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When is a dynamical system mean sensitive?

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Abstract. This article is devoted to studying which conditions imply that a topological dynamical system is mean sensitive and which do not. Among other things, we show that every uniquely ergodic, mixing system with positive entropy is mean sensitive. On the other hand, we provide an example of a transitive system which is cofinitely sensitive or Devaney chaotic with positive entropy but fails to be mean sensitive. As applications of our theory and examples, we negatively answer an open question regarding equicontinuity/sensitivity dichotomies raised by Tu, we introduce and present results of locally mean equicontinuous systems and we show that mean sensitivity of the induced hyperspace does not imply that of the phase space.

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1. Introduction

A pair (X, T) is called a *topological dynamical system* (or simply *t.d.s.*) if X is a compact metric space with metric d and $T : X \to X$ is a continuous map. A Borel probability measure μ in X is *ergodic* if it is invariant under T and every invariant set has measure zero or one.

A question of interest in topological dynamical systems is when orbits from nearby points deviate. *Sensitive dependence on initial conditions* (or briefly *sensitivity*) appeared as the first explicit mathematical definition to describe this turbulent behaviour [**30**]. According to the work of Auslander and Yorke [**3**], we say that a t.d.s. (X, T) is *sensitive* if there exists $\delta > 0$ such that, for any non-empty open subset $U \subset X$, there exist $x, y \in U$ and $n \in \mathbb{N}$ such that $d(T^n x, T^n y) > \delta$. Following from [**2**, Theorem 3.4], we state the following proposition.

PROPOSITION 1.1. *A t.d.s.* (X, T) *is sensitive if and only if there exists a* $\delta > 0$ *such that, for any non-empty open subset* $U \subset X$*, one of the following holds:*

(1) $N_T(U, \delta) := \{n \in \mathbb{Z}_+ : \operatorname{diam}(T^n(U)) > \delta\}$ *is infinite, where* $\operatorname{diam}(\cdot)$ *denotes the diameter of the set;*
(2) $\limsup_{n\to\infty} d(T^n x, T^n y) > \delta$ *for some* $x, y \in U$*; or*
(3) $\limsup_{n\to\infty} \operatorname{diam}(T^n U) > \delta$.

Sensitivity as a form of chaos is very weak. For example, the Sturmian subshift, which is considered as a very rigid system, is sensitive.

Stronger forms of sensitivity have also been studied by adding extra requirements on the set $N_T(U, \delta)$ in (1). This idea was carried out by Moothathu in [**27**]. In particular, he introduced the notion of cofinite sensitivity. That is, a t.d.s. (X, T) is *cofinitely sensitive* if there is a $\delta > 0$ such that, for any non-empty open subset $U \subset X$, $N_T(U, \delta)$ is cofinite, i.e., $\operatorname{diam}(T^i U) \le \delta$ for only finitely many times.

Another interesting approach to strengthen sensitivity (due to its connections with ergodic theory) is to replace the upper limits in (2) and (3) by upper average limits (Cesaro and Banach averages) [**10**, **25**]. Strictly speaking, this is given by the following definition.

Definition 1.2. A t.d.s. (X, T) is *mean sensitive* (or *Banach-mean sensitive*) if there is a $\delta > 0$ such that, for any neighbourhood U of X, there are $x, y \in U$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) > \delta \left(\text{ or } \limsup_{N-M\to\infty} \frac{1}{N-M} \sum_{i=M}^{N-1} d(T^i x, T^i y) > \delta \right).$$

A t.d.s. (X, T) is *diam-mean sensitive* if there is a $\delta > 0$ such that, for every neighbourhood U of X,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \operatorname{diam}(T^i U) > \delta.$$

When considering the opposite side of sensitivity (respectively, mean sensitivity, Banach-mean sensitivity, diam-mean sensitivity), the notions of equicontinuous (respectively, mean equicontinuous, Banach-mean equicontinuity, diam-mean equicontinuous) points and systems appear correspondingly ([**10**, **13**, **25**], respectively); see §2 for definitions.

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Mean sensitivity/equicontinuity has been studied recently because it turned out to be a useful concept to describe/characterize ergodic theoretic properties such as when a system has a discrete spectrum and when its maximal equicontinuous factor is an isomorphism with topological notions [**7**, **10**, **25**]. The main concern of this paper is to explore which conditions imply mean sensitivity and which do not.

The following implications follow from the definitions.

$$\begin{array}{ccc} \text{mean sensitivity} & \Rightarrow & \text{Banach-mean sensitivity} \\ \Downarrow & & \\ \text{cofinite sensitivity} & \Rightarrow & \text{diam-mean sensitivity} \end{array}$$

From examples in [**10**, **25**], we know that each implication is strict. Note that the first row represents 'point' forms of sensitivity, where points are used in the definition, and, in the second row, we write 'diameter' forms of sensitivity when the sensitivity is measured with the diameter of the orbit of open sets. Motivated by this, we ask the following question. Does cofinite sensitivity (which is the strongest form of 'diameter' sensitivity) imply Banach-mean sensitivity (the weakest form of 'point' sensitivity)? This question has a negative answer.

THEOREM 1.3. *There exists a transitive cofinitely sensitive t.d.s.* (X, T) *which is Banach-mean equicontinuous and hence not Banach-mean sensitive.*

Other popular forms of chaos are positive entropy, mixing, Li-Yorke chaos and Devaney chaos. A system is *Devaney chaotic* if it is transitive and the set of periodic points is dense. For the definition of Li-Yorke chaos, see [**6**] . Considerable time has been spent figuring out which conditions of chaos are stronger than others. For example, it is known that positive topological entropy, Devaney chaos and weak mixing implies Li-Yorke chaos [**6**, **17**, **19**].

In this paper we show the following theorem.

THEOREM 1.4. *There exists a Devaney chaotic t.d.s. with positive topological entropy that is almost mean equicontinuous and hence not mean sensitive.*

Note that in [**25**] a transitive non-mean sensitive system with positive entropy was constructed, and also there exist mixing systems that are not mean sensitive.

For clarity, in summary, the conditions that do not imply mean sensitivity are:

- topological mixing [**25**];
- minimality plus a maximal equicontinuous factor which is not one-to-one [**7**];
- cofinite sensitivity (Theorem 1.3);
- Devaney chaos plus positive topological entropy (Theorem 1.4); and
- positive entropy plus unique ergodicity (Theorem 3.15).

For the positive side, the only known conditions that are stronger than mean sensitivity are:

- minimality plus positive topological entropy [**10**, **25**];
- ergodic measure with full support and not purely discrete spectrum.

We add other properties to the list. The reader is referred to §3.3 for details.

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THEOREM 1.5. *The following conditions imply mean sensitivity.*

- *Topologically mixing, unique ergodicity and positive topological entropy.*
- *Minimality and topological weak mixing.*
- *Transitivity and shadowing property with positive topological entropy.*

There are several applications of the theory and counterexamples we constructed in the following areas: (1) equicontinuity/sensitivity dichotomies (2) locally mean equicontinuous (LME) systems and (3) dynamical theory of hyperspaces.

As for dichotomies, it was shown in [**3**] that a minimal t.d.s. is either equicontinuous or sensitive. This result was generalized by showing that every E-system is either equicontinuous or sensitive (see [**13**, Theorem 1.3] or [**16**, Theorem 4.6]). Recall that a t.d.s. is an *E-system* if it is transitive and there exists an invariant measure with full support.

For mean notions, we know that a minimal t.d.s. is either mean equicontinuous or mean sensitive [**10**, **25**]. In [**32**, Question 3.7.3], Tu wondered if each E-system is either mean sensitive or mean equicontinuous. Since every Devaney chaotic t.d.s. is an E-system and mean equicontinuous systems have zero entropy [**10**, **25**], we obtain a negative answer to the above question as a corollary of Theorem 1.4.

COROLLARY 1.6. *There exist E-systems which are neither mean sensitive nor mean equicontinuous.*

Note that, in [**10**, Theorem 13], it was shown that a strongly transitive system (this property is stronger than E-system) is either mean sensitive or mean equicontinuous. Here we show that systems which are transitive and have dense minimal points (so called *M-systems*) are either Banach-mean equicontinuous or Banach-mean sensitive (Corollary 3.11), and systems which are transitive and have the shadowing property (stronger than M-systems) are either mean sensitive or mean equicontinuous (Corollary 3.18).

In [**15**], Glasner and Weiss introduced the concept of local equicontinuity. A t.d.s. (X, T) is *locally equicontinuous (LE)* if, for every $x \in X$, $\overline{\mathrm{orb}(x, T)}$ is almost equicontinuous. Inspired by this, we say that a t.d.s. (X, T) is *LME* if, for every $x \in X$, $\overline{\mathrm{orb}(x, T)}$ is almost mean equicontinuous. In §4.1, we prove an application of Theorem 1.4 by showing the following theorem.

THEOREM 1.7. *Similarly to LE systems, LME systems have zero topological entropy. Nonetheless, contrary to LE systems, ergodic measures on LME systems may be supported on non-minimal subsystems.*

In 'point' forms of sensitivity (such as mean sensitivity), we care about the behaviour of points. In 'diameter' forms of sensitivity, we pay attention to the behaviour of sets. There is a general way to study the behaviour of sets: the study on hyperspatial dynamical systems. Given a t.d.s. (X, T), we can naturally induce a t.d.s. $(K(X), T_K)$, where $K(X)$ is the hyperspace consisting of all non-empty closed subsets of X and endowed with the Hausdorff metric (cf. [**29**]). Bauer and Sigmund in [**5**] initiated a systematic study on the connections between dynamics of (X, T) and $(K(X), T_K)$. In particular, they proved that

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(X, T) is weakly mixing if and only if $K(X)$ is weakly mixing. Later, Banks [**4**] showed that on $(K(X), T_K)$ weak mixing is equivalent to transitivity. Very recently, Wu *et al* [**34**] summarized the connections on $\mathcal{F}$-sensitivity (where $\mathcal{F}$ is a Furstenberg family), and, in particular, they showed that $\mathcal{F}$-sensitivity of $(K(X), T_K)$ implies that of (X, T), and that the converse is also true when, additionally, $\mathcal{F}$ is a filter. We refer the interested reader to [**23**, **24**, **26**] for further connections.

In this paper, we will study how mean forms of sensitivity and equicontinuity behave on hyperspaces. Following the definition and a bit of work (similarly, as in [**34**, Corollary 1]) it is not hard to see that if $(K(X), T_K)$ is diam-mean sensitive, then so is (X, T). Nonetheless, using the example in the proof of Theorem 1.3, we show that for mean sensitivity this does not happen.

THEOREM 1.8. *There exists a transitive t.d.s.* (X, T) *with zero topological entropy that is Banach-mean equicontinuous (and thus not mean sensitive) such that* $(K(X), T_K)$ *is mean sensitive and has positive topological entropy.*

Note that, in [**14**], Glasner and Weiss constructed the first example of a t.d.s. (X, T) that is minimal and has zero topological entropy but $(K(X), T_K)$ has positive topological entropy. These results indicate that induced dynamics on hyperspaces admit more complicated behaviours than dynamics on the phase space.

We also show that, on the hyperspace, mean equicontinuity and diam-mean equicontinuity are equivalent (Corollary 4.9); for mean sensitivity, we have the following result.

THEOREM 1.9. *Let* (X, T) *be a t.d.s. and let* $(K(X), T_K)$ *be its induced hyperspatial t.d.s. Consider the following statements.*

(1) $(K(X), T_K)$ *is diam-mean sensitive.*
(2) $(K(K(X)), T_K)$ *is diam-mean sensitive.*
(3) $(K(X), T_K)$ *is mean sensitive.*
(4) $(K(K(X)), T_K)$ *is mean sensitive.*

Then (1) and (2) are equivalent. If, additionally, (X, T) *is weakly mixing, then (1)–(4) are all equivalent.*

2. *Preliminaries*

Throughout this paper, we denote by $\mathbb{Z}_+$ and $\mathbb{N}$ the sets of non-negative integers and natural numbers, respectively.

