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PROVABILITY LOGICS RELATIVE TO A FIXED EXTENSION OF PEANO ARITHMETIC

TAISHI KURAHASHI

Abstract. Let T and U be any consistent theories of arithmetic. If T is computably enumerable, then the provability predicate $\mathsf{Pr}_\tau(x)$ of T is naturally obtained from each Σ_1 definition $\tau(v)$ of T. The provability logic $\mathsf{PL}_\tau(U)$ of τ relative to U is the set of all modal formulas which are provable in U under all arithmetical interpretations where $\Box$ is interpreted by $\mathsf{Pr}_\tau(x)$. It was proved by Beklemishev based on the previous studies by Artemov, Visser, and Japaridze that every $\mathsf{PL}_\tau(U)$ coincides with one of the logics GL_α, D_β, S_β, and GL^-_β, where α and β are subsets of ω and β is cofinite.

We prove that if U is a computably enumerable consistent extension of Peano Arithmetic and L is one of GL_α, D_β, S_β, and GL^-_β, where α is computably enumerable and β is cofinite, then there exists a Σ_1 definition $\tau(v)$ of some extension of $I\Sigma_1$ such that $\mathsf{PL}_\tau(U)$ is exactly L.

§1. Introduction. Let T be any computably axiomatized extension of the theory $I\Sigma_1$. A Σ_1 formula $\tau(v)$ is said to be a Σ_1 definition of T if $\{n \in \omega : \mathbb{N} \models \tau(\overline{n})\}$ is exactly the set of all Gödel numbers of axioms of T. For each Σ_1 definition $\tau(v)$ of T, let $\mathsf{Pr}_\tau(x)$ be the Σ_1 formula naturally expressing the relation "x is provable from the theory defined by τ". A mapping from the set of all propositional variables to the set of sentences of arithmetic is said to be an arithmetical interpretation. Each arithmetical interpretation f is uniquely extended to the mapping f_τ whose domain is the set of all modal formulas so that f_τ commutes with each propositional connective and $f_\tau(\Box A)$ is $\mathsf{Pr}_\tau(\ulcorner f_\tau(A) \urcorner)$. The set $\mathsf{PL}_\tau(T) = \{A : T \vdash f_\tau(A)$ for all arithmetical interpretations $f\}$ is called the provability logic of $\tau(v)$. Solovay proved in [11] the arithmetical completeness theorem of the propositional modal logic GL with respect to arithmetical interpretations, that is, if T is Σ_1-sound, then $\mathsf{PL}_\tau(T)$ is exactly GL.

Moreover, Solovay proved the arithmetical completeness theorem of the propositional modal logic S, that is, if T is sound, then for any modal formula A, $\mathsf{S} \vdash A$ if and only if $\mathsf{TA} \vdash f_\tau(A)$ for all arithmetical interpretations f, where TA is the set of all sentences of arithmetic which are true in $\mathbb{N}$. From this theorem, it is natural to generalize the notion of provability logic. Let U be an extension of $I\Sigma_1$ which is not necessarily computably enumerable. The set $\mathsf{PL}_\tau(U) = \{A : U \vdash f_\tau(A)$ for all arithmetical interpretations $f\}$ is called the provability logic of $\tau(v)$ relative to U. This notion was introduced by Artemov [1] and Visser [12] independently.

The classification theorem of propositional provability logics was proved by Beklemishev [4] based on the previous studies by Artemov [1, 2], Visser [12, 13],

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and Japaridze [7], that is, all possible propositional provability logics $\mathsf{PL}_\tau(U)$ were listed and classified. More precisely, the classification theorem states that every provability logic $\mathsf{PL}_\tau(U)$ coincides with one of the logics GL_α, D_β, S_β, and GL_β^-, where α and β are subsets of ω and β is cofinite (see also [3]). Moreover, Beklemishev [4] proved that if $\tau(v)$ satisfies an appropriate condition, then for each logic L from GL_α, D_β, S_β and GL_β^-, there exists an extension U of Peano Arithmetic PA such that $\mathsf{PL}_\tau(U)$ is exactly L.

On the other hand, little is known about $\mathsf{PL}_\tau(U)$ when we fix a theory U even if $U = \mathsf{PA}$. For example, it follows from Solovay's first theorem that $\mathsf{PL}_\tau(\mathsf{PA}) = \mathsf{GL}$ for any Σ_1 definition $\tau(v)$ of PA. Also by the proof of Solovay's second theorem, if $\tau(v)$ is a natural Σ_1 definition of $I\Sigma_1$, then $\mathsf{PL}_\tau(\mathsf{PA}) = \mathsf{S}$. Beklemishev [4] proved that for each natural number n, there exists a Σ_1 definition $\tau(v)$ of some theory such that $\mathsf{PL}_\tau(\mathsf{PA}) = \mathsf{GL}_\beta^-$ where $\beta = \{k : k \geq n+1\}$.

Then a natural question arises: Which propositional modal logic can be the provability logic $\mathsf{PL}_\tau(\mathsf{PA})$ for some $\tau(v)$? For this question, we pay attention to a theorem by Jeroslow. Jeroslow [8] proved that there exists a Σ_1 definition $\tau(v)$ of some theory T such that PA is deductively equivalent to $T + \mathsf{Con}_\tau$ where Con_τ is the consistency statement defined by using $\mathsf{Pr}_\tau(x)$. In this case, it can be shown that $\mathsf{PL}_\tau(\mathsf{PA})$ is the logic GL_α where $\alpha = \{0\}$. This approach can be generalized, and can be used to answer the above question.

In this article, by using Jeroslow's method, we prove that for each computably axiomatized consistent extension U of PA and each logic L from GL_α, D_β, S_β and GL_β^- where α is computably enumerable and β is cofinite, there exists a Σ_1 definition $\tau(v)$ of some extension of $I\Sigma_1$ such that $\mathsf{PL}_\tau(U)$ is exactly L.

In Section 3, we discuss the case of GL_α where α is computably enumerable. Section 4 is devoted the case of GL_β^- where β is cofinite. In Section 5, we give a method of constructing Σ_1 definitions having several desired properties. This method is used in Sections 6 and 7. In Section 6, we prove that for each cofinite β, there exists a Σ_1 definition $\tau(v)$ of $I\Sigma_1$ such that $\mathsf{PL}_\tau(U) = \mathsf{S}_\beta$. In Section 7, we firstly prove that there exists a computably axiomatized extension T_1 of $I\Sigma_1$ such that $T_1 + \mathsf{RFN}_{\Sigma_1}(T_1)$ is deductively equivalent to U by using Jeroslow's method, where $\mathsf{RFN}_{\Sigma_1}(T_1)$ is the uniform Σ_1 reflection principle of T_1. Then we prove that for each cofinite β, there exists a Σ_1 definition $\tau(v)$ of T_1 such that $\mathsf{PL}_\tau(U)$ is D_β. Finally, we discuss some refinements of our results in Section 8.

§2. Preliminaries.

2.1. Provability predicates. We say that a set of sentences a *theory*. Throughout this article, we consider only theories extending $I\Sigma_1$ in the language of first-order arithmetic $\mathcal{L}_A$, where $I\Sigma_1$ is obtained from Peano Arithmetic PA by restricting induction axioms to Σ_1 formulas (see [6]). Let ω be the set of all natural numbers. For each $n \in \omega$, $\overline{n}$ denotes the numeral for n. For each $\mathcal{L}_A$-formula φ, $\ulcorner\varphi\urcorner$ denotes the numeral corresponding to the Gödel number of φ. The standard model of arithmetic is denoted by $\mathbb{N}$. The theory $\{\varphi : \mathbb{N} \models \varphi\}$ is denoted by TA that is called the *true arithmetic*.

We say a formula $\tau(v)$ is a *definition* of a theory T if $\{n : \mathbb{N} \models \tau(\overline{n})\}$ is the set of all Gödel numbers of axioms of T. It is known that if T is computably enumerable,

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then there exists a Σ_1 definition of T. For each Σ_1 (resp. $\Delta_1(I\Sigma_1)$) definition $\tau(v)$ of T, we can naturally associate a Σ_1 (resp. $\Delta_1(I\Sigma_1)$) formula $\mathsf{Prf}_\tau(x, y)$ saying that "y is a proof of x from the theory defined by $\tau(v)$" (see [5]). Let $\mathsf{Pr}_\tau(x)$ be the Σ_1 formula $\exists y \mathsf{Prf}_\tau(x, y)$ that is called the *provability predicate* of $\tau(v)$. For each Σ_1 definition $\tau(v)$ of T, let $(\tau|z)(v)$ be the formula $\tau(v) \wedge v < z$. For each natural number k, $(\tau|\overline{k})(v)$ is a Σ_1 definition of a finite subtheory of T. Also let $(\tau + \varphi)(v)$ be the formula $\tau(v) \vee v = \ulcorner \varphi \urcorner$ which is a Σ_1 definition of the theory $T + \varphi$.

The following facts are basic properties of provability predicates (see [5, 10]).

FACT 2.1. *Let $\tau(v)$ and $\tau'(v)$ be any formulas. Then $I\Sigma_1 \vdash \forall v(\tau(v) \to \tau'(v)) \to \forall x(\mathsf{Pr}_\tau(x) \to \mathsf{Pr}_{\tau'}(x))$.*

FACT 2.2. *Let $\tau(v)$ be any Σ_1 definition of T.*

1. *If $T \vdash \forall x \varphi(x)$, then $I\Sigma_1 \vdash \forall x \mathsf{Pr}_\tau(\ulcorner \varphi(\dot{x}) \urcorner)$.*
2. *$I\Sigma_1 \vdash \forall x(\mathsf{Pr}_\tau(\ulcorner \varphi(\dot{x}) \to \psi(\dot{x}) \urcorner) \to (\mathsf{Pr}_\tau(\ulcorner \varphi(\dot{x}) \urcorner) \to \mathsf{Pr}_\tau(\ulcorner \psi(\dot{x}) \urcorner)))$.*
3. *If $\varphi(x)$ is Σ_1, then $I\Sigma_1 \vdash \forall x(\varphi(x) \to \mathsf{Pr}_\tau(\ulcorner \varphi(\dot{x}) \urcorner))$.*
4. *$I\Sigma_1 \vdash \forall x(\mathsf{Pr}_\tau(\ulcorner \mathsf{Pr}_\tau(\ulcorner \varphi(\dot{x}) \urcorner) \to \varphi(\dot{x}) \urcorner) \to \mathsf{Pr}_\tau(\ulcorner \varphi(\dot{x}) \urcorner))$.*
5. *$I\Sigma_1 \vdash \forall x(\mathsf{Pr}_{\tau+\varphi}(x) \leftrightarrow \mathsf{Pr}_\tau(\ulcorner \varphi \to \dot{x} \urcorner))$.*

Here $\ulcorner \varphi(\dot{x}) \urcorner$ is a primitive recursive term corresponding to a function computing the Gödel number of $\varphi(\overline{n})$ from an input n. Clauses 3 and 4 are referred as formalized Σ_1-completeness and formalized Löb's theorem, respectively.

FACT 2.3 (Craig's trick). *For each Σ_1 definition $\tau(v)$ of a theory T, there exists a $\Delta_1(I\Sigma_1)$ formula $\mathsf{Prf}^+_\tau(x, y)$ such that $I\Sigma_1 \vdash \forall x(\mathsf{Pr}_\tau(x) \leftrightarrow \exists y \mathsf{Prf}^+_\tau(x, y))$.*

For each Σ_1 definition $\tau(v)$ of T, let Con_τ be the Π_1 sentence $\neg \mathsf{Pr}_\tau(\ulcorner \overline{0} = \overline{1} \urcorner)$ expressing the consistency of T.

FACT 2.4 (The FGH-principle (see [14])). *Let π be any Π_1 sentence such that $T \vdash \pi \to \mathsf{Con}_\tau$. Then there exists a Σ_1 sentence ψ such that $T \vdash \pi \leftrightarrow \neg \mathsf{Pr}_\tau(\ulcorner \psi \urcorner)$.*

We define the sequence $\{\mathsf{Con}^n_\tau\}_{n \in \omega}$ of Π_1 sentences as follows:

1. Con^0_τ is the sentence $\overline{0} = \overline{0}$;
2. Con^{n+1}_τ is the sentence $\neg \mathsf{Pr}_\tau(\ulcorner \neg \mathsf{Con}^n_\tau \urcorner)$.

