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Dualizing complexes for commutative Noetherian rings

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1. *Introduction.* The theory of dualizing complexes of Grothendieck and Hartshorne ((5), chapter V) has turned out to be a useful tool even in commutative algebra. For instance, Peskine and Szpiro used dualizing complexes in their (partial) solution of Bass's conjecture concerning finitely-generated (f.-g.) modules of finite injective dimension over a Noetherian local ring ((7), chapitre I, § 5); and the present author first obtained the results in (9) by using dualizing complexes.

Hartshorne's account of dualizing complexes in (5) involves the concepts of triangulated category, derived category, and localization of categories. While working on (9), I discovered that it is possible to construct a satisfying theory of dualizing complexes for commutative Noetherian rings which avoids use of triangulated categories, derived categories, and localization of categories, and which instead places great emphasis on the concept of quasi-isomorphism of complexes of modules over such a ring. The purpose of this paper is to present the essential ingredients of this approach: it seems to me that this approach reaches significant results on dualizing complexes comparatively quickly, and, as the methods used are standard ones for workers in homological commutative algebra, this account might be of benefit to other workers in that field.

This account should not be regarded as an exhaustive treatment of dualizing complexes, but rather as an indication of a method of approach. Thus, although the climax of the present account is the 'uniqueness theorem' for dualizing complexes, the interested reader should be able to develop the theory beyond this point using the techniques of the paper. It is of course true that Hartshorne's work in (5) has provided both stimulus and guide-lines for the present work; furthermore, there are a few instances in the theory where the two accounts are sufficiently similar for a reference to (5) to be adequate explanation. Nevertheless, this rather down-to-earth approach is very different from Hartshorne's, and it is offered in the hope that other workers in commutative algebra might find it helpful.

2. *Dualizing complexes: the basic definitions.* All rings considered in this paper will be commutative and Noetherian, and will have non-zero identities. A, B, C will denote (such) rings throughout the paper. $\mathscr{C}(A)$ will denote the category of all A-modules and A-homomorphisms; $\mathscr{F}(A)$ will denote the full subcategory of $\mathscr{C}(A)$ whose objects are the f.-g. A-modules.

(2·1) The additive A-category ((6), 3·3) of all complexes of A-modules and morphisms (i.e. translations) of complexes will be denoted by $\mathscr{Y}(A)$. Capital letters followed by a dot, as in $X^{\cdot}$, will generally be used to denote objects of $\mathscr{Y}(A)$ (or $\mathscr{Y}(B)$

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or $\mathscr{Y}(C)$); if $X^\cdot \in \mathscr{Y}(A)$, the nth term of $X^\cdot$ will be denoted by X^n, and the nth differentiation of $X^\cdot$ by $d_X^n.: X^n \to X^{n+1}$. If M is an A-module, we shall also denote by M the complex of A-modules which has M as 0th term and all its other terms zero. A morphism of complexes will generally be denoted by a small letter followed by a dot. If $u^\cdot: X^\cdot \to Y^\cdot$ is a morphism in $\mathscr{Y}(A)$, then $u^\cdot$ is a family of A-homomorphisms, the nth member of which will be denoted by $u^n: X^n \to Y^n$. (Also $d_Y^n..u^n = u^{n+1}.d_X^n.$ for all integers n, of course.)

A complex $X^\cdot \in \mathscr{Y}(A)$ will be said to be *bounded above* (resp. *bounded below*) if there exists an integer n_0 such that $X^n = 0$ for all $n > n_0$ (resp. $n < n_0$). $Y^\cdot$ is *bounded* if it is both bounded above and bounded below. $\mathscr{Y}^+(A)$ (resp. $\mathscr{Y}^-(A)$, $\mathscr{Y}^b(A)$) denotes the full subcategory of $\mathscr{Y}(A)$ whose objects are the complexes which are bounded below (resp. bounded above, bounded). $\mathscr{Y}_c(A)$ (resp. $\mathscr{Y}_c^+(A), \mathscr{Y}_c^-(A), \mathscr{Y}_c^b(A)$) will denote the full subcategory of $\mathscr{Y}(A)$ (resp. $\mathscr{Y}^+(A), \mathscr{Y}^-(A), \mathscr{Y}^b(A)$) whose objects are those complexes in $\mathscr{Y}(A)$ (resp. $\mathscr{Y}^+(A), \mathscr{Y}^-(A), \mathscr{Y}^b(A)$) all of whose cohomology modules are finitely-generated.

Suppose $T: \mathscr{C}(A) \times \mathscr{C}(B) \to \mathscr{C}(C)$ is an additive functor. Chapter IV, § 5, of Cartan–Eilenberg (3) and § 6·2 of Northcott (6) show how T induces an additive functor (also denoted by T) from $\mathscr{Y}(A) \times \mathscr{Y}(B)$ to $\mathscr{Y}(C)$, having precisely the same variances as the original functor. Suppose, for example, that T is contravariant in the first variable and covariant in the second; then, if $X^\cdot \in \mathscr{Y}(A)$ and $Y^\cdot \in \mathscr{Y}(B)$, the nth term of $T(X^\cdot, Y^\cdot)$ is given by

$$T^n(X^\cdot, Y^\cdot) = \sum_{p \in Z} T(X^p, Y^{n+p}) \quad \text{(direct sum)},$$

and the nth differentiation in this complex is such that its restriction to $T(X^p, Y^{n+p})$ is $T(d_X^{p-1}, Y^{n+p}) + (-1)^p T(X^p, d_Y^{n+p})$. ($Z$ denotes the set of integers.)

A complex $L^\cdot \in \mathscr{Y}(A)$ will be called *injective* (resp. *flat*) if, for all integers n, L^n is injective (resp. flat). Very relevant to the theory of dualizing complexes is the construction of morphisms of functors

$$\text{Identity} \to \text{Hom}_A([\text{Hom}_A(\ , I^\cdot)], I^\cdot) \quad \text{(from } \mathscr{Y}(A) \text{ to } \mathscr{Y}(A))$$

and

$$(\ ) \otimes_A [\text{Hom}_A(I^\cdot, J^\cdot)] \to \text{Hom}_A([\text{Hom}_A(\ , I^\cdot)], J^\cdot) \quad \text{(from } \mathscr{Y}(A) \text{ to } \mathscr{Y}(A)),$$

where $I^\cdot$ and $J^\cdot$ are injective complexes in $\mathscr{Y}^b(A)$. Fundamental to the construction of these morphisms is the following lemma.

(2·2) LEMMA. *Suppose X, L, M are A-modules.*

(i) *There exists a (unique) A-homomorphism*

$$\xi(X, L, M): X \otimes_A \text{Hom}_A(L, M) \to \text{Hom}_A(\text{Hom}_A(X, L), M)$$

such that, for $x \in X$, $f \in \text{Hom}_A(L, M)$ and $g \in \text{Hom}_A(X, L)$,

$$[\xi(X, L, M)(x \otimes f)](g) = f(g(x)).$$

Furthermore, the $\xi(X, L, M)$ constitute a morphism of functors (from $\mathscr{C}(A) \times \mathscr{C}(A) \times \mathscr{C}(A)$ to $\mathscr{C}(A)$):

$$(\ ) \otimes_A \text{Hom}_A(\ ,\ ) \to \text{Hom}_A(\text{Hom}_A(\ ,\ ),\ ).$$

(ii) *If M is injective, $\xi(X, L, M)$ is an isomorphism whenever X is f.-g.*

(iii) *If L and M are both injective, then $\text{Hom}_A(L, M)$ is a flat A-module.*

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Proof. This is straightforward, and left to the reader.

(2·3) PROPOSITION. *Suppose $I^\cdot$ and $J^\cdot$ are injective complexes in $\mathscr{Y}^b(A)$.*

(i) *There is a morphism of functors (from $\mathscr{Y}(A)$ to $\mathscr{Y}(A)$)*

$$\eta(\ ;I^\cdot,J^\cdot)^\cdot\colon (\ )\otimes_A[\mathrm{Hom}_A(I^\cdot,J^\cdot)]\to\mathrm{Hom}_A([\mathrm{Hom}_A(\ ,I^\cdot)],J^\cdot),$$

which is such that $\eta(M;I^\cdot,J^\cdot)^\cdot$ is an isomorphism whenever M is a f.-g. A-module.

(ii) *Let $\tau\colon Z\to\{0,1\}\subseteq Z$ be the mapping defined by $\tau(n)=0,1,1,0$ resp. according as the integer n is congruent to $0,1,2,3$ resp. modulo 4. Let $\alpha^0\colon A\to\sum_{p\in Z}\mathrm{Hom}_A(I^p,I^p)$ be the A-homomorphism defined by the rule: the component of $\alpha^0(a)$ (for an element $a\in A$) in $\mathrm{Hom}_A(I^p,I^p)$ shall be multiplication by $(-1)^{\tau(p)}.a$ (for each $p\in Z$). Then α^0 can be included as the 0th member of a morphism of complexes*

$$\alpha^\cdot(=\alpha_A(I^\cdot)^\cdot)\colon A\to\mathrm{Hom}_A(I^\cdot,I^\cdot),$$

and $\alpha^\cdot$ induces a morphism of functors (from $\mathscr{Y}(A)$ to $\mathscr{Y}(A)$)

$$\phi(\ ;I^\cdot)^\cdot\colon\text{Identity}\to(\ )\otimes_A[\mathrm{Hom}_A(I^\cdot,I^\cdot)].$$

The morphism of functors (from $\mathscr{Y}(A)$ to $\mathscr{Y}(A)$)

$$\eta(\ ;I^\cdot,I^\cdot)^\cdot.\phi(\ ;I^\cdot)^\cdot\colon\text{Identity}\to\mathrm{Hom}_A([\mathrm{Hom}_A(\ ,I^\cdot)],I^\cdot),$$

obtained by composition, will be denoted by $\theta(\ ;I^\cdot)^\cdot$.

