteristic of $B \bigcap \partial N$.

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