Moreover, we can construct a Π_1 formula $\mathsf{Con}_\tau(z)$ with only one free variable z satisfying the following conditions:

- $I\Sigma_1 \vdash \forall z(\mathsf{Con}_\tau(z + 1) \leftrightarrow \neg \mathsf{Pr}_\tau(\ulcorner \neg \mathsf{Con}_\tau(\dot{z}) \urcorner))$;
- $I\Sigma_1 \vdash \mathsf{Con}^n_\tau \leftrightarrow \mathsf{Con}_\tau(\overline{n})$ for each $n \in \omega$.

DEFINITION 2.5. Let $\tau(v)$ be a Σ_1 definition of a theory T. We say that the *height* of $\tau(v)$ is ∞ if the theory $T + \{\mathsf{Con}^n_\tau : n \in \omega\}$ is consistent.

For each natural number $n \geq 1$, let $\mathsf{True}_{\Sigma_n}(x)$ be a Σ_n formula satisfying that for any Σ_n formula $\varphi(x)$, $I\Sigma_1 \vdash \forall x(\varphi(x) \leftrightarrow \mathsf{True}_{\Sigma_n}(\ulcorner \varphi(\dot{x}) \urcorner))$ (see [10]). Moreover, we assume

$$I\Sigma_1 \vdash \forall x(\Sigma_n(x) \to \mathsf{Pr}_\sigma(\ulcorner \dot{x} \leftrightarrow \mathsf{True}_{\Sigma_n}(\dot{x}) \urcorner)),$$

where $\Sigma_n(x)$ is a Σ_1 formula naturally expressing the relation "x is a Σ_n sentence", and $\sigma(v)$ is a natural Σ_1 definition of $I\Sigma_1$. Let $\mathsf{RFN}_{\Sigma_n}(\tau)$ be the sentence $\forall x(\Sigma_n(x) \wedge \mathsf{Pr}_\tau(x) \to \mathsf{True}_{\Sigma_n}(x))$ that is called the *uniform Σ_n reflection principle* of $\tau(v)$. For each Σ_n formula $\varphi(x)$, we assume $I\Sigma_1 \vdash \forall x \Sigma_n(\ulcorner \varphi(\dot{x}) \urcorner)$. Thus $I\Sigma_1 + \mathsf{RFN}_{\Sigma_n}(\tau) \vdash \forall x(\mathsf{Pr}_\tau(\ulcorner \varphi(\dot{x}) \urcorner) \to \varphi(x))$ holds. Let $\mathsf{RFN}(\tau)$ be the set $\{\mathsf{RFN}_{\Sigma_n}(\tau) : n \geq 1\}$.

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FACT 2.6 (See [10]). *For any computably axiomatized extension T of* PA, *there exists a* Σ_1 *definition* $\tau(v)$ *of* T *such that* $T \vdash \mathsf{RFN}(\tau|\overline{p})$ *for all* $p \in \omega$.

2.2. The classification of provability logics. The axioms of propositional modal logic GL are as follows:

1. All propositional tautologies in the language of propositional modal logic;
2. $\Box(p \to q) \to (\Box p \to \Box q)$;
3. $\Box(\Box p \to p) \to \Box p$.

The inference rules of GL are Modus Ponens, Necessitation, and Substitution.

For each set X of modal formulas, let $\mathsf{GL}X$ be the logic whose axioms are all theorems of GL and all elements of X, and whose inference rules are Modus Ponens and Substitution. We identify each axiomatic system of modal logic with the set of all its theorems.

Let A be any modal formula and let $n \in \omega$. We define the modal formula $\Box^n A$ recursively as follows: $\Box^0 A$ is A; $\Box^{n+1} A$ is $\Box\Box^n A$.

DEFINITION 2.7. Let F_n be the modal formula $\Box^{n+1}\bot \to \Box^n \bot$.

DEFINITION 2.8. Let α be a subset of ω, and β be a cofinite subset of ω.

1. $\mathsf{GL}_\alpha = \mathsf{GL}\{F_n : n \in \alpha\}$.
2. $\mathsf{GL}_\beta^- = \mathsf{GL}\{\bigvee_{n \notin \beta} \neg F_n\}$.
3. $\mathsf{D} = \mathsf{GL}\{\neg\Box\bot, \Box(\Box p \vee \Box q) \to (\Box p \vee \Box q)\}$.
4. $\mathsf{S} = \mathsf{GL}\{\Box p \to p\}$.
5. $\mathsf{D}_\beta = \mathsf{D} \cap \mathsf{GL}_\beta^-$.
6. $\mathsf{S}_\beta = \mathsf{S} \cap \mathsf{GL}_\beta^-$.

Notice that for each cofinite $\beta \subseteq \omega$, $\mathsf{GL}_\beta \subseteq \mathsf{D}_\beta \subseteq \mathsf{S}_\beta \subseteq \mathsf{GL}_\beta^-$.

A mapping f from the set of all propositional variables to a set of $\mathcal{L}_A$-sentences is called an *arithmetical interpretation*. Let $\tau(v)$ be any Σ_1 definition of some theory. Each arithmetical interpretation f is uniquely extended to the mapping f_τ from the set of all modal formulas to a set of $\mathcal{L}_A$-sentences so that the following conditions hold:

1. $f_\tau(p)$ is $f(p)$ for any propositional variable p;
2. $f_\tau(\bot)$ is $\overline{0} = \overline{1}$;
3. f_τ commutes with each propositional connective;
4. $f_\tau(\Box A)$ is $\mathsf{Pr}_\tau(\ulcorner f_\tau(A) \urcorner)$.

The set $\mathsf{PL}_\tau(U) = \{A : U \vdash f_\tau(A) \text{ for all arithmetical interpretations } f\}$ is called the *provability logic* of $\tau(v)$ relative to U. Solovay proved the following landmark theorem of propositional provability logic.

THEOREM 2.9 (Solovay's arithmetical completeness theorem [11]). *Let T be any computably enumerable extension of $I\Sigma_1$ and $\tau(v)$ be any Σ_1 definition of T.*

1. *If T is Σ_1-sound, then* $\mathsf{PL}_\tau(T) = \mathsf{GL}$.
2. *If T is sound, then* $\mathsf{PL}_\tau(\mathsf{TA}) = \mathsf{S}$.

In classifying possible propositional provability logics, the notion of trace plays an essential role.

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DEFINITION 2.10. Let A be any modal formula and L be any modal logic. The *trace* $\mathsf{tr}(A)$ of A is the set $\{n \in \omega : A$ is false at the root of some finite rooted transitive irreflexive Kripke model of height $n\}$. The trace $\mathsf{tr}(L)$ of L is the set $\bigcup\{\mathsf{tr}(A) : L \vdash A\}$.

The following proposition is useful to determine the trace of each provability logic.

PROPOSITION 2.11. *Let f be any arithmetical interpretation and $\alpha \subseteq \omega$. The following are equivalent*:

1. $\mathsf{tr}(\mathsf{PL}_\tau(U)) = \alpha$.
2. *For any $n \in \omega$, $U \vdash f_\tau(F_n)$ if and only if $n \in \alpha$.*

PROOF. This follows from the fact that for any $n \in \omega$, $n \in \mathsf{tr}(\mathsf{PL}_\tau(U))$ if and only if $\mathsf{PL}_\tau(U) \vdash F_n$ (Corollary 47 in [3]). ⊣

Throughout this article, we assume that let f be some fixed arithmetical interpretation. Notice that $f_\tau(F_n)$ is the sentence $\neg\mathsf{Con}_\tau^{n+1} \to \neg\mathsf{Con}_\tau^n$ regardless of the choice of f because F_n contains no propositional variables.

The following theorem is a major achievement of the research of propositional provability logics due to Beklemishev [4] based on the studies by Artemov [1, 2], Visser [12, 13], and Japaridze [7]. Our results in the present article are heavily relying on this theorem.

THEOREM 2.12 (The classification theorem). *Suppose that $L = \mathsf{PL}_\tau(U)$ for some U and some Σ_1 definition $\tau(v)$ of a theory T.*

1. *If $\mathsf{tr}(L) = \alpha$ and α is not cofinite, then $L = \mathsf{GL}_\alpha$.*
2. *If $\mathsf{tr}(L) = \beta$ and β is cofinite, then L is either one of GL_β, D_β, S_β and GL_β^- (notice $\mathsf{GL}_\beta \subseteq \mathsf{D}_\beta \subseteq \mathsf{S}_\beta \subseteq \mathsf{GL}_\beta^-$).*

Beklemishev showed that all provability logics are exhibited by the logics $\mathsf{PL}_\tau(U)$ for some fixed $\tau(v)$.

THEOREM 2.13 (Beklemishev (Theorem 4 in [4])). *Let $\tau(v)$ be a Σ_1 definition of some extension of* PA, *whose height is ∞. Then for each logic L from GL_α, D_β, S_β and GL_β^-, there exists an extension U of* PA *such that $\mathsf{PL}_\tau(U)$ is exactly L.*

In this article, we investigate the provability logics $\mathsf{PL}_\tau(U)$ for a fixed computably axiomatized consistent extension U of PA. The following list exhibits already known results for the case of $U = \mathsf{PA}$.

1. $\mathsf{PL}_\tau(\mathsf{PA}) = \mathsf{GL}$ for any Σ_1 definition $\tau(v)$ of PA (Solovay).
2. $\mathsf{PL}_\tau(\mathsf{PA}) = \mathsf{S}$ for a natural Σ_1 definition $\tau(v)$ of $I\Sigma_1$ (Example 61 in [3]).
3. For each $n \in \omega$, there exists a Σ_1 definition $\sigma(v)$ of a theory such that $\mathsf{PL}_\sigma(\mathsf{PA}) = \mathsf{GL}_\beta^-$ where $\beta = \{k : k \geq n+1\}$ (Beklemishev (Theorem 5 in [4])).
4. There exists a Σ_1 definition $\tau(v)$ of a theory such that $\mathsf{PL}_\tau(\mathsf{PA}) = \mathsf{GL}_\alpha$ where $\alpha = \{0\}$ (Jeroslow [8]).

In fact, Jeroslow proved the following result.

FACT 2.14 (Theorem 1.3 in [8]). *For any computably enumerable consistent extension U of* PA, *there exists a Σ_1 definition $\tau(v)$ of some extension T_0 of $I\Sigma_1$ such that the theories U and $T_0 + \mathsf{Con}_\tau$ are deductively equivalent.*

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We obtain the last clause in the list as a consequence of this fact as follows: Let $\tau(v)$ be a Σ_1 definition of a theory T_0 such that PA and $T_0 + \mathsf{Con}_\tau$ are deductively equivalent. Then for any $n \in \omega$, $\mathsf{PA} \vdash f_\tau(F_n)$ if and only if $n = 0$. By Proposition 2.11, $\mathrm{tr}(\mathsf{PL}_\tau(\mathsf{PA})) = \{0\}$. Since $\{0\}$ is not cofinite, $\mathsf{PL}_\tau(\mathsf{PA}) = \mathsf{GL}_{\{0\}}$ by the classification theorem.

In this article, we generalize this kind of argument. Then we finally prove that if U is a computably axiomatized consistent extension of PA and L is one of GL_α, D_β, S_β, and GL^-_β, where α is computably enumerable and β is cofinite, then there exists a Σ_1 definition $\tau(v)$ of some extension T of $I\Sigma_1$ such that $\mathsf{PL}_\tau(U) = L$.

2.3. On Σ_1 definitions. If T is Σ_1-sound, then the height of each Σ_1 definition of T is ∞. Beklemishev proved that even if T is not Σ_1-sound, there exists a Σ_1 definition of T with the height ∞ if T is computably axiomatized.

FACT 2.15 (Beklemishev (Lemma 7 in [4])). *Let T be any computably axiomatized consistent extension of* PA. *Then there exists a Σ_1 definition of T whose height is ∞.*

We prove a strengthening of Beklemishev's result. Our proof is based on an alternative proof of Fact 2.15 given in [9].