Proof. If $X^\cdot\in\mathscr{Y}(A)$, the complex $X^\cdot\otimes_A[\mathrm{Hom}_A(I^\cdot,J^\cdot)]$ is obtained by first forming $\mathrm{Hom}_A(I^\cdot,J^\cdot)=L^\cdot$, and then forming $X^\cdot\otimes_A L^\cdot$. Similar remarks apply to

$$\mathrm{Hom}_A([\mathrm{Hom}_A(X^\cdot,I^\cdot)],J^\cdot).$$

(i) However, $(\ )\otimes_A\mathrm{Hom}_A(\ ,\ )$ and $\mathrm{Hom}_A(\mathrm{Hom}_A(\ ,\ ),\ )$ are both additive functors from $\mathscr{C}(A)\times\mathscr{C}(A)\times\mathscr{C}(A)$ to $\mathscr{C}(A)$, and both are covariant in the first and third variables and contravariant in the second variable. Section 5 of chapter IV of Cartan–Eilenberg (3) shows how these functors induce additive functors from

$$\mathscr{Y}(A)\times\mathscr{Y}(A)\times\mathscr{Y}(A)\quad\text{to}\quad\mathscr{Y}(A),$$

which we shall denote by $(\ )\otimes_A\mathrm{Hom}_A(\ ,\ )$ and $\mathrm{Hom}_A(\mathrm{Hom}_A(\ ,\ ),\ )$. It is also easy to see from this Cartan–Eilenberg theory that the morphism of functors ξ of (2·2)(i) induces a morphism of functors (from $\mathscr{Y}(A)\times\mathscr{Y}(A)\times\mathscr{Y}(A)$ to $\mathscr{Y}(A)$)

$$\eta_2(\ ,\ ,\ )^\cdot\colon(\ )\otimes_A\mathrm{Hom}_A(\ ,\ )\to\mathrm{Hom}_A(\mathrm{Hom}_A(\ ,\ ),\ ),$$

from which we obtain a morphism of functors (from $\mathscr{Y}(A)$ to $\mathscr{Y}(A)$)

$$\eta_2(\ ,I^\cdot,J^\cdot)^\cdot\colon(\ )\otimes_A\mathrm{Hom}_A(I^\cdot,J^\cdot)\to\mathrm{Hom}_A(\mathrm{Hom}_A(\ ,I^\cdot),J^\cdot).$$

It follows from (2·2)(ii) that $\eta_2(M,I^\cdot,J^\cdot)^\cdot$ is an isomorphism of complexes whenever M is a f.-g. A-module.

Finally, since $I^\cdot$ and $J^\cdot$ are bounded, we can easily construct isomorphisms of functors (from $\mathscr{Y}(A)$ to $\mathscr{Y}(A)$)

$$\eta_1(\ )^\cdot\colon(\ )\otimes_A[\mathrm{Hom}_A(I^\cdot,J^\cdot)]\to(\ )\otimes_A\mathrm{Hom}_A(I^\cdot,J^\cdot)$$

and

$$\eta_3(\ )^\cdot\colon\mathrm{Hom}_A(\mathrm{Hom}_A(\ ,I^\cdot),J^\cdot)\to\mathrm{Hom}_A([\mathrm{Hom}_A(\ ,I^\cdot)],J^\cdot).$$

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Define $\eta(\ ;I^\cdot,J^\cdot)^\cdot$ to be the composition $\eta_3(\ )^\cdot.\eta_2(\ ,I^\cdot,J^\cdot)^\cdot.\eta_1(\ )^\cdot$. Then $\eta(M;I^\cdot,J^\cdot)^\cdot$ is an isomorphism of complexes whenever M is a f.-g. A-module.

(ii) Since $I^\cdot$ is bounded, the definition of α^0 is sensible, and α^0 becomes an A-homomorphism. To see that α^0 can be included in a morphism of complexes as claimed, it is only necessary to check that $d_{F^\cdot}^0.\alpha^0 = 0$, where $F^\cdot$ is used to denote the complex $\mathrm{Hom}_A(I^\cdot,I^\cdot)$. This is easy once it has been noted that $n+\tau(n)+\tau(n+1)$ is odd (for all integers n).

If $X^\cdot \in \mathscr{Y}(A)$, define $\phi(X^\cdot;I^\cdot)^\cdot: X^\cdot \to X^\cdot \otimes_A [\mathrm{Hom}_A(I^\cdot,I^\cdot)]$ to be the composition of the natural isomorphism $X^\cdot \to X^\cdot \otimes_A A$ and the morphism

$$X^\cdot \otimes_A A \to X^\cdot \otimes_A [\mathrm{Hom}_A(I^\cdot,I^\cdot)]$$

induced by $\alpha^\cdot$. The $\phi(\ ;I^\cdot)^\cdot$ clearly constitute a morphism of functors as required.

Note. The author concedes that the signs $(-1)^{\tau(p)}$ present in the definition in part (ii) are ungainly; and that use of Hartshorne's definition of $\mathrm{Hom}_A(X^\cdot,Y^\cdot)$ in (5) rather than that of Cartan–Eilenberg would eliminate them; but points out that use of Hartshorne's definition would have led to serious complications concerning signs in part (i).

(2·4) FURTHER DEFINITIONS. If n is an integer, $H^n: \mathscr{Y}(A) \to \mathscr{C}(A)$ will denote the nth cohomology functor. So, if $X^\cdot \in \mathscr{Y}(A)$, then $H^n(X^\cdot) = \ker d_{X^\cdot}^n / \mathrm{im}\, d_{X^\cdot}^{n-1}$. A morphism of complexes $u^\cdot: X^\cdot \to Y^\cdot$ is said to be a *quasi-isomorphism* (abbreviated *quism*) if, for all integers n, $H^n(u^\cdot): H^n(X^\cdot) \to H^n(Y^\cdot)$ is an isomorphism.

A *dualizing complex for* A is an injective complex $I^\cdot \in \mathscr{Y}_c^b(A)$ (so that $I^\cdot$ is bounded and all its cohomology modules are f.-g.) for which the morphism of complexes

$$\theta(X^\cdot;I^\cdot)^\cdot: X^\cdot \to \mathrm{Hom}_A([\mathrm{Hom}_A(X^\cdot,I^\cdot)],I^\cdot)$$

of (2·3)(ii) is a quism whenever $X^\cdot \in \mathscr{Y}_c^b(A)$. In order to develop this concept, some further definitions and theory are required.

Suppose k is an integer. $\{k\}$ will denote the *shift functor* from $\mathscr{Y}(A)$ to $\mathscr{Y}(A)$ defined as follows. If $X^\cdot \in \mathscr{Y}(A)$, $[\{k\}X^\cdot]^n = X^{n+k}$ for all integers n, while $d_{\{k\}X^\cdot}^n = (-1)^k d_{X^\cdot}^{n+k}$. (So the effect of $\{1\}$ (resp. $\{-1\}$) on $X^\cdot$ is to shift it one place to the left (resp. right) and change the sign of the differentiation.) If $u^\cdot: X^\cdot \to Y^\cdot$ in $\mathscr{Y}(A)$, then

$$[\{k\}u^\cdot]^n = u^{n+k}: X^{n+k} \to Y^{n+k}.$$

$\{1\}$ and $\{-1\}$ are inverse automorphisms of $\mathscr{Y}(A)$.

We shall indicate that a diagram of morphisms in $\mathscr{Y}(A)$ commutes by placing '(C)' before or after it. Also, '(E)' placed before or after a sequence of morphisms in $\mathscr{Y}(A)$ will indicate that the sequence is exact ((6), § 4·2).

Suppose $u^\cdot: X^\cdot \to Y^\cdot$ is a morphism of complexes. The *mapping cone of* $u^\cdot$, $Mc(u^\cdot)$, is the complex defined as follows. $Mc(u^\cdot)^n = X^{n+1} \oplus Y^n$ $(= [\{1\}X^\cdot]^n \oplus Y^n)$, and the nth differentiation in $Mc(u^\cdot)$ is the homomorphism

$$(\text{from} \quad [\{1\}X^\cdot]^n \oplus Y^n \quad \text{to} \quad [\{1\}X^\cdot]^{n+1} \oplus Y^{n+1})$$

represented by the matrix

$$\begin{bmatrix} d_{\{1\}X^\cdot}^n & 0 \\ [\{1\}u^\cdot]^n & d_{Y^\cdot}^n \end{bmatrix} = \begin{bmatrix} -d_{X^\cdot}^{n+1} & 0 \\ u^{n+1} & d_{Y^\cdot}^n \end{bmatrix}.$$

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(Thus, if $(x, y) \in X^{n+1} \oplus Y^n$, $d^n_{Mc(u^\cdot)}(x, y) = (-d^{n+1}_{X^\cdot}(x), u^{n+1}(x) + d^n_{Y^\cdot}(y))$. It is easy to check that $Mc(u^\cdot)$ is indeed a complex.) There is an obvious exact sequence in $\mathscr{Y}(A)$:

$$(E)\ 0 \to Y^\cdot \to Mc(u^\cdot) \to \{1\}X^\cdot \to 0.$$

The resulting long exact sequence of cohomology modules:

$$(E) \ldots \to H^{i-1}(Mc(u^\cdot)) \to H^{i-1}(\{1\}X^\cdot) \to H^i(Y^\cdot) \to H^i(Mc(u^\cdot)) \to H^i(\{1\}X^\cdot) \to H^{i+1}(Y^\cdot) \to \ldots$$

will be of considerable use. Note that $H^{i-1}(\{1\}X^\cdot) = H^i(X^\cdot)$, and it is straightforward to check that, for each integer i, the connecting homomorphism

$$H^{i-1}(\{1\}X^\cdot) = H^i(X^\cdot) \to H^i(Y^\cdot)$$

is precisely $H^i(u^\cdot)$.

(2·5) Observation. *A morphism of complexes $u^\cdot: X^\cdot \to Y^\cdot$ is a quism if and only if $Mc(u^\cdot)$ is an exact complex.*

(2·6) Remark. *Suppose $(E)\ 0 \to X^\cdot \xrightarrow{u^\cdot} Y^\cdot \xrightarrow{v^\cdot} W^\cdot \to 0$ is an exact sequence in $\mathscr{Y}(A)$. Then $v^\cdot$ induces a quism $Mc(u^\cdot) \to W^\cdot$.*

(2·7) Definition. Suppose S (resp. T): $\mathscr{Y}(A) \to \mathscr{Y}(B)$ is a covariant (resp. contravariant) functor. We say that S (resp. T) is *mapping-consistent* if

(i) S (resp. T) is additive;

(ii) $S.\{1\} = \{1\}.S$ (resp. $T.\{1\} = \{-1\}.T$);

(iii) whenever $u^\cdot: X^\cdot \to Y^\cdot$ is a morphism in $\mathscr{Y}(A)$, then the exact sequence

$$(E)\ 0 \to Y^\cdot \to Mc(u^\cdot) \to \{1\}X^\cdot \to 0$$

induces the exact sequence in $\mathscr{Y}(B)$:

$$(E)\ 0 \to S(Y^\cdot) \to S(Mc(u^\cdot)) \to S(\{1\}X^\cdot) \to 0$$
$$(\text{resp. } (E)\ 0 \to T(\{1\}X^\cdot) \to T(Mc(u^\cdot)) \to T(Y^\cdot) \to 0),$$

which in addition has the property that, for all integers n, the resulting connecting homomorphism:

$$H^{n-1}(S(\{1\}X^\cdot)) = H^{n-1}(\{1\}S(X^\cdot)) = H^n(S(X^\cdot)) \to H^n(S(Y^\cdot))$$
$$(\text{resp. } H^n(T(Y^\cdot)) \to H^{n+1}(T(\{1\}X^\cdot)) = H^{n+1}(\{-1\}T(X^\cdot)) = H^n(T(X^\cdot)))$$

is precisely $H^n(S(u^\cdot))$ (resp. $H^n(T(u^\cdot))$);

(iv) whenever $X^\cdot \in \mathscr{Y}(A)$ is exact, $S(X^\cdot)$ (resp. $T(X^\cdot)$) in $\mathscr{Y}(B)$ is also exact.

