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p-adic cocycles and their regulator maps

by

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Abstract

We derive a power series formula for the p-adic regulator on the higher dimensional algebraic K-groups of number fields. This formula is designed to be well suited to computer calculations and to reduction modulo powers of p.

Key Words: p-adic regulator, algebraic K-group, local field
Mathematics Subject Classification 2010: 19F27, 14F20, 14G20

1. Introduction

Let F be a p-adic local field. Then a p-adic regulator is a homomorphism of the form, for $s \geq 2$,

$$R_F : K_{2s-1}(\mathcal{O}_F) \cong K_{2s-1}(F) \longrightarrow F.$$

There are lots of such p-adic regulators in the literature (see, for example, [13] §5). They have a number of important uses in arithmetic and geometry.

Borel's construction of the Archimedean regulator [1] uses the van Est isomorphism between continuous and Lie algebra cohomology. In [5] Huber and Kings use an idea originally due to Wagoner [16] which substitutes the p-adic Lazard isomorphism [8] for the van Est isomorphism to construct a p-adic regulator. Using the Bloch-Kato exponential [5] shows that their p-adic regulator coincides with the étale regulator of [14]. A p-adic regulator due to Karoubi is constructed using cyclic homology and relative K-theory ([6]; [7]). An alternative construction of Karoubi's regulator is given by Hamida in [4] as an explicit p-adic integral.

In ([2] II) we used R.H. Fox's free differential calculus to design an algorithm (implemented in C) to construct explicit homology cycles for the general linear group whose Borel regulators were calculated by power series algorithm (implemented in MAPLE) designed from the explicit formula given in [3]. In the course of working on [2] we noticed that our power series also converged p-adically, giving rise to an elementary, complete account of the regulator of [4] which culminates in R_F of Corollary 4.3. Independently this construction was introduced in [15] and used to show that the regulators of [4] and [5] coincide up to a non-zero rational factor.

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Our original motivation for developing the details of the p-adic regulator was similar to our motivation for [2], namely that the power series makes possible an algorithm for evaluating the p-adic valuation of the regulators on homology classes in the general linear group of number rings such as those given by the algorithm of [2]. In addition, for p-adic fields the power series approach potentially offers a way round the p-adic analytic continuation problems which can sometimes bedevil the calculation of p-adic polylogarithms. Also the values of the p-adic regulator of a p-adic local field F land in a fractional ideal of F and a powers series formula can simply be truncated to evaluate the regulator modulo a power of p. In all other respects it should be clear to the reader that our approach has nothing to add to the more sophisticated methods of [5], [6] and [15].

We are very grateful to the referee and the editor for suggesting a number of improvements to our exposition.

2. Functions on p-adic power series

Definition 2.1 Let F be a p-adic local field and let $\mathcal{O}_F$ denote its valuation ring. Let N be a positive integer and let $M_N\mathcal{O}_F$ denote the ring of $N \times N$ matrices with entries in $\mathcal{O}_F$ topologised with the p-adic topology. Fix a positive odd integer $2s-1$ with $s \geq 2$. Let $E(dx_0,\ldots,dx_{2s-1})$ denote the $\mathcal{O}_F$-exterior algebra on symbols $dx_0,\ldots,dx_{2s-1}$ so that $dx_i \wedge dx_j = -dx_j \wedge dx_i$ when $i \neq j$ and $dx_i \wedge dx_i = 0$. Consider the $\mathcal{O}_F$-algebra

$$\hat{\mathcal{A}} = M_N\mathcal{O}_F[[x_0,x_1,\ldots,x_{2s-1}]] \otimes_{\mathcal{O}_F} E(dx_0,\ldots,dx_{2s-1})$$

and set

$$\mathcal{A} = \hat{\mathcal{A}}/\simeq,$$

the quotient of $\hat{\mathcal{A}}$ by the ideal generated by $1 - \sum_{i=0}^{2s-1} x_i$ and $\sum_{i=0}^{2s-1} dx_i$. Setting $|\underline{a}| = a_0 + a_1 + \ldots\ldots + a_{2s-1}$, let $f \in \mathcal{A}$ have the form