LEMMA 2.16. *Let T be any computably axiomatized consistent extension of* PA. *Then there exists a Σ_1 definition $\tau(v)$ of T such that $T + \forall z \mathsf{Con}_\tau(z)$ is consistent.*

PROOF. Let $\tau'(v)$ be a Σ_1 definition of T such that $T \vdash \mathsf{RFN}(\tau'|\overline{p})$ for all $p \in \omega$. The existence of such a Σ_1 definition is guaranteed by Fact 2.6. By Fixed Point Lemma (cf. [10]), we obtain a Σ_1 formula $\tau(v)$ satisfying the following equivalence:

$$I\Sigma_1 \vdash \forall v(\tau(v) \leftrightarrow \tau'(v) \wedge \forall y \leq v \neg \mathsf{Prf}^+_{\tau'}(\ulcorner \neg \forall z \mathsf{Con}_\tau(z) \urcorner, y)).$$

Towards a contradiction, we suppose that $T + \forall z \mathsf{Con}_\tau(z)$ is inconsistent. Then there exists p such that $I\Sigma_1 \vdash \mathsf{Prf}^+_{\tau'}(\ulcorner \neg \forall z \mathsf{Con}_\tau(z) \urcorner, \overline{p})$, and thus $I\Sigma_1 \vdash \tau(v) \to \tau'(v) \wedge v < \overline{p}$. Then $I\Sigma_1 \vdash \mathsf{Pr}_\tau(x) \to \mathsf{Pr}_{\tau'|\overline{p}}(x)$. We obtain $I\Sigma_1 \vdash \mathsf{Pr}_\tau(\ulcorner \neg \mathsf{Con}_\tau(\dot{z}) \urcorner) \to \mathsf{Pr}_{\tau'|\overline{p}}(\ulcorner \neg \mathsf{Con}_\tau(\dot{z}) \urcorner)$. Since $T \vdash \mathsf{RFN}(\tau'|\overline{p})$, we have $T \vdash \mathsf{Pr}_{\tau'|\overline{p}}(\ulcorner \neg \mathsf{Con}_\tau(\dot{z}) \urcorner) \to \neg \mathsf{Con}_\tau(z)$. Hence $T \vdash \mathsf{Pr}_\tau(\ulcorner \neg \mathsf{Con}_\tau(\dot{z}) \urcorner) \to \neg \mathsf{Con}_\tau(z)$. This means $T \vdash \mathsf{Con}_\tau(z) \to \mathsf{Con}_\tau(z+1)$. Since obviously $T \vdash \mathsf{Con}_\tau(0)$, we obtain $T \vdash \forall z \mathsf{Con}_\tau(z)$ by Π_1 induction. This contradicts the consistency of T. Therefore $T + \forall z \mathsf{Con}_\tau(z)$ is consistent.

Then $\forall y \leq \overline{n} \neg \mathsf{Prf}^+_{\tau'}(\ulcorner \neg \forall z \mathsf{Con}_\tau(z) \urcorner, y)$ is true for any $n \in \omega$. This shows that $\tau(v)$ is also a Σ_1 definition of T. ⊣

Throughout this article, we fix a computably axiomatized consistent extension U of PA. Also we fix Σ_1 definitions $\sigma(v)$ and $\mu(v)$ of $I\Sigma_1$ and U, respectively. We may assume that $\sigma(v)$ and $\mu(v)$ satisfy the following conditions:

1. $\mathsf{PA} \vdash \mathsf{RFN}(\sigma)$;
2. the theory $U + \forall z \mathsf{Con}_\mu(z)$ is consistent;
3. $I\Sigma_1 \vdash \forall v(\sigma(v) \to \mu(v))$.

§3. The logics GL_α. In this section, we prove that for any computably enumerable set α of natural numbers, there exists a Σ_1 definition $\tau_\alpha(v)$ of some extension $T(\alpha)$ of $I\Sigma_1$ such that $\mathsf{PL}_{\tau_\alpha}(U)$ is exactly the logic GL_α. Notice that this result is best possible because for any $\alpha \subseteq \omega$, if $\mathsf{PL}_\tau(U) = \mathsf{GL}_\alpha$ for some $\tau(v)$, then α is computably enumerable.

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In this section, we fix a computably enumerable set α of natural numbers. Then we have a Δ_0 formula $\delta(x, y)$ such that for any $n \in \omega$, $n \in \alpha$ if and only if $\mathbb{N} \models \exists y \delta(\overline{n}, y)$. Let $\tau(v)$ be any Σ_1 formula. Then by Fixed Point Lemma, we can effectively find a Σ_1 formula $\alpha_\tau(v)$ satisfying the following equivalence:

$$I\Sigma_1 \vdash \forall v(\alpha_\tau(v) \leftrightarrow \exists y(\delta(v, y) \wedge \forall z \leq y \neg \mathsf{Prf}_\tau^+(\ulcorner \neg \alpha_\tau(\dot{v}) \rightarrow (\neg \mathsf{Con}_\tau(\dot{v}+1) \rightarrow \neg \mathsf{Con}_\tau(\dot{v}))\urcorner, z))).$$

LEMMA 3.1. *Let $\tau(v)$ be a Σ_1 definition of a theory T, whose height is ∞. Then the formula $\alpha_\tau(v)$ satisfies the following two conditions for any natural number n:*

1. *If $n \in \alpha$, then $\mathbb{N} \models \alpha_\tau(\overline{n})$.*
2. *If $T \vdash \neg\alpha_\tau(\overline{n}) \rightarrow (\neg \mathsf{Con}_\tau^{n+1} \rightarrow \neg \mathsf{Con}_\tau^n)$, then $n \in \alpha$.*

PROOF. 1. Suppose $n \in \alpha$. Let $k \in \omega$ be such that $I\Sigma_1 \vdash \delta(\overline{n}, \overline{k}) \wedge \forall y < \overline{k} \neg \delta(\overline{n}, y)$. Then

$$I\Sigma_1 \vdash \alpha_\tau(\overline{n}) \leftrightarrow \forall z \leq \overline{k} \neg \mathsf{Prf}_\tau^+(\ulcorner \neg \alpha_\tau(\overline{n}) \rightarrow (\neg \mathsf{Con}_\tau(\overline{n+1}) \rightarrow \neg \mathsf{Con}_\tau(\overline{n}))\urcorner, z).$$

Suppose, towards a contradiction, $\mathbb{N} \models \neg\alpha_\tau(\overline{n})$. Then

$$\mathbb{N} \models \exists z \leq \overline{k} \mathsf{Prf}_\tau^+(\ulcorner \neg \alpha_\tau(\overline{n}) \rightarrow (\neg \mathsf{Con}_\tau(\overline{n+1}) \rightarrow \neg \mathsf{Con}_\tau(\overline{n}))\urcorner, z).$$

By the Σ_1-completeness of T, we have

$$T \vdash \exists z \leq \overline{k} \mathsf{Prf}_\tau^+(\ulcorner \neg \alpha_\tau(\overline{n}) \rightarrow (\neg \mathsf{Con}_\tau(\overline{n+1}) \rightarrow \neg \mathsf{Con}_\tau(\overline{n}))\urcorner, z).$$

Therefore $T \vdash \neg\alpha_\tau(\overline{n})$.

If $T \vdash \neg \mathsf{Con}_\tau^{n+1} \rightarrow \neg \mathsf{Con}_\tau^n$, then $T \vdash \neg \mathsf{Con}_\tau^n$ by Löb's theorem. This contradicts our assumption that the height of $\tau(v)$ is ∞. Hence $T \nvdash \neg \mathsf{Con}_\tau^{n+1} \rightarrow \neg \mathsf{Con}_\tau^n$. Since $T \vdash \neg\alpha_\tau(\overline{n})$, we also have $T \nvdash \neg\alpha_\tau(\overline{n}) \rightarrow (\neg \mathsf{Con}_\tau^{n+1} \rightarrow \neg \mathsf{Con}_\tau^n)$. Then $T \nvdash \neg\alpha_\tau(\overline{n}) \rightarrow (\neg \mathsf{Con}_\tau(\overline{n+1}) \rightarrow \neg \mathsf{Con}_\tau(\overline{n}))$. Thus

$$\mathbb{N} \models \forall z \leq \overline{k} \neg \mathsf{Prf}_\tau^+(\ulcorner \neg \alpha_\tau(\overline{n}) \rightarrow (\neg \mathsf{Con}_\tau(\overline{n+1}) \rightarrow \neg \mathsf{Con}_\tau(\overline{n}))\urcorner, z).$$

This is a contradiction.

2. Suppose $T \vdash \neg\alpha_\tau(\overline{n}) \rightarrow (\neg \mathsf{Con}_\tau^{n+1} \rightarrow \neg \mathsf{Con}_\tau^n)$. Then

$$T \vdash \mathsf{Prf}_\tau^+(\ulcorner \neg \alpha_\tau(\overline{n}) \rightarrow (\neg \mathsf{Con}_\tau(\overline{n+1}) \rightarrow \neg \mathsf{Con}_\tau(\overline{n}))\urcorner, \overline{p})$$

for some p. If $n \notin \alpha$, then $T \vdash \forall y < \overline{p} \neg \delta(\overline{n}, y)$. Thus

$$T \vdash \exists z(\mathsf{Prf}_\tau^+(\ulcorner \neg \alpha_\tau(\overline{n}) \rightarrow (\neg \mathsf{Con}_\tau(\overline{n+1}) \rightarrow \neg \mathsf{Con}_\tau(\overline{n}))\urcorner, z) \wedge \forall y < z \neg \delta(\overline{n}, y)).$$

Hence $T \vdash \neg\alpha(\overline{n})$. Then $T \vdash \neg \mathsf{Con}_\tau^{n+1} \rightarrow \neg \mathsf{Con}_\tau^n$. This contracts the assumption that the height of $\tau(v)$ is ∞. We conclude $n \in \alpha$. ⊣

For any formulas $\gamma(v)$ and $\tau(v)$, define $\xi_{\gamma,\tau}$ to be the sentence

$$\forall z(\alpha_\tau(z) \wedge \mathsf{Pr}_\gamma(\ulcorner \neg \mathsf{Con}_\tau(\dot{z})\urcorner) \rightarrow \neg \mathsf{Con}_\tau(z)).$$

Since the formula $\alpha_\tau(v)$ is effectively obtained from $\tau(v)$, the sentence $\xi_{\gamma,\tau}$ is also effectively calculable from $\gamma(v)$ and $\tau(v)$. By Fixed Point Lemma, let $\tau_\alpha(v)$ be a Σ_1 formula satisfying the following equivalence:

$$I\Sigma_1 \vdash \forall v(\tau_\alpha(v) \leftrightarrow \sigma(v) \vee \exists x(v = \ulcorner \xi_{\sigma,\tau_\alpha} \rightarrow \dot{x}\urcorner \wedge \mu(x))),$$

where $\ulcorner \xi_{\sigma,\dot{y}} \rightarrow \dot{x}\urcorner$ is a primitive recursive term corresponding to a function calculating the Gödel number of the sentence $\xi_{\sigma,\tau} \rightarrow \varphi$ from an input (φ, τ). Let $T(\alpha)$

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be the theory $I\Sigma_1 + \{\xi_{\sigma,\tau_\alpha} \to \varphi : \varphi \in U\}$, then $\tau_\alpha(v)$ is a Σ_1 definition of the theory $T(\alpha)$. Notice that $T(\alpha)$ is a subtheory of U.

THEOREM 3.2. *The theories U and $T(\alpha) + \xi_{\tau_\alpha,\tau_\alpha}$ are deductively equivalent.*

PROOF. First, we prove $T(\alpha) + \xi_{\tau_\alpha,\tau_\alpha} \vdash U$. Since $I\Sigma_1 \vdash \forall v(\sigma(v) \to \tau_\alpha(v))$ by the choice of $\tau_\alpha(v)$, we have $I\Sigma_1 \vdash \forall x(\mathsf{Pr}_\sigma(x) \to \mathsf{Pr}_{\tau_\alpha}(x))$. Then we obtain $I\Sigma_1 \vdash \xi_{\tau_\alpha,\tau_\alpha} \to \xi_{\sigma,\tau_\alpha}$. By the definition of $T(\alpha)$, $T(\alpha) \vdash \xi_{\sigma,\tau_\alpha} \to \varphi$ for all $\varphi \in U$. We conclude $T(\alpha) + \xi_{\tau_\alpha,\tau_\alpha} \vdash U$.