(2·8) Remarks. *Suppose S (resp. T): $\mathscr{Y}(A) \to \mathscr{Y}(B)$ is a covariant (resp. contravariant) mapping-consistent functor. Then:*

(i) $S.\{-1\} = \{-1\}.S$ (*resp.* $T.\{-1\} = \{1\}.T$);

(ii) *S (resp. T) preserves quisms, i.e. whenever $u^\cdot: X^\cdot \to Y^\cdot$ is a quism in $\mathscr{Y}(A)$, then $S(u^\cdot)$ (resp. $T(u^\cdot)$) is a quism in $\mathscr{Y}(B)$.* (To see this, use (2·5).)

We are particularly interested in functors on $\mathscr{Y}(A)$ obtained (in the manner indicated in (2·1)) from additive functors on categories of modules.

(2·9) Proposition. *Suppose $T: \mathscr{C}(A) \times \mathscr{C}(C) \to \mathscr{C}(B)$ is an additive functor which is covariant in the second variable, and suppose $W^\cdot$ is a fixed complex in $\mathscr{Y}(C)$. Then*

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$T(\ , W^\cdot)\colon \mathscr{Y}(A) \to \mathscr{Y}(B)$ *possesses properties* (2·7) (i), (ii), (iii) *of mapping-consistent functors.*

Furthermore, the composition of any finite number of functors obtained in this way also possesses properties (2·7) (i), (ii), (iii) *of mapping-consistent functors.*

Proof. That such a functor possesses property (2·7) (i) is immediate; that it also possesses property (2·7) (ii) is easy to check. Property (2·7) (iii) is more complicated.

We deal first with the case in which T is covariant in the first variable. Let S denote the functor $T(\ , W^\cdot)\colon \mathscr{Y}(A) \to \mathscr{Y}(B)$. Suppose $u^\cdot\colon X^\cdot \to Y^\cdot$ is a morphism in $\mathscr{Y}(A)$. For each integer i we obtain from the exact sequence

$$(E)\ 0 \to Y^\cdot \to Mc(u^\cdot) \to \{1\}X^\cdot \to 0$$

in $\mathscr{Y}(A)$, the split exact sequence of ith terms:

$$(E)\ 0 \to Y^i \underset{q^i}{\overset{v^i}{\rightleftarrows}} X^{i+1} \oplus Y^i \underset{w^i}{\overset{p^i}{\rightleftarrows}} X^{i+1} \to 0,$$

where v^i, w^i denote the canonical injections, and p^i, q^i denote the canonical projections. Since T is additive, application of the functor $T(\ , W^{n-i})$ (for a given integer n) yields a split exact sequence in $\mathscr{C}(B)$:

$$(E)\ 0 \to T(Y^i, W^{n-i}) \underset{\longleftarrow}{\longrightarrow} T(X^{i+1} \oplus Y^i, W^{n-i}) \underset{\longleftarrow}{\longrightarrow} T(X^{i+1}, W^{n-i}) \to 0.$$

Let $\phi_{i,n-i}\colon T(X^{i+1} \oplus Y^i, W^{n-i}) \to T(X^{i+1}, W^{n-i}) \oplus T(Y^i, W^{n-i})$ be the resulting isomorphism. $\left(\text{So } \phi_{i,n-i} \text{ can be represented by the matrix } \begin{bmatrix} T(p^i, W^{n-i}) \\ T(q^i, W^{n-i}) \end{bmatrix}.\right)$ The direct sum (taken over all $i \in Z$) of the $\phi_{i,n-i}$ gives rise to an isomorphism

$$\theta^n\colon T^n(Mc(u^\cdot), W^\cdot) \to Mc(T(u^\cdot, W^\cdot))^n$$

in $\mathscr{C}(B)$. A straightforward check shows that the family of B-isomorphisms $\{\theta^n\}_{n \in Z}$ actually constitutes an isomorphism of complexes

$$\theta^\cdot\colon S(Mc(u^\cdot)) \to Mc(S(u^\cdot))$$

which is such that

$$\begin{array}{ccccccccc} (E)\ 0 & \to & S(Y^\cdot) & \to & S(Mc(u^\cdot)) & \to & S(\{1\}X^\cdot) & \to & 0 \\ & & \| & & \downarrow \theta^\cdot & & \| & & \\ (C)(E)\ 0 & \to & S(Y^\cdot) & \to & Mc(S(u^\cdot)) & \to & \{1\}S(X^\cdot) & \to & 0. \end{array} \tag{1}$$

The resulting morphism of long exact sequences of cohomology modules shows that S does indeed possess property (2·7) (iii).

Secondly, suppose T is contravariant in the first variable; let

$$U = T(\ , W^\cdot)\colon \mathscr{Y}(A) \to \mathscr{Y}(B).$$

The reasoning for this case is similar: if $u^\cdot\colon X^\cdot \to Y^\cdot$ is a morphism in $\mathscr{Y}(A)$, one obtains an isomorphism of complexes $\psi^\cdot\colon U(Mc(u^\cdot)) \to Mc(\{-1\}U(u^\cdot))$ for which

$$\begin{array}{ccccccccc} (E)\ 0 & \longrightarrow & U(\{1\}X^\cdot) & \longrightarrow & U(Mc(u^\cdot)) & \longrightarrow & U(Y^\cdot) & \to & 0 \\ & & \| & & \downarrow \psi^\cdot & & \| & & \\ (C)(E)\ 0 & \to & \{-1\}U(X^\cdot) & \to & Mc(\{-1\}U(u^\cdot)) & \to & U(Y^\cdot) & \to & 0. \end{array} \tag{2}$$

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Finally, suppose A_1 and A_2 are rings and $V: \mathscr{Y}(A_1) \to \mathscr{Y}(A_2)$ is a functor which is the composition of a finite number of functors obtained in the manner described in the statement of the proposition. It is immediate from the above that V possesses properties (2·7) (i) and (ii); use commutative diagrams of the types (1) and (2) to see that V also possesses property (2·7) (iii).

(2·10) COROLLARY: EXAMPLES OF MAPPING-CONSISTENT FUNCTORS.

(i) *Suppose $I^\cdot$ is an injective complex in $\mathscr{Y}^b(A)$. Then* $\mathrm{Hom}_A(\ , I^\cdot): \mathscr{Y}(A) \to \mathscr{Y}(A)$ *is mapping-consistent.*

(ii) *Suppose $F^\cdot$ is a flat complex in $\mathscr{Y}^b(A)$. Then* $(\ ) \otimes_A F^\cdot: \mathscr{Y}(A) \to \mathscr{Y}(A)$ *is mapping-consistent.*

(iii) *Suppose $f: A \to B$ is a ring homomorphism such that B when regarded as an A-module by means of f is flat. Then* $(\ ) \otimes_A B: \mathscr{Y}(A) \to \mathscr{Y}(B)$ *is mapping-consistent.*

(iv) *The composition of any finite number of functors of the types described in parts* (i), (ii), *and* (iii) *above is also mapping-consistent.*

Proof. (i) In view of (2·9), it remains only to show that, whenever $X^\cdot \in \mathscr{Y}(A)$ is an exact complex, then $\mathrm{Hom}_A(X^\cdot, I^\cdot)$ is exact also. But $\mathrm{Hom}_A(X^\cdot, I^\cdot)$ is the single complex associated with the double complex ((3), chapter IV, §§ 4, 5):

$$\begin{array}{ccccc}
 & \vdots & & \vdots & \\
 & \uparrow & & \uparrow & \\
\dots \to & \mathrm{Hom}_A(X^{-(p+1)}, I^q) & \longrightarrow & \mathrm{Hom}_A(X^{-(p+1)}, I^{q+1}) & \to \dots \\
 & \uparrow{\scriptstyle d_1^{p,q}} & & \uparrow & \\
\dots \to & \mathrm{Hom}_A(X^{-p}, I^q) & \xrightarrow[d_2^{p,q}]{} & \mathrm{Hom}_A(X^{-p}, I^{q+1}) & \to \dots \\
 & \uparrow & & \uparrow & \\
 & \vdots & & \vdots &
\end{array}$$

in which $d_1^{p,q} = \mathrm{Hom}_A(d_{X^\cdot}^{-(p+1)}, I^q)$ and $d_2^{p,q} = (-1)^p \mathrm{Hom}_A(X^{-p}, d_{I^\cdot}^q)$. But $I^\cdot$ is an injective bounded complex, so if $X^\cdot$ is exact, this double complex has all its columns exact, and only a finite number of these columns are non-zero. To see that the associated single complex, i.e. $\mathrm{Hom}_A(X^\cdot, I^\cdot)$, is also exact, either use a spectral sequence argument ((3), chapter XV, § 6), or use a diagram-chase similar to that used by Douglas in § 2 of (4).

(ii) This is proved similarly.

(iii) If M is an arbitrary A-module and N is an arbitrary B-module, then N can be regarded as an A-module by means of f, and the resulting $M \otimes_A N$ has a natural structure as a B-module: in fact, $\otimes_A$ becomes an additive functor from $\mathscr{C}(A) \times \mathscr{C}(B)$ to $\mathscr{C}(B)$. The result is now an immediate consequence of (2·9) applied to this functor, together with the fact that B is A-flat.

(iv) (2·9) provides the only non-trivial part of the proof of this.

In particular, (2·10) shows that, if $I^\cdot$ is an injective complex in $\mathscr{Y}^b(A)$, then $\mathrm{Hom}_A([\mathrm{Hom}_A(\ , I^\cdot)], I^\cdot): \mathscr{Y}(A) \to \mathscr{Y}(A)$ is mapping-consistent; this indicates the relevance of this concept to dualizing complexes.

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3. *Existence of dualizing complexes.* The following technical result (which is based on proposition I.7.1(i) of Hartshorne(5)), will help us to establish a simple criterion for recognizing dualizing complexes.

