for any odd integers X, Y. Since

$$x-\theta y-\alpha = \tfrac{1}{2}\{(2x+1)-\theta(2y+1)\},$$

the lemma implies that $$|(x-\theta y-\alpha)(x-\theta' y-\alpha')| \geqslant \tfrac{1}{2}k \tag{5}$$

for all integers x, y, where θ' and α' denote the algebraic conjugates of θ and α. In proving (3), we can suppose without loss of generality that $x = \theta y+\alpha+\xi$, where $|\xi| < \frac{1}{4}$. Using (5), we obtain

$$|(x-\theta y-\alpha)y| \geqslant \tfrac{1}{2}k\left|\frac{y}{x-\theta' y-\alpha'}\right| \geqslant \tfrac{1}{2}k\,\frac{|y|}{(\theta-\theta')|y|+(\alpha-\alpha')+\frac{1}{4}}.$$

Since $\theta-\theta' = 2\sqrt{(k^2+1)}$ and $\alpha-\alpha' = \sqrt{(k^2+1)}$, the conclusion (3) follows if

$$|y| \geqslant \tfrac{1}{2}(k-1)$$

and k is sufficiently large.

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A NOTE ON THE INTERSECTION THEOREM FOR IDEALS

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Let R be a commutative ring (with a unit element) in which the ascending chain condition for ideals is satisfied, and let $\mathfrak{a}$ be an ideal. The following theorem is well known.

Theorem. *In order that* $\bigcap_{i=1}^{\infty} \mathfrak{a}^i = (0)$ *it is necessary and sufficient that no zero divisor shall be congruent modulo* $\mathfrak{a}$ *to the unit element.*

A special case of this important theorem was proved by Krull ((3); see also (2)), and in the form stated above it has been established by Chevalley ((1), Lemma 2, p. 692). We now give two refinements which appear to be new.

Theorem 1. *In order that* $x \in \bigcap_{i=1}^{\infty} \mathfrak{a}^i$ *it is necessary and sufficient that* $x = xa$ *for at least one element* $a \in \mathfrak{a}$.

Proof. We may suppose that $\mathfrak{a} \neq (1)$. Let S consist of all elements of the form $1-a$ where $a \in \mathfrak{a}$, then S is multiplicatively closed. Further, we have $x = xa$ for at least one $a \in \mathfrak{a}$, if, and only if, $cx = 0$ for at least one $c \in S$. Denote the set of all such x by $\mathfrak{n}$, then $\mathfrak{n}$ is an ideal, and, since $\mathfrak{a} \neq (1)$, $\mathfrak{n}$ does not contain the unit element and so is a proper ideal. Theorem 1 states that $\bigcap_{i=1}^{\infty} \mathfrak{a}^i = \mathfrak{n}$. Now if $x \in \mathfrak{n}$ then $x = ax$ for some $a \in \mathfrak{a}$; hence

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$x = a^i x$ for all i, and therefore $\mathfrak{n} \subseteq \mathfrak{a}^i$ for all i. If we put $R^* = R/\mathfrak{n}$ and $\mathfrak{a}^* = \mathfrak{a}/\mathfrak{n}$, our problem reduces to proving that $\bigcap_{i=1}^{\infty} (\mathfrak{a}^*)^i = (0)$. By the theorem of Krull-Chevalley, this will follow if we show that there is no zero divisor congruent to the unit element modulo $\mathfrak{a}^*$. Assume that $1 \equiv c^* \pmod{\mathfrak{a}^*}$ and that $c^* x^* = 0$. Then if c and x are representatives of c^* and x^* respectively, we have $c \in S$ and $cx \in \mathfrak{n}$; consequently we can find $c' \in S$ such that $c'cx = 0$. Since S is multiplicatively closed, this shows that $x \in \mathfrak{n}$, and therefore $x^* = 0$. The proof is now complete.

In order to make the notation of the next theorem as suggestive as possible, we use $1 - \mathfrak{a}$ to denote the set S.

THEOREM 2. *If* $(0) = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_n$, *where* $\mathfrak{q}_j$ *is* $\mathfrak{p}_j$*-primary, and if* $\mathfrak{p}_j$ *meets* $1 - \mathfrak{a}$ *for* $m+1 \leqslant j \leqslant n$ *but not for* $1 \leqslant j \leqslant m$, *then*

$$\bigcap_{i=1}^{\infty} \mathfrak{a}^i = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_m.$$

Proof. If we keep our previous notations and use the results already obtained, it will be sufficient to prove that $\mathfrak{n} = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_m$. Let $x \in \mathfrak{n}$, then with a suitable $c \in S$ we have $cx = 0 \in \mathfrak{q}_j$, and if $1 \leqslant j \leqslant m$ we have $c \notin \mathfrak{p}_j$; consequently $x \in \mathfrak{q}_j$ for $1 \leqslant j \leqslant m$. Now suppose that $y \in \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_m$. For each $j > m$ we can choose c_j so that $c_j \in \mathfrak{p}_j \cap S$, and then, if N is large, $y(c_{m+1} \ldots c_n)^N \in \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_n = (0)$, i.e. $y(c_{m+1} \ldots c_n)^N = 0$. Since S is multiplicatively closed this shows that $y \in \mathfrak{n}$ and completes the proof.

COROLLARY. $\bigcap_{i=1}^{\infty} \mathfrak{a}^i$ *is always an isolated component of the zero ideal.*

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