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WEAK CONVERGENCE OF NONLINEAR TRANSFORMATIONS OF INTEGRATED PROCESSES: THE MULTIVARIATE CASE

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We consider weak convergence of sample averages of nonlinearly transformed stochastic triangular arrays satisfying a functional invariance principle. A fundamental paradigm for such processes is constituted by integrated processes. The results obtained are extensions of recent work in the literature to the multivariate and non-Gaussian case. As admissible nonlinear transformation, a new class of functionals (so-called locally p-integrable functions) is introduced that adapts the concept of locally integrable functions in Pötscher (2004, *Econometric Theory* 20, 1–22) to the multidimensional setting.

1. INTRODUCTION

In the asymptotic theory of nonlinear regression a central role is played by functionals of the form $n^{-1}\sum_{k=1}^{n} f(n^{-1/2}U_k)$. In particular, in time series regression with integrated processes, $U = (U_k)_{k\in\mathbb{N}}$ typically is a stochastic process such that $n^{-1/2}U_{[nt]}, 0 \le t \le 1$, converges weakly (in the space $D[0,1]$ of càdlàg functions) to some limit process $X = (X(t))_{0\le t\le 1}$ such as Brownian motion (BM). To determine the asymptotic distribution of estimators or test statistics, it is of primary importance to obtain the limiting distribution of such functionals for a large class of functions f. For continuous functions, the continuous mapping theorem (CMT) in its extended form implies that

$$\frac{1}{n}\sum_{k=1}^{n} f(n^{-1/2}U_k) \xrightarrow{d} \int_0^1 f(X(t))\,dt, \tag{1.1}$$

at least as long as the limit process X has continuous paths. In recent years, several approaches have been made to establish (1.1) for wider classes of functions. Park and Phillips (1999, 2001) show that (1.1) actually holds for what they call "regular" functions[1] (which comprise piecewise continuous functions); de Jong (2004) establishes (1.1) for functions that may have finitely many poles. The most general

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results that seem to have been obtained so far can be found in Pötscher (2004) and Berkes and Horváth (2006). In the former, it is shown that to obtain (1.1) it suffices to require the function f to be *locally integrable* (cf. Section 2, Definition), a property that is shared by the classes of functions admitted in Park and Phillips (2001) and also in de Jong (2004). This condition is minimal in the sense that—at least in the case where X is a BM—the integral in (1.1) may be shown to exist if and only if f is locally integrable. The limit process X in Pötscher (2004) is confined to Brownian motion. In Berkes and Horváth (2006), more general Gaussian limit processes are admitted. Local integrability of f is replaced by the more restrictive local integrability of $|x|^{\alpha-1} f(x)$, where $0 < \alpha \leq 1$ is such that the standard deviation $\sigma(t)$ of $X(t)$ satisfies $\sigma(t)^{-\alpha} \in L^1([0,1], dt)$. Moreover, diagonal schemes (U_{nk}) are admitted, generalizing the normalized sums $n^{-1/2}U_k$. In both approaches, it is required that the U_{nk} possess densities (with respect to Lebesgue measure) satisfying certain boundedness conditions. More important, all these approaches deal with *one-dimensional* processes and *Gaussian limit processes.* Both restrictions are crucial in one or the other argument. For example, the argument linking existence of the integral in (1.1) and the condition of local integrability—basically the occupation times formula for Brownian local time—does not extend to higher dimensions, and so additional assumptions are needed to establish existence and finiteness of the integral. Also, the uniform boundedness condition imposed on the densities of the process $(k^{-1/2}U_k)$ in Pötscher (2004) is useful only in the scalar case (cf. Remark 2.2). If X is no longer Gaussian but any other (still one-dimensional) continuous semimartingale, the occupation times formula (in a modified form) can still be used, but certain restrictions must be imposed on the predictable quadratic variation of X.

Here we shall establish

$$\frac{1}{n}\sum_{k=1}^{n} f(U_{nk}) \xrightarrow{d} \int_0^1 f(X(t))\,dt \tag{1.2}$$

for a large class of r-dimensional triangular arrays $(U_{nk}), k = 1, \ldots, n, n \in \mathbb{N}$, such that $(U_{n,[nt]}), 0 \leq t \leq 1$, converges weakly to X. Non-Gaussian limit processes X are admitted. Following are some examples, of which the second stands for a multivariate situation and the third for a non-Gaussian limit distribution. Further interesting examples in the one-dimensional Gaussian case pertaining to fractional BM can be found in Berkes and Horváth (2006).

Example 1

Let (ξ_i) be a sequence of random variables for which a functional central limit theorem holds, i.e., $n^{-1/2}S_{[n\cdot]} \Rightarrow B$, where $S_k = \sum_{i=1}^{k} \xi_i$, B is standard BM, and $\Rightarrow$ denotes weak convergence in $D[0,1]$. Then, introducing the sample mean $\overline{\xi}_n = n^{-1}S_n$, the partial sums of the deviations $U_{nk} = n^{-1/2}\sum_{i=1}^{k}(\xi_i - \overline{\xi}_n)$ converge weakly to a *Brownian bridge* (BB), i.e.,[2]

$$U_{n,[nt]} \Rightarrow B^0(t) = B(t) - tB(1).$$

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Example 2

Consider the VAR model in error correction form

$$\Delta Y_k = \Pi Y_{k-1} + \varepsilon_k, \tag{1.3}$$

where Π is a $d \times d$ matrix and (ε_k) is d-dimensional pure white noise. Singularity of Π is equivalent to the characteristic polynomial of (1.3) having a unit root, in which case the process $Y = (Y_k)$ will be nonstationary. In such a situation, one may be interested in testing whether Y is cointegrated and in determining the cointegrating rank (cf. Johansen, 1995). Under the null hypothesis $H(r)$ that Y is cointegrated with cointegrating rank $d_0 < d$, Π may be written in the form $\Pi = \alpha\beta'$, where α and β are both $d \times d_0$ matrices and β contains the d_0 cointegrating vectors. To judge the performance of such tests a standard procedure is to evaluate their power under *local alternatives*. In our context, the local alternative would be

$$H_n(d_0, d_1) : \Pi = \Pi_n = \alpha\beta' + \frac{1}{n}\alpha_1\beta_1', \tag{1.4}$$

where the $d \times d_1$ matrix β_1 contains d_1 extra cointegrating vectors that enter in the process with very small adjustment coefficients $n^{-1}\alpha_1$. To derive the asymptotic distribution of the likelihood ratio statistic, we need to know the asymptotic behavior of the solution process (Y_k^n) of (1.3) under (1.4). It turns out that, for $U_{nk} = n^{-1/2}\alpha_\perp' Y_k^n$, it holds that

$$U_{n,[nt]} \Rightarrow X(t) = \alpha_\perp' \int_0^t \exp[A(t-s)]\, dB(s),$$

where $\alpha_\perp$ and A are $d \times (d - d_0)$ and $d \times d$ matrices, respectively, related to $\alpha, \beta, \alpha_1, \beta_1$ and B is d-dimensional Brownian motion. In particular, $\alpha_\perp$ is orthogonal to $\alpha : \alpha_\perp'\alpha = 0$. The term X is an $r = (d - d_0)$–dimensional *Ornstein–Uhlenbeck* (OU) process and satisfies the linear stochastic differential equation

$$dX(t) = AX(t)\, dt + \alpha_\perp'\, dB(t).$$

Example 3

This is concerned with convergence of Markov chains to diffusions. More precisely, let X be a solution of the stochastic differential equation (SDE)

$$dX(t) = b(X(t))\, dt + \sigma(X(t))\, dB_t, \qquad 0 \le t \le 1. \tag{1.5}$$

For each $n \in \mathbb{N}$, perform the discretization $h = h_n = 1/n$ and consider the stochastic difference equation

$$\Delta X_k^n = b(X_{k-1}^n)h + \sigma(X_{k-1}^n)\varepsilon_k^n, \qquad k = 1, \ldots, n,$$
$$X_0^n = x.$$

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The ε_k^n are assumed independent and identically distributed (i.i.d.) with zero mean and variance h, the drift $b(x)$, and the diffusion coefficient $\sigma(x)$ continuous. Then, for each n, $U_{nk} = X_k^n$, $k = 0, 1, \ldots, n$, is a homogeneous Markov chain. Under certain conditions on the transition probabilities, which are satisfied for a large class of distributions of the ε_k^n (including $\mathcal{N}(0, h)$ and Bernoulli with jump size $\sqrt{h}$), the corresponding processes X_n will converge weakly to the solution X of the SDE (1.5), provided the latter has a *unique weak* solution. Equivalently, if the *martingale problem for* b and $a = \sigma\sigma'$ is well posed. The latter will be the case if, e.g., $b(x)$ and $a(x)$ are bounded measurable functions (boundedness may be weakened) such that $a(x)$ is continuous and satisfies the uniform ellipticity condition

$$u'a(x)u \geq \lambda |u|^2 \qquad \forall u, x \in \mathbb{R}^r \tag{1.6}$$

for some $\lambda > 0$. The interested reader may find the complete story in Strook and Varadhan (1979). In particular, convergence of the Markov chains X_k^n to the solution of (1.5) is treated in chapter 11.2. For the special case where $\sigma(x) \equiv \sigma$, corresponding results may be found in Christopeit (1983).

In the present paper, we establish (1.2) for a class of function that is suited for the multivariate case and allows for limit processes other than Brownian motion. In the one-dimensional setting, locally integrable functions are covered as a special case.