We prove $U \vdash T(\alpha) + \xi_{\tau_\alpha,\tau_\alpha}$. For this purpose, it suffices to prove $\mathsf{PA} \vdash \xi_{\tau_\alpha,\tau_\alpha}$. First, we prove

$$I\Sigma_1 \vdash \mathsf{Pr}_{\tau_\alpha}(x) \to \mathsf{Pr}_\sigma(\ulcorner \neg\xi_{\sigma,\tau_\alpha} \to \dot{x} \urcorner). \tag{1}$$

Reason in $I\Sigma_1$: Suppose $T(\alpha) \vdash \varphi$. Then there are $\psi_0, \ldots, \psi_{l-1} \in U$ such that $I\Sigma_1 \vdash \bigwedge_{i<l}(\xi_{\sigma,\tau_\alpha} \to \psi_i) \to \varphi$. Since $I\Sigma_1 \vdash \neg\xi_{\sigma,\tau_\alpha} \to \bigwedge_{i<l}(\xi_{\sigma,\tau_\alpha} \to \psi_i)$, we have $I\Sigma_1 \vdash \neg\xi_{\sigma,\tau_\alpha} \to \varphi$. This shows (1).

Since

$$I\Sigma_1 \vdash \alpha_{\tau_\alpha}(z) \wedge \mathsf{Pr}_\sigma(\ulcorner \neg\mathsf{Con}_{\tau_\alpha}(\dot{z}) \urcorner) \wedge \mathsf{Con}_{\tau_\alpha}(z) \to \neg\xi_{\sigma,\tau_\alpha},$$

we have

$$I\Sigma_1 \vdash \mathsf{Pr}_\sigma(\ulcorner \alpha_{\tau_\alpha}(\dot{z}) \wedge \mathsf{Pr}_\sigma(\ulcorner \neg\mathsf{Con}_{\tau_\alpha}(\dot{z}) \urcorner) \wedge \mathsf{Con}_{\tau_\alpha}(\dot{z}) \to \neg\xi_{\sigma,\tau_\alpha} \urcorner).$$

By combining this with (1), we obtain

$$I\Sigma_1 \vdash \mathsf{Pr}_{\tau_\alpha}(x) \to \mathsf{Pr}_\sigma(\ulcorner \alpha_{\tau_\alpha}(\dot{z}) \wedge \mathsf{Pr}_\sigma(\ulcorner \neg\mathsf{Con}_{\tau_\alpha}(\dot{z}) \urcorner) \wedge \mathsf{Con}_{\tau_\alpha}(\dot{z}) \to \dot{x} \urcorner).$$

Since $\alpha_{\tau_\alpha}(z)$ is a Σ_1 formula, $I\Sigma_1 \vdash \alpha_{\tau_\alpha}(z) \to \mathsf{Pr}_\sigma(\ulcorner \alpha_{\tau_\alpha}(\dot{z}) \urcorner)$ by formalized Σ_1-completeness. Thus

$$I\Sigma_1 \vdash \alpha_{\tau_\alpha}(z) \wedge \mathsf{Pr}_{\tau_\alpha}(x) \to \mathsf{Pr}_\sigma(\ulcorner \mathsf{Pr}_\sigma(\ulcorner \neg\mathsf{Con}_{\tau_\alpha}(\dot{z}) \urcorner) \wedge \mathsf{Con}_{\tau_\alpha}(\dot{z}) \to \dot{x} \urcorner).$$

By substituting $\ulcorner \neg\mathsf{Con}_{\tau_\alpha}(\dot{z}) \urcorner$ to x, we have

$$I\Sigma_1 \vdash \alpha_{\tau_\alpha}(z) \wedge \mathsf{Pr}_{\tau_\alpha}(\ulcorner \neg\mathsf{Con}_{\tau_\alpha}(\dot{z}) \urcorner) \to \mathsf{Pr}_\sigma(\ulcorner \mathsf{Pr}_\sigma(\ulcorner \neg\mathsf{Con}_{\tau_\alpha}(\dot{z}) \urcorner) \wedge \mathsf{Con}_{\tau_\alpha}(\dot{z}) \to \neg\mathsf{Con}_{\tau_\alpha}(\dot{z}) \urcorner).$$

Then

$$I\Sigma_1 \vdash \alpha_{\tau_\alpha}(z) \wedge \mathsf{Pr}_{\tau_\alpha}(\ulcorner \neg\mathsf{Con}_{\tau_\alpha}(\dot{z}) \urcorner) \to \mathsf{Pr}_\sigma(\ulcorner \mathsf{Pr}_\sigma(\ulcorner \neg\mathsf{Con}_{\tau_\alpha}(\dot{z}) \urcorner) \to \neg\mathsf{Con}_{\tau_\alpha}(\dot{z}) \urcorner).$$

By formalized Löb's theorem, we obtain

$$I\Sigma_1 \vdash \alpha_{\tau_\alpha}(z) \wedge \mathsf{Pr}_{\tau_\alpha}(\ulcorner \neg\mathsf{Con}_{\tau_\alpha}(\dot{z}) \urcorner) \to \mathsf{Pr}_\sigma(\ulcorner \neg\mathsf{Con}_{\tau_\alpha}(\dot{z}) \urcorner).$$

Thus

$$\mathsf{PA} \vdash \alpha_{\tau_\alpha}(z) \wedge \mathsf{Pr}_{\tau_\alpha}(\ulcorner \neg\mathsf{Con}_{\tau_\alpha}(\dot{z}) \urcorner) \to \neg\mathsf{Con}_{\tau_\alpha}(z)$$

because $\mathsf{PA} \vdash \mathsf{RFN}_{\Sigma_1}(\sigma)$. We have proved $\mathsf{PA} \vdash \xi_{\tau_\alpha,\tau_\alpha}$. Therefore U and $T(\alpha) + \xi_{\tau_\alpha,\tau_\alpha}$ are deductively equivalent. ⊣

LEMMA 3.3. *The theory $T(\alpha) + \forall z\mathsf{Con}_{\tau_\alpha}(z)$ is consistent.*

PROOF. Since $I\Sigma_1 \vdash \forall v(\exists x(v = \ulcorner \xi_{\sigma,\tau_\alpha} \to \dot{x} \urcorner \wedge \mu(x)) \to \mathsf{Pr}_\mu(v))$ and $I\Sigma_1 \vdash \forall v(\sigma(v) \to \mu(v))$, we obtain $I\Sigma_1 \vdash \forall v(\tau_\alpha(v) \to \mathsf{Pr}_\mu(v))$ by the choice of $\tau_\alpha(v)$. Thus $I\Sigma_1 \vdash \forall x(\mathsf{Pr}_{\tau_\alpha}(x) \to \mathsf{Pr}_\mu(x))$.

Let $\chi(z)$ be the formula $\mathsf{Con}_\mu(z) \to \mathsf{Con}_{\tau_\alpha}(z)$. First, we prove $I\Sigma_1 \vdash \forall z\mathsf{Pr}_\sigma(\ulcorner \chi(\dot{z}) \urcorner)$ by using Σ_1 induction. The sentence $\mathsf{Pr}_\sigma(\ulcorner \chi(0) \urcorner)$ is obviously provable in $I\Sigma_1$. Since $I\Sigma_1 \vdash \forall x(\mathsf{Pr}_\sigma(x) \to \mathsf{Pr}_{\tau_\alpha}(x))$,

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$$I\Sigma_1 \vdash \mathsf{Pr}_\sigma(\ulcorner \chi(\dot{z})\urcorner) \to \mathsf{Pr}_{\tau_\alpha}(\ulcorner \chi(\dot{z})\urcorner),$$

and hence

$$I\Sigma_1 \vdash \mathsf{Pr}_\sigma(\ulcorner \chi(\dot{z})\urcorner) \to (\mathsf{Pr}_{\tau_\alpha}(\ulcorner \neg \mathsf{Con}_{\tau_\alpha}(\dot{z})\urcorner) \to \mathsf{Pr}_{\tau_\alpha}(\ulcorner \neg \mathsf{Con}_\mu(\dot{z})\urcorner)).$$

Since $I\Sigma_1 \vdash \forall x(\mathsf{Pr}_{\tau_\alpha}(x) \to \mathsf{Pr}_\mu(x))$, we obtain

$$I\Sigma_1 \vdash \mathsf{Pr}_\sigma(\ulcorner \chi(\dot{z})\urcorner) \to (\mathsf{Pr}_{\tau_\alpha}(\ulcorner \neg \mathsf{Con}_{\tau_\alpha}(\dot{z})\urcorner) \to \mathsf{Pr}_\mu(\ulcorner \neg \mathsf{Con}_\mu(\dot{z})\urcorner)).$$

This implies $I\Sigma_1 \vdash \mathsf{Pr}_\sigma(\ulcorner \chi(\dot{z})\urcorner) \to \chi(z+1)$, and thus $I\Sigma_1 \vdash \mathsf{Pr}_\sigma(\ulcorner \mathsf{Pr}_\sigma(\ulcorner \chi(\dot{z})\urcorner) \to \chi(\dot{z}+1)\urcorner)$. We obtain

$$I\Sigma_1 \vdash \mathsf{Pr}_\sigma(\ulcorner \chi(\dot{z})\urcorner) \to \mathsf{Pr}_\sigma(\ulcorner \chi(\dot{z}+1)\urcorner)$$

because $I\Sigma_1 \vdash \mathsf{Pr}_\sigma(\ulcorner \chi(\dot{z})\urcorner) \to \mathsf{Pr}_\sigma(\ulcorner \mathsf{Pr}_\sigma(\ulcorner \chi(\dot{z})\urcorner)\urcorner)$. By Σ_1 induction, we obtain $I\Sigma_1 \vdash \forall z \mathsf{Pr}_\sigma(\ulcorner \chi(\dot{z})\urcorner)$.

Since $I\Sigma_1 \vdash \mathsf{Pr}_\sigma(\ulcorner \chi(\dot{z})\urcorner) \to \chi(z+1)$, we obtain $I\Sigma_1 \vdash \forall z \chi(z+1)$. Also $I\Sigma_1 \vdash \chi(0)$, and hence $I\Sigma_1 \vdash \forall z \chi(z)$. This means $I\Sigma_1 \vdash \forall z(\mathsf{Con}_\mu(z) \to \mathsf{Con}_{\tau_\alpha}(z))$. Thus $I\Sigma_1 \vdash \forall z \mathsf{Con}_\mu(z) \to \forall z \mathsf{Con}_{\tau_\alpha}(z)$. Then $T(\alpha) + \forall z \mathsf{Con}_{\tau_\alpha}(z)$ is consistent because it is a subtheory of the consistent theory $U + \forall z \mathsf{Con}_\mu(z)$. ⊣

Therefore $\tau_\alpha(v)$ is a Σ_1 definition of $T(\alpha)$ with the height ∞. Then, by Lemma 3.1, the following two conditions hold:

1. If $n \in \alpha$, then $\mathbb{N} \models \alpha_{\tau_\alpha}(\overline{n})$.
2. If $T(\alpha) \vdash \neg\alpha_{\tau_\alpha}(\overline{n}) \to (\neg \mathsf{Con}^{n+1}_{\tau_\alpha} \to \neg \mathsf{Con}^n_{\tau_\alpha})$, then $n \in \alpha$.

LEMMA 3.4. $\mathsf{tr}(\mathsf{PL}_{\tau_\alpha}(U)) = \alpha$.

PROOF. By Proposition 2.11, it suffices to prove that for any $n \in \omega$, $n \in \alpha$ if and only if $U \vdash f_{\tau_\alpha}(F_n)$.

If $n \in \alpha$, then $\mathbb{N} \models \alpha_{\tau_\alpha}(\overline{n})$ by Lemma 3.1. Thus $U \vdash \alpha_{\tau_\alpha}(\overline{n})$ by the Σ_1-completeness of U. By Theorem 3.2, $U \vdash \xi_{\tau_\alpha,\tau_\alpha}$. It follows that U proves $\forall z(\alpha_{\tau_\alpha}(z) \to (\neg \mathsf{Con}_{\tau_\alpha}(z+1) \to \neg \mathsf{Con}_{\tau_\alpha}(z)))$ because the sentences $\neg \mathsf{Con}_{\tau_\alpha}(z+1)$ and $\mathsf{Pr}_{\tau_\alpha}(\ulcorner \neg \mathsf{Con}_{\tau_\alpha}(\dot{z})\urcorner)$ are equivalent in $I\Sigma_1$. Hence $U \vdash \neg \mathsf{Con}^{n+1}_{\tau_\alpha} \to \neg \mathsf{Con}^n_{\tau_\alpha}$, and thus $U \vdash f_{\tau_\alpha}(F_n)$.