(3·1) THEOREM. *Suppose F and G are covariant (resp. contravariant) mapping-consistent functors from $\mathscr{Y}(A)$ to $\mathscr{Y}(B)$, and that $\zeta\colon F \to G$ is a morphism of functors having the property that, whenever $X^\cdot \in \mathscr{Y}(A)$, then*

$$\zeta(\{1\}X^\cdot) = \{1\}\zeta(X^\cdot) \quad (\textit{resp.}\ \zeta(\{1\}X^\cdot) = \{-1\}\zeta(X^\cdot)).$$

Suppose furthermore that $\zeta(M)$ is a quasi-isomorphism whenever M is a f.-g. A-module. Then $\zeta(X^\cdot)$ is a quasi-isomorphism whenever $X^\cdot \in \mathscr{Y}^b_c(A)$.

Proof. We give only that for the covariant case: the proof for the contravariant case is similar. We use truncations of complexes (as in (5), chapter I, § 7). If $X^\cdot \in \mathscr{Y}(A)$ and n is an integer, we make the following definitions:

$$\begin{aligned} \sigma_{>n}(X^\cdot) &\quad \text{is the complex} \quad \ldots \to 0 \to \operatorname{im} d^n_{X^\cdot} \to X^{n+1} \to X^{n+2} \to \ldots; \\ \sigma'_{\geqslant n}(X^\cdot) &\quad \text{is the complex} \quad \ldots \to 0 \to \operatorname{coker} d^{n-1}_{X^\cdot} \to X^{n+1} \to X^{n+2} \to \ldots. \end{aligned}$$

Note that there is a quism $\sigma_{>n-1}(X^\cdot) \to \sigma'_{\geqslant n}(X^\cdot)$ and an exact sequence in $\mathscr{Y}(A)$:

$$(E)\ 0 \to \{-n\}H^n(X^\cdot) \to \sigma'_{\geqslant n}(X^\cdot) \to \sigma_{>n}(X^\cdot) \to 0.$$

($\{-n\}H^n(X^\cdot)$ is the complex which has its nth term equal to $H^n(X^\cdot)$, and all its other terms zero.)

Suppose $X^\cdot \in \mathscr{Y}^b_c(A)$: we show, by descending induction on n, that $\zeta(\sigma_{>n}(X^\cdot))$ is a quism for all integers n. For n sufficiently large, $\sigma_{>n}(X^\cdot)$ is the zero complex, so certainly in that case $\zeta(\sigma_{>n}(X^\cdot))$ is a quism. (Recall that F and G are both additive.) Suppose, inductively, that $\zeta(\sigma_{>k}(X^\cdot))$ is a quism for some integer k. From

$$(E)\ 0 \to \{-k\}H^k(X^\cdot) \xrightarrow{u^\cdot} \sigma'_{\geqslant k}(X^\cdot) \to \sigma_{>k}(X^\cdot) \to 0,$$

we obtain ((2·6)) a quism $Mc(u^\cdot) \to \sigma_{>k}(X^\cdot)$, from which it follows (since ((2·8)) F and G both preserve quisms), that $\zeta(Mc(u^\cdot))$ is a quism. Hence, using property (2·7)(iii) of mapping-consistent functors, we have

$$\begin{array}{ccccccc} (E)\ 0 \to & F(\sigma'_{\geqslant k}(X^\cdot)) & \to & F(Mc(u^\cdot)) & \to & F(\{1\}\{-k\}H^k(X^\cdot)) & \to 0 \\ & \downarrow \zeta(\sigma'_{\geqslant k}(X^\cdot)) & & \downarrow \zeta(Mc(u^\cdot)) & & \downarrow \zeta(\{1\}\{-k\}H^k(X^\cdot)) & \\ (C)\,(E)\ 0 \to & G(\sigma'_{\geqslant k}(X^\cdot)) & \to & G(Mc(u^\cdot)) & \to & G(\{1\}\{-k\}H^k(X^\cdot)) & \to 0. \end{array}$$

$\zeta(Mc(u^\cdot))$ is a quism; $\zeta(\{1\}\{-k\}H^k(X^\cdot)) = \{1-k\}\zeta(H^k(X^\cdot))$ is a quism since $H^k(X^\cdot)$ is a f.-g. A-module; consequently, using the 5-lemma on the resulting morphism of the long exact sequences of cohomology modules, $\zeta(\sigma'_{\geqslant k}(X^\cdot))$ is a quism. Now use the quism $\sigma_{>k-1}(X^\cdot) \to \sigma'_{\geqslant k}(X^\cdot)$ to deduce that $\zeta(\sigma_{>k-1}(X^\cdot))$ is a quism.

Now for n sufficiently small, $\sigma_{>n}(X^\cdot) = X^\cdot$. It follows by induction that $\zeta(X^\cdot)$ is a quism.

(3·2) COROLLARY. *Suppose $I^\cdot$ and $J^\cdot$ are injective complexes in $\mathscr{Y}^b(A)$. Then, for all $X^\cdot \in \mathscr{Y}^b_c(A)$, the morphism of complexes of proposition* (2·3)(i)

$$\eta(X^\cdot; I^\cdot, J^\cdot)^\cdot\colon X^\cdot \otimes_A [\operatorname{Hom}_A(I^\cdot, J^\cdot)] \to \operatorname{Hom}_A([\operatorname{Hom}_A(X^\cdot, I^\cdot)], J^\cdot)$$

is a quasi-isomorphism.

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Proof. Consider the morphism of functors (from $\mathscr{Y}(A)$ to $\mathscr{Y}(A)$)

$$\eta(\ ;I^{\cdot},J^{\cdot})^{\cdot}\colon (\ )\otimes_A[\mathrm{Hom}_A(I^{\cdot},J^{\cdot})]\to\mathrm{Hom}_A([\mathrm{Hom}_A(\ ,I^{\cdot})],J^{\cdot}).$$

By (2·2)(iii), $\mathrm{Hom}_A(I^{\cdot},J^{\cdot})$ is a flat complex in $\mathscr{Y}^b(A)$; it follows from (2·10) that both these functors are mapping-consistent. Also ((2·3)(i)), $\eta(M;\ I^{\cdot},J^{\cdot})^{\cdot}$ is an isomorphism whenever M is a f.-g. A-module. Since it is easy to check that, for all $X^{\cdot}\in\mathscr{Y}(A)$,

$$\eta(\{1\}X^{\cdot};\ I^{\cdot},J^{\cdot})^{\cdot}=\{1\}\eta(X^{\cdot};\ I^{\cdot},J^{\cdot})^{\cdot},$$

the result follows from (3·1).

(3·3) THEOREM. *Suppose the functor $F\colon\mathscr{Y}(A)\to\mathscr{Y}(B)$ is mapping-consistent, and that $F(M)\in\mathscr{Y}_c(B)$ whenever M is a f.-g. A-module. Then $F(X^{\cdot})\in\mathscr{Y}_c(B)$ whenever $X^{\cdot}\in\mathscr{Y}_c^b(A)$.*

Proof. This is sufficiently similar to that of (3·1) to be omitted: use truncations of complexes, descending induction, and the fact that whenever $(E)\,N'\to N\to N''$ in $\mathscr{C}(A)$ with N' and N'' f.-g. A-modules, then N is f.-g. also.

(3·4) LEMMA. (i) *Suppose $F^{\cdot}$ is a flat complex in $\mathscr{Y}_c^-(A)$. Then $M\otimes_A F^{\cdot}\in\mathscr{Y}_c^-(A)$ whenever M is a f.-g. A-module.*

(ii) *Suppose $I^{\cdot}$ is an injective complex in $\mathscr{Y}_c^+(A)$. Then $\mathrm{Hom}_A(M,I^{\cdot})\in\mathscr{Y}_c^+(A)$ whenever M is a f.-g. A-module.*

Proof. (i) It is clear that $A\otimes_A F^{\cdot}\in\mathscr{Y}_c^-(A)$, and it follows easily that $P\otimes_A F^{\cdot}\in\mathscr{Y}_c^-(A)$ whenever P is a f.-g. free A-module (since $\otimes_A\colon\mathscr{C}(A)\times\mathscr{C}(A)\to\mathscr{C}(A)$ and (for all integers n) $H^n\colon\mathscr{Y}(A)\to\mathscr{C}(A)$ are all additive).

$F^{\cdot}$ is bounded above: suppose t is an integer such that $F^i=0$ for all $i>t$. Now suppose M is an arbitrary f.-g. A-module; M can be included in an exact sequence

$$(E)\,0\to N\to P\to M\to 0$$

in $\mathscr{F}(A)$ in which P is f.-g. and free. Since $F^{\cdot}$ is a flat complex,

$$(E)\,0\to N\otimes_A F^{\cdot}\to P\otimes_A F^{\cdot}\to M\otimes_A F^{\cdot}\to 0,$$

whence

$$\begin{aligned}(E)\,\ldots\to H^{i-1}(M\otimes_A F^{\cdot})\to H^i(N\otimes_A F^{\cdot})\to H^i(P\otimes_A F^{\cdot})\to H^i(M\otimes_A F^{\cdot})\\ \to H^{i+1}(N\otimes_A F^{\cdot})\to\ldots\to H^{t-1}(M\otimes_A F^{\cdot})\to H^t(N\otimes_A F^{\cdot})\to H^t(P\otimes_A F^{\cdot})\\ \to H^t(M\otimes_A F^{\cdot})\to 0.\end{aligned}$$

By the above, $H^i(P\otimes_A F^{\cdot})$ is f.-g. for all integers i. It follows that $H^t(M\otimes_A F^{\cdot})$ is f.-g. But M was an arbitrary f.-g. A-module. Since N is f.-g., $H^t(N\otimes_A F^{\cdot})$ is f.-g., whence (since $H^{t-1}(P\otimes_A F^{\cdot})$ is f.-g. anyway) $H^{t-1}(M\otimes_A F^{\cdot})$ is f.-g. Now use descending induction.

(ii) This is proved similarly: as $\mathrm{Hom}_A(\ ,\ )\colon\mathscr{C}(A)\times\mathscr{C}(A)\to\mathscr{C}(A)$ is contravariant in the first variable, one uses ascending induction in this case.

(3·5) COROLLARY. (i) *Suppose $F^{\cdot}$ is a flat complex in $\mathscr{Y}_c^b(A)$. Then $X^{\cdot}\otimes_A F^{\cdot}\in\mathscr{Y}_c^b(A)$ for all $X^{\cdot}\in\mathscr{Y}_c^b(A)$.*

(ii) *Suppose $I^{\cdot}$ is an injective complex in $\mathscr{Y}_c^b(A)$. Then $\mathrm{Hom}_A(X^{\cdot},I^{\cdot})\in\mathscr{Y}_c^b(A)$ for all $X^{\cdot}\in\mathscr{Y}_c^b(A)$.*

(iii) *Suppose $f\colon A\to B$ is a ring homomorphism such that B, when regarded as an A-module by means of f, is flat. Then $X^{\cdot}\otimes_A B\in\mathscr{Y}_c^b(B)$ for all $X^{\cdot}\in\mathscr{Y}_c^b(A)$.*

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Proof. For parts (i) and (ii), use (2·10), (3·3), and (3·4). Part (iii) is clear.