2. WEAK CONVERGENCE FOR LOCALLY *p*-INTEGRABLE FUNCTIONALS

Let $U = (U_{nk}), k = 1, \ldots, n, n \in \mathbb{N}$, be a stochastic triangular array of $\mathbb{R}^r$-valued random vectors all defined on some basic probability space $(\Omega, \mathcal{F}, \mathrm{P})$. As usual, we adjoin with U a sequence of $\mathbb{R}^r$-valued stochastic processes $X_n = (X_n(t))_{t\in[0,1]}$ by setting

$$X_n(t) = U_{nk}, \qquad \frac{k-1}{n} \leq t < \frac{k}{n}, \qquad k = 1, \ldots, n,$$

and $X_n(1) = U_{nn}$. The processes X_n thus defined have right continuous paths with left limits. The definition deviates from common practice, where the associated continuous time processes are defined by $\widetilde{X}_n(t) = U_{n,[nt]}$, where $[a]$ denote the *largest* integer $\leq a$ (with a fictitious initial value U_{n0}—usually $U_{n0} = 0$). This has the advantages that only actually "observed" values $U_{n1}, \ldots, U_{nn}$ (and no fictitious initial value U_{n0}) appear and that

$$\frac{1}{n}\sum_{k=1}^{n} f(U_{nk}) = \int_0^1 f(X_n(t))\,dt \tag{2.1}$$

without any correction for initial and final values. Because our concern is only with "ordinary" (Lebesgue) integrals, this difference in definition of the $X_n(t)$

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does not lead to troubles related to "anticipating" or "nonanticipating" integrands. Moreover, one can show that $X_n \Rightarrow X$ is equivalent to $\widetilde{X}_n \Rightarrow X$ provided X has continuous paths.

Our setting covers as a special case the situation considered in Park and Phillips (1999, 2001) and also Pötscher (2004), namely, arrays of the form $U_{nk} = n^{-1/2}U_k$, where $(U_k)_{k\in\mathbb{N}}$ is a stochastic process and $\widetilde{X}_n(t) = n^{-1/2}U_{[nt]}$. We may view the X_n as random elements taking values in the space $D_r[0,1]$ of $\mathbb{R}^r$-valued càdlàg functions. We identify $D_r[0,1]$ with $D([0,1])^r$ and endow it with the product topology. The latter may be generated by various metrics, e.g., $d_r(x,y) = [d(x_1,y_1)^2 + \cdots + d(x_r,y_r)^2]^{1/2}$ or $d_r'(x,y) = \max\{d(x_1,y_1),\ldots,d(x_r,y_r)\}$ (for $x(t) = (x_1(t),\ldots x_r(t))$ and the same for y). Weak convergence in $D_r[0,1]$ will be denoted by $\Rightarrow$, whereas for convergence in distribution of random variables or vectors we shall use the usual $\xrightarrow{d}$. The term $|x|$ will denote euclidean norm in $\mathbb{R}^r$.

The class of functions for which we shall derive weak convergence (1.2) is specified by the following definition.

DEFINITION. *A measurable function* $f : \mathbb{R}^r \to \mathbb{R}$ *is called* locally p-integrable *(with* $1 \le p < \infty$*) if, for every* $R > 0$,

$$\int_{\{|x|\le R\}} |f(x)|^p \, dx < \infty. \tag{2.2}$$

Note that local p-integrability for some $p > 1$ implies local p'-integrability for every $p' < p$, and so in particular for $p = 1$. For $p = 1$ (and $r = 1$) this is just the definition of a *locally integrable* function in Pötscher (2004). In the *one-dimensional* case, if the limit process X is a *continuous semimartingale* with predictable quadratic variation $\langle X\rangle$, the occupation times formula states that

$$\int_0^1 |f(X(t))| \, d\langle X\rangle_t = \int |f(x)| l_1(x) \, dx, \tag{2.3}$$

where $l_t(x)$ is the local time of X at x. For many limit processes of interest such as BM, BB, and the one-dimensional OU process, $\langle X\rangle_t = \sigma^2 t$ for some constant σ^2. In this case, we find that on the ω set $\Omega_R = \{\omega \in \Omega : \sup_t |X(t,\omega)| \le R\}$

$$\begin{aligned}\int_0^1 |f(X(t))| \, dt &= \sigma^{-2} \int_0^1 |f(X(t))| \, 1_{\{|X(t)|\le R\}} \, dt \\ &= \sigma^{-2} \int_{-R}^{R} |f(x)| l_1(x) \, dx \\ &\le \sigma^{-2} L_R \int_{-R}^{R} |f(x)| \, dx,\end{aligned}$$

where $L_R = \sup_{-R\le x\le R} l_1(x)$. Because local time $l_t(x)$ can be chosen continuous in t and càdlàg in x, this is almost surely (a.s.) a finite number. Hence, for such

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processes, local integrability guarantees existence (and finiteness) of the integral $\int_0^1 f(X(t))\,dt$ with probability one. For limit processes (still one-dimensional) whose predictable quadratic variation is of the more general form

$$\langle X\rangle_t = \int_0^t \sigma^2(s, X(s))\,ds,$$

(2.3) implies that on Ω_R

$$\int_0^1 |f(X(t))|\,dt = \int_0^1 \frac{|f(X(t))|}{\sigma^2(t, X(t))} 1_{\{|X(t)|\le R\}}\,d\langle X\rangle_t\,. \tag{2.4}$$

Hence, if

$$\inf_{0\le t\le 1} \sigma^2(t, X(t,\omega)) > 0 \quad \text{a.s.}, \tag{2.5}$$

it will follow from (2.4) and (2.3) (denoting $s(\omega) = \inf_{0\le t\le 1}\sigma^2(t, X(t,\omega))$) that

$$\int_0^1 |f(X(t))|\,dt \le s^{-2}\int_{-R}^R |f(x)|\,l_1(x)\,dx \le s^{-2}L_R\int_{-R}^R |f(x)|\,dx$$

on Ω_R. Because $\bigcup_R \Omega_R = \Omega$, this shows that the integral $\int_0^1 f(X(t))\,dt$ exists and is finite with probability one. Assumption (2.5) is satisfied, e.g., for geometric BM $X(t)$, where $d\langle X\rangle_t = \sigma^2 X(t)^2\,dt$ and $\inf_{0\le t\le 1} X(t) > 0$. If $\sigma^2(t, X(t))$ is not bounded away from zero with probability one, local p-integrability with $p > 1$ may still yield the desired result. To see this, perform a (p, q)-Hölder estimate (with $p^{-1} + q^{-1} = 1$) on the right-hand side of (2.4) (for simplicity for autonomous $\sigma(x)$):

$$\begin{aligned}
&\int_0^1 \frac{|f(X(t))|}{\sigma^2(X(t))} 1_{\{|X(t)|\le R\}}\,d\langle X\rangle_t \\
&\quad\le \left[\int_0^1 |f(X(t))|^p\,1_{\{|X(t)|\le R\}}\,d\langle X\rangle_t\right]^{1/p} \\
&\qquad\times \left[\int_0^1 \sigma^{-2q}(X(t)) 1_{\{|X(t)|\le R\}}\,d\langle X\rangle_t\right]^{1/q} \\
&\quad= \left[\int_{-R}^R |f(x)|^p\,l_1(x)\,dx\right]^{1/p}\left[\int_{-R}^R \sigma^{-2q}(x) l_1(x)\,dx\right]^{1/q} \\
&\quad\le L_R\left[\int_{-R}^R |f(x)|^p\,dx\right]^{1/p}\left[\int_{-R}^R \sigma^{-2q}(x)\,dx\right]^{1/q}.
\end{aligned}$$

Hence, if f is locally p-integrable with $p > 1$ and σ^{-2} is locally q-integrable, then the integral $\int_0^1 f(X(t))\,dt$ exists and is finite with probability one.

In the *multidimensional* case, no analogue of formula (2.3) seems to exist, and existence of the integral must be ensured in a different way. We shall come back to this point after having stated the assumption to be made.

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Assumption 1. There exists a process $X = (X(t))_{t\in[0,1]}$ with continuous paths such that $X_n \Rightarrow X$ in the product topology of $D_r[0,1]$. Here the X_n are defined as at the beginning of Section 2.

Assumption 2. For each n, k, U_{nk} possesses a density φ_{nk} (with respect to r-dimensional Lebesgue measure) such that

$$\sup_n \frac{1}{n}\sum_{k=1}^{n} \|\varphi_{nk}\|_q < \infty, \tag{2.6}$$

where $\|\varphi\|_q$ denotes L^q-norm (with $\|\varphi\|_\infty = \text{ess-sup}_x\, |\varphi(x)|$) and $p^{-1} + q^{-1} = 1$ (with $q = \infty$ for $p = 1$).

Henceforth, when referring to Assumption 2, we shall denote by C_1 any finite bound for the left-hand side.

Assumption 3. For almost all t (with respect to Lebesgue measure on $[0,1]$) $X(t)$ possesses a density ψ_t (with respect to r-dimensional Lebesgue measure) such that the following inequality holds. For $p > 1$ (with the p from (2.2)),

$$\int_0^1 \left[\int \psi_t^q(x)\,dx\right]^{1/q} dt \le C_2 \tag{2.7a}$$

for some finite constant C_2. For $p = 1$, (2.7a) is replaced by

$$\int_0^1 \psi_t(x)\,dt \le C_2 \quad \text{for all } x. \tag{2.7b}$$

2.1. Remarks Concerning Assumption 2

Remark 2.1. In the important special case where $U_{nk} = n^{-1/2}U_k$, assume that the U_k have densities φ_k. Then the density φ_{nk} of U_{nk} is given by $\varphi_{nk}(x) = n^{r/2}\varphi_k(n^{1/2}x)$, and $\|\varphi_{nk}\|_q = n^{r/2p}\|\varphi_k\|_q$. Hence, in this case, Assumption 2 amounts to the following requirement.

Assumption 2′. The U_k possess densities φ_k (with respect to r-dimensional Lebesgue measure) that satisfiy

$$\sup_n n^{(r/2p)-1}\sum_{k=1}^{n} \|\varphi_k\|_q \le C_1. \tag{2.6′}$$

Because the sum $\sum_{k=1}^n \|\varphi_k\|_q$ is strictly increasing, Assumption 2′ can only be satisfied if $r \le 2p$.

Remark 2.2. Consider the following assumption (which, for $r = 1$ and $p = 1$, is Assumption 2.2 in Pötscher, 2004).

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Assumption 2″. The random vectors $k^{-1/2}U_k$ possess densities h_k such that $\sup_k \|h_k\|_q \le M < \infty$.