If $U \vdash f_{\tau_\alpha}(F_n)$, then $T(\alpha) \vdash \xi_{\tau_\alpha,\tau_\alpha} \to f_{\tau_\alpha}(F_n)$ by Theorem 3.2. Let η_n be the sentence $\forall z(z \neq \overline{n} \to (\neg \mathsf{Con}_{\tau_\alpha}(z+1) \to \neg \mathsf{Con}_{\tau_\alpha}(z)))$. Then $I\Sigma_1 \vdash \eta_n \wedge \neg\alpha_{\tau_\alpha}(\overline{n}) \to \xi_{\tau_\alpha,\tau_\alpha}$. Thus

$$T(\alpha) \vdash \eta_n \wedge \neg\alpha_{\tau_\alpha}(\overline{n}) \to f_{\tau_\alpha}(F_n). \tag{2}$$

Since $I\Sigma_1 \vdash \overline{n} > z \wedge \neg \mathsf{Con}_{\tau_\alpha}(z+1) \to \neg \mathsf{Con}^n_{\tau_\alpha}$, we have $I\Sigma_1 \vdash \overline{n} > z \wedge \neg \mathsf{Con}_{\tau_\alpha}(z+1) \to f_{\tau_\alpha}(F_n)$. Also $I\Sigma_1 \vdash \overline{n} < z \wedge \mathsf{Con}_{\tau_\alpha}(z) \to f_{\tau_\alpha}(F_n)$ because $I\Sigma_1 \vdash \overline{n} < z \wedge \mathsf{Con}_{\tau_\alpha}(z) \to \mathsf{Con}^{n+1}_{\tau_\alpha}$. Hence we have $I\Sigma_1 \vdash z \neq \overline{n} \wedge \neg \mathsf{Con}_{\tau_\alpha}(z+1) \wedge \mathsf{Con}_{\tau_\alpha}(z) \to f_{\tau_\alpha}(F_n)$. Thus

$$I\Sigma_1 \vdash \neg\eta_n \to f_{\tau_\alpha}(F_n). \tag{3}$$

By (2) and (3), we obtain $T(\alpha) \vdash \neg\alpha_{\tau_\alpha}(\overline{n}) \to f_{\tau_\alpha}(F_n)$. This means $T(\alpha) \vdash \neg\alpha_{\tau_\alpha}(\overline{n}) \to (\neg \mathsf{Con}^{n+1}_{\tau_\alpha} \to \neg \mathsf{Con}^n_{\tau_\alpha})$. By Lemma 3.1, we conclude $n \in \alpha$. ⊣

THEOREM 3.5. $\mathsf{PL}_{\tau_\alpha}(U) = \mathsf{GL}_\alpha$.

PROOF. By Lemma 3.4, $\mathsf{tr}(\mathsf{PL}_{\tau_\alpha}(U)) = \alpha$. If α is not cofinite, then $\mathsf{PL}_{\tau_\alpha}(U) = \mathsf{GL}_\alpha$ by the classification theorem.

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If α is cofinite, then $\mathsf{PL}_{\tau_\alpha}(U)$ is one of GL_α, D_α, S_α, and GL^-_α by the classification theorem.

Since the sentence $\forall z \mathsf{Con}_{\tau_\alpha}(z)$ is Π_1 and $I\Sigma_1 \vdash \forall z \mathsf{Con}_{\tau_\alpha}(z) \to \mathsf{Con}_{\tau_\alpha}$, there exists a Σ_1 sentence ψ such that $I\Sigma_1 \vdash \forall z \mathsf{Con}_{\tau_\alpha}(z) \leftrightarrow \neg \mathsf{Pr}_{\tau_\alpha}(\ulcorner \psi \urcorner)$ by the FGH-principle.

Towards a contradiction, suppose

$$U \vdash \bigwedge_{n \notin \alpha} f_{\tau_\alpha}(F_n) \to (\mathsf{Pr}_{\tau_\alpha}(\ulcorner \mathsf{Pr}_{\tau_\alpha}(\ulcorner \psi \urcorner) \urcorner) \to \mathsf{Pr}_{\tau_\alpha}(\ulcorner \psi \urcorner)).$$

Then

$$U \vdash \bigwedge_{n \notin \alpha} f_{\tau_\alpha}(F_n) \to (\forall z \mathsf{Con}_{\tau_\alpha}(z) \to \neg \mathsf{Pr}_{\tau_\alpha}(\ulcorner \neg \forall z \mathsf{Con}_{\tau_\alpha}(z) \urcorner)).$$

Since $T(\alpha) + \xi_{\tau_\alpha, \tau_\alpha} \vdash U$ by Theorem 3.2, and $\forall z \mathsf{Con}_{\tau_\alpha}(z)$ implies the sentences $\xi_{\tau_\alpha, \tau_\alpha}$ and $\bigwedge_{n \notin \alpha} f_{\tau_\alpha}(F_n)$, we obtain

$$T(\alpha) + \forall z \mathsf{Con}_{\tau_\alpha}(z) \vdash \neg \mathsf{Pr}_{\tau_\alpha}(\ulcorner \neg \forall z \mathsf{Con}_{\tau_\alpha}(z) \urcorner).$$

Therefore $T(\alpha) + \forall z \mathsf{Con}_{\tau_\alpha}(z) \vdash \mathsf{Con}_{\tau_\alpha + \forall z \mathsf{Con}_{\tau_\alpha}(z)}$. Since $\tau_\alpha(v)$ is a Σ_1 definition of $T(\alpha)$, $T(\alpha) + \forall z \mathsf{Con}_{\tau_\alpha}(z)$ is inconsistent by the second incompleteness theorem. This contradicts Lemma 3.3.

We have proved

$$U \nvdash \bigwedge_{n \notin \alpha} f_{\tau_\alpha}(F_n) \to (\mathsf{Pr}_{\tau_\alpha}(\ulcorner \mathsf{Pr}_{\tau_\alpha}(\ulcorner \psi \urcorner) \urcorner) \to \mathsf{Pr}_{\tau_\alpha}(\ulcorner \psi \urcorner)).$$

Hence the modal formula $\bigwedge_{n \notin \alpha} F_n \to (\Box\Box p \to \Box p)$, which is a theorem of D_α, is not in $\mathsf{PL}_{\tau_\alpha}(U)$. Therefore $\mathsf{D}_\alpha \nsubseteq \mathsf{PL}_{\tau_\alpha}(U)$. We conclude $\mathsf{PL}_{\tau_\alpha}(U) = \mathsf{GL}_\alpha$. ⊣

§4. **The logics GL^-_β.** In this section, we fix a cofinite $\beta \subseteq \omega$. We prove the existence of a Σ_1 definition $\tau(v)$ of an extension of $I\Sigma_1$ such that $\mathsf{PL}_\tau(U) = \mathsf{GL}^-_\beta$.

By Theorem 3.5, the Σ_1 definition $\tau_\beta(v)$ of the theory $T(\beta)$ satisfies $\mathsf{PL}_{\tau_\beta}(U) = \mathsf{GL}_\beta$. Let $k = \max\{n : n \notin \beta\}$, and $\tau'_\beta(v)$ be the Σ_1 formula $\tau_\beta(v) \vee v = \ulcorner \neg \mathsf{Con}^k_{\tau_\beta} \urcorner$. Then $\tau'_\beta(v)$ is a Σ_1 definition of the theory $T(\beta)' = T(\beta) + \neg \mathsf{Con}^k_{\tau_\beta}$.

LEMMA 4.1. $U \vdash \neg \mathsf{Con}^{k+1}_{\tau'_\beta}$.

PROOF. Since $I\Sigma_1 \vdash \forall x(\mathsf{Pr}_{\tau_\beta}(x) \to \mathsf{Pr}_{\tau'_\beta}(x))$, we have $I\Sigma_1 \vdash \neg \mathsf{Con}^n_{\tau_\beta} \to \neg \mathsf{Con}^n_{\tau'_\beta}$ for any $n \in \omega$. Hence $T(\beta)' \vdash \neg \mathsf{Con}^k_{\tau'_\beta}$ because $T(\beta)' \vdash \neg \mathsf{Con}^k_{\tau_\beta}$ by the definition of $T(\beta)'$. Then $U \vdash \mathsf{Pr}_{\tau'_\beta}(\ulcorner \neg \mathsf{Con}^k_{\tau'_\beta} \urcorner)$ and thus $U \vdash \neg \mathsf{Con}^{k+1}_{\tau'_\beta}$. ⊣

LEMMA 4.2. *For all $n \leq k$, $I\Sigma_1 \vdash \mathsf{Con}^n_{\tau'_\beta} \leftrightarrow \mathsf{Con}^n_{\tau_\beta}$.*

PROOF. In our proof of Lemma 4.1, we showed $I\Sigma_1 \vdash \mathsf{Con}^n_{\tau'_\beta} \to \mathsf{Con}^n_{\tau_\beta}$ for any $n \in \omega$. Thus it suffices to prove that $I\Sigma_1 \vdash \neg \mathsf{Con}^n_{\tau'_\beta} \to \neg \mathsf{Con}^n_{\tau_\beta}$ holds for any $n \leq k$.

We prove $I\Sigma_1 \vdash \neg \mathsf{Con}^n_{\tau'_\beta} \to \neg \mathsf{Con}^n_{\tau_\beta}$ by induction on $n \leq k$. The case $n = 0$ is obvious. Let $n + 1 \leq k$ and suppose $I\Sigma_1 \vdash \neg \mathsf{Con}^n_{\tau'_\beta} \to \neg \mathsf{Con}^n_{\tau_\beta}$. Since $T(\beta)'$

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proves this sentence, $I\Sigma_1 \vdash \mathsf{Pr}_{\tau'_\beta}(\ulcorner\neg\mathsf{Con}^n_{\tau'_\beta}\urcorner) \to \mathsf{Pr}_{\tau'_\beta}(\ulcorner\neg\mathsf{Con}^n_{\tau_\beta}\urcorner)$, and thus $I\Sigma_1 \vdash \neg\mathsf{Con}^{n+1}_{\tau'_\beta} \to \mathsf{Pr}_{\tau'_\beta}(\ulcorner\neg\mathsf{Con}^n_{\tau_\beta}\urcorner)$. By the definition of $\tau'_\beta(v)$, we have

$$I\Sigma_1 \vdash \neg\mathsf{Con}^{n+1}_{\tau'_\beta} \to \mathsf{Pr}_{\tau_\beta}(\ulcorner\neg\mathsf{Con}^k_{\tau_\beta} \to \neg\mathsf{Con}^n_{\tau_\beta}\urcorner).$$

Since $n < k$, we have $I\Sigma_1 \vdash \mathsf{Pr}_{\tau_\beta}(\ulcorner\neg\mathsf{Con}^n_{\tau_\beta}\urcorner) \to \neg\mathsf{Con}^k_{\tau_\beta}$, and thus

$$I\Sigma_1 \vdash \neg\mathsf{Con}^{n+1}_{\tau'_\beta} \to \mathsf{Pr}_{\tau_\beta}(\ulcorner\mathsf{Pr}_{\tau_\beta}(\ulcorner\neg\mathsf{Con}^n_{\tau_\beta}\urcorner) \to \neg\mathsf{Con}^n_{\tau_\beta}\urcorner).$$

By the formalized Löb's theorem,

$$I\Sigma_1 \vdash \neg\mathsf{Con}^{n+1}_{\tau'_\beta} \to \mathsf{Pr}_{\tau_\beta}(\ulcorner\neg\mathsf{Con}^n_{\tau_\beta}\urcorner).$$

Then we obtain

$$I\Sigma_1 \vdash \neg\mathsf{Con}^{n+1}_{\tau'_\beta} \to \neg\mathsf{Con}^{n+1}_{\tau_\beta}.$$

This completes the proof. ⊣

LEMMA 4.3. $\mathsf{tr}(\mathsf{PL}_{\tau'_\beta}(U)) = \beta$.

PROOF. By Proposition 2.11, it suffices to prove that for any $n \in \omega$, $n \in \beta$ if and only if $U \vdash f_{\tau'_\beta}(F_n)$. We distinguish the following three cases.

