Polynomial series expansion for optimisation of wing plane structures in idealised critical flutter conditions

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ABSTRACT

A numerical procedure, which utilises polynomial power series expansions for the optimisation of multipanel wing structures in idealised critical flutter conditions, is introduced and developed. It arises from the Rayleigh-Ritz method and employes trial polynomial describing functions both for the flexural displacement and for the thickness variation over the multipanel surface. An idealised structural plate model, according to the Kirchhoff's theory, together with a linearised supersonic aerodynamic approach, are supposed. The classical Euler-Lagrange optimality criterion, based on variational principles, has been utilised for the optimisation operations, where by imposing the stationary conditions of the Lagrangian functional expression, a nonlinear algebraic equations system is obtained, whose solution is found by an appropriate algorithm. By utilising an iterative process it is possible to reach the reference structure critical conditions, with an optimised thickness distribution throughout the multipanel surface. The final part of the work consists in searching the minimum weight of the multipanel planform wing structure with optimised thickness profile *vs* the flutter frequency, considered as a variable imput parameter, for fixed flutter speed and equal to the critical one of the reference uniform structure.

NOMENCLATURE

E	Young's modulus
$f^{(a)}$	aerodynamic distributed force over the multipanel structure surface
f_i	trial describing functions of the dimensionless flexural displacement behaviour over the wing plane structure
g_{l_t}	trial describing functions of the nondimensional thickness behaviour over the wing plane structure
h	wing plate thickness
I_o	second moment of inertial area per unit length of the reference plate structure
k_{ij}, k_{nj}	stiffness matrix elements
$k_{ij}^{(l_{t3})}$	generic components of the stiffness matrix elements
k_{ij_*}	matrix elements which take into account both the structural and the aerodymic forces presence
L	cantilever length of the wing structure
m_{ij}, m_{nj}	mass matrix elements
$m_{ij}^{(l_t)}$, $m_{nj}^{(l_t)}$	generic components of the mass matrix elements
N	whole number of the degrees of freedom
N_a, $N_b + 1$	numbers of the coefficients and describing functions in the flexural displacement series expansions along the axes x and y, respectively
N_t	whole number of the coefficients and describing functions in the nondimensional thickness series expansion
N_{tx}, N_{ty}	number of the coefficients and describing functions in the non-dimensional thickness series expansions along axes x and y, respectively
N_{t3}	whole number of the coefficients and describing functions in the series expansion of the nondimensional thickness cubic power
N_{t3x}, N_{t3y}	number of the coefficients and describing functions in the series expansion of the nondimensional thickness cubic power behaviour along the axes x and y, respectively
N_w	whole number of the coefficients and describing functions in the flexural displacement series expansion
$\mathbf{P}(x, y)$	generic point on the multipanel structure surface
q	dynamic pressure
S	wing plane structure surface
S_d	dimensionless wing plane structure surface

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t_{l_t} generic coefficient of the nondimensional thickness series expansion
$t_{l_{t3}}$ generic coefficient of the nondimensional thickness cubic series expansion
t_o thickness value of the reference uniform multipanel structure
T dimensionless relative component of the thickness
T_{qm} mean square value of the nondimensional thickness
w flexural displacement
w_i generic coefficient of the nondimensional flexural displacement series expansion
W nondimensional flexural displacement
x, y plane structure reference system
X, Y dimensionless coordinates of the plane structure reference system

Greek symbols

δ_{ij} Kronecker's delta
λ_i generic Lagrange multiplier
ρ volumetric mass density
σ dimensional dynamic pressure parameter
σ_d dimensionless dynamic pressure parameter
ϕ_i generic multivariable equation
ω angular frequency
ω_d dimensionless angular frequency parameter