2.1. *Subsets of* $\mathbb{Z}_+$.

Let F be a subset of $\mathbb{Z}_+$. The *density* and *upper Banach density* of F are defined by

$$\overline{D}(F) = \limsup_{n\to\infty} \frac{\#\{F \cap [0, n-1]\}}{n}$$

and

$$BD^*(F) = \limsup_{N-M\to\infty} \frac{\#\{F \cap [M, N-1]\}}{N-M} = \limsup_{n\to\infty} \left\{ \sup_{N-M=n} \frac{\#\{F \cap [M, N-1]\}}{n} \right\},$$

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where $\#\{\cdot\}$ denotes the cardinality of the set. It is clear that $\overline{D}(F) \leq BD^*(F)$ for any $F \subset \mathbb{Z}_+$.

We say that a subset $F \subset \mathbb{Z}_+$ is *syndetic* if there is an $n \in \mathbb{N}$ such that $F \cap \{m, m+1, \ldots, m+n\} \neq \emptyset$ for any $m \in \mathbb{Z}_+$, and we say it is *thick* if there are $n_i \to \infty$ such that $F \supset \bigcup_{i=1}^{\infty} \{n_i, n_i+1, \ldots, n_i+i\}$.

2.2. *Topological dynamics.* Let (X, T) be a t.d.s., let $x \in X$ and let U be a neighbourhood of x. We denote the orbit of x by $\mathrm{orb}(x, T) = \{x, Tx, \ldots\}$, its orbit closure by $\overline{\mathrm{orb}(x, T)}$ and the return times of x to U by $N(x, U) = \{n \in \mathbb{Z}_+ : T^n x \in U\}$.

We say that a point x is *periodic* if $T^n x = x$ for some $n \in \mathbb{Z}_+$, *transitive* if $\overline{\mathrm{orb}(x, T)} = X$, *recurrent* if $N(x, U)$ is non-empty for any neighbourhood U of x and *minimal* if $N(x, U)$ is syndetic for any neighbourhood U of x.

A t.d.s. (X, T) is a *transitive system* if, for any non-empty open sets $U, V \subset X$, there exists $n \in \mathbb{N}$ such that $T^n U \cap V \neq \emptyset$; it is *weakly mixing* if the product t.d.s. $X \times X$ is transitive; and it is a *minimal system* if every point of X is transitive. It is well known that if x is recurrent, then $\overline{\mathrm{orb}(x, T)}$ is transitive.

Given a t.d.s. (X, T), we denote by $M(X, T)$ the collection of all T-invariant Borel probability measures on X. It is well known that $M(X, T)$ is always non-empty. Let $\mu \in M(X, T)$. We define its *support* by $\mathrm{supp}(\mu) = \{x \in X : \mu(U) > 0 \text{ for any neighbourhood } U \text{ of } x\}$.

We say that a t.d.s. (X, T) is:

- *Devaney chaotic* if it is transitive and the set of periodic points is dense;
- an *M-system* if it is transitive and the set of minimal points is dense; and
- an *E-system* if it is transitive and there exists an invariant measure with full support.

Every Devaney chaotic system is an M-system and each M-system is an E-system.

Let (X, T) and (Y, S) be two t.d.s. and let $\pi : X \to Y$ be a continuous function. We say that π is a *factor map* if π is surjective and satisfies that $\pi \circ T = S \circ \pi$. In this case, we say that X is an *extension* of Y or Y is a *factor* of X. The factor π is said to be *almost one-to-one* if there exists a residual subset (i.e., contains a dense G_δ set) G such that $\pi^{-1}(\pi(x)) = \{x\}$ for all $x \in G$.

We refer any reader not familiar with topological entropy to the textbook [**33**].

2.3. *Various forms of equicontinuity and sensitivity.* We have defined various forms of sensitivity in §1.

Let (X, T) be a t.d.s. and let $x \in X$. We say that the point $x \in X$ is:

- an *(ordinary) equicontinuous point* if, for any $\varepsilon > 0$, there is a $\delta > 0$ such that, for every $y \in X$ with $d(x, y) < \delta$, $d(T^n x, T^n y) < \varepsilon$ for all $n \in \mathbb{Z}_+$;
- a *mean equicontinuous point* if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for every $y \in X$ with $d(x, y) < \delta$, $\limsup_{n\to\infty} (1/n) \sum_{i=0}^{n-1} d(T^i x, T^i y) < \varepsilon$;
- a *Banach-mean equicontinuous point* if, for any $\varepsilon > 0$, there is a $\delta > 0$ such that, for every $y \in X$ with $d(x, y) < \delta$, $\limsup_{N-M\to\infty} (1/(N-M)) \sum_{i=M}^{N-1} d(T^i x, T^i y) < \varepsilon$; and

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- a *diam-mean equicontinuous point* if, for any $\varepsilon > 0$, there is a neighbourhood U of x such that $\limsup_{n\to\infty} (1/n) \sum_{i=0}^{n-1} \operatorname{diam}(T^i U) < \varepsilon$.

A t.d.s. (X, T) is *Q-equicontinuous* (where $Q =$ ordinary or mean or Banach-mean or diam-mean) if all points in X are Q-equicontinuous; and it is *almost Q-equicontinuous* if the set of Q-equicontinuous points is residual.

Remark 2.1.

(1) When every point is Q-equicontinuous (where $Q =$ ordinary or mean or Banach-mean or diam-mean), by compactness it is easy to check that the δ is independent of the choice of x.

(2) With the usual triangle inequality argument, we can make clear the difference between mean equicontinuity and diam-mean equicontinuity, i.e.:
   - $x \in X$ is a mean equicontinuous point if and only if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\sup_{y_1, y_2 \in U} \limsup_{n\to\infty} (1/n) \sum_{i=0}^{n-1} d(T^i y_1, T^i y_2) < \varepsilon$, where $U = \{z : d(z, x) < \delta\}$; and
   - $x \in X$ is a diam-mean equicontinuity point if and only if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\limsup_{n\to\infty} \sup_{y_1, y_2 \in U} (1/n) \sum_{i=0}^{n-1} d(T^i y_1, T^i y_2) < \varepsilon$, where $U = \{z : d(z, x) < \delta\}$.

(3) According to Fomin [**9**], a t.d.s. (X, T) is *mean-L-stable* if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $d(x, y) < \delta$, then $\overline{D}(\{i \geq 0 : d(T^i x, T^i y) \geq \varepsilon\}) < \varepsilon$. It was shown in [**25**] that a t.d.s. is mean-L-stable if and only if it is mean equicontinuous. Actually, the proof of such equivalence comes from a general argument which can be used to show that every mean condition has an equivalent density condition. To be precise, one only needs to show the following easy observation. Let $M \geq 0$ and let $\{a_i\}$ be a sequence of reals with $0 \leq a_i \leq M$.
   - If $\limsup_{n\to\infty} (1/n) \sum_{i=0}^{n-1} a_i \leq \delta$, then $\overline{D}(\{i : a_i \geq \sqrt{\delta}\}) \leq \sqrt{\delta}$.
   - If $\overline{D}(\{i \geq 0 : a_i \geq \delta\}) \leq \delta$, then $\limsup_{n\to\infty} (1/n) \sum_{i=0}^{n-1} a_i \leq (M + 1)\delta$.

(4) There exist almost equicontinuous transitive t.d.s.s (X, T) which are not mean equicontinuous [**25**].

(5) Let (X, T) be an almost equicontinuous t.d.s. which is not mean equicontinuous. By [**13**, Theorem 1.3], such a system is not an E-system. Thus if A is the closure of the union of the supports of all invariant measures, then A is a proper closed invariant subset of X. Consider the factor map $\pi : X \to Y$ that maps A to a point p. Observe that A contains no transitive points and that the restricted map $\pi' = \pi|_{X\setminus A} : X \setminus A \to Y \setminus \{p\}$ is a homeomorphism. This means that π is an almost one-to-one map. Besides, it is easy to see that $Y \times Y$ is uniquely ergodic (i.e., it has only one invariant measure) with the only invariant measure concentrated at the point (p, p). By [**25**, Corollary 3.7], the t.d.s. on Y is mean equicontinuous. This shows that every almost equicontinuous transitive t.d.s. is an almost one-to-one extension of a mean equicontinuous one. For a comparison, note that factors of almost equicontinuous systems may not even be almost mean equicontinuous (see, e.g., [**25**, Remark 4.4]).

For transitive and minimal systems, we can state dichotomies between an equicontinuous side and a sensitive side. It is known that each transitive t.d.s. is either Q-sensitive (where $Q =$ ordinary or mean or Banach-mean or diam-mean) or almost

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Q-equicontinuous, and that a minimal t.d.s. is either Q-sensitive or Q-equicontinuous (see [**1**, **10**, **25**], respectively). In this paper, we will present more characterizations of dichotomy. To make it easier for the reader, we briefly sketch the proof of the dichotomy theorem. If (X, T) is transitive and $U \subset X$ is a non-empty open set, then either:

(i) U contains a Q-equicontinuous point; or

(ii) there are no Q-equicontinuous points in U.

If (ii) holds, then there exists a transitive point $x \in U$ which is not a Q-equicontinuous point. It is easy to see that Tx is not a Q-equicontinuous point. Using a triangle inequality argument (e.g., see [**25**, Proposition 5.3]), one can show that (X, T) is Q-sensitive. If (i) holds, then all transitive points in U are Q-equicontinuous points and so are all the transitive points in $T^{-i}U$, $i = 1, 2, 3, \ldots$. Since $\bigcup_{i=1}^{\infty} T^{-i}U$ contains all the transitive points, it means that every transitive point is a Q-equicontinuous point and then (X, T) is almost Q-equicontinuous.

Remark 2.2.

(1) The set of transitive points is equal to the set of equicontinuous points of a transitive almost equicontinuous t.d.s. (see, e.g., [**1**]). This implies that a transitive equicontinuous t.d.s. must be minimal. On the other hand, there are transitive non-minimal mean equicontinuous t.d.s.s [**25**]. In the constructed example, the set of its non-transitive mean equicontinuous points forms a dense subset. Remark 2.1(4) above provides another example where some, but not all, of the non-transitive points are mean equicontinuous points (just note that Y is mean equicontinuous and that the restricted factor map $\pi|_{X\setminus\pi^{-1}(U)} : X \setminus \pi^{-1}(U) \to Y \setminus U$ is a uniform isomorphism for any open set U which contains p).

(2) There are some easy conditions that imply that a t.d.s. is not mean equicontinuous. Since every transitive mean equicontinuous t.d.s. is uniquely ergodic (see, e.g., [**25**]), we obtain that the following families are not mean equicontinuous:
- transitive systems with more than one minimal subsystem, e.g., non-minimal M-system; and
- non-minimal E-systems (since it has at least two invariant measure: the one with full support and the one concentrated on a minimal proper subset).

2.4. *Symbolic dynamics.* Let $\Sigma_2^+ = \{0, 1\}^{\mathbb{N}}$ be endowed with the Cantor product topology (given by the discrete topology on $\{0, 1\}$). A compatible metric on Σ_2^+ is defined by $d(x, y) = 0$ if $x = y$; otherwise $d(x, y) = 1/i$ with $i = \min\{j \in \mathbb{N} : x_j \neq y_j\}$. We have that Σ_2^+ is compact, and the *shift map* $\sigma : \Sigma_2^+ \to \Sigma_2^+$, defined by $\sigma(x)_n = x_{n+1}$ for $n \in \mathbb{N}$, is continuous. We often refer to (Σ_2^+, σ) as the *full shift* and to any compact σ-invariant subsystem $X \subset \Sigma_2^+$ as a *subshift*.