1. $n < k$: By Lemma 3.4, $n \in \beta$ if and only if $U \vdash f_{\tau_\beta}(F_n)$. By Lemma 4.2, $I\Sigma_1 \vdash f_{\tau_\beta}(F_n) \leftrightarrow f_{\tau'_\beta}(F_n)$. Therefore $n \in \beta$ if and only if $U \vdash f_{\tau'_\beta}(F_n)$.
2. $n = k$: Since $k \notin \beta$, $U \nvdash f_{\tau_\beta}(F_k)$ by Lemma 3.4. Thus $U \nvdash \neg\mathsf{Con}^k_{\tau_\beta}$. By Lemma 4.2, $U \nvdash \neg\mathsf{Con}^k_{\tau'_\beta}$. Since $U \vdash \neg\mathsf{Con}^{k+1}_{\tau'_\beta}$ by Lemma 4.1, we also have $U \nvdash \neg\mathsf{Con}^{k+1}_{\tau'_\beta} \to \neg\mathsf{Con}^k_{\tau'_\beta}$. Therefore $U \nvdash f_{\tau'_\beta}(F_k)$.
3. $n > k$: In this case, $n \in \beta$. Also $U \vdash f_{\tau'_\beta}(F_n)$ because $U \vdash \neg\mathsf{Con}^{k+1}_{\tau'_\beta}$ by Lemma 4.1. ⊣

THEOREM 4.4. $\mathsf{PL}_{\tau'_\beta}(U) = \mathsf{GL}^-_\beta$.

PROOF. By Lemma 4.3, we obtain $\mathsf{tr}(\mathsf{PL}_{\tau'_\beta}(U)) = \beta$. By the classification theorem, $\mathsf{PL}_{\tau'_\beta}(U)$ is one of GL_β, D_β, S_β and GL^-_β. Notice that $\Box^{k+1}\bot \in \mathsf{PL}_{\tau'_\beta}(U)$ by Lemma 4.1. Since $\mathsf{S}_\beta \subseteq \mathsf{S}$ and $\mathsf{S} \vdash \neg\Box^{k+1}\bot$, we have $\mathsf{PL}_{\tau'_\beta}(U) \nsubseteq \mathsf{S}_\beta$. We conclude $\mathsf{PL}_{\tau'_\beta}(U) = \mathsf{GL}^-_\beta$. ⊣

§5. A method of obtaining Σ_1 definitions with some properties. In this section, we give a method of obtaining Σ_1 definitions satisfying desirable conditions. We will use this method in the next two sections.

In this section, we also fix a cofinite subset β of ω. Let $\gamma(v)$ be any Σ_1 definition of some theory T. Then let $\gamma^*_\beta(v)$ be a Σ_1 formula satisfying the following equivalence:

$$I\Sigma_1 \vdash \forall v \left(\gamma^*_\beta(v) \leftrightarrow \gamma(v) \vee \bigvee_{m \notin \beta} (\neg\mathsf{Con}^{m+1}_\mu \wedge v = \ulcorner\neg\mathsf{Con}^m_{\gamma^*_\beta}\urcorner) \right).$$

We prove the following theorem.

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THEOREM 5.1. *Let $i \geq 1$. Suppose that a Σ_1 definition $\gamma(v)$ of a theory T satisfies the following conditions*:

1. $I\Sigma_1 \vdash \forall x(\mathsf{Pr}_\gamma(x) \to \mathsf{Pr}_\mu(x))$,
2. $U \vdash \mathsf{Pr}_\gamma(\ulcorner\varphi\urcorner) \to \varphi$ *for all* Σ_i *sentences* φ.

Then the formula $\gamma^*_\beta(v)$ is a Σ_1 definition of T satisfying the following conditions:

1. $I\Sigma_1 \vdash \forall x(\mathsf{Pr}_\gamma(x) \to \mathsf{Pr}_{\gamma^*_\beta}(x))$,
2. $I\Sigma_1 \vdash \mathsf{Con}^{k+1}_\mu \to \forall x(\mathsf{Pr}_{\gamma^*_\beta}(x) \to \mathsf{Pr}_\gamma(x))$ *for* $k = \max\{n : n \notin \beta\}$,
3. *for any* $n \in \omega$, $n \in \beta$ *if and only if* $U \vdash f_{\gamma^*_\beta}(F_n)$,
4. *for any* Σ_i *sentence* φ, $U \vdash \bigwedge_{m \notin \beta} f_{\gamma^*_\beta}(F_m) \to (\mathsf{Pr}_{\gamma^*_\beta}(\ulcorner\varphi\urcorner) \to \varphi)$,
5. $U \nvdash \bigvee_{m \notin \beta} \neg f_{\gamma^*_\beta}(F_m)$.

Let $\gamma(v)$ satisfy the assumptions of Theorem 5.1. We prove Theorem 5.1 by showing the following several lemmas.

LEMMA 5.2. *For any $n \notin \beta$, $I\Sigma_1 \vdash \neg\mathsf{Con}^{n+1}_\mu \to \neg\mathsf{Con}^{n+1}_{\gamma^*_\beta}$.*

PROOF. Let $n \notin \beta$. Since β is cofinite, we have

$$I\Sigma_1 \vdash \neg\mathsf{Con}^{n+1}_\mu \to \bigvee_{m \notin \beta} (\neg\mathsf{Con}^{m+1}_\mu \wedge \ulcorner\neg\mathsf{Con}^n_{\gamma^*_\beta}\urcorner = \ulcorner\neg\mathsf{Con}^m_{\gamma^*_\beta}\urcorner).$$

By the definition of $\gamma^*_\beta(v)$, $I\Sigma_1 \vdash \neg\mathsf{Con}^{n+1}_\mu \to \gamma^*_\beta(\ulcorner\neg\mathsf{Con}^n_{\gamma^*_\beta}\urcorner)$. Then $I\Sigma_1 \vdash \neg\mathsf{Con}^{n+1}_\mu \to \mathsf{Pr}_{\gamma^*_\beta}(\ulcorner\neg\mathsf{Con}^n_{\gamma^*_\beta}\urcorner)$, and hence $I\Sigma_1 \vdash \neg\mathsf{Con}^{n+1}_\mu \to \neg\mathsf{Con}^{n+1}_{\gamma^*_\beta}$. ⊣

LEMMA 5.3. *For any $n \in \omega$, $I\Sigma_1 \vdash \mathsf{Con}^n_\mu \to \mathsf{Con}^n_{\gamma^*_\beta}$.*

PROOF. We prove by induction on n. The case $n = 0$ is trivial. Suppose $I\Sigma_1 \vdash \mathsf{Con}^n_\mu \to \mathsf{Con}^n_{\gamma^*_\beta}$. By the definition of $\gamma^*_\beta(v)$, we have

$$I\Sigma_1 \vdash \mathsf{Con}^{n+1}_\mu \wedge \gamma^*_\beta(v) \to \left(\gamma(v) \vee \bigvee_{m \notin \beta, m \geq n+1} v = \ulcorner\neg\mathsf{Con}^m_{\gamma^*_\beta}\urcorner\right) \tag{4}$$

because $m < n+1$ implies $I\Sigma_1 \vdash \mathsf{Con}^{n+1}_\mu \to \mathsf{Con}^{m+1}_\mu$. For each $m \geq n+1$, $I\Sigma_1 \vdash \neg\mathsf{Con}^{n+1}_{\gamma^*_\beta} \to \neg\mathsf{Con}^m_{\gamma^*_\beta}$ and $I\Sigma_1 \vdash \neg\mathsf{Con}^m_{\gamma^*_\beta} \to \mathsf{Pr}_\gamma(\ulcorner\neg\mathsf{Con}^m_{\gamma^*_\beta}\urcorner)$. Thus

$$I\Sigma_1 \vdash \neg\mathsf{Con}^{n+1}_{\gamma^*_\beta} \to \left(\bigvee_{m \notin \beta, m \geq n+1} v = \ulcorner\neg\mathsf{Con}^m_{\gamma^*_\beta}\urcorner \to \mathsf{Pr}_\gamma(v)\right).$$

Moreover, since $I\Sigma_1 \vdash \gamma(v) \to \mathsf{Pr}_\gamma(v)$, we obtain

$$I\Sigma_1 \vdash \neg\mathsf{Con}^{n+1}_{\gamma^*_\beta} \to \left(\left(\gamma(v) \vee \bigvee_{m \notin \beta, m \geq n+1} v = \ulcorner\neg\mathsf{Con}^m_{\gamma^*_\beta}\urcorner\right) \to \mathsf{Pr}_\gamma(v)\right). \tag{5}$$

By (4) and (5), $I\Sigma_1 \vdash \mathsf{Con}^{n+1}_\mu \wedge \neg\mathsf{Con}^{n+1}_{\gamma^*_\beta} \to (\gamma^*_\beta(v) \to \mathsf{Pr}_\gamma(v))$. Therefore

$$I\Sigma_1 \vdash \mathsf{Con}^{n+1}_\mu \wedge \neg\mathsf{Con}^{n+1}_{\gamma^*_\beta} \to \forall x(\mathsf{Pr}_{\gamma^*_\beta}(x) \to \mathsf{Pr}_\gamma(x)).$$

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Since $I\Sigma_1 \vdash \neg\mathsf{Con}^{n+1}_{\gamma^*_\beta} \to \mathsf{Pr}_{\gamma^*_\beta}(\ulcorner\neg\mathsf{Con}^{n}_{\gamma^*_\beta}\urcorner)$,

$$I\Sigma_1 \vdash \mathsf{Con}^{n+1}_{\mu} \wedge \neg\mathsf{Con}^{n+1}_{\gamma^*_\beta} \to \mathsf{Pr}_{\gamma}(\ulcorner\neg\mathsf{Con}^{n}_{\gamma^*_\beta}\urcorner).$$

By induction hypothesis, we have $I\Sigma_1 \vdash \mathsf{Pr}_{\gamma}(\ulcorner\neg\mathsf{Con}^{n}_{\gamma^*_\beta}\urcorner) \to \mathsf{Pr}_{\gamma}(\ulcorner\neg\mathsf{Con}^{n}_{\mu}\urcorner)$, and hence

$$I\Sigma_1 \vdash \mathsf{Con}^{n+1}_{\mu} \wedge \neg\mathsf{Con}^{n+1}_{\gamma^*_\beta} \to \mathsf{Pr}_{\gamma}(\ulcorner\neg\mathsf{Con}^{n}_{\mu}\urcorner).$$

Since $I\Sigma_1 \vdash \forall x(\mathsf{Pr}_{\gamma}(x) \to \mathsf{Pr}_{\mu}(x))$,

$$I\Sigma_1 \vdash \mathsf{Con}^{n+1}_{\mu} \wedge \neg\mathsf{Con}^{n+1}_{\gamma^*_\beta} \to \mathsf{Pr}_{\mu}(\ulcorner\neg\mathsf{Con}^{n}_{\mu}\urcorner).$$

Then $I\Sigma_1 \vdash \mathsf{Con}^{n+1}_{\mu} \wedge \neg\mathsf{Con}^{n+1}_{\gamma^*_\beta} \to \neg\mathsf{Con}^{n+1}_{\mu}$, and we finally obtain

$$I\Sigma_1 \vdash \mathsf{Con}^{n+1}_{\mu} \to \mathsf{Con}^{n+1}_{\gamma^*_\beta}.$$

This completes our proof. ⊣

LEMMA 5.4. *For any $n \notin \beta$, $U \nvdash f_{\gamma^*_\beta}(F_n)$.*

PROOF. Let $n \notin \beta$. Then by Lemma 5.2, $I\Sigma_1 \vdash \neg\mathsf{Con}^{n+1}_{\mu} \to \neg\mathsf{Con}^{n+1}_{\gamma^*_\beta}$. Also by Lemma 5.3, $I\Sigma_1 \vdash \neg\mathsf{Con}^{n}_{\gamma^*_\beta} \to \neg\mathsf{Con}^{n}_{\mu}$. Therefore $I\Sigma_1 \vdash f_{\gamma^*_\beta}(F_n) \to f_{\mu}(F_n)$. Since $U \nvdash \neg\mathsf{Con}^{n}_{\mu}$, we have $U \nvdash f_{\mu}(F_n)$ by Löb's theorem. Hence $U \nvdash f_{\gamma^*_\beta}(F_n)$. ⊣

LEMMA 5.5. *For any Σ_i sentence φ, $U \vdash \bigwedge_{m\notin\beta} f_{\gamma^*_\beta}(F_m) \to (\mathsf{Pr}_{\gamma^*_\beta}(\ulcorner\varphi\urcorner) \to \varphi)$.*