Then the φ_k in Assumption 2′ are given by $\varphi_k(x) = k^{-r/2}h_k\ (k^{-1/2}x)$, with $\|\varphi_k\|_q = k^{-r/2p}\|h_k\|_q$. Hence

$$A_n = n^{(r/2p)-1}\sum_{k=1}^{n}\|\varphi_k\|_q \le Mn^{(r/2p)-1}\sum_{k=1}^{n}k^{-r/2p}. \tag{2.6″}$$

For $r < 2p$, the sum on the right-hand side of (2.6″) is $O(n^{1-r/2p})$, so that $A_n = O(1)$. Consequently, in this case, condition (A2″) is *sufficient* for (A2′). For $r \ge 2p$, the term on the right-hand side of (2.5″) diverges. Therefore, the concept of local integrability ($p = 1$) together with Assumption 2″ (i.e., the setting considered in Pötscher, 2004) does only work in the one-dimensional setting ($r = 1$).

2.2. Remarks Concerning Assumption 3

Remark 3.1. As we have seen earlier, many one-dimensional limit processes of importance such as BM, BB, OU process, and geometric BM satisfy (2.5), which turned out to be a sufficient condition for existence and finiteness of the integral in (1.1) for locally p-integrable functions. Therefore, for this latter purpose, Assumption 3 could be dispensed with in the one-dimensional case. However, in any case, Assumption 3 is needed for a certain approximation result (cf. Lemma A.4 in Appendix A), and so this assumption must be required anyway.

Remark 3.2. Assumption 3 is satisfied for the following processes:

(a) (multivariate) BM;
(b) BB;
(c) (multivariate) OU process;
(d) geometric BM;
(e) more general: one-dimensional Gaussian processes with variance $\sigma^2(t)$ satisfying

$$\int_0^1 \frac{dt}{\sigma(t)} < \infty; \tag{2.8}$$

(f) diffusion limit processes as in Example 3 in the Introduction.

Proofs of these assertions are given in Appendix C. Case (e) covers one-dimensional BM and, more generally, fractional BM with parameter $0 < H < 1$ and also the cases (b), (c) (univariate case), and (d). Some further examples related to fractional BM and a discussion of condition (2.8) can be found in Berkes and Horváth (2006). With Example 3, we leave the scope of Gaussian processes, and methods from the theory of partial differential equations or diffusion processes have to be brought into the game to verify estimates such as (2.7). At the end of Appendix C, we will give a brief sketch of results of this type.

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Remark 3.3. We still have to make sure that Assumption 3 guarantees existence and finiteness of the integral $\int_0^1 f(X(t))\,dt$ (in particular in the multivariate case where we cannot resort to an occupation times formula). To this end note that, as a consequence of Assumption 3, for locally p-integrable f (and with Ω_R as before)

$$\begin{aligned}
\mathrm{E}1_{\Omega_R}\int_0^1 |f(X(t))|\,dt &\le \int_0^1 \mathrm{E}\,|f(X(t))|\,1_{\{|X(t)|\le R\}}\,dt \\
&= \int_0^1 \int_{\{|x|\le R\}} |f(x)|\,\psi_t(x)\,dx\,dt \\
&\le \left[\int_{\{|x|\le R\}} |f(x)|^p\,dx\right]^{1/p} \int_0^1 \left[\int \psi_t^q(x)\,dx\right]^{1/q} dt \\
&\le C_2\left[\int_{\{|x|\le R\}} |f(x)|^p\,dx\right]^{1/p}
\end{aligned}$$

for $p > 1$ and

$$\mathrm{E}1_{\Omega_R}\int_0^1 |f(X(t))|\,dt \le \int_{\{|x|\le R\}} |f(x)| \int_0^1 \psi_t(x)\,dt\,dx \le C_2 \int_{\{|x|\le R\}} |f(x)|\,dx$$

for $p = 1$. Hence, in both cases, the expected value on the left-hand side is finite, implying that the integral $\int_0^1 |f(X(t))|\,dt$ is finite a.s. on Ω_R. Now let $R \nearrow \infty$ to see that $\int_0^1 |f(X(t))|\,dt$ is a.s. finite. Actually, in the one-dimensional case with $X = B = \mathrm{BM}$, local integrability (i.e., $p = 1$) is equivalent to existence and almost sure finiteness of the integral $\int_0^1 f(B(t))\,dt$. This follows from a theorem of Engelbert and Schmidt (cf. Karatzas and Shreve, 1988, Prop. 6.27 plus Prob. 6.29).

Remark 4. Not surprisingly, Assumptions 2 and 3 are closely related. This becomes evident if we consider the densities $\psi_{nt}(x)$ of $X_n(t)$. By definition of the X_n, it holds that $\psi_{nt}(x) = \varphi_{nk}(x)$ for $(k-1)/n \le t < k/n$. Now, in general, weak convergence $X_n \Rightarrow X$ will not imply convergence of the densities. In situations, however, where this is the case, i.e., where $\psi_{nt}(x) \to \psi_t(x)$ (say, a.s. with respect to Lebesgue measure), we have that (for $p > 1$)

$$\begin{aligned}
\int_0^1 \left[\int \psi_t^q(x)\,dx\right]^{1/q} dt &= \int_0^1 \left[\int \lim_{n\to\infty} \psi_{nt}^q(x)\,dx\right]^{1/q} dt \\
&\le \underline{\lim}_{n\to\infty} \int_0^1 \left[\int \psi_{nt}^q(x)\,dx\right]^{1/q} dt \\
&= \underline{\lim}_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \left[\int \varphi_{nk}^q(x)\,dx\right]^{1/q}
\end{aligned}$$

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$$= \underline{\lim}_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \|\varphi_{nk}\|_q$$

$$< \infty.$$

In other words, Assumption 2 implies Assumption 3 (in this special situation). A similar result is true for $p = 1$ (i.e., $q = \infty$).

The main result of this section is the following theorem.

THEOREM 2.1. *Under Assumptions 1–3, (1.2) is true for every locally p-integrable function f.*

In certain situations (cf. Section 3), the conditions made are not quite sufficient to establish (2.6) but rather

$$\sup_{n\geq K} \frac{1}{n} \sum_{k=K}^{n} \|\varphi_{nk}\|_q < \infty \qquad \textbf{(2.6*)}$$

for some integer K. (A similar problem arises in Pötscher, 2004.) Call Assumption 2 with (2.6) replaced by (2.6*) Assumption 2*. Then the following result is true.

THEOREM 2.1*. *If, in Theorem 2.1, Assumption 2 is replaced by Assumption 2*, then*

$$\frac{1}{n} \sum_{k=K}^{n} f(U_{nk}) \xrightarrow{d} \int_0^1 f(X(t))\, dt. \qquad \textbf{(1.2*)}$$

If, under (2.6*), one would nevertheless like to establish the original convergence (1.2), one has to consider the means

$$S_n = \frac{1}{n} \sum_{k=1}^{K} f(U_{nk})$$

separately and show that $S_n = o_P(1)$. Because, for fixed k, $U_{nk} = x + o_P(1)$ in the (most common) case where $X(t)$ has fixed initial value x, i.e., $\mathrm{P}(X(0) = x) = 1$, this will depend on the well-behavedness of the function f near x. More details can be found in Pötscher (2004).

Remark 5. An inspection of the proof of Theorem 2.1 (see Appendix A) shows that the assertion remains true if the following changes in the definition of local p-integrability and the assumptions are performed. Let $w : \mathbb{R}^r \to \mathbb{R}$ be a positive Borel function (a *weight function*).

- In (2.2), replace dx by $\mu(dx) = w(x)^p dx$. In other words, $f(x)w(x)$ is locally p-integrable.
- In Assumption 2, replace $\|\varphi_{nk}\|_q$ by $\|\varphi_{nk}\|_q^{(w)} = [\int \varphi_{nk}^q(x) w(x)^{-q}\, dx]^{1/q}$, with $\|\varphi_{nk}\|_\infty^{(w)} = \text{ess-sup}_x\, \varphi_{nk}(x) w(x)^{-1}$.

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- In Assumption 3 (formula (2.7a)), replace dx by $w(x)^{-q}\,dx$. In case $p = 1$, replace (2.7b) by $\sup_x w(x)^{-1} \int_0^1 \psi_t(x)\,dt < \infty$.

Introducing the weight function w allows a certain trade-off between the behavior of f and that of the densities. For example, poles of the densities may be admitted at the price of reducing the class of functions f for which (1.2) remains valid. This is in the spirit of Berkes and Horváth (2006), where local integrability of $|x|^{\alpha-1} f(x)$ (i.e., $w(x) = |x|^{\alpha-1}$) is required.

3. INTEGRATED PROCESSES

An important special case is provided by triangular arrays of the form

$$U_{nk} = \sum_{j=0}^{k-1} A_{nj} u_{k-j}, \tag{3.1}$$

where (u_s) is a linear process:

$$u_s = \sum_{\nu=0}^{\infty} \Gamma_\nu \varepsilon_{s-\nu}. \tag{3.2}$$

(For simplicity we assume initial value zero, the general case offering no additional difficulties.) The u_s and ε_s are of dimension d, so that A_{nj} and Γ_ν are $r \times d$ and $d \times d$ matrices, respectively. Typically, $r \le d$, which we shall assume henceforth. The ε_s are assumed i.i.d. with mean zero and positive definite covariance matrix Σ with minimal eigenvalue $\lambda > 0$. For $A_{nj} = n^{-\alpha} A$, (3.1) is an integrated (vector) process. The rate of convergence α depends on the kind of linear processes admitted in (3.2). In particular, we shall have to distinguish processes u with *short memory* (autoregressive moving average–type [ARMA-type]), where the coefficient matrices Γ_ν decline geometrically fast, and those with *long memory,* for which the Γ_ν tend to zero at polynomial speed. To be definite, our approach will be designed so as to cover the following *paradigms.*

(I) $A_{nj} = n^{-1/2} I_r$ (i.e., $r = d$), $\sum_{n=0}^{\infty} |\Gamma_n| < \infty$. This is the "classical" integrated process. For later reference,

$$n^{1/2} A_{nj} = I. \tag{3.3-I}$$

As is well known (cf. Johansen, 1995), the limit process is $X(t) = \Gamma(1)\Sigma^{1/2} B(t)$, where B is r-dimensional standard BM and $\Gamma(1) = \sum_{n=0}^{\infty} \Gamma_n$.