$$f = \sum_{\underline{a}=(a_0,\ldots,a_{2s-1})} \sum_{u=0}^{2s-1} f(\underline{a},u)\, p^{e|\underline{a}|} x_0^{a_0} \cdots x_{2s-1}^{a_{2s-1}}\, dx_0 \wedge \cdots \wedge \hat{dx_u} \wedge \cdots dx_{2s-1}$$

with each a_j an integer greater than or equal to zero and $f(\underline{a},u) \in M_N\mathcal{O}_F$. Define a function, justified by Proposition 2.2,

$$\Phi_{2s-1} : \mathcal{A} \longrightarrow F$$

by the formula

$$\Phi_{2s-1}(f) = \sum_{\underline{a}=(a_0,\ldots,a_{2s-1})} \sum_{u=0}^{2s-1} (-1)^u \mathrm{Trace}\, f(\underline{a},u)\, p^{e|\underline{a}|} \frac{a_0! \cdot a_1! \cdots a_{2s-1}!}{(|\underline{a}| + 2s - 1)!}.$$

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Hence $\Phi_{2s-1}(f)$ term-by-term substitutes

$$(-1)^u f(\underline{a},u)\, p^{e|\underline{a}|}\, \frac{a_0!\cdot a_1!\cdots a_{2s-1}!}{(|\underline{a}|+2s-1)!}$$

for

$$f(\underline{a},u)\, p^{e|\underline{a}|}\, x_0^{a_0}\cdots x_{2s-1}^{a_{2s-1}}\, dx_0\wedge\cdots\wedge \hat{dx_u}\wedge\cdots dx_{2s-1}$$

and then takes the trace of the resulting matrix in $M_N F$.

Proposition 2.2 *The series $\Phi_{2s-1}(f)$ of Definition 2.1 converges p-adically in F for all $e\geq 1$ if p is odd and for all $e\geq 2$ if $p=2$.*

Proof: The series $\Phi_{2s-1}(f)$ is defined on $\mathcal{A}$, not merely on $\hat{\mathcal{A}}$, because a simple calculation shows that it is equal to the term-by-term integration of f over the standard $(2s-1)$-simplex. This point of view will be important in §4.7, where the series is related to Hamida's p-adic regulator formula [4].

If v_p is the p-adic valuation on the rational numbers and $[x]$ denotes the integer part of x, then

$$v_p(l!)=\sum_{i=1}^{\infty}[\frac{l}{p^i}]=\frac{l-\alpha(l)}{p-1}.$$

Here, if $l=\sum_{j\geq 0} b_j p^j$ with each b_j an integer in the range $0\leq b_j\leq p-1$, we set $\alpha(l)=\sum_{j\geq 0} b_j$. Therefore

$$\begin{aligned}
&v_p(p^{e|\underline{a}|}\,\tfrac{a_0!\cdot a_1!\cdots a_{2s-1}!}{(|\underline{a}|+2s-1)!})\\
&=e|\underline{a}|+v_p(a_0!\cdot a_1!\cdots a_{2s-1}!)-\tfrac{|\underline{a}|+2s-1-\alpha(|\underline{a}|+2s-1)}{p-1}\\
&\geq (e-\tfrac{1}{p-1})|\underline{a}|-\tfrac{2s-1}{p-1}
\end{aligned}$$

which tends to infinity as the monomial $x_0^{a_0}\cdots x_{2s-1}^{a_{2s-1}}$ tends to zero in the power series topology (that is, as $|\underline{a}|$ tends to infinity). □

Remark 2.3 The integer e appearing in Definition 2.1 is chosen merely in order to ensure that the series $\Phi_{2s-1}(f)$ of converges p-adically in F. The values in F of $\Phi_{2s-1}(f)$ vary by a non-zero factor according to the choice of e. However, in Definition 4.1, the formula for a p-adic regulator as a convergent series also contains a scalar factor depending on the choice of e. The result is a p-adic series formula in Definition 4.4 which is independent of e and equal to that of [4].