Fix $n \in \mathbb{N}$. We call $w \in \{0, 1\}^n$ a *word of length* n and write $|w| = n$. Let $\mathfrak{o}(w) = \#\{i \in \mathbb{N} : w_i = 1\}$ be the number of occurrences of symbol 1 in w. For any two words $u = u_1u_2 \ldots u_n$ and $v = v_1v_2 \ldots v_m$, we define the *concatenation* of u, v by $uv = u_1u_2 \ldots u_nv_1v_2 \ldots v_m$ or, equivalently, by $u \sqcup v$. In the same manner, we define by u^m the concatenation of m copies of u for some $m \in \mathbb{N}$ and by u^∞ the infinite concatenation of u. Let X be a subshift of Σ_2^+ and let $x = x_1x_2 \cdots \in X$. For any $t, j \in \mathbb{Z}_+$, we denote

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$x_{[t,t+j]} = x_t x_{t+1} \dots x_{t+j}$. We say that a word $w = w_1 w_2 \dots w_n$ appears in x at position t if $x_{t+j-1} = w_j$ for $j = 1, 2, \dots, n$, i.e., $x_{[t,t+n-1]} = w$. By $\mathcal{L}(X)$, we mean the *language* of subshift X, which is the set consisting of all words that can appear in some $x \in X$, and by $\mathcal{L}_n(X)$, we mean the set of all words of length n in $\mathcal{L}(X)$. For any word $u \in \mathcal{L}_n(X)$, its *cylinder set* is defined by $[u] = \{x \in X : x_1 x_2 \dots x_n = u\}$. Note that the cylinder sets $\{[u] : u \in \mathcal{L}(X)\}$ form a basis of the topology of X. A subshift is transitive if and only if it contains a transitive point.

3. *Main results*

3.1. *Proof of Theorem 1.3.* In this section, we will prove Theorem 1.3, i.e., we construct a subshift which is cofinitely sensitive but not Banach-mean sensitive. The notation we need was summarized in §2.

We are working with the non-common notation of using $A \sqcup B = AB$ to denote the concatenation of two words and $\bigsqcup_{i=a}^{b} A^i$ to denote the concatenation of the words $\{A^i\}_{i=a}^{b}$. By convention, if $a > b$, we assume that $\bigsqcup_{i=a}^{b} A^i$ is an empty word.

To begin with, let $A_1 = 111$ and $B_1 = 000$. For $n \geq 2$, we recursively write

$$A_n = A_{n-1} 0^{k_{n-1}} B_{n-1} 0^{k_{n-1}} A_{n-1}$$

and

$$\begin{aligned} B_n &= \bigsqcup_{i=1}^{|A_n|} A_{n-1} 0^{i-1} 1 0^{|A_n|-i} \sqcup A_{n-1} 0^{|A_n|} \\ &= A_{n-1} 1 0^{|A_n|-1} \dots A_{n-1} 0^{i-1} 1 0^{|A_n|-i} \dots A_{n-1} 0^{|A_n|-1} 1 A_{n-1} 0^{|A_n|}, \end{aligned}$$

where the sequence $\{k_n\}$ is required to satisfy that $k_n \geq n(2|A_n| + |B_n|)$.

Let $x = \lim_{n\to\infty} A_n 0^\infty$ and $X = \overline{\operatorname{orb}(x, \sigma)}$. By construction, x is clearly a recurrent point and, consequently, X is a transitive t.d.s. Before proceeding, we describe several properties of X with the following lemmas.

LEMMA 3.1. *For any $m > n$ and $\omega \in \mathcal{L}_t(A_m) \cup \mathcal{L}_t(B_m)$ with $t(= t_n) = |A_n| + 2k_n + |B_n|$,*

$$\mathfrak{o}(\omega) = \#\{i \in \mathbb{N} : \omega_i = 1\} \leq |A_n| + |B_n|.$$

Proof. We will prove this using induction on m.

Assume that $m = n + 1$. If $\omega \in \mathcal{L}_t(A_{n+1})$, we clearly have $\mathfrak{o}(\omega) < |A_n| + |B_n|$. This is because ω can only be either of the form

$$A_n 0^{k_n} B_n 0^{k_n}, \quad 0^{k_n} B_n 0^{k_n} A_n$$

or

$$(A_n)_{[|A_n|-l,|A_n|]} 0^{k_n} B_n 0^{k_n} (A_n)_{[1,|A_n|-l-1]}, \quad 0 \leq l < |A_n| - 1.$$

If $\omega \in \mathcal{L}_t(B_{n+1})$, then

$$\begin{aligned} B_{n+1} &= \bigsqcup_{i=1}^{|A_{n+1}|} A_n 0^{i-1} 1 0^{|A_{n+1}|-i} \sqcup A_n 0^{|A_{n+1}|} \\ &= A_n 1 0^{|A_{n+1}|-1} \dots A_n 0^{i-1} 1 0^{|A_{n+1}|-i} \dots A_n 0^{|A_{n+1}|-1} 1 A_n 0^{|A_{n+1}|}. \end{aligned}$$

With this, we obtain that $\mathfrak{o}(\omega) \leq |A_n| + 1 < |A_n| + |B_n|$.

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Assume this for any $n < l \le m-1$ and $\omega \in \mathcal{L}_t(A_l) \cup \mathcal{L}_t(B_l)$. We conclude that

$$\mathfrak{o}(\omega) \le |A_n| + |B_n|.$$

If $\omega \in \mathcal{L}_t(A_m)$, then

$$\begin{aligned}\omega \in \mathcal{L}_t(A_{m-1}) \cup \mathcal{L}_t(A_{m-1}0^{k_{m-1}}) \cup \mathcal{L}_t(0^{k_{m-1}}A_{m-1})\\ \cup \mathcal{L}_t(B_{m-1}) \cup \mathcal{L}_t(B_{m-1}0^{k_{m-1}}) \cup \mathcal{L}_t(0^{k_{m-1}}B_{m-1}).\end{aligned}$$

Thus we also conclude that $\mathfrak{o}(\omega) \le |A_n| + |B_n|$.

Finally, assume that $\omega \in \mathcal{L}_t(B_m)$. We express

$$B_m = \bigsqcup_{i=1}^{|A_m|} A_{m-1}0^{i-1}10^{|A_m|-i} \sqcup A_{m-1}0^{|A_m|}.$$

Since each A_l $(l > n)$ starts and ends with A_n,

$$\begin{aligned}\omega \in \mathcal{L}_t(A_{m-1}) &\cup \bigcup_{i=1}^{|A_m|} \mathcal{L}_t(0^{i-1}10^{|A_m|-i}) \cup \mathcal{L}_t(0^{|A_m|})\\ &\cup \bigcup_{i=1}^{|A_m|}\bigcup_{p=1}^{t-1} \mathcal{L}_t(0^{i-1}10^{|A_m|-i}(A_{n+1})_{[1,p]})\\ &\cup \bigcup_{i=1}^{|A_m|}\bigcup_{q=0}^{t-2} \mathcal{L}_t((A_{n+1})_{[|A_{n+1}|-q,|A_{n+1}|]}0^{i-1}10^{|A_m|-i})\\ &\cup \bigcup_{q=0}^{t-2} \mathcal{L}_t((A_{n+1})_{[|A_{n+1}|-q,|A_{n+1}|]}0^{|A_m|}).\end{aligned}$$

Once again, we conclude that $\mathfrak{o}(\omega) \le |A_n| + |B_n|$. □

Let $y \in X$. We define $E_y := \{i \in \mathbb{Z}_+ : y_{i+1} = 1\}$.

LEMMA 3.2. *For any $y \in X$, $BD^*(E_y) = 0$.*

Proof. We will prove this result in two cases.

Assume that $y \in \mathrm{orb}(x, \sigma)$. Then $y = \sigma^a x$ is a transitive point for some $a \in \mathbb{Z}_+$. Note that, for each $n > t_1 = |A_1| + 2k_1 + |B_1|$, there is a unique $m(= m(n)) \in \mathbb{N}$ such that $t_m \le n = r_m t_m + s_m < t_{m+1}$ with $t_m = |A_m| + 2k_m + |B_m|$, $r_m \in \mathbb{N}$ and $0 \le s_m \le t_m - 1$. Thus by Lemma 3.1, for any interval $[M, N-1]$ with length n,

$$\begin{aligned}\frac{\#\{E_y \cap [M, N-1]\}}{n} &= \frac{\mathfrak{o}(y_{[M+1,M+n]})}{n}\\ &\le \frac{\mathfrak{o}(y_{[M+1,M+(r_m+1)t_m]})}{r_m t_m}\\ &\le \frac{(r_m+1)(|A_m| + |B_m|)}{r_m(|A_m| + 2k_m + |B_m|)}\\ &\le \frac{(r_m+1)(|A_m| + |B_m|)}{r_m[|A_m| + 2m(2|A_m| + |B_m|) + |B_m|]}.\end{aligned}$$

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Considering this and the fact that $m \to \infty$ when $n \to \infty$, we conclude that

$$BD^*(E_y) = \limsup_{n\to\infty}\left\{\sup_{N-M=n}\frac{\#\{E_y \cap [M, N-1]\}}{n}\right\} \le \limsup_{m\to\infty}\frac{(r_m+1)(|A_m|+|B_m|)}{r_m[|A_m|+2m(2|A_m|+|B_m|)+|B_m|]} = 0.$$

Now assume that $y \in X \setminus \mathrm{orb}(x, \sigma)$. There exists a sequence $\{a_i\}_{i=1}^{\infty} \subset \mathbb{Z}_+$ such that $\sigma^{a_i}x \to y$. Hence, for each word $y_{[M+1,M+n]}$ with $M \in \mathbb{Z}_+$, $n \in \mathbb{N}$, there is an $a_{b_n} \in \mathbb{N}$ such that

$$y_{[1,M+n]} = \sigma^{a_{b_n}} x_{[1,M+n]} = x_{[a_{b_n}+1,a_{b_n}+M+n]}.$$

Using Lemma 3.1 (and a similar argument to that used before) we conclude that

$$BD^*(E_y) = \limsup_{n\to\infty}\left\{\sup_{N-M=n}\frac{\mathfrak{o}(x_{[a_{b_n}+M+1,a_{b_n}+M+n]})}{n}\right\} \le \limsup_{m\to\infty}\frac{(r_m+1)(|A_m|+|B_m|)}{r_m[|A_m|+2m(2|A_m|+|B_m|)+|B_m|]} = 0. \qquad \square$$

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let X be the subshift defined above. First, we will prove that X is cofinitely sensitive. Let $U \subset X$ be a non-empty open set. Since $x \in X$ is a transitive point, there exists $m \in \mathbb{N}$ such that $\sigma^m x \in U$. Furthermore, there exists $s \in \mathbb{N}$ such that the cylinder $[x_{[m+1,m+s]}] \subset U$. By the construction, we can pick an $i \in \mathbb{N}$ such that $x_{[m+1,m+s]}$ is a subword of A_i. Without loss of generality, we can further assume the word A_i ends with $x_{[m+1,m+s]}$ (choose large enough s, if necessary). Also, observe that each word A_{i_1} ends with A_{i_2} and appears in B_{i_3} for any $i_2 < i_1 < i_3 \in \mathbb{N}$. Thus it is not hard to see that the sequences $z_j = x_{[m+1,m+s]}0^j10^\infty$ ($j = 0, 1, 2, \ldots$) are eventually the points of X and, furthermore,that $\{z_j : j \in \mathbb{Z}_+\} \subset [x_{[m+1,m+s]}] \subset U$. Consequently,

$$N_T(U, 1/2) = \{n \in \mathbb{Z}_+ : \mathrm{diam}(\sigma^n U) > 1/2\} \supset \mathbb{Z}_+ \setminus [0, m+s],$$

and the cofinite sensitivity follows.