Special symbols

∂ partial differentiation
$I^{(T^2)}$ mean square thickness expression
$\mathcal{J}(\ )$ integral of a generic polynomial over the multipanel surface
$\mathrm{L_a}$ Lagrangian functional
$\mathrm{H}_2(\mathrm{S})$ space of the square summable functions on the surface S
T kinetic energy expression
U_{ft} flexural-torsional plate strain energy expression
$\mathrm{U}^{(a)}$ work performed by the aerodynamic distributed forces
$\mathrm{U}^{(s)}$ work performed by the structural distributed forces
W plane structure dimensionless relative weight
W_{l_t} component of the series expansion of the plane structure relative weigh

Subscripts

i, j subscripts referring to the functions and coefficients of the flexural displacement series expansion
i_a, j_a, i_b, j_b subscripts referring to the functions and coefficients in the flexural displacement polynomial series expansions along the axes x and y, respectively
l_t, m_t, n_t subscripts appearing in the thickness series expansion
l_{tx}, m_{tx}, n_{tx} l_{ty}, m_{ty}, n_{ty} subscripts appearing in the series expansion of the thickness behaviour along the axes x and y, respectively
l_{t3} subscript appearing in the series expansion of the non-dimensional thickness cubic power
l_{t3x}, l_{t3y} subscripts appearing in the series expansions of the non-dimensional thickness cubic power along the axes x and y, respectively

Superscripts

(l_{t3}) superscript referring to a generic component of the stiffness matrix elements
(l_t) superscript referring to a generic component of the mass matrix elements

1.0 INTRODUCTION

The problem of optimisation of panels under flutter speed constraints has been a subject of wide research and a major interest for several authors. Weisshaar[1] proposed a numerical solution of the aeroelastic optimisation of simply supported panels of sandwich construction in high supersonic flow. Its numerical model is limited to the one-dimensional case. Pierson[2] presented a numerical technique for determining the minimum weight conditions of a simply supported one-dimensional panel, where an aeroelastic eigenvalue characterising the flutter speed is held constant and with constant in-plane stresses. He considered the nondimensional frequency parameter as an additional problem variable. Santini *et al*[3] developed an optimisation model of a beam in critical flutter conditions, which consists mainly of a step-by-step numerical integration through the axial coordinate. The main problem to be solved is a two-point boundary value problem of a differential system with many variables[4,5]. Also an idealised approach based on the finite element method (FEM), has been employed by Santini *et al*[6] for beams and plates optimisation with constraints of dynamic or aeroelastic nature.

Librescu *et al*[7] dealt with the minimum weight design of orthotropic panels subjected to a supersonic flutter speed constraint and with a uniform in-plane loadings. The Euler-Lagrange problem[8] was reduced to three fourth-order simultaneous partial differential equations in three dependent variables (flexural displacement, Lagrange multiplier and thickness), in two different boundary conditions cases. By using the standard Galerkin's procedure the system of differential equations is transformed into a set of nonlinear algebraic equations.

Tizzi[9,10] utilised the same integration process of Santini[3], with an original and efficient technique to solve the two-point problem. Similar to the Pierson model[2], the frequency is considered as a variable input parameter.

Lillico *et al*[11] and Taylor *et al*[12] developed a computer program to obtain minimum mass design of wing structures in composite materials, subject to constraints on flutter speed, divergence speed and material stress. Unsteady aerodynamic loads are calculated from the strip theory, with an option which utilises the lifting surface theory. The problem is reduced to a one-dimensional model along the spanwise direction.

In this paper, an idealised plate model according to the Kirchhoff's theory[13], and a supersonic linearised aerodynamic approach[14] are employed for a multipanel planform wing structure. A numerical procedure[15-17], which arises from the Rayleigh-Ritz method[18-19], and which utilises polynomial power series expansions both for the flexural displacement and for the thickness behaviour function, is introduced and developed.

The main interest of the research herein reported focuses on the thickness profile of the lighest multipanel planform structure, which can bear at most the critical dynamic pressure of the reference uniform structure with the same in-plane geometry but with a fixed minimum value of the thickness throughout the whole surface.