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3. A p-adic cocycle

Definition 3.1 Let $X_0, X_1, \ldots, X_{2s-1}$ be matrices lying in $M_N\mathcal{O}_F$ with $s \geq 2$. Denote the $2s$-tuple $(X_0, X_1, \ldots, X_{2s-1})$ by $\underline{X}$. If $\mathcal{A}$ is the algebra introduced in Definition 2.1, let

$$\nu(\underline{X}) = 1 + p^e \sum_{i=0}^{2s-1} X_i x_i \in \mathcal{A}.$$

Hence $\nu(\underline{X})$ is invertible in $\mathcal{A}$ with

$$\nu(\underline{X})^{-1} = 1 + \sum_{i \geq 1} (-1)^i p^{e \cdot i} B^i$$

where $B = \sum_{i=0}^{2s-1} X_i x_i$. The derivative $d\nu(\underline{X}) = p^e dB = p^e \sum_{i=0}^{2s-1} X_i dx_i$ also lies in $\mathcal{A}$ and so

$$\nu(\underline{X})^{-1} d\nu(\underline{X}) \in \mathcal{A}.$$

Furthermore $(\nu(\underline{X})^{-1} d\nu(\underline{X}))^{2s-1}$ is homogeneous of weight $2s-1$ in the differentials dx_i so that we have

$$\Phi_{2s-1}((\nu(\underline{X})^{-1} d\nu(\underline{X}))^{2s-1}) \in F.$$

Denote by $G_{N,e}F$ the closed subgroup of $GL_N\mathcal{O}_F$ consisting of matrices which are congruent to the identity modulo p^e. With the p-adic topology on $G_{N,e}F$ the map

$$\tilde{\Phi}_{2s-1} : (1 + p^e X_0, 1 + p^e X_1, \ldots, 1 + p^e X_{2s-1}) \mapsto \Phi_{2s-1}((\nu(\underline{X})^{-1} d\nu(\underline{X}))^{2s-1})$$

lies in $\mathrm{Map}_{cts}((G_{N,e}F)^{2s}, F)$, the p-adically continuous functions from the $2s$-fold cartesian product of $G_{N,e}F$ to F.

Theorem 3.2 *(i) If $Y_1, Y_2 \in G_{N,e}F$ then*

$$\tilde{\Phi}_{2s-1}(Y_1(1 + p^e X_0)Y_2, \ldots, Y_1(1 + p^e X_{2s-1})Y_2)$$

$$= \tilde{\Phi}_{2s-1}(1 + p^e X_0, \ldots, 1 + p^e X_{2s-1}).$$

Similarly, if $Y \in GL_N\mathcal{O}_F$,

$$\tilde{\Phi}_{2s-1}(Y(1 + p^e X_0)Y^{-1}, \ldots, Y(1 + p^e X_{2s-1})Y^{-1})$$

$$= \tilde{\Phi}_{2s-1}(1 + p^e X_0, \ldots, 1 + p^e X_{2s-1}).$$

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(ii) If F/E is a Galois extension and $\sigma \in \mathrm{Gal}(F/E)$ then

$$\sigma(\tilde{\Phi}_{2s-1}(1+p^e X_0,\ldots,1+p^e X_{2s-1})) = \tilde{\Phi}_{2s-1}(1+p^e \sigma X_0,\ldots,1+p^e \sigma X_{2s-1}).$$

(iii) The function $\tilde{\Phi}_{2s-1}$ is a $(2s-1)$-dimensional p-adically continuous cocycle on $G_{N,e}F$ with values in the trivial $G_{N,e}F$-module F.