Let $\varepsilon > 0$ and $\eta \in (0, \varepsilon)$. For η, there is a $K \in \mathbb{N}$ such that $1/(K+1) \le \eta < 1/K$. For any pair $(y_1, y_2) \in X \times X$, we can use Lemma 3.2 to obtain $BD^*(E_{y_1} \cup E_{y_2}) = 0$. This implies that if $F = \{i \in \mathbb{Z}_+ : d(\sigma^i y_1, \sigma^i y_2) \ge \eta\}$, then

$$BD^*(F) \le \limsup_{N-M\to\infty}\frac{2K\#\{(E_{y_1} \cup E_{y_2}) \cap [M, N-1]\}}{N-M} = 0.$$

Hence, similarly to Remark 2.1(3),

$$\begin{aligned}\limsup_{N-M\to\infty}\frac{1}{N-M}\sum_{i=M}^{N-1} d(\sigma^i y_1, \sigma^i y_2) &\le \limsup_{N-M\to\infty}\frac{1}{N-M}(\#\{F \cap [M, N-1]\}\cdot \mathrm{diam}(X) + \eta\cdot(N-M)) \\ &\le BD^*(F)\cdot\mathrm{diam}(X) + \eta < \varepsilon.\end{aligned}$$

Since the choice of pair (y_1, y_2) is arbitrary, X is Banach-mean equicontinuous and thus is not Banach-mean sensitive. $\square$

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3.2. *Proof of Theorem 1.4.* In this section, we show that positive topological entropy and Devaney chaos does not imply mean sensitivity. As a corollary, we show that mean sensitivity and mean equicontinuity have no dichotomy for E-systems. It is fair to mention that the following construction is inspired by the techniques in [**25**] and [**24**]. Nonetheless, the proof for this result has several extra technical issues.

We start with an arbitrarily binary minimal subshift (Y, σ). Pick a point $y = y_1 y_2 \ldots \in Y$ and denote the word $C_n = y_1 \ldots y_n$ for each $n \in \mathbb{N}$. Set $A_1 = 101$, $B_1 = C_1$. Now we recursively define

$$A_{n+1} = A_n 0^{k_n} B_n 0^{k_n} A_n$$

and

$$\begin{aligned} B_{n+1} &= C_{n+1} \bigsqcup_{i=1}^{n} (A_i 0^{|A_{i+1}|})^{n+1-i} \\ &= C_{n+1} (A_1 0^{|A_2|})^n (A_2 0^{|A_3|})^{n-1} \ldots (A_n 0^{|A_{n+1}|})^1, \end{aligned}$$

where the sequence $\{k_n\}$ satisfies that:

(1) $k_m/(|A_n| + 2k_n + |B_n|) > |B_m|/|B_n|$ for all $1 \le n < m$; and

(2) $k_n \ge n(2|A_n| + |B_n|)$.

Let $x = \lim_{n\to\infty} A_n 0^\infty$ and $X = \overline{\mathrm{orb}(x, \sigma)}$. From the construction, it is clear that (Y, σ) is a minimal subshift of (X, σ), and the system (X, σ) is transitive and has dense periodic points (of the form $\sigma^t(A_n 0^{|A_{n+1}|})^\infty$). From this, we obtain the following result.

PROPOSITION 3.3. *(X, σ) is a Devaney chaotic t.d.s.*

We will later prove that the subshift (X, σ) satisfies the following property.

PROPOSITION 3.4. *(X, σ) is almost mean equicontinuous.*

From the construction and Propositions 3.3 and 3.4, we obtain the following proposition

PROPOSITION 3.5. *In the full shift (Σ_2^+, σ), every minimal subshift (Y, σ) is contained in a Devaney chaotic subshift (X, σ) that is almost mean equicontinuous.*

Note that since the system in Proposition 3.5 is transitive but not uniquely ergodic, by Remark 2.2(2) it can not be mean equicontinuous.

Now we can prove Theorem 1.4.

Proof of Theorem 1.4. Let (Y, σ) be a minimal subshift with positive topological entropy. Using the fixed Y, we construct a subshift (X, σ), as in Proposition 3.5. Then (X, σ) is Devaney chaotic and almost mean equicontinuous, and so it is not mean sensitive [**10**, **25**]. Besides, (X, σ) also has positive topological entropy because it contains (Y, σ) as a subsystem. This implies that Theorem 1.4 holds, which completes the proof. □

Consequently, we obtain the proof of Corollary 1.6.

Proof of Corollary 1.6. Note that every Devaney chaotic t.d.s. is an E-system and that mean equicontinuous systems have zero entropy [**10**, **25**]. Then by combining this with Theorem 1.4, the corollary follows. One can also yield the corollary from Proposition 3.5. □

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So it remains to prove Proposition 3.4. For this, we need more information about the structure of (X, σ), which will be obtained in the following series of lemmas. To help the reader to get a good understanding of these lemmas, we will describe the relations between them: Lemma 3.9 is exactly what we need in the proof of Proposition 3.4; Lemmas 3.6, 3.7 and 3.8 are used to prove Lemma 3.9.

Observe that each A_j starts and ends with A_i for all $1 \le i < j \in \mathbb{N}$. Let $m \ge n+1$. Using that $A_{n+1} = A_n 0^{k_n} B_n 0^{k_n} A_n$, one can check that there exists an $M \in \mathbb{N}$ and a sequence $\{m_i\}_{i=1}^{M}$ with $n \le m_i \le m-1$ such that

$$\begin{aligned} A_m &= \bigsqcup_{i=1}^{M} A_n 0^{k_{m_i}} B_{m_i} 0^{k_{m_i}} \sqcup A_n \\ &= A_n 0^{k_n} B_n 0^{k_n}\ A_n 0^{k_{n+1}} B_{n+1} 0^{k_{n+1}} \dots A_n 0^{k_n} B_n 0^{k_n} A_n. \end{aligned} \tag{3.1}$$

LEMMA 3.6. *For each $m \ge n$ and $1 \le s \le |A_n| + 2k_m + |B_m|$,*

$$\mathfrak{o}((A_n 0^{k_m} B_m 0^{k_m})_{[1,s]}) \le \max\left\{1, \frac{s}{|A_n| + 2k_n + |B_n|}\right\} \cdot (|A_n| + |B_n|).$$

Proof. It is clearly true for $m = n$. Assume that $m > n$. The result is easy to check for $s \in [1, |A_n| + k_m]$. Now, since the sequence $\{k_n\}$ satisfies condition (1), $k_m > 2k_n$ and $|B_m| < (|B_n| \cdot k_m)/(|A_n| + 2k_n + |B_n|)$. This leads to

$$\begin{aligned} \mathfrak{o}(A_n 0^{k_m} B_m 0^{k_m}{}_{[1,s]}) &\le |A_n| + |B_m| \le |A_n| + \frac{|B_n| \cdot k_m}{|A_n| + 2k_n + |B_n|} \\ &\le \frac{(|A_n| + k_m)(|A_n| + |B_n|)}{|A_n| + 2k_n + |B_n|} \\ &\le \frac{s}{|A_n| + 2k_n + |B_n|}(|A_n| + |B_n|) \end{aligned}$$

for all $s > |A_n| + k_m$, which completes the proof. □

LEMMA 3.7. *For each $m \ge n+1$:*

(1) $\mathfrak{o}(A_m) \le (|A_m|/(|A_n| + 2k_n + |B_n|) + 1) \cdot (|A_n| + |B_n|)$*;*

(2) $\mathfrak{o}(A_m 0^{|A_{m+m'}|}) = \mathfrak{o}(A_m) \le ((|A_m| + |A_{m+m'}|)/(|A_n| + 2k_n + |B_n|)) \cdot (|A_n| + |B_n|)$ *for any $m' \ge 1$; and*

(3) $\mathfrak{o}(A_n 0^{|A_m|}{}_{[1,s]}) \le |A_n| + |B_n|$ *for any $1 \le s \le |A_n| + |A_m|$.*

Proof. (3) is trivial and obviously (1) $\Longrightarrow$ (2). So we only need to show (1).

Let $M \in \mathbb{N}$ and $\{m_i\}_{i=1}^{M}$ be such that $n \le m_i \le m-1$ and

$$A_m = \bigsqcup_{i=1}^{M} A_n 0^{k_{m_i}} B_{m_i} 0^{k_{m_i}} \sqcup A_n.$$

Observe that, for each $r \ge n$,

$$|A_n 0^{k_r} B_r 0^{k_r}| = |A_n| + 2k_r + |B_r| \ge |A_n| + 2k_n + |B_n|.$$

By Lemma 3.6,

$$\mathfrak{o}(A_n 0^{k_r} B_r 0^{k_r}) \le \frac{|A_n 0^{k_r} B_r 0^{k_r}|}{|A_n| + 2k_n + |B_n|} \cdot (|A_n| + |B_n|),$$

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so

$$\begin{aligned}\mathfrak{o}(A_m) &= \sum_{i=1}^{M} \mathfrak{o}(A_n 0^{k_{m_i}} B_{m_i} 0^{k_{m_i}}) + \mathfrak{o}(A_n) \\ &\le \sum_{i=1}^{M} \frac{|A_n 0^{k_{m_i}} B_{m_i} 0^{k_{m_i}}|}{|A_n| + 2k_n + |B_n|} \cdot (|A_n| + |B_n|) + (|A_n| + |B_n|) \\ &\le \left(\frac{|A_m|}{|A_n| + 2k_n + |B_n|} + 1 \right) \cdot (|A_n| + |B_n|),\end{aligned}$$

which completes the proof. □

Since y is a minimal point, the word 0 or 1 appears in y syndetically. This implies that there exists $N \in \mathbb{N}$ such that A_N does not appear in y. We will use this N in the following lemma.

LEMMA 3.8. *Let $n \ge N$, $m \ge n+1$ and $1 \le t \le |B_m|$ such that $(B_m)_{[t,t+|A_n|-1]} = A_n$. Then, for each $0 \le s \le |B_m| + k_m - t$,*

$$\mathfrak{o}((B_m 0^{k_m})_{[t,t+s]}) \le \left(\frac{s+1}{|A_n| + 2k_n + |B_n|} + 1 \right) \cdot (|A_n| + |B_n|).$$

Proof. By definition,

$$B_m = C_m \bigsqcup_{i=1}^{m-1} (A_i 0^{|A_{i+1}|})^{m-i}.$$

Although the choice of N, A_n does not appear as a subword of $C_m \bigsqcup_{i=1}^{n-1} (A_i 0^{|A_{i+1}|})^{m-i}$, nonetheless, for each $n \le i \le m-1$, A_n is contained in $A_i 0^{|A_{i+1}|}$.

By hypothesis, $(B_m)_{[t,t+|A_n|-1]} = A_n$. There exist $n \le r \le m-1$ and $0 \le l < m-r$ such that t locates in some A_r of B_m, i.e.,

$$\left| C_m \bigsqcup_{i=1}^{r-1} (A_i 0^{|A_{i+1}|})^{m-i} \sqcup (A_r 0^{|A_{r+1}|})^{l} \right| < t \le \left| C_m \bigsqcup_{i=1}^{r-1} (A_i 0^{|A_{i+1}|})^{m-i} \sqcup (A_r 0^{|A_{r+1}|})^{l+1} \right|.$$

Also there exists $M \in \mathbb{N}$ and a sequence $\{m_i\}_{i=1}^{M}$ with $n \le m_i \le r-1$ such that

$$A_r = \bigsqcup_{i=1}^{M} A_n 0^{k_{m_i}} B_{m_i} 0^{k_{m_i}} \sqcup A_n.$$

We assume that there exists $0 \le a \le M$ such that

$$t = \left| C_m \bigsqcup_{i=1}^{r-1} (A_i 0^{|A_{i+1}|})^{m-i} \sqcup (A_r 0^{|A_{r+1}|})^{l} \right| + \left| \bigsqcup_{i=1}^{a} A_n 0^{k_{m_i}} B_{m_i} 0^{k_{m_i}} \right| + 1.$$

This may not be the case, as A_n appears also as subwords of some B_{m_i}. We will explain later what to do if this is not the case.