(II) The second paradigm is provided by Example 2 in the Introduction. The solution of (1.3) for $\Pi = \Pi_n$ with

$$\Pi_n = \alpha\beta' + \frac{1}{n}\alpha_1\beta_1' = \Pi^{(0)} + \frac{1}{n}\Pi^{(1)}$$

is given by

$$Y_k^n = \sum_{j=0}^{k-1} A_n^j \varepsilon_{k-j},$$

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where $A_n = I_d + \Pi_n$. Because $U_{nk} = n^{-1/2}\alpha'_\perp Y_{nk}$, we have $A_{nj} = n^{-1/2}\alpha'_\perp A_n^j$ and $u = \varepsilon$, i.e., $\Gamma_n = \delta_n I$ (unit impulse at $n = 0$). Note that $\alpha'_\perp \Pi^{(0)} = 0$. The dimension of A_{nj} and $\alpha'_\perp$ is $r \times d$, with $r = d - d_0 < d$. As already stated in the Introduction, the limit process X is an r-dimensional OU process.

In what follows, we shall need the following simple relation between these matrices:

$$n^{1/2}A_{nj} = \alpha'_\perp I + \frac{1}{n}\Delta_{nj}, \tag{3.3-II}$$

with residuals given by

$$\Delta_{nj} = \alpha'_\perp \Pi^{(1)} \sum_{i=0}^{j-1} (I + \Pi_n)^i . \tag{3.4}$$

This is easily proved by induction.

(III) This is (the multivariate version of) Example 2.7 in Berkes and Horváth (2006). In this case, $A_{nj} = n^{-H}A, 1/2 < H < 1$. Hence

$$n^H A_{nj} = A. \tag{3.3-III}$$

The linear process u is allowed to have long memory, i.e., $\Gamma_n \sim n^{-\beta}\Gamma$ with β and H related by $H = 3/2 - \beta, 1/2 < \beta < 1$ (we shall make this more precise subsequently). The limit process X is fractional BM.

In all three examples, the right-hand side of (3.2) converges a.s. and in L^2. Using (3.1) and (3.2), we express the U_{nk} in terms of the ε_j:

$$U_{nk} = \sum_{l=0}^{k-1} C_{nl}\varepsilon_{k-l} + \sum_{l=k}^{\infty} C_{nl}\varepsilon_{k-l} = U_{nk}^+ + U_{nk}^-. \tag{3.5}$$

Note that U_{nk}^+ and U_{nk}^- are independent. A simple calculation shows that

$$C_{nl} = \sum_{j=0}^{l\wedge(k-1)} A_{nj}\Gamma_{l-j}. \tag{3.6}$$

Let Φ_{nk} denote the characteristic function (c.f.) of U_{nk} and ϕ the c.f. of the ε_s. By virtue of (3.5),

$$|\Phi_{nk}(\upsilon)| = \left|\mathbf{E}\exp\left[i\upsilon' U_{nk}^+\right]\exp\left[i\upsilon' U_{nk}^-\right]\right| \le \left|\mathbf{E}\exp\left[i\upsilon' U_{nk}^+\right]\right| = \prod_{l=0}^{k-1}\left|\phi\left(C'_{nl}\upsilon\right)\right|. \tag{3.7}$$

Hence, if the C_{nl} are regular square matrices, integrability of $\phi(\upsilon)$ (i.e., $\int|\phi(\upsilon)|\,d\upsilon < \infty$) will imply integrability of the $\Phi_{nk}(\upsilon)$ (by means of the transformation formula for integrals) and, a forteriori (by the inversion formula), the existence

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of continuous bounded densities φ_{nk} for the U_{nk}. In view of our paradigmatic scenarios, however, we would like to admit nonsquare C_{nl} and also find weaker integrability conditions for ϕ. Actually, our objective is to find conditions on ϕ that will ensure Assumption 2:

$$\sup_n \frac{1}{n} \sum_{k=1}^{n} \|\varphi_{nk}\|_q < \infty. \tag{3.8}$$

Suppose that $\Phi_{nk} \in L_r^1$ (henceforth, L_n^p will denote the space of Borel functions h on $\mathbb{R}^n$ subject to $\int_{\mathbb{R}^n} |h(v)|^p \, dv < \infty$, with norm $\|\bullet\|_{L_n^p}$; the shorthand notation $\|\bullet\|_p$ will be reserved for $\|\bullet\|_{L_r^p}$ as in Section 2). Then, by the (multivariate) inversion formula,[3] (continuous and bounded) densities φ_{nk} do exist and are given by

$$\varphi_{nk}(x) = (2\pi)^{-r} \int_{\mathbb{R}^r} e^{-iv'x} \Phi_{nk}(v) \, dv.$$

As a consequence, denoting $m_{nk} = (2\pi)^{-r} \|\Phi_{nk}\|_1$,

$$\varphi_{nk}(x) \le (2\pi)^{-r} \int_{\mathbb{R}^r} |\Phi_{nk}(v)| \, dv = m_{nk},$$

$$\int_{\mathbb{R}^r} \varphi_{nk}(x)^q \, dx = \int_{\mathbb{R}^r} \varphi_{nk}(x)^{q-1} \varphi_{nk}(x) \, dx \le m_{nk}^{q-1}$$

for $1 < q < \infty$. Therefore,

$$\|\varphi_{nk}\|_q \le m_{nk}^{1/p} \tag{3.9}$$

(which is also valid for $p = 1, q = \infty$). Hence, a sufficient condition for (3.8) to hold is

$$\sup_n \frac{1}{n} \sum_{k=1}^{n} \|\Phi_{nk}\|_1^{1/p} < \infty. \tag{3.10}$$

In the discussion that follows, we shall present conditions on the c.f. ϕ of the ε_s together with conditions on the asymptotic behavior of the C_{nl} that will ensure (3.10), the link being provided by (3.9). Actually, the assumptions to be made allow us only to establish the following weaker form of (3.10):

$$\sup_{n \ge K} \frac{1}{n} \sum_{k=K}^{n} \|\Phi_{nk}\|_1^{1/p} < \infty \tag{3.10*}$$

for some integer K. Correspondingly, we can only conclude that Assumption 2* is satisfied, i.e.,

$$\sup_{n \ge K} \frac{1}{n} \sum_{k=K}^{n} \|\varphi_{nk}\|_q < \infty. \tag{2.6*}$$

Our first condition to establish (3.10*) is nothing but the multivariate version of condition (3.4) in Pötscher (2004).

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Assumption 4. ϕ is in $L_d^m(dv)$ for some nonnegative integer $m \geq 1$:

$$\int_{\mathbb{R}^d} |\phi(v)|^m \, dv < \infty.$$

Instead of an integer m, one may admit any *real* number $\mu \geq 1$, but—as the proof of Proposition 3.1 will show—if it is assumed that $\phi \in L_d^\mu(dv)$ with $\mu \in (m-1, m]$ and $m \in \mathbb{Z}$, then we may as well require the weaker condition $\phi \in L_d^m(dv)$.

As to the conditions on C_{nl}, we let ourselves be guided by the three examples in the Introduction. By virtue of (3.7), the C_{nl} need only be considered for $l \leq k-1$, where

$$C_{nl} = \sum_{j=0}^{l} A_{nj} \Gamma_{l-j}.$$

(I) $C_{nl} = n^{-1/2} \sum_{j=0}^{l} \Gamma_j$. Putting $C = \Gamma(1) = \sum_{j=0}^{\infty} \Gamma_j$, this means that

$$n^{1/2} C_{nl} = C + R_{nl} \tag{3.11-I}$$

with $|R_{nl}| \leq \sum_{j=l+1}^{\infty} |\Gamma_j|$. In the one-dimensional case, it is required that $\Gamma(1) \neq 0$. The corresponding condition in the multivariate setting will be regularity of C.

(II) By (3.4), for white noise input $u = \varepsilon$ (as considered in Johansen, 1995), $\Gamma_n = \delta_n I$ and $n^{1/2} A_{nj} = \alpha_\perp' I + (1/n)\Delta_{nj}$, so that

$$n^{1/2} C_{nl} = n^{1/2} A_{nl} = \alpha_\perp' I + \frac{1}{n} \Delta_{nl} = C + R_{nl} \tag{3.11-II}$$

with

$$C = \alpha_\perp', \qquad R_{nl} = \frac{1}{n} \Delta_{nl} = \frac{1}{n} \alpha_\perp' C_1 \sum_{i=0}^{l-1} (I + \Pi_n)^i .$$

By construction (cf. Johansen, 1995), $\alpha_\perp' \alpha_\perp$ has (full) rank r. In general, $|I + \Pi_n| \leq c$ for some constant $c > 1$, and hence $|R_{nl}| \leq n^{-1} L c^l$ uniformly in n for some constant L. In Johansen (1995) it is assumed that all eigenvalues of the matrix $A = I + \Pi^{(0)}$ have modulus less than one, so that the same will be true for A_n for n large enough. Still, this is not sufficient for $c \leq 1$, and so, without further information, it may not be concluded that $R_{nl} \to 0$. If, however, it can be shown in a special setting that $c \leq 1$, then Δ_{nl} is uniformly bounded and $n^{1/2} C_{nl} \to C$.

(III) $\Gamma_n \sim n^{-\beta} \Gamma$, such that $\sum_{j=0}^{n} \Gamma_j \sim n^{1-\beta}$ (i.e., $n^{\beta-1} \sum_{j=0}^{n} \Gamma_j \to \Gamma$). Then (cf. (3.3-III))

$$n^H C_{nl} = \sum_{j=0}^{l} n^H A_{nj} \Gamma_{l-j} = A \sum_{j=0}^{l} \Gamma_j,$$

$$n^H l^{\beta-1} C_{nl} = C + R_l, \qquad C = A\Gamma, \qquad R_l = o(1). \tag{3.11-III}$$

Here C should have full rank r.

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In view of the relations (3.3), we shall make the following assumption.

Assumption 5. There exist numbers $1/2 \le \alpha < 1$, $-1/2 < \gamma \le 0$, such that

$$\underline{\lim}_{n,l\to\infty} n^{2\alpha} l^{2\gamma} \lambda_{\min}\left(C_{nl} C'_{nl}\right) \ge \delta > 0,$$

$$\overline{\lim}_{n,l\to\infty} n^{2\alpha} l^{2\gamma} \lambda_{\max}\left(C_{nl} C'_{nl}\right) \le M < \infty.$$

Here $\lambda_{\min}$ ($\lambda_{\max}$) denotes the smallest (largest) eigenvalue. The $\underline{\lim}$ is to be understood for $l \le n$; i.e., the precise meaning of the assumption is as follows: There exists a positive number δ such that for all $0 < \kappa < 1$ an integer $K = K(\kappa)$ may be found such that $n^{2\alpha} l^{2\gamma} \lambda_{\min}\left(C_{nl} C'_{nl}\right) \ge \kappa\delta$ for all n, l satisfying $n \ge l \ge K$. Similarly for the $\overline{\lim}$ (with $\kappa > 1$).