PROOF. Let $n \notin \beta$. Then $I\Sigma_1 \vdash \neg\mathsf{Con}^{n+1}_{\mu} \to \neg\mathsf{Con}^{n+1}_{\gamma^*_\beta}$ by Lemma 5.2. Then $I\Sigma_1 \vdash f_{\gamma^*_\beta}(F_n) \to (\neg\mathsf{Con}^{n+1}_{\mu} \to \neg\mathsf{Con}^{n}_{\gamma^*_\beta})$. Since $I\Sigma_1 \vdash \neg\mathsf{Con}^{n}_{\gamma^*_\beta} \to \mathsf{Pr}_{\gamma}(\ulcorner\neg\mathsf{Con}^{n}_{\gamma^*_\beta}\urcorner)$, we have

$$I\Sigma_1 \vdash f_{\gamma^*_\beta}(F_n) \to ((\neg\mathsf{Con}^{n+1}_{\mu} \wedge v = \ulcorner\neg\mathsf{Con}^{n}_{\gamma^*_\beta}\urcorner) \to \mathsf{Pr}_{\gamma}(v)).$$

By combining this with $I\Sigma_1 \vdash \gamma(v) \to \mathsf{Pr}_{\gamma}(v)$, we have

$$I\Sigma_1 \vdash \bigwedge_{m\notin\beta} f_{\gamma^*_\beta}(F_m) \to \left(\left(\gamma(v) \vee \bigvee_{m\notin\beta}(\neg\mathsf{Con}^{m+1}_{\mu} \wedge v = \ulcorner\neg\mathsf{Con}^{m}_{\gamma^*_\beta}\urcorner)\right) \to \mathsf{Pr}_{\gamma}(v)\right).$$

By the definition of $\gamma^*_\beta(v)$, we obtain $I\Sigma_1 \vdash \bigwedge_{m\notin\beta} f_{\gamma^*_\beta}(F_m) \to (\gamma^*_\beta(v) \to \mathsf{Pr}_{\gamma}(v))$, and hence

$$I\Sigma_1 \vdash \bigwedge_{m\notin\beta} f_{\gamma^*_\beta}(F_m) \to \forall x(\mathsf{Pr}_{\gamma^*_\beta}(x) \to \mathsf{Pr}_{\gamma}(x)).$$

By the assumption, $U \vdash \mathsf{Pr}_{\gamma}(\ulcorner\varphi\urcorner) \to \varphi$ for any Σ_i sentence φ. Therefore

$$U \vdash \bigwedge_{m\notin\beta} f_{\gamma^*_\beta}(F_m) \to (\mathsf{Pr}_{\gamma^*_\beta}(\ulcorner\varphi\urcorner) \to \varphi)$$

for any Σ_i sentence φ. ⊣

LEMMA 5.6. *For any $n \in \beta$, $U \vdash f_{\gamma^*_\beta}(F_n)$.*

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PROOF. Let $n \in \beta$. Since $\neg\mathsf{Con}^n_{\gamma^*_\beta}$ is a Σ_1 sentence, $U \vdash \bigwedge_{m\notin\beta} f_{\gamma^*_\beta}(F_m) \to f_{\gamma^*_\beta}(F_n)$ by Lemma 5.5. On the other hand, since $I\Sigma_1 \vdash \neg f_{\gamma^*_\beta}(F_m) \to f_{\gamma^*_\beta}(F_n)$ for any $m \neq n$, we have $I\Sigma_1 \vdash \bigvee_{m\notin\beta} \neg f_{\gamma^*_\beta}(F_m) \to f_{\gamma^*_\beta}(F_n)$ because $n \in \beta$. Therefore we obtain $U \vdash f_{\gamma^*_\beta}(F_n)$. ⊣

LEMMA 5.7. $U \nvdash \bigvee_{m\notin\beta} \neg f_{\gamma^*_\beta}(F_m)$.

PROOF. Since $I\Sigma_1 \vdash \bigvee_{m\notin\beta} \neg f_{\gamma^*_\beta}(F_m) \to \neg\mathsf{Con}^{k+1}_{\gamma^*_\beta}$ for $k = \max\{n : n \notin \beta\}$, we have

$$I\Sigma_1 \vdash \bigvee_{m\notin\beta} \neg f_{\gamma^*_\beta}(F_m) \to \neg\mathsf{Con}^{k+1}_{\mu}$$

by Lemma 5.3. Since $U \nvdash \neg\mathsf{Con}^{k+1}_{\mu}$, we conclude $U \nvdash \bigvee_{m\notin\beta} \neg f_{\gamma^*_\beta}(F_m)$. ⊣

PROOF OF THEOREM 5.1. Since $U + \forall z\mathsf{Con}_\mu(z)$ is consistent and $\forall z\mathsf{Con}_\mu(z)$ is a Π_1 sentence, $\mathbb{N} \models \forall z\mathsf{Con}_\mu(z)$. From this, we have $\mathbb{N} \models \forall v(\gamma(v) \leftrightarrow \gamma^*_\beta(v))$. Thus $\gamma^*_\beta(v)$ is also a Σ_1 definition of T. Clauses 1 and 2 are obvious by the definition of $\gamma^*_\beta(v)$. Clause 3 follows from Lemmas 5.4 and 5.6. Clause 4 is Lemma 5.5. Clause 5 is Lemma 5.7. ⊣

§6. The logics S_β. In this section, we prove that for any cofinite $\beta \subseteq \omega$, there exists a Σ_1 definition $\tau(v)$ of $I\Sigma_1$ such that $\mathsf{PL}_\tau(U) = \mathsf{S}_\beta$. Recall that $\sigma(v)$ is a Σ_1 definition of $I\Sigma_1$ satisfying that $I\Sigma_1 \vdash \forall x(\mathsf{Pr}_\sigma(x) \to \mathsf{Pr}_\mu(x))$ and $U \vdash \mathsf{RFN}(\sigma)$. Therefore by Theorem 5.1, the Σ_1 definition $\sigma^*_\beta(v)$ of $I\Sigma_1$ satisfies the following conditions:

$$\text{for any } n \in \omega, n \in \beta \text{ if and only if } U \vdash f_{\sigma^*_\beta}(F_n), \tag{6}$$

$$\text{for any } \mathcal{L}_A\text{-sentence } \varphi,\ U \vdash \bigwedge_{m\notin\beta} f_{\sigma^*_\beta}(F_m) \to (\mathsf{Pr}_{\sigma^*_\beta}(\ulcorner\varphi\urcorner) \to \varphi), \tag{7}$$

$$U \nvdash \bigvee_{m\notin\beta} \neg f_{\gamma^*_\beta}(F_m). \tag{8}$$

THEOREM 6.1. *For any cofinite $\beta \subseteq \omega$, $\mathsf{PL}_{\sigma^*_\beta}(U) = \mathsf{S}_\beta$.*

PROOF. By (6) and Proposition 2.11, $\mathsf{tr}(\mathsf{PL}_{\sigma^*_\beta}(U)) = \beta$. Then by the classification theorem, $\mathsf{PL}_{\sigma^*_\beta}(U)$ is one of GL_β, D_β, S_β, and GL^-_β. By (7), the modal formula $\bigwedge_{m\notin\beta} F_m \to (\Box p \to p)$ is in $\mathsf{PL}_{\sigma^*_\beta}(U)$. Since $\mathsf{D}_\beta \subseteq \mathsf{D}$, $\mathsf{D} \vdash \bigwedge_{m\notin\beta} F_m$ and $\mathsf{D} \nvdash \Box p \to p$, we have $\mathsf{PL}_{\sigma^*_\beta}(U) \nsubseteq \mathsf{D}_\beta$. Hence $\mathsf{PL}_{\sigma^*_\beta}(U)$ is either S_β or GL^-_β.

Also $\mathsf{PL}_{\sigma^*_\beta}(U) \neq \mathsf{GL}^-_\beta$ because $\bigvee_{m\notin\beta} \neg F_m \notin \mathsf{PL}_{\sigma^*_\beta}(U)$ by (8). We conclude $\mathsf{PL}_{\sigma^*_\beta}(U) = \mathsf{S}_\beta$. ⊣

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§7. The logics D_β. In this section, we prove that for each cofinite $\beta \subseteq \omega$, there exists a Σ_1 definition $\tau(v)$ of some extension T of $I\Sigma_1$ such that $\mathsf{PL}_\tau(U) = \mathsf{D}_\beta$. First, we prove that there exists a theory T_1 such that U is deductively equivalent to $T_1 + \mathsf{RFN}_{\Sigma_1}(T_1)$ by using Jerosolw's method.

THEOREM 7.1. *There exist a theory T_1 and a Σ_1 definition $\delta(v)$ of T_1 satisfying the following conditions*:

1. $I\Sigma_1 \vdash \forall x(\mathsf{Pr}_\delta(x) \to \mathsf{Pr}_\mu(x))$,
2. *the theories U and $T_1 + \mathsf{RFN}_{\Sigma_1}(\delta)$ are deductively equivalent*.

PROOF. Let $\delta(v)$ be the Σ_1 formula $\sigma(v) \vee \exists x(v = \ulcorner \mathsf{RFN}_{\Sigma_1}(\sigma) \to \dot{x} \urcorner \wedge \mu(x))$. Then $\delta(v)$ is a Σ_1 definition of the theory $T_1 = I\Sigma_1 + \{\mathsf{RFN}_{\Sigma_1}(\sigma) \to \varphi : \varphi \in U\}$. Notice that T_1 is a subtheory of U.

1. Since it is easy to see $I\Sigma_1 \vdash \delta(v) \to \mathsf{Pr}_\mu(v)$, we obtain $I\Sigma_1 \vdash \forall x(\mathsf{Pr}_\delta(x) \to \mathsf{Pr}_\mu(x))$.

2. Since $I\Sigma_1 \vdash \forall x(\mathsf{Pr}_\sigma(x) \to \mathsf{Pr}_\delta(x))$, $I\Sigma_1 \vdash \mathsf{RFN}_{\Sigma_1}(\delta) \to \mathsf{RFN}_{\Sigma_1}(\sigma)$. Hence $T_1 + \mathsf{RFN}_{\Sigma_1}(\delta) \vdash U$ by the choice of T_1.

It suffices to prove $\mathsf{PA} \vdash \mathsf{RFN}_{\Sigma_1}(\delta)$. An intuitive proof is as follows[1]: Since T_1 is a subtheory of $I\Sigma_1 + \neg\mathsf{RFN}_{\Sigma_1}(\sigma)$, we have $\mathsf{PA} \vdash \mathsf{RFN}_{\Sigma_1}(\sigma + \neg\mathsf{RFN}_{\Sigma_1}(\sigma)) \to \mathsf{RFN}_{\Sigma_1}(\delta)$. By the second incompleteness theorem for 1-provability (see [3]), $\mathsf{PA} \vdash \mathsf{RFN}_{\Sigma_1}(\sigma) \leftrightarrow \mathsf{RFN}_{\Sigma_1}(\sigma + \neg\mathsf{RFN}_{\Sigma_1}(\sigma))$. Since $\mathsf{PA} \vdash \mathsf{RFN}_{\Sigma_1}(\sigma)$, we obtain $\mathsf{PA} \vdash \mathsf{RFN}_{\Sigma_1}(\delta)$.