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This implies that

$$(B_m 0^{k_m})_{[t,t+s]} = \Bigg(\bigsqcup_{i=a+1}^{M} A_n 0^{k_{m_i}} B_{m_i} 0^{k_{m_i}} \sqcup A_n 0^{|A_{r+1}|} \sqcup (A_r 0^{|A_{r+1}|})^{m-r-l-1} \bigsqcup_{i=r+1}^{m-1} (A_i 0^{|A_{i+1}|})^{m-i} \sqcup 0^{k_m} \Bigg)_{[0,s]}. \tag{3.2}$$

Depending on the location of $t+s$, $(B_m 0^{k_m})_{[t,t+s]}$ can be rewritten as the concatenations

$$(B_m 0^{k_m})_{[t,t+s]} = \bigsqcup_{j=1}^{9} E_j,$$

where, for each $\ell = 1, \ldots, 9$, E_ℓ is either empty, or satisfies that:

(1) $E_1 = \bigsqcup_{i=a+1}^{M} A_n 0^{k_{m_i}} B_{m_i} 0^{k_{m_i}}$;
(2) $E_2 = A_n 0^{|A_{r+1}|}$;
(3) $E_3 = (A_r 0^{|A_{r+1}|})^u$ for some $0 \le u \le m-r-l-1$;
(4) $E_4 = \bigsqcup_{i=r+1}^{v-1} (A_i 0^{|A_{i+1}|})^{m-i}$ for some $r+2 \le v \le m-1$;
(5) $E_5 = (A_v 0^{|A_{v+1}|})^w$ for some $0 \le w \le m-v$;
(6) $E_6 = \bigsqcup_{i=1}^{p} A_n 0^{k_{b_i}} B_{b_i} 0^{k_{b_i}}$ for some $p \in \mathbb{N}$ and $\{b_i\}_{i=1}^{p}$ with $n \le b_i \le m-2$;
(7) $E_7 = A_n 0^{k_c} B_{k_c} 0^{k_c}{}_{[1,e_1]}$ for some $c \ge n$ and $1 \le e_1 \le |A_n| + |B_{k_c}| + 2k_c$;
(8) $E_8 = A_n 0^{|A_d|}{}_{[1,e_2]}$ for some $d \ge n$ and $1 \le e_2 \le |A_n| + |A_d|$; and
(9) $E_9 = A_n 0^{|A_m|} 0^{k_m}{}_{[1,e_3]}$ for some $1 \le e_3 \le |A_m| + |A_n| + k_m$.

Now we make some comments about the decomposition for better understanding. Observing the expression of 3.2, if $t+s$ locates in some $A_t 0^{|A_{t+1}|}$ $(n \le t \le m-1)$, then $\bigsqcup_{i=1}^{5} E_i$ is the left part determined by the chosen $A_t 0^{|A_{t+1}|}$. Since $A_t 0^{|A_{t+1}|} = \bigsqcup_{i=1}^{q} A_n 0^{k_{m_i}} B_{m_i} 0^{k_{m_i}} \sqcup A_n 0^{|A_{t+1}|}$ for some $q \in \mathbb{Z}_+$ and a sequence $\{m_i\}_{i=1}^{q}$ with $n \le m_i \le t-1$, $\bigsqcup_{i=6}^{9} E_i$ is the part from the beginning of the chosen $A_t 0^{|A_{t+1}|}$ to the location of $t+s$. E_7, E_8, E_9 are the possible cases that are left over from the division and one and only one of them is non-empty. The idea of the decomposition is to expand $B_m 0^{k_m}$ into concatenations of the form $A_n 0^{k_{m_i}} B_{m_i} 0^{k_{m_i}}$ and $A_n 0^{|A_j|}$.

Once $B_m 0^{k_m}$ is broken as the union of E_i, as above, we can make use of Lemmas 3.6 and 3.7 to show the following.

(1) $\mathfrak{o}(E_1) = \sum \mathfrak{o}(A_n 0^{k_{m_i}} B_{m_i} 0^{k_{m_i}}) \le |E_1|/(|A_n| + 2k_n + |B_n|) \cdot (|A_n| + |B_n|)$.
(2) By Lemma 3.7(1), $\mathfrak{o}(E_2) = \mathfrak{o}(A_n) \le |E_2|/(|A_n| + 2k_n + |B_n|) \cdot (|A_n| + |B_n|)$.
(3) By Lemma 3.7(2), $\mathfrak{o}(E_3) \le |E_3|/(|A_n| + 2k_n + |B_n|) \cdot (|A_n| + |B_n|)$.
(4) By Lemma 3.7(2), $\mathfrak{o}(E_4) \le |E_4|/(|A_n| + 2k_n + |B_n|) \cdot (|A_n| + |B_n|)$.
(5) By Lemma 3.7(2), $\mathfrak{o}(E_5) \le |E_5|/(|A_n| + 2k_n + |B_n|) \cdot (|A_n| + |B_n|)$.
(6) $\mathfrak{o}(E_6) = \sum \mathfrak{o}(A_n 0^{k_{b_i}} B_{b_i} 0^{k_{b_i}}) \le |E_6|/(|A_n| + 2k_n + |B_n|) \cdot (|A_n| + |B_n|)$.
(7) By Lemma 3.6, $\mathfrak{o}(E_7) \le \max\{1, |E_7|/(|A_n| + 2k_n + |B_n|)\} \cdot (|A_n| + |B_n|)$.
(8) $\mathfrak{o}(E_8) \le \mathfrak{o}(A_n) \le |A_n| + |B_n|$.
(9) $\mathfrak{o}(E_9) \le \mathfrak{o}(A_n) \le |A_n| + |B_n|$.

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Hence (if $E_i = \emptyset$, then $\mathfrak{o}(E_i) = 0$; otherwise using the above inequalities)

$$\mathfrak{o}((B_m 0^{k_m})_{[t,t+s]}) = \sum_{i=1}^{6} \mathfrak{o}(E_i) + \max\{\mathfrak{o}(E_7), \mathfrak{o}(E_8), \mathfrak{o}(E_9)\}$$
$$\leq \left(\frac{s+1}{|A_n| + 2k_n + |B_n|} + 1\right) \cdot (|A_n| + |B_n|).$$

If the extra assumption of t does not hold, we can continue the (eventually finite) process of expanding B_{m_i}; this process will only provide words of the same type as considered before.

This finishes the proof. □

LEMMA 3.9. *If $j - i > |A_n|$, $n \geq N$ and $x_{[i,i+|A_n|-1]} = A_n$, then*

$$\mathfrak{o}(x_{[i,j-1]}) \leq \left(\frac{j-i}{|A_n| + 2k_n + |B_n|} + 2\right)(|A_n| + |B_n|).$$

Proof. Observe that A_n ($n \geq N$) does not appear in y and that $x_{[i,i+|A_n|-1]}$ is equal to A_n. We have to consider different cases depending of the locations of i and j.

(1) If there exists $p \in \mathbb{N}$ and a sequence $\{m_i\}_{i=1}^{p}$ with $m_i \geq n$ such that

$$x_{[i,j-1]} = \bigsqcup_{i=1}^{p-1} A_n 0^{k_{m_i}} B_{m_i} 0^{k_{m_i}} \sqcup (A_n 0^{k_{m_p}} B_{m_p} 0^{k_{m_p}})_{[1,s_1]}$$

for some $1 \leq s_1 < |A_n| + 2k_{m_p} + |B_{m_p}|$, then by Lemma 3.6 it is not hard to see that

$$\mathfrak{o}(x_{[i,j-1]}) \leq \left(\frac{j-i}{|A_n| + 2k_n + |B_n|} + 1\right)(|A_n| + |B_n|).$$

Hence the inequality follows.

(2) If there exist $m \geq n+1$ and $1 \leq t \leq |B_m|$ such that the initial position of $x_{[i,j-1]}$ is exactly the tth position of B_m and $(B_m)_{[t,t+|A_n|-1]} = A_n$, then two cases are involved:

(a) if $|A_n| < j - i \leq |B_m| + k_m - t + 1$, then $x_{[i,j-1]}$ can be expressed as

$$x_{[i,j-1]} = (B_m 0^{k_m})_{[t,t+s_2]}$$

for some $0 \leq s_2 \leq |B_m| + k_m - t$ and by Lemma 3.8 we conclude this case;

(b) if $j - i > |B_m| + k_m - t + 1$, then we can write $x_{[i,j-1]}$ as

$$x_{[i,j-1]} = (B_m 0^{k_m})_{[t,|B_m|+k_m]} \bigsqcup_{i=1}^{q-1} A_n 0^{k_{m_i}} B_{m_i} 0^{k_{m_i}} \sqcup (A_n 0^{k_{m_q}} B_{m_q} 0^{k_{m_q}})_{[1,s_3]},$$

where $q \in \mathbb{N}$, $\{m_i\}_{i=1}^{q}$ is a sequence with $m_i \geq n$, and $1 \leq s_3 < |A_n| + 2k_{m_q} + |B_{m_q}|$. By applying Lemmas 3.6 and 3.8, we finish this case and so the whole proof is complete. □

Now we are going to prove Proposition 3.4.

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Proof of Proposition 3.4. To show that X is almost mean equicontinuous, we will show that the transitive point x is mean equicontinuous.

Let $\varepsilon > 0$. There exists $K \in \mathbb{N}$ such that, for any $a, b \in X$, if $a_{[1,K]} = b_{[1,K]}$, then $d(a, b) < \varepsilon/5$. Since k_n satisfies (2), i.e., $k_n \geq n(2|A_n| + |B_n|)$, we conclude that

$$\lim_{n\to\infty} \frac{|A_n| + |B_n|}{|A_n| + |B_n| + 2k_n} = 0.$$

Considering this, there exists $m \in \mathbb{N}$ large enough such that

$$\frac{2K(|A_m| + |B_m|)}{|A_m| + |B_m| + 2k_m} < \frac{\varepsilon}{4}.$$

Let $z \in [A_m]$. If there exists a $k \in \mathbb{Z}_+$ such that $\sigma^k z = 0^\infty$, then

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\sigma^i z, 0^\infty) = 0.$$

Otherwise, $\sigma^k z \neq 0^\infty$ for all $k \in \mathbb{Z}_+$. For any $i \in \mathbb{N}$, there is an $l_i \in \mathbb{N}$ such that $z_{[1,1+i]} = x_{[l_i, l_i+i]}$. Since z starts with A_m, then by Lemma 3.9 and the choice of K,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\sigma^i z, 0^\infty) &\leq \frac{\varepsilon}{5} \limsup_{n\to\infty} \frac{1}{n} \#\{0 \leq i \leq n-1 : (\sigma^i z)_{[1,K]} = 0^K\} \\
&\quad + \limsup_{n\to\infty} \frac{1}{n} \#\{0 \leq i \leq n-1 : (\sigma^i z)_{[1,K]} \neq 0^K\} \\
&< \frac{\varepsilon}{4} + \limsup_{n\to\infty} \frac{1}{n} \cdot 2K \cdot \mathfrak{o}(z_{[1,n]}) \\
&< \frac{\varepsilon}{4} + \limsup_{n\to\infty} \frac{1}{n} \cdot 2K \cdot \left[\frac{n}{|A_m| + 2k_m + |B_m|} (|A_m| + |B_m|) \right] \\
&\quad + \limsup_{n\to\infty} \frac{1}{n} \cdot 4K \cdot (|A_m| + |B_m|) \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.
\end{aligned}$$

This implies that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\sigma^i x, \sigma^i z) < \varepsilon,$$

for any $z \in [A_m]$ and hence x is a mean equicontinuous point, which completes the whole proof. □

Inspired by Corollary 1.6, we naturally ask if every E (or M, P)-system is either Banach-mean/diam-mean sensitive or Banach-mean/diam-mean equicontinuous. We conjecture that this question is still not true for E-systems in general, and we leave it open here. In the following, we will show that for M-systems the dichotomy theorem is always true for Banach-mean notions.