Finally, we need one more assumption about the c.f. ϕ that allows us to control the behavior of $\phi(n^\alpha l^\gamma C'_{nl} u)$.

Assumption 6. $|\phi(v)| < 1$ for $v \ne 0$. If $r < d$, there exists a measurable function $\rho : \mathbb{R}^+ \to [0, 1]$ such that $\rho(0) = 1$, $\rho(s) < 1$ for $s > 0$, $\int_0^\infty \rho(s)\, ds < \infty$, and $|\phi(v)| \le \rho(|v|)$ for all $v \in \mathbb{R}^d$.

In the one-dimensional case, Assumption 6 is equivalent to requiring that the ε_j do not have a lattice distribution.

PROPOSITION 3.1. *For U_{nk} given by (3.1) and (3.2), Assumptions 4–6 are sufficient for the U_{nk} to have densities φ_{nk} satisfying (3.10*) provided the parameters r, p, α, γ satisfy the following conditions:*

$$\begin{aligned} &\alpha \le |\gamma| + \frac{1}{2} \quad \text{if } p > r\left(|\gamma| + \frac{1}{2}\right), \\ &\alpha < |\gamma| + \frac{1}{2} \quad \text{if } p = r\left(|\gamma| + \frac{1}{2}\right), \\ &r\alpha \le p \quad \text{if } p < r\left(|\gamma| + \frac{1}{2}\right). \end{aligned} \tag{3.12}$$

Note that $\alpha > |\gamma| + 1/2$ is not a feasible constellation because also the third case implies that $\alpha < |\gamma| + 1/2$. For $r = 1$, because $1/2 \le |\gamma| + 1/2 < 1$, the first case applies with any $p \ge 1$. In all three of the preceding examples, the condition $\alpha \le |\gamma| + 1/2$ is satisfied ($\alpha = 1/2, \gamma = 0$ in I and II; in III, $\alpha = H = 3/2 - \beta \in (1/2, 1), \gamma = \beta - 1 = 1/2 - H$, and so $\alpha + \gamma = 1/2$).

Usually, r, α, and γ will be given, and the problem is to find the smallest (or the infimum of those) p such that (3.12) is satisfied, corresponding to the largest class of locally p-integrable functions. For this purpose, we have to read (3.12) backward. For $r \ge 2$, the infimum is $p_0 = r\alpha$ (a number ≥ 1). If $\alpha < |\gamma| + 1/2$, then $p = p_0$ is a feasible choice. If $\alpha = |\gamma| + 1/2$, then $p_0 = r(|\gamma| + 1/2)$. This

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constellation is not feasible (middle case in (3.12)), and so in this case every $p > p_0$ will do but not p_0 itself.

NOTES

1. As pointed out by a referee, the class of "regular" functions in Park and Phillips (1999) actually coincides with the class of locally Riemann integrable functions.

2. Though, strictly speaking, this is not correct, we shall often use notations such as $U_{n,[nt]} \Rightarrow X(t)$ or $X_n(t) \Rightarrow X(t)$ to indicate weak convergence in $D[0,1]$ (instead of replacing the time index t by a dot). The use of the arrow $\Rightarrow$ should make it clear that *functional* weak convergence is meant and not just convergence in distribution of the random variables $X_n(t)$ for fixed t, etc., for which we shall reserve the symbol $\xrightarrow{d}$.

3. I have not found a ready to use reference for the multivariate inversion formula for L^1 c.f.s. Starting from the general inversion formula (cf. Kawata, 1972, Thm. 8.9.1) one may, however, proceed as in the one-dimensional case to derive the formula in Section 3 (cf. Kawata, 1972, proof of Thm. 4.4.2; Shiryaev, 1996, Sect. 12, proof of Thm. 3b).

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APPENDIX A: Proof of Theorem 2.1

LEMMA A.1. *Let f be locally p-integrable. Then, for each $\varepsilon > 0$, there exists a continuous function f_ε such that*

$$\int |f(x) - f_\varepsilon(x)|^p \, dx < \varepsilon.$$

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Proof. For $N = 1, 2, \ldots$, consider the open annulus $A_N = \{x \in \mathbb{R}^r : N - 1 < |x| < N\}$. Let f_N denote the restriction of f to A_N (i.e., $f_N : A_N \to \mathbb{R}$ is given by $f_N(x) = f(x)$). By standard approximation results (cf. Hewitt and Stromberg, 1969, Thm. 13.21), there exist continuous functions $f_{N,\varepsilon} : A_N \to \mathbb{R}$ with compact support (in A_N) such that

$$\int_{A_N} \left| f_N(x) - f_{N,\varepsilon}(x) \right|^p dx < \varepsilon 2^{-N}.$$

Define $f_\varepsilon : \mathbb{R}^m \to \mathbb{R}$ by

$$f_\varepsilon(x) = \sum_{N=1}^{\infty} f_{N,\varepsilon}(x) 1_{S_N}(x)$$

(i.e., in particular $f_\varepsilon(x) = 0$ near the boundary of each annulus). Then f_ε is continuous and

$$\int |f(x) - f_\varepsilon(x)|^p dx = \sum_{N=1}^{\infty} \int_{A_N} \left| f_N(x) - f_{N,\varepsilon}(x) \right|^p dx < \varepsilon.$$

■

Introduce

$$Z_n = n^{-1} \sum_{k=1}^{n} f(U_{nk}), \qquad Z_{n,\varepsilon} = n^{-1} \sum_{k=1}^{n} f_\varepsilon(U_{nk}),$$

$$Z_\varepsilon = \int_0^1 f_\varepsilon(X(t))\, dt, \qquad Z = \int_0^1 f(X(t))\, dt.$$

LEMMA A.2. *Under Assumption 2,*

$$\sup_n \mathrm{E} \left| Z_n - Z_{n,\varepsilon} \right| \le C_1 \varepsilon^{1/p}.$$

Proof.
$$\mathrm{E} \left| Z_n - Z_{n,\varepsilon} \right| = \mathrm{E} \left| n^{-1} \sum_{k=1}^{n} f(U_{nk}) - n^{-1} \sum_{k=1}^{n} f_\varepsilon(U_{nk}) \right|$$

$$\le n^{-1} \sum_{k=1}^{n} \mathrm{E} |f(U_{nk}) - f_\varepsilon(U_{nk})|$$

$$= n^{-1} \sum_{k=1}^{n} \int |f(x) - f_\varepsilon(x)| \varphi_{nk}(x)\, dx. \tag{A.1}$$

By Lemma A.1, the right-hand side in (A.1) may be estimated for $p = 1$ by

$$n^{-1} \sum_{k=1}^{n} \|\varphi_{nk}\|_\infty \int |f(x) - f_\varepsilon(x)|\, dx \le \varepsilon n^{-1} \sum_{k=1}^{n} \|\varphi_{nk}\|_\infty \le C_1 \varepsilon$$

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and for $p > 1$ by

$$n^{-1}\sum_{k=1}^{n}\left[\int |f(x)-f_\varepsilon(x)|^p\,dx\right]^{1/p}\left[\int \varphi_{nk}^q(x)\,dx\right]^{1/q}$$
$$\le \varepsilon^{1/p} n^{-1}\sum_{k=1}^{n}\|\varphi_{nk}\|_q$$
$$\le C_1\varepsilon^{1/p}$$

uniformly in n. ■

LEMMA A.3. *Let $h:\mathbb{R}^r\to\mathbb{R}$ be a continuous function and P a probability measure on $D_r[0,1]$ such that $P(C_r[0,1])=1$ (where $C_r[0,1]$ denotes the subset of continuous functions). Then the mapping $H: D_r[0,1]\to\mathbb{R}$ defined by*

$$H(x)=\int_0^1 h(x(t))\,dt \tag{A.2}$$

is measurable and continuous with respect to the Skorokhod topology except on a set of P measure zero.

Proof. Note first that the integral in (A.2) is well defined and finite because the paths $x(t), 0\le t\le 1$, are bounded in $\mathbb{R}^r$ and h is bounded on bounded sets. As to measurability, this follows from the argument in Billingsley (1968, App. II [Measurability of Some Mappings]) if h is bounded. But every continuous function $h(x)$ may be approximated by (continuous) bounded functions $h_N(x)$ such that

$$H_N(x)=\int_0^1 h_N(x(t))\,dt\to\int_0^1 h(x(t))\,dt=H(x).$$

Hence H is measurable as limit of a sequence of measurable functions. As to almost sure continuity, we shall show that the set of continuity points of H contains $C_r[0,1]$. To this end, consider any $x\in C_r[0,1]$ and a sequence $x_n\in D_r[0,1]$ such that $x_n\to x$ in the (product) Skorokhod topology. Because x is continuous, this implies uniform convergence $\|x_n-x\|\to 0$. As a consequence, all paths $x_n(t), 0\le t\le 1, n=1,2,\ldots$, lie in some bounded set B. Hence, by bounded convergence (because $|h(x_n(t))|\le \sup_{z\in B}|h(z)|$), $\int_0^1 h(x_n(t))\,dt\to\int_0^1 h(x(t))\,dt$. ■

COROLLARY A.1. *Under Assumption 1, for every $\varepsilon>0$,*

$$Z_{n,\varepsilon}\xrightarrow{d} Z_\varepsilon. \tag{A.3}$$

Proof. By (2.1), (A.3) is the same as

$$\int_0^1 f_\varepsilon(X_n(t))\,dt\xrightarrow{d}\int_0^1 f_\varepsilon(X(t))\,dt.$$

By Assumption 1, the induced measure $P=\mathrm{P}X^{-1}$ has the property that $P(C_r[0,1])=1$. The assertion follows then from Lemma A.3 and the CMT (in its extended form). ■

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Though we shall not need it, let us note in passing that the assertion of Lemma A.3 remains valid if h is a Borel function that is bounded on bounded sets and continuous except on a set of (r-dimensional) Lebesgue measure zero and if—in addition—the one-dimensional distributions $P\pi_t^{-1}$ are absolutely continuous with respect to Lebesgue measure for almost all t.