Proof: It is elementary to check that the formula for $\Phi_{2s-1}(f)$ given in §2 coincides with applying the integral over Δ^{2s-1} term by term to the monomial differential $(2s-1)$-forms in the power series f. By this observation, part (iii) is proved using Stokes' Theorem, as in [3]. In fact, one needs only the elementary case of Stokes' Theorem applied to a monomial $(2s-1)$-form, say

$$\omega = x_0^{a_0} x_1^{a_1} x_2^{a_2} \ldots x_{2s-1}^{a_{2s-1}} x_{2s}^{a_{2s}} dx_0 \wedge \ldots \wedge \hat{dx_u} \wedge \ldots \wedge \hat{dx_v} \wedge \ldots dx_{2s}$$

with $0 \leq u < v \leq 2s$ integrated over the standard $2s$-simplex. Part (i) is also proved as in [3] and part (ii) is obvious. □

4. The p-adic regulator

Definition 4.1 As in §2.1, let F be a p-adic local field and let $\mathcal{O}_F$ denote its valuation ring. From the localisation sequence for algebraic K-theory [10] and the vanishing of even K-groups of finite fields [11] we have an isomorphism

$$K_{2s-1}(\mathcal{O}_F) \cong K_{2s-1}(F)$$

for all $s \geq 2$. Let Hur denote the Hurewicz homomorphism to the integral homology of the infinite general linear group, with the discrete topology,

$$\mathrm{Hur} : K_{2s-1}(\mathcal{O}_F) \longrightarrow H_{2s-1}(GL\mathcal{O}_F;\mathbb{Z}).$$

When N is large the inclusion induces an isomorphism

$$H_{2s-1}(GL_N\mathcal{O}_F;\mathbb{Z}) \xrightarrow{\cong} H_{2s-1}(GL\mathcal{O}_F;\mathbb{Z}).$$

To be precise this is true for $N \geq \max(4s-1, 2s-1+sr(\mathcal{O}_F))$ where $sr(\mathcal{O}_F)$ is Bass's stable rank of $\mathcal{O}_F$ [9].

Choosing N large we define

$$R_{N,F} : H_{2s-1}(GL_N\mathcal{O}_F;\mathbb{Z}) \longrightarrow F$$

to be equal to the composition of the transfer map

$$H_{2s-1}(GL_N\mathcal{O}_F;\mathbb{Z}) \xrightarrow{Tr} H_{2s-1}(G_{N,e}F;\mathbb{Z})$$

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with the homomorphism

$$\frac{1}{[GL_N\mathcal{O}_F : G_{N,e}F]} \cdot < [\tilde{\Phi}_{2s-1}], - >: H_{2s-1}(G_{N,e}F;\mathbb{Z}) \longrightarrow F$$

given by pairing a discrete homology class with the continuous cohomology class of Theorem 3.2 (iii) and dividing by the index of $G_{N,e}F$ in $GL_N\mathcal{O}_F$. Explicitly, if d,ϵ are the residue degree and ramification index of $F/\mathbb{Q}_p$

$$[GL_N\mathcal{O}_F : G_{N,e}F] = |GL_N\mathbb{F}_{p^d}|p^{N^2d(e\epsilon-1)}.$$

Proposition 4.2 *For large N the homomorphism $R_{N,F}$ is equal to the homomorphism*

$$H_{2s-1}(GL_N\mathcal{O}_F;\mathbb{Z}) \xrightarrow{i_*} H_{2s-1}(GL_{N+1}\mathcal{O}_F;\mathbb{Z}) \overset{R_{N+1,F}}{\longrightarrow} F$$

where i_* is induced by the inclusion map.

Proof: Dualising the Double Coset Formula of ([12] p.19) we have the homology version for $J,H \subseteq G$, subgroups of finite index.

$$H_{2s-1}(J;\mathbb{Z}) \xrightarrow{i_*} H_{2s-1}(G;\mathbb{Z}) \xrightarrow{Tr} H_{2s-1}(H;\mathbb{Z})$$

$$= \textstyle\sum_{z\in J\backslash G/H} H_{2s-1}(J;\mathbb{Z}) \xrightarrow{Tr} H_{2s-1}(J\bigcap zHz^{-1};\mathbb{Z})$$

$$\overset{(z^{-1}-z)_*}{\longrightarrow} H_{2s-1}(z^{-1}Jz\bigcap H;\mathbb{Z}) \xrightarrow{i_*} H_{2s-1}(H;\mathbb{Z}).$$