THEOREM 3.10. *Each Banach-mean equicontinuous M-system (X, T) is minimal.*

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Proof. If X is a non-minimal M-system, by [**31**, Theorem 4] there exists a $\delta > 0$ such that, for any non-empty open subset $U \subset X$, there is a pair $(x, y) \in U \times U$ satisfying that the set $F = \{n \in \mathbb{Z}_+ : d(T^n x, T^n y) > \delta\}$ is a thick set. Note that the upper Banach density of each thick set is exactly one. Similarly to Remark 2.1(3),

$$\limsup_{N-M\to\infty} \frac{1}{N-M} \sum_{i=M}^{N-1} d(T^n x, T^n y) > \limsup_{N-M\to\infty} \frac{1}{N-M} \sum_{i=M}^{N-1} (\delta \#\{F \cap [M, N-1]\})$$
$$= \delta \cdot BD^*(F) = \delta.$$

That is, (X, T) is Banach-mean sensitive, which is a contradiction. □

Combining [**25**, Propsition 6.1(5)] and Theorem 3.10, we immediately have the following corollary.

COROLLARY 3.11. *Let (X, T) be an M-system. Then it is either Banach-mean sensitive or Banach-mean equicontinuous.*

3.3. *Conditions that imply mean sensitivity.* In this section, we are looking for sufficient conditions under which a t.d.s. is mean sensitive. First, we focus on a special class of transitive and ergodic systems. To begin with, we need some results and definitions introduced in [**11**].

Let (X, T) be a t.d.s., let μ be an ergodic measure on X and let $f \in L^2(\mu)$. We say that f is an *almost periodic function* if $\{U_T^j f : j \in \mathbb{Z}_+\}$ is a compact subset of $L^2(\mu)$. An ergodic system (X, μ, T) is weakly mixing if and only if every almost periodic function is constant. On the other hand, (X, μ, T) has a discrete spectrum if and only if every $f \in L^2(\mu)$ is almost periodic (these two results are due to Halmos and Von-Neumann; for detailed proofs, see [**33**]).

Now assume that f is a continuous function on X. We say that (X, T) is *f-mean sensitive* if there exists $\varepsilon > 0$ such that, for any non-empty open subset $U \subset X$, we can find $x, y \in U$ satisfying

$$d_f(x, y) := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} |f(T^j x) - f(T^j y)|^2 d\mu > \varepsilon.$$

When considering the opposite side, we say that $x \in X$ is an *f-mean equicontinuous point* if, for any $\varepsilon > 0$, there is $\delta > 0$ such that if $d(x, y) < \delta$, then $d_f(x, y) < \varepsilon$. (X, T) is said to be an *almost f-mean equicontinuous system* if there is a point which is both transitive and f-mean equicontinuous. Similarly to the proof of [**11**, Theorem 2.14], we obtain the following lemma.

LEMMA 3.12. *Let (X, T) be a t.d.s. Then (X, T) is almost mean equicontinuous if and only if it is almost f-mean equicontinuous for every continuous function f.*

LEMMA 3.13. *Let (X, T) be a transitive t.d.s., let μ be an ergodic measure on X and let $f \in L^2(\mu)$ be a non-almost periodic function. If, for any non-empty open subset $A \subset X$ and any infinite subset $S \subset \mathbb{Z}_+$, there are $s \neq t \in S$ and a generic point $z \in X$ for μ*

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(with respect to continuous functions) such that $z \in T^{-s}A \cap T^{-t}A$, then (X, T) is mean sensitive.

Proof. Continuous functions are dense in $L^2(X, \mu)$ and the set of non-almost periodic functions is open [**11**, Theorem 1.13], so there exists a continuous function $f \in L^2(X, \mu)$ that is not almost periodic. This implies that it is not totally bounded, and hence there exists $\varepsilon > 0$ and an infinite subset $S \subset \mathbb{Z}_+$ such that

$$\int |U^i f - U^j f|^2 \, d\mu \geq \varepsilon \quad \text{for every } i \neq j \in S.$$

For any non-empty open subset $A \subset X$, there exists $s \neq t \in S$ and a generic point $z \in X$ for μ such that $z \in T^{-s}A \cap T^{-t}A$. Define $g(x) := |U_T^s f(x) - U_T^t f(x)|^2$. Since f is continuous, so is g. As z is a generic point for μ, for the continuous function g,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} g(T^j z) = \int g \, d\mu \geq \varepsilon.$$

Let $p := T^s z \in A$ and $q := T^t z \in A$. Then

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} |f(T^j p) - f(T^j q)|^2 &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} |f(T^{j+s} z) - f(T^{j+t} z)|^2 \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} g(T^j z) > \varepsilon. \end{aligned}$$

That is, for any non-empty open subset $A \subset X$, there are $p, q \in A$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} |f(T^j p) - f(T^j q)|^2 > \varepsilon.$$

Following the definition, we know that (X, T) is f-mean sensitive. By [**11**, Theorem 2.13], (X, T) is not almost f-mean equicontinuous, and by Lemma 3.12 (X, T) is also not almost mean equicontinuous. Now using the dichotomy theorem in [**10**, **25**], (X, T) is mean sensitive. □

In the following theorem, (2) and (3) are corollaries from previously known results. We write them together for comparison.

THEOREM 3.14. *Let (X, T) be a transitive t.d.s. and μ be an ergodic measure on X. (X, T) is mean sensitive if one of the following holds.*

(1) *(X, T) is uniquely ergodic, topologically mixing and has positive topological entropy.*
(2) *μ has full support and (X, μ, T) does not have discrete spectrum (in particular, if it is non-trivial and measurably weakly mixing or has positive entropy with respect to μ).*
(3) *(X, T) is minimal and has positive topological entropy.*

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Proof. (1) Since (X, T) is uniquely ergodic (with the unique ergodic measure μ) and has positive topological entropy, then it has no discrete spectrum, and hence there is a non-almost periodic function $f \in L^2(\mu)$ (see, e.g., [**11**, **33**]). Since (X, T) is topologically strongly mixing, for any non-empty open subset $A \subset X$ and any infinite subset $S \subset \mathbb{Z}_+$, it is clear that $\{T^n A : n \in S\}$ is dense in X. This implies that there are $s \neq t \in S$ such that $T^{-s}A \cap T^{-t}A \neq \emptyset$. By the unique ergodicity of (X, T), we know that each non-empty open subset $T^{-s}A \cap T^{-t}A$ contains a generic point for μ. Using Lemma 3.13, (X, T) is mean sensitive.

(2) Since μ has full support, every open set has positive measure. In [**10**], it is shown that if (X, μ, T) does not have discrete spectrum, then there exists $\delta > 0$ such that, for every set of positive measure $U \subset X$, there are $x, y \in U$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) > \delta.$$

(3) This result can be found in [**10**, **25**]. Note that, for any minimal t.d.s. (X, T), each ergodic measure on X has full support. Using the variation principle and (2), it is not hard to see that (X, T) is mean sensitive. □

What we can see is that positive entropy plus a strong form of topological ergodicity implies mean sensitivity. We do not know if every transitive (or even weakly mixing) uniquely ergodic t.d.s. with positive entropy is mean sensitive. Note that without the transitivity condition, this question has a negative answer.

THEOREM 3.15. *There exists a uniquely ergodic t.d.s. with positive entropy that is not sensitive.*

Proof. Let $Y \subset \{0, 1\}^{\mathbb{N}}$ be a minimal uniquely ergodic subshift with positive topological entropy. Let $X = Y \cup \{2, 3\}^{\mathbb{N}}$, $y \in Y$ and let $T : X \to X$ be defined as

$$Tx = \begin{cases} \sigma(x) & \text{if } x \in Y, \\ y & \text{otherwise.} \end{cases}$$

It is not hard to see that (X, T) is uniquely ergodic and that it is not sensitive (using the open set $\{2, 3\}^{\mathbb{N}}$). □

The following theorem can be deduced from results in [**25**].

THEOREM 3.16. *Every non-trivial minimal topologically weakly mixing t.d.s. is mean sensitive.*

Proof. Let (X, T) be a non-trivial minimal topological weakly mixing t.d.s. Assume that it is not mean sensitive. Then, from [**25**, Corollary 5.5] or [**10**, Theorem 8], (X, T) is mean equicontinuous.

Let $f_{\text{eq}} : X \to X_{\text{eq}}$ be the maximal equicontinuous factor map. Then, by [**25**, Corollary 3.6], f_{eq} is proximal, i.e., $f_{\text{eq}}(x) = f_{\text{eq}}(y)$ if and only if

$$\liminf_{n\to\infty} d(T^n x, T^n y) = 0.$$

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Since (X, T) is minimal and weakly mixing, X has no non-trivial equicontinuous factors, i.e., X_{eq} is a singleton. It then follows that X is proximal, which contradicts the assumption that (X, T) is non-trivial minimal. We conclude that (X, T) is mean sensitive. □

Now we will study systems with the shadowing property. Recall that a sequence $\{x_n\}_{n=0}^{\infty}$ is a *δ-pseudo-orbit* for T if $d(x_{n+1}, Tx_n) < \delta$ for all $n \in \mathbb{Z}_+$, and it is *ϵ-traced* by a point $x \in X$ if $d(T^n x, x_n) < \epsilon$ for all $n \in \mathbb{Z}_+$. We say that a t.d.s. (X, T) has the *shadowing property* if, for any $\varepsilon > 0$, we can find a $\delta > 0$ such that each δ-pseudo-orbit for T is ε-traced by some point of X.

It is known that a transitive system with the shadowing property is either equicontinuous or sensitive (see, e.g., [**28**, Theorem 6]). Now we show that if the transitive system with shadowing property has positive topological entropy, then it is mean sensitive. This result is inspired by [**22**].

To prove it, we need the notions of a sensitive and a distal pair. A pair $(x_1, x_2) \in X^2 \setminus \Delta_2$ is called a *sensitive pair* [**35**] if, for any neighbourhood U_i of x_i $(i = 1, 2)$ and any non-empty open subset U of X, there exist $k \in \mathbb{N}$ and $y_i \in U$ such that $T^k y_i \in U_i$ for $i = 1, 2$, and is called a *distal pair* (see, e.g., [**8**]) if $\liminf_{n\to\infty} d(T^n x_1, T^n x_2) > 0$.

THEOREM 3.17. *Let (X, T) be a transitive t.d.s. with the shadowing property. If (X, T) has positive entropy, then it is mean sensitive.*

Proof. Following from the proof of Lemma 3.3 in [**22**], we know that there exist a sensitive and distal pair (v_1, v_2) in X^2. By the definition of distal pair, there are $\varepsilon > 0$ and $m \in \mathbb{N}$ such that

$$d(T^\ell v_1, T^\ell v_2) > 3\varepsilon$$

for all $\ell \geq m$. For this ε, choose $\delta > 0$ such that each δ -pseudo-orbit can be ε-traced. Let U, W be non-empty open subsets of X with $d(W, X \setminus U) > \varepsilon$, and let V_i be neighbourhoods of v_i with $\operatorname{diam}(V_i) < \delta$ for $i = 1, 2$. Since (v_1, v_2) is a sensitive pair, there are $x_1, x_2 \in W$ and $n \in \mathbb{N}$ such that

$$T^n x_1 \in V_1 \quad \text{and} \quad T^n x_2 \in V_2.$$

Fix $i \in \{1, 2\}$. It is easy to see that the sequence

$$x_i, Tx_i, \ldots, T^{n-1}x_i, v_i, Tv_i \ldots$$

is a δ-pseudo-orbit. Then there exist $y_i \in X$ such that

$$d(x_i, y_i) < \varepsilon \quad \text{and} \quad d(T^{k-n} v_i, T^k y_i) < \varepsilon, \quad k = n, n+1, \ldots.$$

This implies that $y_i \in U$ since $d(W, X \setminus U) > \varepsilon$, and that

$$d(T^k y_1, T^k y_2) > \varepsilon$$

for all $k \geq m + n$. Hence

$$\limsup_{n\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} d(T^n y_1, T^n y_2) > \varepsilon,$$

which shows that (X, T) is mean sensitive. □

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The following corollary is a consequence of this.