We can now establish a satisfactory criterion for recognizing dualizing complexes.

(3·6) THEOREM. *Consider an injective complex $I^\cdot \in \mathscr{Y}_c^b(A)$, and the morphism of functors*

$$\theta(\ ;I^\cdot)^\cdot : \text{Identity} \to \text{Hom}_A([\text{Hom}_A(\ ,I^\cdot)],I^\cdot)$$

of proposition (2·3)(ii). *$I^\cdot$ is a dualizing complex for A (i.e., $\theta(X^\cdot;I^\cdot)^\cdot$ is a quism for all $X^\cdot \in \mathscr{Y}_c^b(A)$) if and only if the morphism of complexes*

$$\alpha^\cdot (= \alpha_A(I^\cdot)^\cdot) : A \to \text{Hom}_A(I^\cdot,I^\cdot)$$

of proposition (2·3)(ii) *is a quasi-isomorphism.*

Proof. With the notation of (2·3), $\alpha^\cdot$ is a quism if and only if

$$\phi(A;I^\cdot)^\cdot : A \to A \otimes_A [\text{Hom}_A(I^\cdot,I^\cdot)]$$

is a quism. Also ((3·2)),

$$\eta(X^\cdot;I^\cdot,I^\cdot)^\cdot : X^\cdot \otimes_A [\text{Hom}_A(I^\cdot,I^\cdot)] \to \text{Hom}_A([\text{Hom}_A(X^\cdot,I^\cdot)],I^\cdot)$$

is a quism for all $X^\cdot \in \mathscr{Y}_c^b(A)$. It is now immediate that if $I^\cdot$ is a dualizing complex for A, then $\alpha^\cdot$ is a quism.

Suppose, conversely, that $\alpha^\cdot$ is a quism. Consider the morphism of functors (from $\mathscr{Y}(A)$ to $\mathscr{Y}(A)$)

$$\phi(\ ;I^\cdot)^\cdot : \text{Identity} \to (\ ) \otimes_A [\text{Hom}_A(I^\cdot,I^\cdot)];$$

we plan to apply (3·1) to this morphism. It is clear that both functors concerned are mapping-consistent and covariant (for $\text{Hom}_A(I^\cdot,I^\cdot)$ is a flat complex in $\mathscr{Y}^b(A)$); also, it is easy to see that, for all $X^\cdot \in \mathscr{Y}(A)$, $\phi(\{1\}X^\cdot;I^\cdot)^\cdot = \{1\}\phi(X^\cdot;I^\cdot)^\cdot$; it remains to check that $\phi(M;I^\cdot)^\cdot$ is a quism whenever M is a f.-g. A-module.

Since $\alpha^\cdot$ is a quism, $\phi(A;I^\cdot)^\cdot$ is a quism. It follows easily (using induction and the 5-lemma) that, if P is a f.-g. free A-module, then $\phi(P;I^\cdot)^\cdot$ is a quism. Suppose M is an arbitrary f.-g. A-module; M can be included in an exact sequence

$$(E)\ 0 \to N \to P \to M \to 0$$

in $\mathscr{F}(A)$ in which P is f.-g. and free. Denote $\text{Hom}_A(I^\cdot,I^\cdot)$ by $F^\cdot$; since $F^\cdot$ is a flat complex, we obtain

$$\begin{array}{ccccccccc} (E)\ 0 & \longrightarrow & N & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi(N;I)^\cdot} & & \downarrow{\scriptstyle (\phi P;I)^\cdot} & & \downarrow{\scriptstyle \phi(M;I)^\cdot} & & \\ (C)\ (E)\ 0 & \to & N \otimes_A F^\cdot & \to & P \otimes_A F^\cdot & \to & M \otimes_A F^\cdot & \to & 0 \end{array}$$

in $\mathscr{Y}(A)$. It now follows from consideration of the resulting morphism of the long exact sequences of cohomology modules that $\phi(M;I^\cdot)^\cdot$ is a quism. (Use the facts that $\phi(P;I^\cdot)^\cdot$ is a quism, that $F^\cdot$ is bounded, and that M is an arbitrary f.-g. A-module, so that the validity of a statement about M implies the validity of the corresponding statement about N.)

Therefore, by theorem (3·1), $\phi(X^\cdot;I^\cdot)^\cdot$ is a quism for all $X^\cdot \in \mathscr{Y}_c^b(A)$. Since $\eta(X^\cdot;I^\cdot,I^\cdot)^\cdot$ is also a quism for all $X^\cdot \in \mathscr{Y}_c^b(A)$, it follows that $\theta(X^\cdot;I^\cdot)^\cdot$ is a quism for all $X^\cdot \in \mathscr{Y}_c^b(A)$, i.e., that $I^\cdot$ is a dualizing complex for A.

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Remark. If $I^\cdot$ is a dualizing complex for A, then it can be shown that

$$\theta(X^\cdot; I^\cdot)^\cdot \colon X^\cdot \to \mathrm{Hom}_A([\mathrm{Hom}_A(X^\cdot, I^\cdot)], I^\cdot)$$

is a quism not only for all $X^\cdot \in \mathscr{D}_c^b(A)$, but also for all (unbounded) complexes $X^\cdot \in \mathscr{D}_c(A)$. This extra property can be established using the ideas developed above together with a suitable analogue of proposition I.7.1 (ii) and (iii) of Hartshorne (5).

(3·7) COROLLARY. *Suppose A is a Gorenstein ring of finite Krull dimension n, so that, when considered as a module over itself, A has finite injective dimension equal to n ((2)); let*

$$(E)\ 0 \to A \xrightarrow{d} A^0 \to A^1 \to \ldots \to A^n \to 0 \to \ldots$$

be a minimal injective resolution for A. Let $I^\cdot$ be the complex

$$\ldots \to 0 \to A^0 \to A^1 \to \ldots \to A^n \to 0 \to \ldots$$

(in which it is to be understood that A^0 is the 0th term). Then $I^\cdot$ is a dualizing complex for A.

Proof. $I^\cdot$ is certainly an injective complex in $\mathscr{D}_c^b(A)$. Also, d induces a quism $\delta^\cdot \colon A \to I^\cdot$. Since $\mathrm{Hom}_A(\ , I^\cdot) \colon \mathscr{D}(A) \to \mathscr{D}(A)$ preserves quisms ((2·8)),

$$\mathrm{Hom}_A(\delta^\cdot, I^\cdot) \colon \mathrm{Hom}_A(I^\cdot, I^\cdot) \to \mathrm{Hom}_A(A, I^\cdot)$$

is a quism. It is now straightforward to check that the composition

$$A \xrightarrow{\alpha^\cdot} \mathrm{Hom}_A(I^\cdot, I^\cdot) \xrightarrow{\mathrm{Hom}_A(\delta^\cdot, I^\cdot)} \mathrm{Hom}_A(A, I^\cdot) \to I^\cdot$$

(where the last morphism is the obvious natural isomorphism) is just $\delta^\cdot$, which is a quism. Hence $\alpha^\cdot$ is a quism. Now use (3·6).

(3·8) LEMMA. (Cartan–Eilenberg (3), chapter II, proposition 6·1 *a*.) *Suppose $f \colon A \to B$ is a ring homomorphism, and I is an injective A-module. Then $\mathrm{Hom}_A(B, I)$, when regarded as a B-module in the natural way, is an injective B-module.*

Proof. $\mathrm{Hom}_B(\ , \mathrm{Hom}_A(B, I))$ and $\mathrm{Hom}_A(\ \otimes_B B, I)$ are naturally equivalent functors from $\mathscr{C}(B)$ to $\mathscr{C}(B)$.

(3·9) THEOREM. *Suppose $f \colon A \to B$ is a ring homomorphism such that B, when regarded as an A-module by means of f, is finitely-generated. Suppose A has a dualizing complex $J^\cdot$ given by:*

$$J^\cdot \colon \ldots 0 \to J^t \to J^{t+1} \to \ldots \to J^{t+n} \to 0 \to 0 \ldots.$$

Then the complex $I^\cdot$ of B-modules and homomorphisms given by:

$$I^\cdot \colon \ldots 0 \to \mathrm{Hom}_A(B, J^t) \to \mathrm{Hom}_A(B, J^{t+1}) \to \ldots \to \mathrm{Hom}_A(B, J^{t+n}) \to 0 \to \ldots$$

(in which it is to be understood that $\mathrm{Hom}_A(B, J^t)$ is the t-th term) is a dualizing complex for B.

Proof. The $\mathrm{Hom}_A(B, J^k)$ have natural B-module structures, and when they are considered as such, the mappings $\mathrm{Hom}_A(B, J^k) \to \mathrm{Hom}_A(B, J^{k+1})$ become B-homomorphisms. As B is a f.-g. A-module, and $J^\cdot$ is an injective complex in $\mathscr{D}_c^b(A)$, (3·4) (ii) shows that $\mathrm{Hom}_A(B, J^\cdot) \in \mathscr{D}_c^b(A)$, so $I^\cdot \in \mathscr{D}_c^b(B)$. Also, (3·8) shows that $I^\cdot$ is an injective complex.

Since $J^\cdot$ is a dualizing complex for A, and B is a f.-g. A-module,

$$\theta(B; J^\cdot)^\cdot \colon B \to \mathrm{Hom}_A([\mathrm{Hom}_A(B, J^\cdot)], J^\cdot)$$

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is a quism in $\mathscr{Y}(A)$. But $\mathrm{Hom}_A([\mathrm{Hom}_A(B, J^\cdot)], J^\cdot)$ has a natural structure as a complex in $\mathscr{Y}(B)$, and, furthermore, there are natural isomorphisms in $\mathscr{Y}(B)$:

$$\lambda^\cdot: \mathrm{Hom}_A([\mathrm{Hom}_A(B, J^\cdot)], J^\cdot) \to \mathrm{Hom}_A([[\mathrm{Hom}_A(B, J^\cdot)] \otimes_B B], J^\cdot)$$

and

$$\mu^\cdot: \mathrm{Hom}_A([[\mathrm{Hom}_A(B, J^\cdot)] \otimes_B B], J^\cdot) \to \mathrm{Hom}_B([\mathrm{Hom}_A(B, J^\cdot)], [\mathrm{Hom}_A(B, J^\cdot)]).$$

(The second of these isomorphisms is induced by the natural equivalence between the functors $\mathrm{Hom}_A(\ \otimes_B B,\ )$ and $\mathrm{Hom}_B(\ , \mathrm{Hom}_A(B,\ ))$ from $\mathscr{C}(B) \times \mathscr{C}(A)$ to $\mathscr{C}(B)$.) A straightforward computation now shows that the composition

$$\mu^\cdot . \lambda^\cdot . \theta(B; J^\cdot)^\cdot: B \to \mathrm{Hom}_B(I^\cdot, I^\cdot)$$

is just $\alpha_B(I^\cdot)^\cdot$ (in $\mathscr{Y}(B)$); it follows that $\alpha_B(I^\cdot)^\cdot$ is a quism in $\mathscr{Y}(B)$, and so, by (3·6), $I^\cdot$ is a dualizing complex for B.