We wish to apply this to the case in which $J = GL_N\mathcal{O}_F$, $G = GL_{N+1}\mathcal{O}_F$ and $H = G_{N+1,e}F \lhd G$. In this case $zHz^{-1} = H$ and so

$$H_{2s-1}(J;\mathbb{Z}) \xrightarrow{i_*} H_{2s-1}(G;\mathbb{Z}) \xrightarrow{Tr} H_{2s-1}(H;\mathbb{Z})$$

$$= \textstyle\sum_{z\in J\backslash G/H} H_{2s-1}(J;\mathbb{Z}) \xrightarrow{Tr} H_{2s-1}(J\bigcap H;\mathbb{Z})$$

$$\xrightarrow{i_*} H_{2s-1}(H;\mathbb{Z}) \overset{(z^{-1}-z)_*}{\longrightarrow} H_{2s-1}(H;\mathbb{Z}).$$

From the second part of Theorem 3.2 (i)

$$Res^{G_{N+1,e}F}_{G_{N,e}F}((z^{-1}-z)^*[\Phi_{2s-1}]) \in H^{2s-1}_{cts}(G_{N,e}F;\mathbb{Z})$$

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is equal to $[\Phi_{2s-1}]$ for $G_{N,e}F$. Therefore

$$[GL_{N+1}\mathcal{O}_F : G_{N+1,e}F]R_{N+1,F}\cdot i_*$$

$$= [GL_{N+1}\mathcal{O}_F : GL_N\mathcal{O}_F \cdot G_{N+1,e}F][GL_N\mathcal{O}_F; G_{N,e}F]R_{N,F}$$

$$= [GL_{N+1}\mathcal{O}_F : GL_N\mathcal{O}_F \cdot G_{N+1,e}F][GL_N\mathcal{O}_F G_{N+1,e}F; G_{N+1,e}F]R_{N,F}$$

so that $R_{N,F} = R_{N+1,F}\cdot i_*$. □

Corollary 4.3 *For large N the homomorphism*

$$H_{2s-1}(GL\mathcal{O}_F;\mathbb{Z}) \xrightarrow{i_*^{-1}} H_{2s-1}(GL_N\mathcal{O}_F;\mathbb{Z}) \xrightarrow{R_{N,F}} F$$

is independent of N.

Definition 4.4 Define a homomorphism

$$\hat{R}_F : H_{2s-1}(GL\mathcal{O}_F;\mathbb{Z}) \longrightarrow F$$

by the formula

$$\hat{R}_F = \frac{(-1)^s(s-1)!}{(2s-2)!(2s-1)!}R_{N,F}i_*^{-1},$$

in the notation of Corollary 4.3, where N is a large positive integer.

Theorem 4.5 *In the notation of Definitions 4.1 and 4.4 the composition*

$$R_F : K_{2s-1}(F) \cong K_{2s-1}(\mathcal{O}_F) \xrightarrow{\text{Hur}} H_{2s-1}(GL\mathcal{O}_F;\mathbb{Z}) \xrightarrow{\hat{R}_F} F$$

is equal to the p-adic regulator homomorphism defined by Hamida in [4].

Remark 4.6 Using an explicit p-adically analytic cocycle it is shown in [15] that the Karoubi-Hamida p-adic regulator, which equals R_F by Theorem 4.5, coincides up to a non-zero rational factor with the Wagoner-Huber-Kings p-adic regulator of [16] and [5].

4.7 *Proof of Theorem 4.5*

First we should point out that [4] gives an explicit formula for a p-adic regulator only in the case when $F = \mathbb{Q}_p$. However the sketched proof showing that this construction is well-defined and coincides with Karoubi's cyclic homology p-adic regulator applies equally well for general F. The regulator of [4] is defined by composing the Hurewicz homomorphism with the homomorphism, for large N,

$$R : H_{2s-1}(GL_N F;\mathbb{Z}) \longrightarrow F$$

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which is induced by sending a $2s$-tuple of matrices $(Y_0,\ldots,Y_{2s-1})$ in the bar resolution for $GL_N F$ to the integral

$$\frac{(-1)^s(s-1)!}{(2s-2)!(2s-1)!}\text{Trace}\int_{\Delta^{2s-1}}(\nu^{-1}d\nu)^{2s-1}$$

where $\nu = \sum_{i=0}^{2s-1} x_i Y_i$ where the x_i's are the barycentric coordinates. The verification that this integral converges p-adically for a general $2s$-tuple is quite delicate and is carried out in the Appendix to [15].