COROLLARY 3.18. *Each transitive t.d.s. with the shadowing property is either mean equicontinuous or mean sensitive.*

4. *Applications*

In this section, we show applications of our previous results. Corollary 1.6 has already been explained in §3.2 . The other applications need further explanation.

4.1. *Local mean equicontinuity.* In [**15**], Glasner and Weiss introduced the concept of local equicontinuity. A t.d.s. (X, T) is LE if, for every $x \in X$, $\overline{\text{orb}(x, T)}$ is almost equicontinuous. Inspired by this, we define local mean equicontinuity.

Definition 4.1. A t.d.s. (X, T) is LME if, for every $x \in X$, $\overline{\text{orb}(x, T)}$ is almost mean equicontinuous.

In general, LE systems have zero topological entropy. This comes from the fact that almost equicontinuous systems have zero topological entropy.

We know that almost mean equicontinuous systems may have positive topological entropy (see the example in §3.2) but, nonetheless, LME systems always have zero topological entropy.

THEOREM 4.2. *Every LME t.d.s. has zero topological entropy*

Proof. Assume that (X, T) has positive topological entropy. This implies that there exists a point $x \in X$ and an ergodic measure μ such that μ has positive entropy and $\text{supp}(\mu) = \overline{\text{orb}(x, T)}$. By Theorem 3.14(2), $\overline{\text{orb}(x, T)}$ is mean sensitive. Hence (X, T) is not locally mean equicontinuous. □

Theorem 1.3 in [**15**] says that if (X, T) is locally equicontinuous, then every invariant ergodic probability measure on X is supported on a minimal subsystem . This is not true for LME systems (as described in Theorem 1.7).

THEOREM 4.3. *There exists a non-minimal E-system that is LME.*

Proof. Consider the example constructed in §3.2 with $(Y, \sigma) = \{0^\infty\}$. We already knew it is a non-minimal E -system (since it is Devaney chaotic, as Proposition 3.3 stated); we will show that it is LME. Let x be the transitive point constructed in §3.2 and $y \in X$. If $y \in \text{orb}(x, \sigma)$, then $\overline{\text{orb}(y, \sigma)} = X$ is almost mean equicontinuous (this is proved in §3.2). If $y \notin \text{orb}(x, \sigma)$, then, observing the construction, y is either a periodic point or a limit point with the form $B0^\infty$, where B is a subword of some A_n (furthermore, the possible B is the suffix of A_n). This implies that $\overline{\text{orb}(y, \sigma)}$ is a finite set and hence is almost mean equicontinuous, which completes the whole proof. □

Now we are going to prove Theorem 1.7.

Proof of Theorem 1.7. The first part is Theorem 4.2. The second part follows from Theorem 4.3, since it is clear that $X = \text{supp}(\mu)$ is not equal to the union of its minimal subsystems (in fact, X is the closure of the union of its minimal subsystems). □

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4.2. *Hyperspaces.* Let X be a compact metric space. We define the hyperspace $K(X)$ as the space of non-empty closed subsets of X equipped with the *Hausdorff metric*, d_H, which is defined by

$$\begin{aligned} d_H(A, B) &= \max\left\{\max_{x\in A}\min_{y\in B} d(x, y), \max_{y\in B}\min_{x\in A} d(x, y)\right\} \\ &= \inf\{\varepsilon > 0 : B(A, \varepsilon) \supset B,\ B(B, \varepsilon) \supset A\} \end{aligned}$$

for $A, B \in K(X)$ and $B(A, \varepsilon) = \bigcup_{a\in A} B(a, \varepsilon)$. The Hausdorff metric, d_H, can induce a topology on $K(X)$ known as the the *Vietoris topology*.

Let $U_1, \ldots, U_n$ be non-empty open subsets of X and fix $n \in \mathbb{N}$.

$$\langle U_1, \ldots, U_n\rangle = \left\{A \in K(X) : A \subset \bigcup_{i=1}^{n} U_i \text{ and } A \cap U_i \neq \emptyset \text{ for each } i = 1, \ldots, n\right\}.$$

The family

$$\{\langle U_1, \ldots, U_n\rangle : U_1, \ldots, U_n \text{ are non-empty open subsets of } X, n \in \mathbb{N}\}$$

forms a basis for the Vietoris topology. With this topology, $K(X)$ is compact.

Given a continuous map $T : X \to X$, we induce a continuous map $T_K : K(X) \to K(X)$ by

$$T_K(C) = TC \quad \text{for } C \in K(X).$$

We know that $(K(X), T_K)$ is a t.d.s. We refer any reader not familiar with hyperspace to [**29**] for more details.

For $n \in \mathbb{N}$, we define $K_n(X) := \{A \in K(X) : |A| \leq n\}$ and $K_\infty(X) := \bigcup_{n\geq 1} K_n(X)$. The following facts are easy to check.

LEMMA 4.4. *Let (X, T) be a t.d.s. Then:*
(1) *$K_n(X)$ is closed and $K_\infty(X)$ is dense in $K(X)$ [**5**, Lemma 2]; and*
(2) *if Y is a dense subset of X, then $K_\infty(Y)$ is dense in $K(X)$.*

4.2.1. *Proof of Theorem 1.8.* As mentioned before, every generalized diam-form of sensitivity can be inherited from the hyperspace $K(X)$ to the phase space X (see, for example, [**34**]). Nonetheless, we will prove Theorem 1.8 which says that this is not the case for (Banach) mean sensitivity.

Proof of Theorem 1.8. Let (X, σ) be the example constructed in the proof of Theorem 1.3 (in §3.1). By Theorem 1.3, (X, σ) is Banach-mean equicontinuous and hence it has zero topological entropy (see, for example, [**25**, Theorem 3.8]). Now we show that $(K(X), \sigma_K)$ is mean sensitive.

Let $\mathcal{U}$ be a non-empty open subset of $K(X)$. Considering that $K_\infty(X)$ is dense in $K(X)$, $\mathrm{orb}(x, \sigma)$ is dense in X, and by Lemma 4.4 there is a point

$$P = \{p_1, \ldots, p_m\} \in K_\infty(\mathrm{orb}(x, \sigma)) \cap \mathcal{U}$$

for some $m \in \mathbb{N}$. Let $\varepsilon > 0$ be such that $B_{d_H}(P, \varepsilon) \subset \mathcal{U}$. For each $1 \leq k \leq m$, there exists $N_k > 0$ such that $1/N_k < \varepsilon$. Since each p_k lies on the orbit of x under a shift map, we can

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see $(p_k)_{[0,N_k]}$ as a subword for a certain A_{n_k}. Without loss of generality, we can assume that the word A_{n_k} ends with $(p_k)_{[0,N_k]}$ (choose large enough N_k, if necessary). By the construction, each word A_{i_1} ends with A_{i_2} and appears in B_{i_3} for any $i_2 < i_1 < i_3 \in \mathbb{N}$.

Now we construct a closed subset Q_k of X as follows.

$$\begin{aligned} Q_k = \{&(p_k)_{[0,N_k]}10^{|A_{n_k}|-1} \ldots, \\ &(p_k)_{[0,N_k]}010^{|A_{n_k+1}|-2} \ldots, \ldots, \\ &(p_k)_{[0,N_k]}0^j10^{|A_{n_k+j}|-(j+1)} \ldots, \ldots, \\ &(p_k)_{[0,N_k]}0^\infty\}. \end{aligned}$$

One can check that $d_H(\{p_k\}, Q_k) < \varepsilon$ and that, for each $i > N_k$, there is a $q \in Q_k$ such that $(p_k)_i \neq q_i$. Thus

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d_H(\sigma_K^i\{p_k\}, \sigma_K^i Q_k) = \limsup_{n\to\infty} \frac{1}{n}\sum_{i=N_k+1}^{n-1} 1 = 1.$$

Let $Q := \bigcup_{k=1}^m Q_k$ and $N := \max_{1\le k\le m} N_k$. Clearly, $d_H(P, Q) < \varepsilon$ and hence $Q \in \mathcal{U}$. By Lemma 3.2, $\overline{D}(E_{p_k}) = 0$ for each $k = 1, \ldots, m$. This implies that if

$$G = \{i \in \mathbb{N} : (p_k)_i = 0 \text{ for all } k = 1, \ldots, m\},$$

then $\overline{D}(G) = 1$. Note that, for each $i \in G \cap (N, +\infty)$, we can choose a $q' \in Q$ with $q'_i = 1$. This implies that $d_H(\sigma_K^i P, \sigma_K^i Q) = 1$, and hence we conclude that

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d_H(\sigma_K^i P, \sigma_K^i Q) &\ge \limsup_{n\to\infty} \frac{1}{n}\sum_{i=N+1}^{n-1} d_H(\sigma_K^i P, \sigma_K^i Q) \\ &\ge \limsup_{n\to\infty} \frac{1}{n}\sum_{i\in[N+1,n-1]\cap G} 1 \\ &= \overline{D}(G) = 1. \end{aligned}$$

This shows that $K(X)$ is mean sensitive.

It remains to show that the topological entropy of $K(X)$ is positive. By the construction, it is not hard to see that (X, σ) contains a subsystem (Y, σ), which is the collection of points that contain at most one 1. That is, $Y = \{x^i \in \Sigma_2^+ : i \in \mathbb{Z}_+\}$ with the shift map, where $x^i = x_1^i x_2^i \ldots x_j^i \ldots$, $i \ge 0$, $j \ge 1$ with $x_j^i = 1$ if $j = i$, and $x_j^i = 0$ if $j \neq i$. Since $(K(Y), \sigma_K)$ has positive topological entropy (see, for instance, [**21**, Theorem 13]) and $K(Y) \subset K(X)$, we conclude that $(K(X), \sigma_K)$ has positive topological entropy. □

On the other hand, we cannot construct such a counter-example for weakly mixing or diam-mean sensitive systems (see Theorem 1.9).

A question in a similar spirit is the following. Can a t.d.s. with zero topological entropy have a hyperspace with uniform positive entropy (u.p.e.) of order n ($n \ge 2$)?

Let $n \ge 2$. We say that $(x_1, \ldots, x_n) \in X^n$ is an *n-entropy tuple* if $\{x_1, \ldots, x_n\}$ is not a singleton and for each pairwise disjoint closed neighbourhoods V_i of x_i, (X, T) has positive topological entropy with respect to the open cover $\mathcal{U} = \{X \setminus V_i : 1 \le i \le n\}$.

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We say that a t.d.s. (X, T) has *uniform positive entropy (or is u.p.e.) of order n* if each $(x_1, \ldots, x_n) \in X^n \setminus \Delta_n$ is an entropy tuple.

The answer to the question is no. It was shown in [**18**, Theorem 8.4] that, for any $n \geq 2$, X is u.p.e. of order n if and only if $K(X)$ is u.p.e. of order n. We will generalize one of the implications of this result in Theorem 4.7 by showing how we can find locally the entropy pairs.

Definition 4.5. Let (X, T) be a t.d.s. and let $\tilde{A} = (A_1, \ldots, A_k)$ be a tuple of subsets of X. We say that a subset $J \subseteq \mathbb{Z}_+$ is an independence set for $\tilde{A}$ (or that $\tilde{A}$ has the independence set J) if, for any non-empty finite subset $I \subseteq J$,

$$\bigcap_{i \in I} T^{-i} A_{s(i)} \neq \emptyset$$

for any $s \in \{1, \ldots, k\}^I$.

The following characterizations appeared in [**20**].