LEMMA A.4. *Under Assumption 3,*

$$\mathrm{E}\,|Z_\varepsilon - Z| \le \varepsilon^{1/p} C_2.$$

Proof. By Lemma A.1, for $p > 1$,

$$\begin{aligned}
\mathrm{E}\,|Z_\varepsilon - Z| &= \mathrm{E}\left| \int_0^1 f_\varepsilon(X(t))\,dt - \int_0^1 f(X(t))\,dt \right| \\
&\le \int_0^1 \mathrm{E}\,|f_\varepsilon(X(t)) - f(X(t))|\,dt \\
&= \int_0^1 \int |f_\varepsilon(x) - f(x)|\,\psi_t(x)\,dx\,dt \\
&\le \int_0^1 \left[\int |f_\varepsilon(x) - f(x)|^p\,dx \right]^{1/p} \left[\int \psi_t^q(x)\,dx \right]^{1/q} dt \\
&\le \varepsilon^{1/p} C_2.
\end{aligned}$$

For $p = 1$,

$$\begin{aligned}
\mathrm{E}\,|Z_\varepsilon - Z| &\le \int_0^1 \int |f_\varepsilon(x) - f(x)|\,\psi_t(x)\,dx\,dt \\
&= \int |f_\varepsilon(x) - f(x)| \left[\int_0^1 \psi_t(x)\,dt \right] dx \\
&\le \varepsilon C_2.
\end{aligned}$$

Proof of Theorem 2.1. Let $\varphi, \varphi_\varepsilon, \varphi_n, \varphi_{n,\varepsilon}$ denote the characteristic functions of $Z, Z_\varepsilon, Z_n, Z_{n,\varepsilon}$, respectively. We have to show that

$$\lim_{n\to\infty} \varphi_n(t) = \varphi(t) \tag{A.4}$$

for all t. Henceforth, fix t.

Step 1. By Lemma A.2, making use of the fact that $|1 - e^{ih}| \le |h|$,

$$\begin{aligned}
\left|\varphi_n(t) - \varphi_{n,\varepsilon}(t)\right| &= \left| \mathrm{E}e^{itZ_n} - \mathrm{E}e^{itZ_{n,\varepsilon}} \right| \\
&\le \mathrm{E}\left| e^{itZ_n} \left(1 - e^{it(Z_{n,\varepsilon} - Z_n)} \right) \right| \\
&\le \mathrm{E}\left| 1 - e^{it(Z_{n,\varepsilon} - Z_n)} \right| \\
&\le |t|\,\mathrm{E}\left| Z_{n,\varepsilon} - Z_n \right| \\
&\le |t|\,C_1 \varepsilon^{1/p}
\end{aligned} \tag{A.5}$$

uniformly in n.

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Step 2. By the same calculation, using Lemma A.4,

$$|\varphi(t)-\varphi_\varepsilon(t)| \le |t|\, C_2 \varepsilon^{1/p}. \tag{A.6}$$

Step 3. By Corollary A.1, for each fixed $\eta > 0$ and $\varepsilon > 0$,

$$\left|\varphi_{n,\varepsilon}(t)-\varphi_\varepsilon(t)\right| < \eta \tag{A.7}$$

for n large enough, say, $n \ge n_0(\eta,\varepsilon)$.

Putting (A.5)–(A.7) together, fix any $\eta > 0$ and chose $\varepsilon > 0$ so small that (still for fixed t)

$$|t|\,(C_1+C_2)\,\varepsilon^{1/p} \le 2\eta.$$

Then, for $n \ge n_0(\eta,\varepsilon)$,

$$|\varphi_n(t)-\varphi(t)| < 3\eta.$$

This shows (A.4). ■

Proof of Theorem 2.1*. Modify the definitions of Z_n and $Z_{n,\varepsilon}$ by summing from index K onward. Then Lemma A.2 remains valid, and only the proof of Corollary A.1 has to be modified insofar as now

$$Z_{n,\varepsilon} = \int_0^1 f_\varepsilon(X_n(t))\,dt + R_{n,\varepsilon}\frac{1}{n}\sum_{n=1}^{K} f_\varepsilon(U_{nk})$$

with $R_{n,\varepsilon} = n^{-1}\sum_{n=1}^{K-1} f_\varepsilon(U_{nk})$. But because $U_{nk} = O_P(1)$ for fixed k, $R_{n,\varepsilon} = o_P(1)$ for every fixed $\varepsilon > 0$. ■

Proof of Remark 5. Lemma A.1 remains valid with $\mu(dx)$ in place of dx. In the proof of Lemma A.2, substitute $1 = w(x)w(x)^{-1}$ under the integral in (A.1) and use the Hölder estimate (for $p > 1$)

$$\begin{aligned}
\text{the right-hand side of (A.1)} &\le n^{-1}\sum_{k=1}^{n}\left[\int |f(x)-f_\varepsilon(x)|^p\,\mu(dx)\right]^{1/p} \\
&\quad\times\left[\int \varphi_{nk}^q(x)w(x)^{-q}\,dx\right]^{1/q} \\
&\le \varepsilon^{1/p} n^{-1}\sum_{k=1}^{n}\|\varphi_{nk}\|_q^{(w)} \\
&\le C_1\varepsilon^{1/p}.
\end{aligned}$$

The same procedure applied to the proof of Lemma A.4 yields

$$\begin{aligned}
\mathrm{E}|Z_\varepsilon - Z| &\le \int_0^1\left[\int |f_\varepsilon(x)-f(x)|^p\,\mu(dx)\right]^{1/p} \\
&\quad\times\left[\int \psi_t^q(x)w(x)^{-q}\,dx\right]^{1/q} dt \\
&\le \varepsilon^{1/p} C_2.
\end{aligned}$$

The rest is as in the proof of Theorem 2.1. The proof for $p = 1$ is obvious. ■

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APPENDIX B: Proof of Proposition 3.1

LEMMA B.1. *Let X be an r-dimensional random vector with c.f. ϕ, $\mathrm{E}X = 0$, $\mathrm{E}|X|^2 < \infty$, $\Sigma = \mathrm{E}XX'$. Let λ denote the minimal eigenvalue of Σ. Then there exists $\Delta \in (0,1)$ such that*

$$|\phi(t)| \le \exp\left[-\frac{\lambda}{8}|t|^2\right]$$

for $t \in R^r$ subject to $|t| \le \Delta$.

Proof. Expansion of the c.f. around zero yields

$$e^{iz} = 1 + iz - \frac{z^2}{2} - \frac{z^2}{2}\left(e^{i\tau z} - 1\right)$$

with $0 < \tau = \tau(z) < 1$. Hence, inserting $z = t'X$ and taking expectations,

$$\begin{aligned}\phi(t) = \mathrm{E}e^{it'X} &= 1 - \frac{t'\Sigma t}{2} - \frac{1}{2}\mathrm{E}\left\{(t'X)^2\left(e^{i\tau t'X} - 1\right)\right\} \\ &= 1 - \frac{t'\Sigma t}{2} + R(t),\end{aligned}$$

$$\begin{aligned}|R(t)| &\le \frac{1}{2}\mathrm{E}(t'X)^2\left|e^{i\tau t'X} - 1\right| \\ &\le \frac{1}{2}|t|^2\mathrm{E}|X|^2\left|e^{i\tau t'X} - 1\right| \\ &= |t|^2 o(1) \quad \text{as } t \to 0\end{aligned}$$

(the latter by bounded convergence). Therefore

$$\phi(t) = 1 - \frac{t'\Sigma t}{2} + R(t) \quad \text{with } R(0) = 0, \qquad R(t) = o(|t|^2) \quad \text{as } t \to 0.$$

Now proceed as in Pötscher (2004, proof of Lem. B.1). For the reader's convenience, we reproduce his arguments here. There exists $0 < \Delta' < 1$ such that, for $|t| \le \Delta'$, it holds that $|R(t)| \le \lambda|t|^2/4 \le 1/2$ as well as $t'\Sigma t/2 < 1$. As a consequence, $|\phi(t)| \le |1 - t'\Sigma t/2| + |R(t)| \le 1 - \lambda|t|^2/2 + \lambda|t|^2/4 = 1 - \lambda|t|^2/4$. Because $\phi(0) = 1$ and $\phi(t)$ is continuous, we may choose $0 < \Delta'' < \Delta'$ such that $|\phi(t)| > 0$ for $|t| < \Delta''$. Taking logarithms,

$$\log|\phi(t)| \le \log\left(1 - \frac{1}{4}\lambda|t|^2\right) = -\frac{1}{4}\lambda|t|^2 + \delta(t) \quad \text{for } |t| \le \Delta'',$$

with $\delta(0) = 0$ and $\delta(t) = o(|t|^2)$ as $t \to 0$. In particular, we may find $0 < \Delta \le \Delta''$ such that $|\delta(t)| \le \lambda|t|^2/8$ for $|t| \le \Delta$ and hence

$$\log|\phi(t)| \le -\frac{1}{8}\lambda|t|^2. \qquad \blacksquare$$

Proof of Proposition 3.1. The proof is an adaptation of the proof of Lemma 3.1 in Pötscher (2004) to the multidimensional case. Our starting point is (3.7):

$$|\Phi_{nk}(v)| \le \prod_{l=0}^{k-1}\left|\phi\left(C_{nl}'v\right)\right|.$$

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For every choice of positive numbers R_{nk},

$$\int_{\mathbb{R}^r} |\Phi_{nk}(v)|\,dv = \int_{|v|\le R_{nk}} \prod_{l=0}^{k-1} \left|\phi\left(C'_{nl}v\right)\right| dv + \int_{|v|> R_{nk}} \prod_{l=0}^{k-1} \left|\phi\left(C'_{nl}v\right)\right| dv$$

$$= I_{nk} + J_{nk}. \tag{B.1}$$

Concerning I_{nk}. Let λ denote the minimal eigenvalue of the covariance matrix Σ of the ε_s and let $\Delta \in (0,1)$ be as in Lemma B.1:

$$|\phi(u)| \le \exp\left[-\frac{\lambda}{8}|u|^2\right] \tag{B.2}$$

for $|v| \le \Delta$. By Assumption 5, for n, l large enough, say, $n, l \ge K'$ with $K' = K(\frac{1}{2})$,

$$n^{2\alpha} l^{2\gamma}\, \lambda_{\min}\left(C_{nl}C'_{nl}\right) \ge \frac{\delta}{2}. \tag{B.3}$$