In the proof of Theorem 4.5 we observed that the formula for $\Phi_{2s-1}(f)$ can be interpreted as term by term integration of the monomial $(2s-1)$-forms appearing in f over the $(2s-1)$-simplex. From this one sees that the construction which we have given uses the same integral as [4], but only in the situation where each Y_i lies in $G_{N,e}F$ in which case we saw in §2 and §3 that it is very easy to show p-adic convergence.

Let $j : G_{N,e}F \longrightarrow GL_N\mathcal{O}_F$ denote the inclusion. The above discussion shows that

$$[GL_N\mathcal{O}_F : G_{N,e}F]R_{N,F} = R\cdot j_*\cdot Tr : H_{2s-1}(GL_N\mathcal{O}_F;\mathbb{Z}) \longrightarrow F$$

and the result follows since $j_*\cdot Tr = [GL_N\mathcal{O}_F : G_{N,e}F]$. □

REFERENCES

1. J.I. Burgos Gil: *The Regulators of Beilinson and Borel*; Centre de Récherches Mathématiques Monograph Sér. **15**, A.M. Soc (2002).
2. Z. Choo, W. Mannan, R. Garcia-Sanchez and V.P. Snaith: Computer calculations of the Borel regulator I and II; arXiv:0908.3765v2[math.KT] 4 Sep 2009 and arXiv:0909.0883v1[math.KT] 4 Sep 2009.
3. N. Hamida: Description explicite du régulateur de Borel; C.R. Acad. Sci. Paris Sr. I Math. **330** (2000) 169-172.
4. N. Hamida: Le régulateur p-adique; C.R. Acad. Sci. Paris Sr. I Math. **342** (2006) 807-812.
5. A. Huber and G. Kings: A p-adic analogue of the Borel regulator and the Bloch-Kato exponential map; preprint DFG-Forschergruppe Regensburg/Leipzig (2006).
6. M. Karoubi: Homologie cyclique et régulateurs en K-théorie algébrique; C.R. Acad. Sci. Paris Sr. I Math. **297** (1983) no.10, 557-560.
7. M. Karoubi: *Homologie cyclique et K-théorie*; Astérisque **149** (1987).

https://doi.org/10.1017/is010008010jkt125 Published online by Cambridge University Press

8. M. Lazard: Groupes analytiques p-adiques; Pub. Math. I.H.E.S. **26** Paris (1965) 389-603.

9. Yu.P. Nesterenko and A.A. Suslin: Homology of the general linear group over a local ring and Milnor's K-theory; (Russian) Izv. Akad. Nauk. SSSR Ser. Mat. **53** (1989), no. 1, 121-146; translation in Math. USSR-Izv. **34** (1990) no. 1, 121-145.

10. D.G. Quillen: Higher Algebraic K-theory I: Battelle K-theory Conf. 1972; Lecture Notes in Math. **371** (1973) 85-147.

11. D.G. Quillen: On the cohomology and K-theory of the general linear groups over a finite field; Annals of Math. **96** (1972) 552-586.

12. V.P. Snaith: *Explicit Brauer Induction (with applications to algebra and number theory)*; Cambridge Studies in Advanced Math. **40** (1994) Cambridge University Press.

13. V.P. Snaith: Local fundamental classes derived from higher dimensional K-groups; Proc. Great Lakes K-theory Conf., Fields Institute Communications Series **16** (A.M.Soc. Publications) 285-324 (1997).

14. C. Soulé: On higher p-adic regulators; Springer-Verlag Lecture Notes in Mathematics **854** (1981) 372-401.

15. G. Tamme: Comparison of the Karoubi regulator and the p-adic Borel regulator; preprint #17 (25 July 2007) DFG-Forschergruppe Regensburg/Leipzig.

16. J.B. Wagoner: Continuous cohomology and p-adic K-theory; Springer-Verlag Lecture Notes in Mathematics **551** (1976) 241-248.

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