LEMMA 4.6. *Let (X, T) be a t.d.s. and let U_1, U_2 be pairwise disjoint subsets. Then:*

(1) *a pair $(x_1, x_2) \in X^2 \setminus \Delta_2$ is an entropy pair if and only if there is an independence set S with positive density for (U, V), where U, V are any non-empty neighbourhoods of x_1, x_2, respectively;*
(2) *a pair (U_1, U_2) has an independence set with positive density if and only if $h_{\text{top}}(T, \{U_1^c, U_2^c\}) > 0$; and*
(3) *$h_{\text{top}}(T, \{U_1^c, U_2^c\}) > 0$ if and only if there is an entropy pair $(x_1, x_2) \in U_1 \times U_2$.*

THEOREM 4.7. *Let (X, T) be a t.d.s. and let $A_1, A_2 \in K(X)$ with $A_1 \cap A_2 = \emptyset$. If (A_1, A_2) is an entropy pair of $(K(X), \sigma_K)$, then there exists an entropy pair, $(x_1, x_2) \in A_1 \times A_2$, of (X, T).*

Proof. Let $n \in \mathbb{N}$. We define $U_1^n := \bigcup_{a_1 \in A_1} B(a_1, 1/n)$ and $U_2^n := \bigcup_{a_2 \in A_2} B(a_2, 1/n)$. Without loss of generality, we can assume that $U_1^1 \cap U_2^1 = \emptyset$. Since (A_1, A_2) is an entropy pair, by Lemma 4.6(1) for each $n \in \mathbb{N}$, there is an independence set $S = S(n)$ with positive density for $(\langle U_1^n \rangle, \langle U_2^n \rangle)$. This implies that, for any finite subset $F \subset S$,

$$\bigcap_{k \in F} T_K^{-k} \langle U_{i_k}^n \rangle \neq \emptyset$$

for any $i_k \in \{1, 2\}^F$. Let $A \in \bigcap_{k \in F} T_K^{-k} \langle U_{i_k}^n \rangle$. We know that $T^k A \subset U_{i_k}^n$ for any $k \in F$, i.e., S is also an independence set with positive density for (U_1^n, U_2^n). By Lemma 4.6(2), $h_{\text{top}}(T, \{(U_1^n)^c, (U_2^n)^c)\}) > 0$. Then using Lemma 4.6(3) there is an entropy pair $(x_1^n, x_2^n) \in U_1^n \times U_2^n$. Assume that $(x_1^n, x_2^n) \to (x_1, x_2)$ when $n \to \infty$ (passing to subsequence, if necessary). It is easy to check that $(x_1, x_2) \in A_1 \times A_2$ is an entropy pair. □

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We leave the following open question. Is every t.d.s. with u.p.e. of order $n (n \in \mathbb{N})$ mean sensitive?

4.2.2. *Relationships on hyperspaces.* The notions of diam-mean equicontinuity and mean equicontinuity are different in general. Actually, in §3.1, we provided a transitive mean equicontinuous t.d.s. which has no diam-mean equicontinuous points. Nonetheless, on hyperspaces, the notions coincide.

THEOREM 4.8. *Let (X, T) be a t.d.s. and let $A \in K(X)$. Then A is diam-mean equicontinuous if and only if it is mean equicontinuous.*

Proof. By definition, every diam-mean equicontinuous point in $K(X)$ is mean equicontinuous.

Now assume that $A \in K(X)$ is mean equicontinuous. By Remark 2.1(2), for every $\varepsilon > 0$, there is a $\delta > 0$ such that, for any $B_1, B_2 \in B_{d_H}(A, \delta)$,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d_H(T_K^i B_1, T_K^i B_2) < \frac{\varepsilon}{2}.$$

Now we choose suitable $0 < \delta' < \delta/2$ and an open subcover $\{B_d(x_1, \delta'), \ldots, B_d(x_m, \delta')\}$ of A such that

$$\mathcal{V} = \langle B_d(x_1, \delta'), \ldots, B_d(x_m, \delta')\rangle \subset \langle \overline{B_d(x_1, \delta')}, \ldots, \overline{B_d(x_m, \delta')}\rangle \subset B_{d_H}(A, \delta).$$

For simplicity, we denote $\mathcal{W} = \langle \overline{B_d(x_1, \delta')}, \ldots, \overline{B_d(x_m, \delta')}\rangle$. Now set $E = \{x_1, x_2, \ldots, x_m\}$ and $F = \bigcup_{i=1}^m \overline{B_d(x_i, \delta')}$. Clearly, $E,\ F \in B_{d_H}(A, \delta)$. Note that

$$\begin{aligned}\operatorname{diam}(T_K^i \mathcal{V}) &\le \sup_{B_1\in\mathcal{W}} d_H(T_K^i B_1, T_K^i E) + \sup_{B_2\in\mathcal{W}} d_H(T_K^i E, T_K^i B_2)\\ &\le d_H(T_K^i F, T_K^i E) + d_H(T_K^i E, T_K^i F) = 2d_H(T_K^i E, T_K^i F)\end{aligned}$$

for all $i \in \mathbb{Z}_+$. Thus

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \operatorname{diam}(T_K^i \mathcal{V}) &\le \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \operatorname{diam}(T_K^i \mathcal{W})\\ &\le 2 \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d_H(T_K^i E, T_K^i F) < \varepsilon.\end{aligned}$$

This implies that A is a diam-mean equicontinuous point. □

As a consequence of Theorem 4.8, we obtain the following corollary.

COROLLARY 4.9. *Let (X, T) be a t.d.s. Then $K(X)$ is diam-mean equicontinuous if and only if $K(X)$ is mean equicontinuous.*

Also, we know that, in general, diam-mean sensitivity and mean sensitivity are different concepts (e.g., see Theorem 1.3). We do not know if, in general, a hyperspace is mean sensitive if and only if it is diam-mean sensitive. Nonetheless, when combining [**25**, Theorem 5.4], [**4**, Theorem 2] and Theorem 4.8, we obtain the following corollary.

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COROLLARY 4.10. *Let (X, T) be a weakly mixing t.d.s. Then $(K(X), T_K)$ is diam-mean sensitive if and only if $(K(X), T_K)$ is mean sensitive.*

We do not know if this characterization holds in general. Theorem 1.9 present a partial characterization.

Recall that a map $\pi : X \to Y$ is called *semi-open* if, for every non-empty open subset $U \subset X$, the interior of $\pi(U)$ in Y is not empty. Let $x \in X$. We say that π *is open at* x if, for every neighbourhood U of x, $\pi(U)$ is a neighbourhood of $\pi(x)$. It is known that π is semi-open if and only if the set $\{x \in X : \pi \text{ is open at } x\}$ is residual [**12**, Lemma 2.1].

LEMMA 4.11. *Let $\phi : K(K(X)) \to K(X)$ be defined by $\mathcal{A} \mapsto D = \bigcup_{A\in\mathcal{A}} A$. Then ϕ is a well-defined semi-open factor map.*

Proof. To see that ϕ is well defined, it suffices to check that $D = \bigcup_{A\in\mathcal{A}} A \in K(X)$ for any given $\mathcal{A} \in K(K(X))$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of D and $x_n \to x$, and let $A_n \in \mathcal{A}$ with $x_n \in A_n$ for each $n \in \mathbb{N}$. Assume that A_n converges to $A \in \mathcal{A} \in K(K(X))$. This implies that $x \in A \subset D$ and thus $D \in K(X)$. Furthermore, we observe that $d_H(\bigcup_{A\in\mathcal{A}} A, \bigcup_{B\in\mathcal{B}} B) \le d_H(\mathcal{A}, \mathcal{B})$ for any $\mathcal{A}, \mathcal{B} \in K(K(X))$, $\phi(\{C\}) = C$ for any $C \in K(X)$, and $T_K \circ \phi = \phi \circ T_K$. Hence ϕ is a factor map.

Now let $\mathcal{U} = \langle \mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_m\rangle$ be a non-empty open subset of $K(K(X))$, where each $\mathcal{U}_j$ is non-empty open in $K(X)$. Without loss of generality, we assume that, for each $1 \le j \le m$, there are non-empty open subsets $U_1^j, \ldots, U_n^j$ such that $\langle U_1^j, \ldots, U_n^j\rangle \subset \mathcal{U}_j$. Let $\mathcal{W} = \langle U_1^1, \ldots, U_n^1, \ldots, U_1^m, \ldots, U_n^m\rangle$. It is not hard to see that $\mathcal{W}$ is non-empty and open in $K(X)$. We now claim that $\mathcal{W} \subset \phi(\mathcal{U})$. Indeed, for any E in $\mathcal{W}$, there are non-empty open subsets V_i^j of X such that $V_i^j \subset \overline{V_i^j} \subset U_i^j$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ and $E \in \langle V_1^1, \ldots, V_n^1, \ldots, V_1^m, \ldots, V_n^m\rangle$. Let $E_i^j = E \cap \overline{V_i^j}$ and $E^j = \bigcup_{i=1}^n E_i^j$. Clearly, $E_i^j \subset U_i^j$, $E^j \in \mathcal{U}_j$ and $E = \bigcup_{j=1}^m E^j$. We define $\mathcal{E} := \{E^1, \ldots, E^m\}$. We know that $\mathcal{E} \in \mathcal{U}$ and $\phi(\mathcal{E}) = E$. This shows ϕ is semi-open. □

LEMMA 4.12. *Let $\pi : (X, T) \to (Y, S)$ be a semi-open factor map. If (Y, S) is diam-mean sensitive, then so is (X, T).*

Proof. The map $\pi^{-1} : Y \to K(X)$ is an upper semi-continuous function, so it possesses a residual subset $Y_0 \subset Y$ of continuous points. Since π is semi-open, by [**12**, Lemma 2.1] $X_0 = \pi^{-1}(Y_0)$ is a residual set of X. Now let U be a non-empty open subset in X. Let $x_0 \in X_0 \cap U$ and $\varepsilon > 0$ be such that $B(x_0, \varepsilon) \subset U$. By the definition of X_0 and the semi-openness of π, there is a continuous point $y_0 \in Y_0 \cap \operatorname{int}(\pi(B(x_0, \varepsilon)))$ for π^{-1} with $y_0 = \pi(x_0)$ (where $\operatorname{int}(\cdot)$ denotes the interior of a given set). Let $\sigma > 0$ be such that $B(y_0, \sigma) \subset \operatorname{int}(\pi(B(x_0, \varepsilon)))$. Since (Y, S) is diam-mean sensitive, similarly to Remark 2.1(3), there is a $t > 0$ such that the upper density of $F := \{i \in \mathbb{Z}_+ : \operatorname{diam}(S^i B(y_0, \sigma)) > 2t\}$ is greater than t. For each $i \in F$, there exists $y_i \in B(y_0, \sigma)$ such that

$$d(S^i y_i, S^i y_0) > t.$$

By the continuity of S^i, there is a $y_i{}' \in Y_0 \cap B(y_0, \sigma)$ such that

$$d(S^i y_i, S^i y_i{}') < d(S^i y_i, S^i y_0) - t,$$

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and hence

$$d(S^i y_i', S^i y_0) > t.$$

Since $y_i' \in Y_0 \cap B(y_0, \sigma) \subset Y_0 \cap \pi(B(x_0, \varepsilon))$, there is an $x_i \in B(x_0, \varepsilon)$ with $y_i' = \pi(x_i)$. Then

$$d(\pi T^i x_i, \pi T^i x_0) > t.$$

By the continuity of π, there is $0 < s < t$ such that

$$d(T^i x_i, T^i x_0) > s,$$

and thus $F \subset \{i \in \mathbb{Z}_+ : \operatorname{diam}(T^i B(x_0, \varepsilon)) > s\}$. Similarly to Remark 2.1(3),

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \operatorname{diam}(T^i U) &\geq \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \operatorname{diam}(T^i B(x_0, \varepsilon)) \\ &\geq s \cdot \overline{D}(F) > st > 0,\end{aligned}$$

which shows that (X, T) is diam-mean sensitive. □

The following lemma is a particular case of [**34**, Corollary 1].

LEMMA 4.13. *Let (X, T) be a t.d.s. If $(K(X), T_K)$ is diam-mean sensitive, then so is (X, T).*

Now we are ready to show Theorem 1.9.

Proof of Theorem 1.9. The equivalence of (1) $\iff$ (2) follows from Lemmas 4.11 , 4.12 and 4.13. When X is weakly mixing, we use Corollary 4.10 to conclude that (1) $\iff$ (2) $\iff$ (3) $\iff$ (4). □

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