(3·7) and (3·9) together establish the existence of a dualizing complex for any ring which can be expressed as a homomorphic image of a Gorenstein ring of finite Krull dimension.

4. *Uniqueness properties of dualizing complexes.* In this section, we shall investigate the extent to which a dualizing complex for a ring A is uniquely determined by A. The first few results in the section will show that it is possible to alter a dualizing complex for A in certain natural ways to produce new dualizing complexes for A.

(4·1) PROPOSITION. *Suppose S is a multiplicatively closed subset (m.c.s.) of A (such that $0 \notin S$), and $I^\cdot$ is an injective complex in $\mathscr{Y}^b(A)$. Then there is a morphism of functors (from $\mathscr{Y}(A)$ to $\mathscr{Y}(S^{-1}A)$)*

$$\pi(\ ; I^\cdot \otimes_A S^{-1}A)^\cdot: [\mathrm{Hom}_A(\ , I^\cdot)] \otimes_A S^{-1}A \to \mathrm{Hom}_{S^{-1}A}(\ \otimes_A S^{-1}A, I^\cdot \otimes_A S^{-1}A)$$

which is such that $\pi(X^\cdot; I^\cdot \otimes_A S^{-1}A)^\cdot$ is a quasi-isomorphism whenever $X^\cdot \in \mathscr{Y}_c^b(A)$.

Proof. If M and N are arbitrary A-modules, then the functor

$$(\ ) \otimes_A S^{-1}A: \mathscr{C}(A) \to \mathscr{C}(S^{-1}A)$$

induces an $S^{-1}A$-homomorphism

$$\pi_1(M, N): \mathrm{Hom}_A(M, N) \otimes_A S^{-1}A \to \mathrm{Hom}_{S^{-1}A}(M \otimes_A S^{-1}A, N \otimes_A S^{-1}A);$$

furthermore, as M and N vary through $\mathscr{C}(A)$, the $\pi_1(M, N)$ constitute a natural transformation of functors; moreover, $\pi_1(M, N)$ is an isomorphism whenever M is f.-g. Now π_1 induces a morphism of functors (from $\mathscr{Y}(A)$ to $\mathscr{Y}(S^{-1}A)$)

$$\pi(\ ; I^\cdot \otimes_A S^{-1}A)^\cdot: [\mathrm{Hom}_A(\ , I^\cdot)] \otimes_A S^{-1}A \to \mathrm{Hom}_{S^{-1}A}(\ \otimes_A S^{-1}A, I^\cdot \otimes_A S^{-1}A),$$

which is such that $\pi(M; I^\cdot \otimes_A S^{-1}A)^\cdot$ is an isomorphism whenever M is f.-g. Also $I^\cdot \otimes_A S^{-1}A$ is an injective complex in $\mathscr{Y}^b(S^{-1}A)$; so, by (2·10), both functors concerned in this morphism are mapping-consistent (and contravariant); in addition, it is easy to check that, for all $X^\cdot \in \mathscr{Y}(A)$, $\pi(\{1\}X^\cdot; I^\cdot \otimes_A S^{-1}A)^\cdot = \{-1\}\pi(X^\cdot; I^\cdot \otimes_A S^{-1}A)^\cdot$. Now use (3·1) to complete the proof.

(4·2) THEOREM. *Suppose $I^\cdot$ is an injective complex in $\mathscr{Y}_c^b(A)$. Then the following statements are equivalent:*

(i) *$I^\cdot$ is a dualizing complex for A;*

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(ii) *for every m.c.s. S of A (for which $0 \notin S$), $S^{-1}(I^\cdot)$ is a dualizing complex for $S^{-1}A$;*
(iii) *for every maximal ideal $\mathfrak{m}$ of A, $(I^\cdot)_{\mathfrak{m}}$ is a dualizing complex for $A_{\mathfrak{m}}$.*

Proof. (i) $\Rightarrow$ (ii). Suppose S is an m.c.s. of A (with $0 \notin S$). It is clearly sufficient to show that $I^\cdot \otimes_A S^{-1}A$ is a dualizing complex for $S^{-1}A$; $I^\cdot \otimes_A S^{-1}A$ is certainly an injective complex in $\mathscr{D}(S^{-1}A)$, and it is also in $\mathscr{D}^b_c(S^{-1}A)$.

Since $I^\cdot$ is a dualizing complex for A, $\alpha^\cdot: A \to \mathrm{Hom}_A(I^\cdot, I^\cdot)$ is a quism; therefore (by (2·10)(iii) and (2·8)(ii)) $\alpha^\cdot \otimes_A S^{-1}A: A \otimes_A S^{-1}A \to [\mathrm{Hom}_A(I^\cdot, I^\cdot)] \otimes_A S^{-1}A$ is a quism in $\mathscr{D}(S^{-1}A)$. Consider the composition

$$S^{-1}A \to A \otimes_A S^{-1}A \to [\mathrm{Hom}_A(I^\cdot, I^\cdot)] \otimes_A S^{-1}A \to \mathrm{Hom}_{S^{-1}A}(I^\cdot \otimes_A S^{-1}A, I^\cdot \otimes_A S^{-1}A)$$

in which the first morphism is the natural isomorphism, the second is $\alpha^\cdot \otimes_A S^{-1}A$, and the third is the quism $\pi(I^\cdot; I^\cdot \otimes_A S^{-1}A)^\cdot$ of (4·1). The composition is a quism, and a straightforward computation shows that this composition is just $\alpha_{S^{-1}A}(I^\cdot \otimes_A S^{-1}A)^\cdot$. It follows from (3·6) that $I^\cdot \otimes_A S^{-1}A$ is a dualizing complex for $S^{-1}A$.

(ii) $\Rightarrow$ (iii) This is immediate.

(iii) $\Rightarrow$ (i) Reasoning similar to that employed in the proof of '(i) $\Rightarrow$ (ii)' shows that, for all maximal ideals $\mathfrak{m}$ of A,

$$\alpha^\cdot \otimes_A A_{\mathfrak{m}}: A \otimes_A A_{\mathfrak{m}} \to [\mathrm{Hom}_A(I^\cdot, I^\cdot)] \otimes_A A_{\mathfrak{m}}$$

is a quism. It follows (using, for example, (1), proposition 3·9) that $\alpha^\cdot$ is a quism, so that ((3·6)) $I^\cdot$ is a dualizing complex for A.

Recall (Swan (10), part II, chapter 7) that if P is a f.-g. projective A-module, then the rank function of P,

$$r_P: \mathrm{Spec}(A) \to Z$$

(where Z denotes the integers), is defined by the rule that, for all $\mathfrak{p} \in \mathrm{Spec}(A)$, $r_P(\mathfrak{p})$ is the rank of the f.-g. free $A_{\mathfrak{p}}$-module $P_{\mathfrak{p}}$. r_P is constant on the connected components of $\mathrm{Spec}(A)$ (in the Zariski topology). The invertible A-modules are precisely the f.-g. projective A-modules of (constant) rank 1; the isomorphism classes of invertible A-modules form a group (called the Picard group of A) with respect to an operation induced from tensor product; in particular ((10), part II, chapter 1), if P is an invertible A-module, the A-homomorphism $\gamma(P): P \otimes_A \mathrm{Hom}_A(P, A) \to A$ defined by (for $x \in P$ and $f \in \mathrm{Hom}_A(P, A)$) $\gamma(P)(x \otimes f) = f(x)$ is an isomorphism.

We can now show that certain variations in dualizing complexes are possible.

(4·3) THEOREM. *Suppose $I^\cdot$ and $J^\cdot$ are injective complexes in $\mathscr{D}^b_c(A)$, that P is an invertible A-module, and that v is an integer. Then:*
(i) *$I^\cdot$ is a dualizing complex for A if and only if $\{v\}I^\cdot$ is.*
(ii) *$I^\cdot$ is a dualizing complex for A if and only if $I^\cdot \otimes_A P$ is.*
(iii) *Suppose there is a quasi-isomorphism $\delta^\cdot: I^\cdot \to J^\cdot$. Then $I^\cdot$ is a dualizing complex for A if and only if $J^\cdot$ is.*

Proof. (i) Clearly, $\{v\}I^\cdot$ is an injective complex in $\mathscr{D}^b_c(A)$. Moreover, the complexes $\mathrm{Hom}_A(I^\cdot, I^\cdot)$ and $\mathrm{Hom}_A(\{v\}I^\cdot, \{v\}I^\cdot)$ have corresponding terms equal: in fact, for each integer n, $\mathrm{Hom}^n_A(I^\cdot, I^\cdot) = \mathrm{Hom}^n_A(\{v\}I^\cdot, \{v\}I^\cdot) = \sum_{i \in Z} \mathrm{Hom}_A(I^i, I^{i+n})$ (direct sum). We construct an isomorphism in $\mathscr{D}(A)$ between these two complexes as follows. For

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each integer n, let $\beta^n\colon \mathrm{Hom}^n_A(I^\cdot, I^\cdot) \to \mathrm{Hom}^n_A(\{v\}I^\cdot, \{v\}I^\cdot)$ be the A-homomorphism which, for each $i \in Z$, has restriction to $\mathrm{Hom}_A(I^i, I^{i+n})$ equal to multiplication by $(-1)^{iv}$. Then β^n is clearly an isomorphism (for each integer n); straightforward computations now show that the β^n constitute an isomorphism of complexes $\beta^\cdot$, and that $\beta^\cdot.\alpha(I^\cdot)^\cdot = (-1)^s\alpha(\{v\}I^\cdot)^\cdot$, where $s = \frac{1}{2}v(v-1)$. It follows that $\alpha(I^\cdot)^\cdot$ is a quism if and only if $\alpha(\{v\}I^\cdot)^\cdot$ is a quism. Now use (3·6).

(ii) This is immediate from (4·2), since, for all maximal ideals $\mathfrak{m}$ of A, $P \otimes_A A_\mathfrak{m} \cong A_\mathfrak{m}$ (as $A_\mathfrak{m}$-modules).