It would be worthwhile to give a detailed proof of $\mathsf{PA} \vdash \mathsf{RFN}_{\Sigma_1}(\delta)$. By the same argument of our proof of (1) in Theorem 3.2, we can prove

$$I\Sigma_1 \vdash \mathsf{Pr}_\delta(x) \to \mathsf{Pr}_\sigma(\ulcorner \neg\mathsf{RFN}_{\Sigma_1}(\sigma) \to \dot{x} \urcorner).$$

Since $I\Sigma_1 \vdash \Sigma_1(x) \to \mathsf{Pr}_\sigma(\ulcorner \dot{x} \leftrightarrow \mathsf{True}_{\Sigma_1}(\dot{x}) \urcorner)$, we have

$$I\Sigma_1 \vdash \Sigma_1(x) \wedge \mathsf{Pr}_\delta(x) \to \mathsf{Pr}_\sigma(\ulcorner \neg\mathsf{RFN}_{\Sigma_1}(\sigma) \to \mathsf{True}_{\Sigma_1}(\dot{x}) \urcorner). \tag{9}$$

By the definition of $\mathsf{RFN}_{\Sigma_1}(\sigma)$, $I\Sigma_1 \vdash \Sigma_1(x) \wedge \mathsf{Pr}_\sigma(x) \wedge \neg\mathsf{True}_{\Sigma_1}(x) \to \neg\mathsf{RFN}_{\Sigma_1}(\sigma)$. Then

$$I\Sigma_1 \vdash \mathsf{Pr}_\sigma(\ulcorner \Sigma_1(\dot{x}) \wedge \mathsf{Pr}_\sigma(\ulcorner \mathsf{True}_{\Sigma_1}(\dot{x}) \urcorner) \wedge \neg\mathsf{True}_{\Sigma_1}(\dot{x}) \to \neg\mathsf{RFN}_{\Sigma_1}(\sigma) \urcorner).$$

By combining this with $I\Sigma_1 \vdash \Sigma_1(x) \to \mathsf{Pr}_\sigma(\ulcorner \Sigma_1(\dot{x}) \urcorner)$, we have

$$I\Sigma_1 \vdash \Sigma_1(x) \to \mathsf{Pr}_\sigma(\ulcorner \mathsf{Pr}_\sigma(\ulcorner \mathsf{True}_{\Sigma_1}(\dot{x}) \urcorner) \wedge \neg\mathsf{True}_{\Sigma_1}(\dot{x}) \to \neg\mathsf{RFN}_{\Sigma_1}(\sigma) \urcorner).$$

Hence by (9),

$$I\Sigma_1 \vdash \Sigma_1(x) \wedge \mathsf{Pr}_\delta(x) \to \mathsf{Pr}_\sigma(\ulcorner \mathsf{Pr}_\sigma(\ulcorner \mathsf{True}_{\Sigma_1}(\dot{x}) \urcorner) \wedge \neg\mathsf{True}_{\Sigma_1}(\dot{x}) \to \mathsf{True}_{\Sigma_1}(\dot{x}) \urcorner).$$

Then

$$I\Sigma_1 \vdash \Sigma_1(x) \wedge \mathsf{Pr}_\delta(x) \to \mathsf{Pr}_\sigma(\ulcorner \mathsf{Pr}_\sigma(\ulcorner \mathsf{True}_{\Sigma_1}(\dot{x}) \urcorner) \to \mathsf{True}_{\Sigma_1}(\dot{x}) \urcorner).$$

By formalized Löb's theorem,

$$I\Sigma_1 \vdash \Sigma_1(x) \wedge \mathsf{Pr}_\delta(x) \to \mathsf{Pr}_\sigma(\ulcorner \mathsf{True}_{\Sigma_1}(\dot{x}) \urcorner).$$

Since $\mathsf{PA} \vdash \mathsf{RFN}_{\Sigma_1}(\sigma)$, we obtain

$$\mathsf{PA} \vdash \Sigma_1(x) \wedge \mathsf{Pr}_\delta(x) \to \mathsf{True}_{\Sigma_1}(x).$$

We have proved $\mathsf{PA} \vdash \mathsf{RFN}_{\Sigma_1}(\delta)$. ⊣

[1] This simple proof was pointed out by the referee.

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Then $\delta(v)$ is a Σ_1 definition of T_1 such that $U \vdash \forall x(\mathsf{Pr}_\delta(x) \to \mathsf{Pr}_\mu(x))$ and $U \vdash \mathsf{RFN}_{\Sigma_1}(\delta)$. By Theorem 5.1, for each cofinite $\beta \subseteq \omega$, $\delta^*_\beta(v)$ is a Σ_1 definition of T_1 such that

$$I\Sigma_1 \vdash \forall x(\mathsf{Pr}_\delta(x) \to \mathsf{Pr}_{\delta^*_\beta}(x)), \tag{10}$$

$$I\Sigma_1 \vdash \mathsf{Con}^{k+1}_\mu \to \forall x(\mathsf{Pr}_{\delta^*_\beta}(x) \to \mathsf{Pr}_\delta(x)) \text{ for } k = \max\{n : n \notin \beta\}, \tag{11}$$

$$\text{for any } n \in \omega, n \in \beta \text{ if and only if } U \vdash f_{\delta^*_\beta}(F_n), \tag{12}$$

$$\text{for any } \Sigma_1 \text{ sentence } \varphi, U \vdash \bigwedge_{m \notin \beta} f_{\delta^*_\beta}(F_m) \to (\mathsf{Pr}_{\delta^*_\beta}(\ulcorner \varphi \urcorner) \to \varphi). \tag{13}$$

THEOREM 7.2. *For any cofinite* $\beta \subseteq \omega$, $\mathsf{PL}_{\delta^*_\beta}(U) = \mathsf{D}_\beta$.

PROOF. By (12) and Proposition 2.11, $\mathsf{tr}(\mathsf{PL}_{\delta^*_\beta}(U)) = \beta$. By the classification theorem, $\mathsf{PL}_{\delta^*_\beta}(U)$ is one of GL_β, D_β, S_β, and GL^-_β. By (13), the modal formula $\bigwedge_{m \notin \beta} F_m \to (\Box\Box p \to \Box p)$ is in $\mathsf{PL}_{\delta^*_\beta}(U)$. Since $\mathsf{GL}_\beta \subseteq \mathsf{GL}_\omega$, $\mathsf{GL}_\omega \vdash \bigwedge_{n \notin \beta} F_n$ and $\mathsf{GL}_\omega \nvdash \Box\Box p \to \Box p$, we have $\mathsf{PL}_{\delta^*_\beta}(U) \neq \mathsf{GL}_\beta$.

Towards a contradiction, we suppose

$$U \vdash \bigwedge_{m \notin \beta} f_{\delta^*_\beta}(F_m) \to (\mathsf{Pr}_{\delta^*_\beta}(\ulcorner \neg \mathsf{RFN}_{\Sigma_1}(\delta^*_\beta) \urcorner) \to \neg \mathsf{RFN}_{\Sigma_1}(\delta^*_\beta)).$$

Since $I\Sigma_1 \vdash \mathsf{RFN}_{\Sigma_1}(\delta^*_\beta) \to \bigwedge_{m \notin \beta} f_{\delta^*_\beta}(F_m)$, we have $U \vdash \mathsf{RFN}_{\Sigma_1}(\delta^*_\beta) \to \neg \mathsf{Pr}_{\delta^*_\beta}(\ulcorner \neg \mathsf{RFN}_{\Sigma_1}(\delta^*_\beta) \urcorner)$. Hence

$$U \vdash \mathsf{RFN}_{\Sigma_1}(\delta^*_\beta) \to \mathsf{Con}_{\delta^*_\beta + \mathsf{RFN}_{\Sigma_1}(\delta^*_\beta)}. \tag{14}$$

By (10), $T_1 \vdash \mathsf{RFN}_{\Sigma_1}(\delta^*_\beta) \to \mathsf{RFN}_{\Sigma_1}(\delta)$. Then $T_1 + \mathsf{RFN}_{\Sigma_1}(\delta^*_\beta) \vdash U$ by Theorem 7.1. From this with (14), we have

$$T_1 + \mathsf{RFN}_{\Sigma_1}(\delta^*_\beta) \vdash \mathsf{Con}_{\delta^*_\beta + \mathsf{RFN}_{\Sigma_1}(\delta^*_\beta)}.$$

Since $\delta^*_\beta(v)$ is a Σ_1 definition of T_1, the theory $T_1 + \mathsf{RFN}_{\Sigma_1}(\delta^*_\beta)$ is inconsistent by the second incompleteness theorem. By (11), $I\Sigma_1 \vdash \mathsf{Con}^{k+1}_\mu \to (\mathsf{RFN}_{\Sigma_1}(\delta) \to \mathsf{RFN}_{\Sigma_1}(\delta^*_\beta))$ for $k = \max\{n : n \notin \beta\}$. Hence the theory $T_1 + \mathsf{RFN}_{\Sigma_1}(\delta) + \mathsf{Con}^{k+1}_\mu$ is also inconsistent. By Theorem 7.1, $U \vdash T_1 + \mathsf{RFN}_{\Sigma_1}(\delta)$, and thus $U + \mathsf{Con}^{k+1}_\mu$ is inconsistent. This is a contradiction.

Hence

$$U \nvdash \bigwedge_{m \notin \beta} f_{\delta^*_\beta}(F_m) \to (\mathsf{Pr}_{\delta^*_\beta}(\ulcorner \neg \mathsf{RFN}_{\Sigma_1}(\delta^*_\beta) \urcorner) \to \neg \mathsf{RFN}_{\Sigma_1}(\delta^*_\beta)),$$

and $\bigwedge_{m \notin \beta} F_m \to (\Box p \to p)$ is not in $\mathsf{PL}_{\delta^*_\beta}(U)$. Since this modal formula is a theorem of S_β, we have $\mathsf{S}_\beta \nsubseteq \mathsf{PL}_{\delta^*_\beta}(U)$. Therefore $\mathsf{PL}_{\delta^*_\beta}(U) = \mathsf{D}_\beta$. ⊣

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§8. **Refinements.** Throughout this article, the theory U is assumed to be a computably axiomatized consistent extension of PA. We close this article by refining our theorems, that is, we show that the assumptions on U can be somewhat weakened. We say that a theory T is Σ_1-*reflexive* if there exists a Σ_1 definition $\tau(v)$ of T such that $T \vdash \mathsf{RFN}_{\Sigma_1}(\tau|\overline{p})$ for all $p \in \omega$. From Fact 2.6, every computably axiomatized extension T of PA is Σ_1-reflexive. Fact 2.6 is used only in the proof of Lemma 2.16, and our proof of Lemma 2.16 is also applicable to computably axiomatized consistent Σ_1-reflexive extensions of $I\Sigma_1$. Notice that the conclusion of Lemma 2.16 is also true for any Σ_1-sound theory. Since our proof of Theorem 5.1 is based on Lemma 2.16, Theorem 5.1 holds for computably axiomatized consistent extensions U of $I\Sigma_1$ which are Σ_1-reflexive or Σ_1-sound.

First, we pay attention to Theorem 6.1. In our proof of Theorem 6.1, Theorem 5.1 is used and the condition $U \vdash \mathsf{RFN}(I\Sigma_1)$ is assumed. The condition $U \vdash \mathsf{RFN}(I\Sigma_1)$ is only used to derive the condition (7), and hence it can be weakened to the local version. That is, U is sufficient to be an extension of $\mathsf{Rfn}(I\Sigma_1) = \{\mathsf{Pr}_{I\Sigma_1}(\ulcorner\varphi\urcorner) \to \varphi : \varphi \text{ is an } \mathcal{L}_A\text{-sentence}\}$. Also by Craig's trick, the assumption that U is computably axiomatized can be weakened. Then we obtain the following refinement of Theorem 6.1.

THEOREM 8.1. *Let U be a computably enumerable consistent extension of $I\Sigma_1 + \mathsf{Rfn}(I\Sigma_1)$ which is either Σ_1-reflexive or Σ_1-sound. For any cofinite $\beta \subseteq \omega$, there exists a Σ_1 definition $\tau(v)$ of $I\Sigma_1$ such that $\mathsf{PL}_\tau(U) = \mathsf{S}_\beta$.*

Notice that in our proofs of Theorems 3.5, 4.4 and 7.2, Lemma 2.16 and the condition $U \vdash \mathsf{RFN}_{\Sigma_1}(I\Sigma_1)$ are used. Therefore we obtain the following theorem.

THEOREM 8.2. *Let U be a computably enumerable consistent extension of $I\Sigma_1 + \mathsf{RFN}_{\Sigma_1}(I\Sigma_1)$ which is either Σ_1-reflexive or Σ_1-sound. Let L be one of GL_α, D_β and GL^-_β where α is computably enumerable and β is cofinite, then there exists a Σ_1 definition $\tau(v)$ of some extension of $I\Sigma_1$ such that $\mathsf{PL}_\tau(U)$ is exactly L.*

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DEPARTMENT OF NATURAL SCIENCE
NATIONAL INSTITUTE OF TECHNOLOGY, KISARAZU COLLEGE
2-11-1 KIYOMIDAI-HIGASHI
KISARAZU, CHIBA 292-0041, JAPAN
E-mail: kurahashi@n.kisarazu.ac.jp

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