Also, $|C'_{nl}v|^2 \le \lambda_{\max}(C_{nl}C'_{nl})|v|^2 \le 2Mn^{-2\alpha}l^{-2\gamma}\ |v|^2$ for $n, l \ge K'' = K(2)$, and so $|v| \le n^{\alpha}k^{\gamma}\,c\Delta$ implies

$$\left|C'_{nl}v\right|^2 \le 2M(k/l)^{2\gamma}\,c^2\Delta^2 \le \Delta^2,$$

if we put $c = (2M)^{-1/2}$ (remember that $\gamma \le 0$, and so $(k/l)^{2\gamma} \le 1$ for $l \le k$). Henceforth, let $K = K' \vee K''$ and put $R_{nk} = n^{\alpha}k^{\gamma}\,c\Delta$. It then follows from the definition of I_{nk} (cf. (B.1)) together with (B.2) and $|\phi| \le 1$ that

$$I_{nk} \le \int_{|v|\le R_{nk}} \prod_{l=K}^{k-1} \left|\phi\left(C'_{nl}v\right)\right| dv \le \int_{|v|\le R_{nk}} \prod_{l=K}^{k-1} \exp\left[-\frac{\lambda}{8}\left|C'_{nl}v\right|^2\right] dv \tag{B.4}$$

for $n, k \ge K$. Because, by (B.3), $\left|C'_{nl}v\right|^2 \ge \lambda_{\min}\left(C_{nl}C'_{nl}\right)|v|^2 \ge (\delta/2)n^{-2\alpha}l^{-2\gamma}$, the right-hand side of (B.4) may be estimated by

$$\int_{|v|\le R_{nk}} \prod_{l=K}^{k-1} \exp\left[-\frac{\lambda\delta}{16}n^{-2\alpha}l^{-2\gamma}\,|v|^2\right] dv$$

$$= \int_{|v|\le R_{nk}} \exp\left[-c_1 n^{-2\alpha} \sum_{l=K}^{k-1} l^{-2\gamma}\,|v|^2\right] dv$$

(new constants c_i will be introduced without further mention). Because $\sum_{l=K}^{k-1} l^{-2\gamma} \ge c_2 k^{1+2|\gamma|}$ for $k \ge K+1$,

$$I_{nk} \le \int_{|v|\le R_{nk}} \exp\left[-c_3 n^{-2\alpha} k^{1+2|\gamma|}\,|v|^2\right] dv. \tag{B.5}$$

Now make the substitution $n^{-\alpha}k^{(1+2|\gamma|)/2}v = u$. Remember that $R_{nk} = n^{\alpha}k^{\gamma}\,c\Delta$ and note that $|v| \le R_{nk} \Leftrightarrow |u| \le k^{1/2}c\Delta$. Then, putting $\theta = (1+2|\gamma|)/2$ for a while,

$$\text{the right-hand side of (B.5)} = n^{r\alpha}k^{-r\theta}\int_{|u|\le k^{1/2}c\Delta} \exp\left[-c_3\,|u|^2\right] du \le c_4 n^{r\alpha}k^{-r\theta}$$

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for $n, k \geq K$, and hence

$$I_{nk}^{1/p} \leq c_5 n^{r\alpha/p} k^{-r\theta/p}. \tag{B.6}$$

Using the fact that

$$\sum_{k=1}^{n} k^{-\eta} \leq \begin{cases} c' n^{1-\eta}, & 0 < \eta < 1, \\ c' \log n, & \eta = 1,\ n \geq 2, \\ c', & \eta > 1, \end{cases}$$

for some constant c', it follows from (B.6) (with $\eta = r\theta/p = r\,(1+2\,|\gamma\,|)\,/2p$) that

$$\begin{aligned} \frac{1}{n} \sum_{k=K}^{n} I_{nk}^{1/p} &\leq c_5 n^{r\alpha/p-1} \sum_{k=1}^{n} k^{-r\delta/p} \\ &\leq \begin{cases} c_6 n^{r\alpha/p-1} n^{1-r(1+2|\gamma\,|)/2p}, & p > r\,(|\gamma\,|+1/2), \\ c_6 n^{r\alpha/p-1} \log n, & p = r\,(|\gamma\,|+1/2), \\ c_6 n^{r\alpha/p-1}, & p < r\,(|\gamma\,|+1/2). \end{cases} \end{aligned} \tag{B.7}$$

The exponent ν of n in the first case is

$$\nu = \frac{r\alpha}{p} - \frac{r\,(1+2\,|\gamma\,|)}{2p} = \frac{r}{p}\left(\alpha - |\gamma\,| - \frac{1}{2}\right).$$

Therefore, for n^{ν} to be bounded, it must hold that $\nu \leq 0$, i.e., $\alpha \leq |\gamma\,| + 1/2$. In the third case, for n^{ν} to be bounded, p must satisfy $r\alpha \leq p$. The second case, $p = r\,(|\gamma\,| + 1/2)$, leads to boundedness if $p > r\alpha$, i.e., $\alpha < |\gamma\,| + 1/2$. All three cases may occur.

Concerning J_{nk}. Going back to (B.1), apply a k-fold Hölder estimate to obtain

$$J_{nk} \leq \int_{|v|>R_{nk}} \prod_{l=K}^{k-1} \left|\phi\left(C_{nl}' v\right)\right| dv \leq \prod_{l=K}^{k-1} \left(\int_{|v|>R_{nk}} \left|\phi\left(C_{nl}' v\right)\right|^{k-K} dv\right)^{1/(k-K)} \tag{B.8}$$

with K as before (cf. (B.3)).

Case (a): $r = d$. For $n, k \geq K$, $n^{\alpha} l^{\gamma}\, C_{nl}$ is regular by Assumption 5. Hence, making the substitution $u = C_{nl}' v$,

$$\int_{|v|>R_{nk}} \left|\phi\left(C_{nl}' v\right)\right|^{k-K} dv = |\det C_{nl}|^{-1} \int_{\left|C_{nl}'^{-1} u\right|>R_{nk}} |\phi\,(u)|^{k-K}\, du. \tag{B.9}$$

By virtue of (B.3),

$$\begin{aligned} |\det C_{nl}| &= \det\left(C_{nl}\, C_{nl}'\right)^{1/2} \\ &\geq \lambda_{\min}\left(C_{nl}\, C_{nl}'\right)^{r/2} \\ &\geq \left(\frac{\delta}{2}\right)^{r/2} n^{-r\alpha} l^{-r\gamma} \end{aligned}$$

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and

$$\left|C_{nl}'^{-1}u\right| = \left[u'\left(C_{nl}'\,C_{nl}\right)^{-1}u\right]^{1/2} \le \lambda_{\max}^{1/2}\left(\left(C_{nl}'\,C_{nl}\right)^{-1}\right)|u|$$

$$= \lambda_{\min}^{-1/2}\left(C_{nl}'\,C_{nl}\right)|u| \le \left(\frac{\delta}{2}\right)^{-1/2} n^{\alpha}l^{\gamma}\,|u|.$$

As a consequence, remembering that $R_{nk} = n^{\alpha}k^{\gamma}\,c\Delta$, it follows from (B.9) that

$$\int_{|v|>R_{nk}} \left|\phi\left(C_{nl}'v\right)\right|^{k-K} dv \le c_1 n^{r\alpha}l^{r\gamma}\int_{|u|>c_2(k/l)^{\gamma}} |\phi(u)|^{k-K}\,du$$

$$\le c_1 n^{r\alpha}K^{r\gamma}\int_{|u|>c_2(k/K)^{\gamma}} |\phi(u)|^{k-K}\,du,$$

where we have made use of the fact that $l^{\gamma} \le K^{\gamma}$ (remember that $\gamma \le 0$) and start the numeration of constants c_i afresh. Hence, from (B.8),

$$J_{nk} \le c_1 n^{r\alpha}K^{r\gamma}\int_{|u|>c_2(k/K)^{\gamma}} |\phi(u)|^{k-K}\,du. \tag{B.10}$$

Put $\phi_k = \sup\left\{|\phi(u)| : |u| > c_2(k/K)^{\gamma}\right\}$. Then, for $k \ge K+m$,

$$J_{nk} \le c_1 n^{r\alpha}K^{r\gamma}\phi_k^{k-K-m}\int_{\mathbb{R}^r} |\phi(u)|^{m}\,du. \tag{B.11}$$

(In this context, let us remark that nothing is gained by admitting any real number $\mu \ge 1$ instead of an integer m in Assumption 4. For if $m-1 < \mu \le m$, then (B.11) would still hold for $k \ge K+m$, with exponent μ under the integral. But then we may from the start require the weaker condition with $\mu = m$.) By Assumption 6, because $c_2 > 0$, it holds that $\phi_k \le \phi_n < 1$ for $k \le n, \phi_n^{-(K+m)} \le \phi_1^{-(K+m)}$ and hence

$$J_{nk} \le c_3 n^{r\alpha}\phi_n^{k-K-m} \le c_4 n^{r\alpha}\phi_n^{k},$$

$$\frac{1}{n}\sum_{k=K+m}^{n} J_{nk}^{1/p} \le c_5 n^{r\alpha/p-1}\left(1-\phi_n^{n}\right) \le c_6 n^{r\alpha/p-1}. \tag{B.12}$$

Boundedness of $n^{-1}\sum_{k=K+m}^{n} J_{nk}^{1/p}$ is thus equivalent to the condition $r\alpha \le p$.