(iii) First use the ideas of Hartshorne ((5), chapter I, 4·4 and 4·5), to construct a morphism of complexes $\gamma^\cdot\colon J^\cdot \to I^\cdot$ such that $Id_{I^\cdot} - \gamma^\cdot.\delta^\cdot$ is homotopic ((5), chapter I, § 2) to zero. ($Id_{I^\cdot}$ denotes the identity morphism of the complex $I^\cdot$.) It follows, since $\delta^\cdot$ is a quism, that $\gamma^\cdot$ is a quism also.

Next, for all $X^\cdot \in \mathscr{Y}(A)$, let $\xi(X^\cdot)^\cdot = \mathrm{Hom}_A(X^\cdot, \gamma^\cdot)\colon \mathrm{Hom}_A(X^\cdot, J^\cdot) \to \mathrm{Hom}_A(X^\cdot, I^\cdot)$. Now $\xi(\ )^\cdot\colon \mathrm{Hom}_A(\ , J^\cdot) \to \mathrm{Hom}_A(\ , I^\cdot)$ is a morphism of functors (from $\mathscr{Y}(A)$ to $\mathscr{Y}(A)$); for all $X^\cdot \in \mathscr{Y}(A)$, it is the case that $\xi(\{1\}X^\cdot)^\cdot = \{-1\}\xi(X^\cdot)^\cdot$; both functors concerned are mapping-consistent ((2·10) (i)); and $\xi(A)^\cdot$ is a quism. Moreover, a simple argument, rather similar to that employed in the proof of (3·6), will show that $\xi(M)^\cdot$ is a quism whenever M is a f.-g. A-module. It therefore follows from (3·1) that $\xi(X^\cdot)^\cdot$ is a quism whenever $X^\cdot \in \mathscr{Y}^b_c(A)$.

In particular, $\mathrm{Hom}_A(J^\cdot, \gamma^\cdot)\colon \mathrm{Hom}_A(J^\cdot, J^\cdot) \to \mathrm{Hom}_A(J^\cdot, I^\cdot)$ is a quism. But, since $\mathrm{Hom}_A(\ , I^\cdot)\colon \mathscr{Y}(A) \to \mathscr{Y}(A)$ is a mapping-consistent functor,

$$\mathrm{Hom}_A(\delta^\cdot, I^\cdot)\colon \mathrm{Hom}_A(J^\cdot, I^\cdot) \to \mathrm{Hom}_A(I^\cdot, I^\cdot)$$

is a quism. Composing,

$$\mathrm{Hom}_A(\delta^\cdot, \gamma^\cdot)\colon \mathrm{Hom}_A(J^\cdot, J^\cdot) \to \mathrm{Hom}_A(I^\cdot, I^\cdot)$$

is a quism: use of this quism in conjunction with the fact that $Id_{I^\cdot} - \gamma^\cdot.\delta^\cdot$ is homotopic to zero enables one to see that $\alpha_A(J^\cdot)^\cdot$ is a quism if and only if $\alpha_A(I^\cdot)^\cdot$ is a quism. Now use (3·6).

Recall that, if B is a ring, then Spec (B) is not connected (in the Zariski topology) if and only if B is isomorphic to the direct product of two other rings. (Note that our restricted use of the word 'ring' excludes any one-element set.) Our main theorem in this section shows that, when the ring concerned has connected prime spectrum, the only possible variations in dualizing complexes are those given in (4·3). Before establishing this 'uniqueness theorem', we need two preparatory results, the second of which is a 'uniqueness theorem' for dualizing complexes over a local ring.

(4·4) LEMMA. *Suppose $X^\cdot$, $F^\cdot$ are complexes in $\mathscr{Y}^b(A)$, and that $F^\cdot$ is flat. If either $X^\cdot$ or $F^\cdot$ is an exact complex, then $X^\cdot \otimes_A F^\cdot$ is exact. However, if neither $X^\cdot$ nor $F^\cdot$ is exact, and r (resp. s) is the largest integer i for which $H^i(X^\cdot)$ (resp. $H^i(F^\cdot)$) $\neq 0$, then*

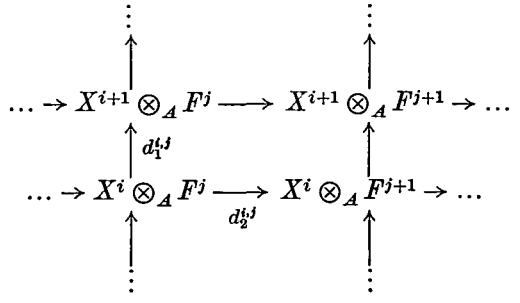
$$H^i(X^\cdot \otimes_A F^\cdot) = 0 \quad \textit{for all} \quad i > r+s,$$

while

$$H^{r+s}(X^\cdot \otimes_A F^\cdot) \cong H^r(X^\cdot) \otimes_A H^s(F^\cdot).$$

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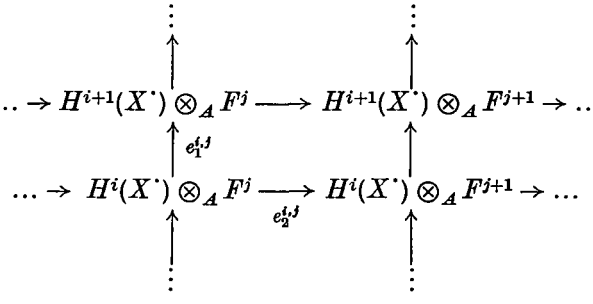
Proof. $X^{\cdot} \otimes_A F^{\cdot}$ is the single complex associated with the double complex



(in which $d_1^{i,j} = d_{X^{\cdot}}^i \otimes_A F^j$ and $d_2^{i,j} = (-1)^i (X^i \otimes_A d_{F^{\cdot}}^j)$); since $X^{\cdot}$ and $F^{\cdot}$ are bounded, the second filtration of this (double) complex is regular, and so convergent. Consider the associated spectral sequence, i.e., the second spectral sequence of the double complex,

$$E_2^{p,q} \underset{p}{\Rightarrow} H^n(X^{\cdot} \otimes_A F^{\cdot})$$

((3), chapter XV, § 6): $E_2^{p,q}$ is isomorphic to the '$H_{II}^{q,p}$' cohomology module associated with the double complex



(in which $e_1^{i,j} = 0$ and $e_2^{i,j} = (-1)^i (H^i(X^{\cdot}) \otimes_A d_{F^{\cdot}}^j)$). (Here, we have used the fact that $F^{\cdot}$ is a flat complex.) It is now clear that if $X^{\cdot}$ is exact, then $E_2^{p,q} = 0$ for all integers p and q, and so $X^{\cdot} \otimes_A F^{\cdot}$ is exact also. We may henceforth suppose that $X^{\cdot}$ is not exact.

$F^{\cdot}$ is bounded: let u be the largest integer i for which $F^i \neq 0$. (We may assume $F^{\cdot}$ is not the zero complex.) Suppose t ($\leqslant u$) is an integer such that $H^i(F^{\cdot}) = 0$ for all $i > t$. Let $Z^t = \ker d_{F^{\cdot}}^t$. From the exact sequence

$$(E)\ 0 \to Z^t \to F^t \to F^{t+1} \to \ldots \to F^u \to 0$$

and the fact that $F^{\cdot}$ is a flat complex, we see that Z^t is a flat A-module, and furthermore that if Y is any A-module, then

$$(E)\ 0 \to Y \otimes_A Z^t \to Y \otimes_A F^t \to \ldots \to Y \otimes_A F^u \to 0$$

is also exact. It follows that $H^t(Y \otimes_A F^{\cdot}) \cong Y \otimes_A H^t(F^{\cdot})$, while $H^i(Y \otimes_A F^{\cdot}) = 0$ for all $i > t$. This shows that, if $F^{\cdot}$ is exact, $E_2^{p,q} = 0$ for all p and q, so $X^{\cdot} \otimes_A F^{\cdot}$ is exact. So we may suppose neither $X^{\cdot}$ nor $F^{\cdot}$ is exact, and that r and s are defined as in the statement of the lemma. It follows (upon taking $t = s$ in the above) that $E_2^{p,q} = 0$ whenever $p > s$; and that $E_2^{s,q} \cong H^q(X^{\cdot}) \otimes_A H^s(F^{\cdot})$. In addition, we know that $E_2^{p,q} = 0$ whenever $q > r$. The result now follows from (3), chapter XV, corollary 5·4.

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(4·5) THEOREM. *Suppose A is a local ring, and $I^\cdot$, $J^\cdot$ are injective complexes in $\mathscr{D}_c^b(A)$ which are both dualizing complexes for A. Then there exist an integer t and a quasi-isomorphism*

$$\delta^\cdot : I^\cdot \to \{t\}J^\cdot.$$

Proof. Let $L^\cdot = \mathrm{Hom}_A(I^\cdot, J^\cdot)$ and $M^\cdot = \mathrm{Hom}_A(J^\cdot, I^\cdot)$: both $L^\cdot$ and $M^\cdot$ are flat complexes in $\mathscr{D}_c^b(A)$ ((2·2) (iii) and (3·5) (ii)). Consider the quism ((3·2))

$$\eta(L^\cdot; J^\cdot, I^\cdot)^\cdot : L^\cdot \otimes_A M^\cdot = L^\cdot \otimes_A [\mathrm{Hom}_A(J^\cdot, I^\cdot)] \to \mathrm{Hom}_A([\mathrm{Hom}_A(L^\cdot, J^\cdot)], I^\cdot).$$

Since $J^\cdot$ is a dualizing complex for A and $I^\cdot \in \mathscr{D}_c^b(A)$,

$$\theta(I^\cdot; J^\cdot)^\cdot : I^\cdot \to \mathrm{Hom}_A([\mathrm{Hom}_A(I^\cdot, J^\cdot)], J^\cdot) = \mathrm{Hom}_A(L^\cdot, J^\cdot)$$

is a quism; this induces a quism

$$\mathrm{Hom}_A[(\mathrm{Hom}_A(L^\cdot, J^\cdot)], I^\cdot) \to \mathrm{Hom}_A(I^\cdot, I^\cdot)$$

(since $I^\cdot$ is injective and in $\mathscr{D}_c^b(A)$). But, since $I^\cdot$ is a dualizing complex for A,

$$\alpha_A(I^\cdot)^\cdot : A \to \mathrm{Hom}_A(I^\cdot, I^\cdot)$$

is a quism. It follows that $H^i(L^\cdot \otimes_A M^\cdot) = 0$ for $i \neq 0$, while $H^0(L^\cdot \otimes_A M^\cdot) \cong A$. We now apply (4·4), from which it follows that neither $L^\cdot$ nor $M^\cdot$ is exact: let t (resp. v) be the largest integer i for which $H^i(L^\cdot)$ (resp. $H^i(M^\cdot)$) $\neq 0$. Again by (4·4), $H^i(L^\cdot \otimes_A M^\cdot) = 0$ for all $i > t+v$, while

$$H^{t+v}(L^\cdot \otimes_A M^\cdot) \cong H^t(L^\cdot) \otimes_A H^v(M^\cdot).$$

But A is local, and $H^t(L^\cdot)$ and $H^v(M^\cdot)$ are both f.-g.; it follows (Serre (8), chapitre I, proposition 10) that $H^t(L^\cdot) \otimes_A H^v(M^\cdot) \neq 0$. Therefore

$$t+v = 0 \quad \text{and} \quad H^t(L^\cdot) \otimes_A H^v(M^\cdot) \cong A.$$

Now use (6), chapter 9, proposition 3, to see that each of $H^t(L^\cdot)$ and $H^v(M^\cdot)$ is a cyclic A-module; since the tensor product of these two modules is isomorphic to A, it follows that they must themselves both be isomorphic to A.

However, as A is projective, we can actually construct a morphism $\sigma^\cdot : \{-t\}A \to L^\cdot$ with the property that $H^t(\sigma^\cdot)$ is an isomorphism. We aim to show that $\sigma^\cdot$ is actually a quism. Suppose that this is not the case; then there must be an integer i other than t for which $H^i(L^\cdot) \neq 0$; let r be the second largest integer i such that $H^i(L^\cdot) \neq 0$.

Next, the long exact cohomology sequence induced by

$$(E)\ 0 \to L^\cdot \to Mc(\sigma^\cdot) \to \{1\}\{-t\}A \to 0 \quad (*)$$

shows that ($Mc(\sigma^\cdot) \in \mathscr{D}_c^b(A)$ and) r is the greatest integer i for which $H^i(Mc(\sigma^\cdot)) \neq 0$. So, by (4·4) and Serre (8), chapitre I, proposition 10, $r-t$ is the greatest integer i for which $H^i(Mc(\sigma^\cdot) \otimes_A M^\cdot) \neq 0$.

However, we have available a second method for studying the cohomology of $Mc(\sigma^\cdot) \otimes_A M^\cdot$: for $M^\cdot$ is a flat complex in $\mathscr{D}_c^b(A)$, so that $(\ ) \otimes_A M^\cdot : \mathscr{D}(A) \to \mathscr{D}(A)$ is a mapping-consistent functor. Hence (*) induces ((2·7) (iii))

$$(E)\ 0 \to L^\cdot \otimes_A M^\cdot \to Mc(\sigma^\cdot) \otimes_A M^\cdot \to \{1\}\{-t\}A \otimes_A M^\cdot \to 0 \quad (\dagger).$$

Now $H^i(L^\cdot \otimes_A M^\cdot) = 0$ for $i \neq 0$, while $H^0(L^\cdot \otimes_A M^\cdot) \cong A$. Also, for all integers i, $H^i(\{-t\}A \otimes_A M^\cdot) \cong H^{i-t}(M^\cdot)$, which is zero for $i > 0$, while $H^0(\{-t\}A \otimes_A M^\cdot) \cong A$.

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Also, from the above, $H^i(Mc(\sigma^\cdot) \otimes_A M^\cdot) = 0$ for all $i > r-t$, and in particular for all $i \geqslant 0$. Consequently, the long exact cohomology sequence induced by (†) shows that there is an exact sequence

$$(E)\ 0 \to H^{-1}(Mc(\sigma^\cdot) \otimes_A M^\cdot) \to A \to A \to 0,$$

and that $H^{i-1}(Mc(\sigma^\cdot) \otimes_A M^\cdot) \cong H^i(\{-t\}A \otimes_A M^\cdot) \cong H^{i-t}(M^\cdot)$ for all $i < 0$. Now use Atiyah–Macdonald (1), chapter 6, exercise 1 (i), to see that $H^{-1}(Mc(\sigma^\cdot) \otimes_A M^\cdot) = 0$, whence $r-t < -1$. Consequently, there must be an integer i other than $-t\ (= v)$ for which $H^i(M^\cdot) \neq 0$; let q be the second largest integer i for which $H^i(M^\cdot) \neq 0$. We can now calculate that $r-t+1 = q+t$. (**)

On reversing the roles of $I^\cdot$ and $J^\cdot$ in the above argument, we obtain

$$q-v+1 = r+v,$$

i.e. $r-t = q+t+1$, a contradiction to (**) above. Hence $\sigma^\cdot: \{-t\}A \to L^\cdot$ is a quism.

To complete the proof, we note that $\sigma^\cdot$ induces a quism

$$\mathrm{Hom}_A(L^\cdot, J^\cdot) \to \mathrm{Hom}_A(\{-t\}A, J^\cdot) = \{t\}\,\mathrm{Hom}_A(A, J^\cdot).$$

But $J^\cdot$ is a dualizing complex for A, so there is a quism

$$I^\cdot \to \mathrm{Hom}_A([\mathrm{Hom}_A(I^\cdot, J^\cdot)], J^\cdot) = \mathrm{Hom}_A(L^\cdot, J^\cdot),$$

while $\mathrm{Hom}_A(A, J^\cdot)$ is isomorphic to $J^\cdot$. Composing, we finally obtain a quism $I^\cdot \to \{t\}J^\cdot$.

(4·6) THEOREM. *Suppose A is a ring having connected spectrum, and $I^\cdot$, $J^\cdot$ are injective complexes in $\mathscr{D}_c^b(A)$ which are both dualizing complexes for A. Then there exist an integer t, an invertible A-module P, and a quasi-isomorphism*

$$\delta^\cdot: I^\cdot \to \{t\}J^\cdot \otimes_A P.$$

Proof. Let $L^\cdot = \mathrm{Hom}_A(I^\cdot, J^\cdot)$ and $M^\cdot = \mathrm{Hom}_A(J^\cdot, I^\cdot)$: both $L^\cdot$ and $M^\cdot$ are flat complexes in $\mathscr{D}_c^b(A)$ ((2·2) (iii) and (3·5) (ii)).

Suppose $\mathfrak{p} \in \mathrm{Spec}(A)$. By (4·1), there is a quism (in $\mathscr{D}(A_\mathfrak{p})$) $L^\cdot_\mathfrak{p} \to \mathrm{Hom}_{A_\mathfrak{p}}(I^\cdot_\mathfrak{p}, J^\cdot_\mathfrak{p})$; by (4·2), $I^\cdot_\mathfrak{p}$ and $J^\cdot_\mathfrak{p}$ are both dualizing complexes for $A_\mathfrak{p}$; therefore, by (4·5), there is an integer $t(\mathfrak{p})$ and a quism (in $\mathscr{D}(A_\mathfrak{p})$) $I^\cdot_\mathfrak{p} \to \{t(\mathfrak{p})\}J^\cdot_\mathfrak{p}$; this induces a quism

$$\{-t(\mathfrak{p})\}\,\mathrm{Hom}_{A_\mathfrak{p}}(J^\cdot_\mathfrak{p}, J^\cdot_\mathfrak{p}) \to \mathrm{Hom}_{A_\mathfrak{p}}(I^\cdot_\mathfrak{p}, J^\cdot_\mathfrak{p})$$

which enables us to calculate the cohomology of $\mathrm{Hom}_{A_\mathfrak{p}}(I^\cdot_\mathfrak{p}, J^\cdot_\mathfrak{p})$.

The indexed set $\{t(\mathfrak{p})\}_{\mathfrak{p} \in \mathrm{Spec}(A)}$ determines a function $t: \mathrm{Spec}(A) \to Z$. Now for $\mathfrak{p} \in \mathrm{Spec}(A)$, $t(\mathfrak{p})$ is the one and only integer i for which $H^i(\mathrm{Hom}_{A_\mathfrak{p}}(I^\cdot_\mathfrak{p}, J^\cdot_\mathfrak{p})) \neq 0$, and so is the one and only integer i for which $H^i(L^\cdot_\mathfrak{p}) \neq 0$; furthermore, $H^{t(\mathfrak{p})}(L^\cdot_\mathfrak{p}) \cong A_\mathfrak{p}$. Use of (3·8) of (9) now shows that, if $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(A)$ are such that $\mathfrak{p} \supseteq \mathfrak{q}$, then $t(\mathfrak{p}) = t(\mathfrak{q})$. It follows, by (3·7) of (9), that $t: \mathrm{Spec}(A) \to Z$ is constant: we denote its constant value also by t.

Thus t is the one and only integer i for which $H^i(L^\cdot) \neq 0$; moreover, the (f.-g.) A-module $H^t(L^\cdot)$ has the property that $[H^t(L^\cdot)]_\mathfrak{p} \cong A_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$. It follows that $H^t(L^\cdot)$ is invertible: denote this module by Q.

Similarly, there is precisely one integer i for which $H^i(M^\cdot) \neq 0$: let v denote this integer. $H^v(M^\cdot)$ will also be invertible; let P denote this module.

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Next, reasoning as in the first part of the proof of (4·5) (using the facts that $I^\cdot$ and $J^\cdot$ are both dualizing complexes for A) we see $H^i(L^\cdot \otimes_A M^\cdot) = 0$ if $i \neq 0$, while

$$H^0(L^\cdot \otimes_A M^\cdot) \cong A.$$

Now use (4·4) (in conjunction with the already established facts that

$$\operatorname{Supp}(H^t(L^\cdot)) = \operatorname{Supp}(H^v(M^\cdot)) = \operatorname{Spec}(A))$$

to see that $t+v = 0$ and $H^t(L^\cdot) \otimes_A H^v(M^\cdot) \cong A$, i.e. $Q \otimes_A P \cong A$. Since P is invertible, it follows that $Q \cong \operatorname{Hom}_A(P, A)$.

Since Q is a projective A-module, we can actually construct a quism

$$\{-t\}\operatorname{Hom}_A(P, A) \to L^\cdot = \operatorname{Hom}_A(I^\cdot, J^\cdot);$$

this induces a quism

$$\operatorname{Hom}_A([\operatorname{Hom}_A(I^\cdot, J^\cdot)], J^\cdot) \to \{t\}\operatorname{Hom}_A([\operatorname{Hom}_A(P, A)], J^\cdot).$$

Since $J^\cdot$ is a dualizing complex for A, there is a quism

$$I^\cdot \to \operatorname{Hom}_A([\operatorname{Hom}_A(I^\cdot, J^\cdot)], J^\cdot).$$

Also, since P is f.-g. and $J^\cdot$ is an injective complex, (2·2) shows that there is an isomorphism of complexes $\operatorname{Hom}_A([\operatorname{Hom}_A(P, A)], J^\cdot) \to P \otimes_A [\operatorname{Hom}_A(A, J^\cdot)]$, and the latter complex is clearly isomorphic to $J^\cdot \otimes_A P$. Composing, we finally obtain a quism

$$I^\cdot \to \{t\}J^\cdot \otimes_A P.$$

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