Case (b): $r < d$. Going back to (B.8) and making use of Assumption 6, we estimate the integral on the right-hand side by

$$\int_{|v|>R_{nk}} \left|\phi\left(C_{nl}'v\right)\right|^{k-K} dv \le \int_{|v|>R_{nk}} \rho\left(\left|C_{nl}'v\right|\right)^{k-K} dv.$$

Making the substitution $v = n^{\alpha}l^{\gamma}\,u$ and remembering that $R_{nk} = n^{\alpha}k^{\gamma}\,c\Delta$, the right-hand side may be further estimated by

$$n^{r\alpha}l^{r\gamma}\int_{|u|>(k/l)^{\gamma}\,c\Delta} \rho\left(\left|n^{\alpha}l^{\gamma}\,C_{nl}'u\right|\right)^{k-K} du. \tag{B.13}$$

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For $n, k \geq K$, $\left|n^{\alpha} l^{\gamma} C'_{nl} u\right| = n^{\alpha} l^{\gamma} \left(u' C_{nl} C'_{nl} u\right)^{1/2} \geq n^{\alpha} l^{\gamma} \lambda_{\min}^{1/2} \left(C_{nl} C'_{nl}\right) |u| \geq \sqrt{\delta/2}$ (by (B.3)). Hence, in view of the properties of the function ρ required in Assumption 6 and because $l^{\gamma} \leq K^{\gamma}$, (B.13) is dominated by

$$n^{r\alpha} K^{r\gamma} \int_{|u| > (k/K)^{\gamma} c\Delta} \rho\left(c_7 |u|\right)^{k-K} du.$$

But this is of the same form as the expression (B.10) and may be treated in the same way.

Note that in both cases the treatment of the second term J_{nk} in (B.1) contributes only the condition $r\alpha \leq p$, which is already contained in (B.7). (In the first two cases, it is automatically fulfilled because $p \geq r\left(|\gamma| + 1/2\right) \geq r\alpha$.) Taking the estimates (B.7) and (B.13) together, we find that (3.10*) is satisfied with K replaced by $K + m$. ■

APPENDIX C: Proof of (2.7) for Various Special Cases

C.1. Proof of (2.7a) for Multivariate Gaussian Processes. Let $X(t) \sim \mathcal{N}(m_t, \Sigma_t)$ with covariance Σ_t satisfying the following condition.

Assumption C1. For $p > 1$, the minimal eigenvalue σ_t^2 of Σ_t satisfies

$$\int_0^1 \sigma_t^{-r/p} dt < \infty.$$

Then, if $\nu_r(x; m, \Sigma)$ denotes the r-dimensional normal density with mean m and covariance Σ,

$$\psi_t(x)^q = (2\pi)^{r(1-q)/2} q^{-r/2} \left[\det \Sigma_t\right]^{(1-q)/2} \nu_r(x; m_t, q^{-1} \Sigma_t),$$

$$\begin{aligned} \left(\int \psi_t(x)^q dx\right)^{1/q} &= (2\pi)^{-r/2p} q^{-r/2q} \left[\det \Sigma_t\right]^{-1/2p} \\ &\leq \text{const.} \times \sigma_t^{-r/p}. \end{aligned}$$

The inequality follows from $\det \Sigma_t \geq \sigma_t^{2r}$.

Assumption (C1) is fulfilled for r-dimensional BM provided $2p > r$. For the OU process considered in Example 2 in the Introduction, the covariance matrix is

$$\Sigma_t = \alpha'_{\perp} \int_0^t e^{(A+A')u} du \alpha_{\perp}.$$

Consider the diagonalization $A + A' = Q \Lambda Q'$, where Q is an orthogonal matrix and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_d)$ is the diagonal martix of eigenvalues of $A + A'$. Then $e^{(A+A')u} = Q e^{\Lambda u} Q'$, and so

$$\int_0^t e^{(A+A')u} du = Q \int_0^t e^{\Lambda u} du Q' = Q M_t Q'.$$

The entries of the matrix $M_t = \int_0^t e^{\Lambda u} du$ are of the form $\lambda^{-1} \left(e^{\lambda t} - 1\right)$ for nonzero eigenvalue λ of Λ and t for zero eigenvalues. Typically, in the setting considered in Example

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2, A is degenerate, and so zero eigenvalues do exist. In this case, the minimal eigenvalue of QM_tQ' is t. Hence, because $x'\Sigma_t x = x'\alpha_\perp' QM_tQ'\alpha_\perp x \geq \lambda_{\min}\left(QM_tQ'\right)x'\alpha_\perp'\alpha_\perp x \geq \lambda_{\min}\left(QM_tQ'\right)\lambda_{\min}\left(\alpha_\perp'\alpha_\perp\right)|x|^2 = ct\,|x|^2$, it follows that $\sigma_t^2 \geq ct$ (note that $c > 0$ because $\alpha_\perp'\alpha_\perp$ is regular). As a consequence, for the OU process the situation is the same as for r-dimensional BM, namely, Assumption (C1) is fulfilled as long as $2p > r$.

For r-dimensional BM, (2.7b) is not satisfied unless $r = 1$. This follows from the substitution $s = 1/t$, which yields

$$\int_0^1 \psi_t(x)\,dt = (2\pi)^{-r/2}\int_1^\infty s^{r/2-2}e^{-|x|^2 s/2}\,ds.$$

For $r = 1$,

$$\int_1^\infty s^{-3/2}e^{-|x|^2 s/2}\,ds \leq 2$$

uniformly in x. For $r \geq 2$, however,

$$\underline{\lim}_{x\to 0}\int_1^\infty s^{r/2-2}e^{-|x|^2 s/2}\,ds \geq \int_1^\infty s^{r/2-2}\underline{\lim}_{x\to 0}e^{-|x|^2 s/2}\,ds$$

$$= \int_1^\infty s^{r/2-2}\,ds = \infty.$$

This settles examples (a) and (c) in Remark 3.2.

C.2. Proof of (2.7b) for One-Dimensional Gaussian Processes as in Remark 3.2 (e). Without restricting generality, we may assume mean zero, i.e., $X(t) \sim \mathcal{N}(0, \sigma(t)^2)$. By the inversion formula for densities,

$$\psi_t(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-ivx}e^{-v^2\sigma(t)^2/2}\,dv.$$

By assumption (2.8) and because

$$\int_{-\infty}^\infty e^{-v^2\sigma(t)^2/2}\,dv = \frac{\sqrt{2\pi}}{\sigma(t)}$$

if $\sigma(t) > 0$, it follows from Tonelli's theorem that

$$\int_0^1 \psi_t(x)\,dt \leq \frac{1}{2\pi}\int_0^1 dt\int_{-\infty}^\infty e^{-v^2\sigma(t)^2/2}\,dv = \frac{1}{\sqrt{2\pi}}\int_0^1 \frac{dt}{\sigma(t)} < \infty$$

uniformly in x. ■

C.3. Proof of (2.7b) for Remark 3.2 (a)–(d) (One-dimensional case). For every case, we verify condition (2.8). Then the assertion follows from (e).

(a) Fractional BM (one-dimensional): $\sigma(t) = t^H$ satisfies (2.8) for $0 < H < 1$.

(b) For BB, $\sigma(t) = \sqrt{t(1-t)}$. Hence

$$\int_0^1 \frac{dt}{\sigma(t)} = \int_0^1 \frac{dt}{\sqrt{t(1-t)}} = \pi.$$

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(c) One-dimensional OU process: $dX(t) = -bX(t)dt + \sigma\, dB(t)$. The variance is

$$\sigma(t)^2 = \frac{\sigma^2}{2b}\left(1 - e^{-2bt}\right).$$

Hence

$$\begin{aligned}\int_0^1 \frac{dt}{\sigma(t)} &= \sigma^{-1}\sqrt{2b}\int_0^1 \frac{dt}{\sqrt{1-e^{-2bt}}} \\ &= \left(2b\sigma^2\right)^{-1/2}\int_0^{1-e^{-2b}} \frac{du}{(1-u)\sqrt{u}} \\ &= \left(2b\sigma^2\right)^{-1/2} \ln\frac{1+\sqrt{u}}{1-\sqrt{u}}\Big|_0^{1-e^{-2b}} \\ &= \left(2b\sigma^2\right)^{-1/2} \ln\frac{1+\sqrt{1-e^{-2b}}}{1-\sqrt{1-e^{-2b}}}.\end{aligned}$$

(d) For geometric BM $dX(t) = X(t)\left[bdt + \sigma\, dB(t)\right], X(0) = x$, the variance is

$$\sigma(t)^2 = x^2 e^{2bt}\left(e^{\sigma^2 t} - 1\right),$$

and therefore (performing the substitution $u = e^{\sigma^2 t} - 1$ and denoting $\overline{u} = e^{\sigma^2} - 1$)

$$\begin{aligned}\int_0^1 \frac{dt}{\sigma(t)} &= x^{-1}\int_0^1 \frac{e^{-2bt}}{\sqrt{e^{\sigma^2 t} - 1}}dt \\ &= \left(x\sigma^2\right)^{-1}\int_0^{\overline{u}} \frac{(u+1)^{-(2b+\sigma^2)/\sigma^2}}{\sqrt{u}}du \\ &< \infty\end{aligned}$$

(note that this is true regardless of the sign of the drift parameter b).

C.4. Condition (2.7) for Example 3. Assume that the coefficients $b(x)$ and $a(x) = \sigma(x)\,\sigma(x)'$ of the SDE (1.5) satisfy the following conditions.

(i) a and b are (globally) bounded and Lipschitz continuous on compact sets;

(ii) in addition, a is globally Hölder continuous and satisfies the uniform ellipticity condition (1.6).

Then the transition function of the solution X of the SDE (1.5) possesses a transition density $\gamma(x,t,y)$ satisfying the bound

$$\gamma(x,t,y) \le Ct^{-r/2}\exp\left[-c\frac{|y-x|^2}{t}\right] \qquad \textbf{(C.1)}$$

for some positive constants c, C. The transition density $\gamma(x,t,y)$ may be obtained as fundamental solution of the parabolic equation $(L - \partial/\partial t)\,\gamma = 0$, where L is the differential operator

$$L = \sum_{i=1}^r b_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^r a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}$$

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(cf. Friedman, 1964; 1975, Chs. 6.4 and 6.5). As a consequence of (C.1), for fixed initial value $x = x_0$, $X(t)$ possesses the density

$$\psi_t(x) = \gamma(x_0, t, x) \leq Ct^{-r/2} \exp\left[-c\frac{|x - x_0|^2}{t}\right] = C' v_r(x; x_0, (t/2c) I).$$

This brings us back to the first case considered in this Appendix, namely, multivariate Gaussian processes, and actually to multivariate BM. As a consequence, (2.7a) is fulfilled if $2p > r$. ■

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