The classical and well known Euler-Lagrange optimality criterion[8] based on variational principles, which is an efficient and powerful tool to find solutions of many complex optimisation problems, has been utilised for the optimisation operations. By using the Lagrange multipliers, as in the model of Librescu[7] and other authors[20,21], an expression of the extended Lagrangian functional has been derived. Imposing the stationary conditions of this with respect to the Lagrangian degrees of freedom leads to a nonlinear algebraic equations system, the solution of which can be obtained by the algorithm C05NBF of the NAG utility package[22], which applies to the Powell's hybrid method properly modified[23].

Starting from the second natural eigenmode the critical conditions of the uniform reference structure are reached, but with an optimised thickness distribution on the surface, through a series of repeated optimisation operations. The final part of the numerical procedure is devoted to search the optimised profile minimum weight *vs* the

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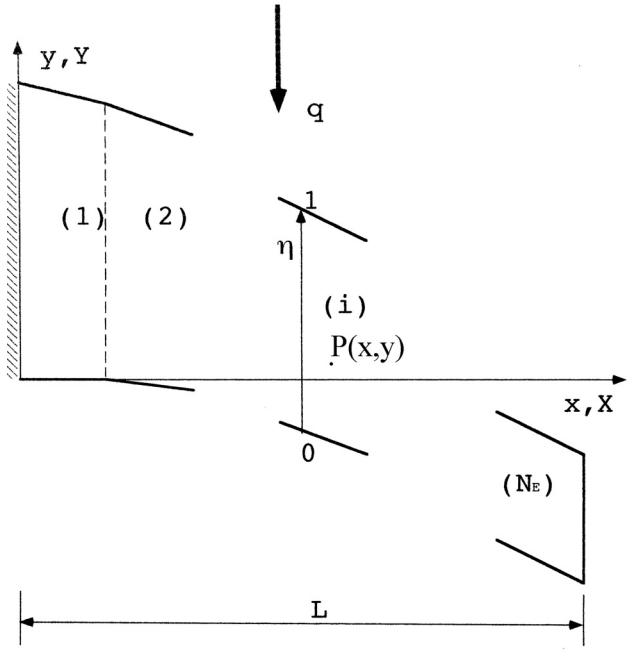


Figure 1. A generic multiplanel wing structure exposed to a high supersonic flow.

frequency, considered as a variable input parameter, for fixed flutter speed and equal to the critical one of the reference multipanel structure with the uniform thickness.

Two illustrative examples utilised for application can be significant tools to evaluate the efficiency of this proposed method.

2.0 MATHEMATICAL MODEL

2.1 General remarks

First we consider a generic plane wing structure, formed by a series of N_E trapezoidal plate segments, connected in serial form along the cantilever length, and exposed to an idealised high supersonic flow with dynamic pressure q, as in Fig. 1.

The coordinates x, y of a generic point $\mathbf{P}(x, y)$ on the multipanel surface and the out-of-plane flexural displacement $w(\mathbf{P})$ can be reformulated in nondimensional form:

$$X=\frac{x}{L} \quad Y=\frac{y}{L} \quad W(\mathbf{P})=\frac{w(\mathbf{P})}{L} \qquad \ldots (1)$$

The Einstein's summation convention for repeated indices has been adopted in all the forthcoming mathematical formulae. The nondimensional flexural displacement $W(\mathbf{P})$ can be expressed by means of a polynomial power series expansion, as:

$$W(\mathbf{P})=w_i f_i(\mathbf{P}) \quad i=1,2,.....N_w \qquad \ldots (2)$$

where $f_i(\mathbf{P})$ are arbitrary polynomial function elements, belonging to the space $H_2(S)$ of the square summable functions on the wing surface S, domain of integration of the multipanel structure, which can be written as:

$$f_i(\mathbf{P})=X^{i_a+1}Y^{i_b} \qquad \ldots (3)$$

$$i_a=1,2....N_a \quad i_b=0,1,2....N_b \quad i=i_b N_a+i_a \quad N_w=N_a(N_b+1) \qquad \ldots (4)$$

and, as in the Ritz method, satisfy only the geometric boundary conditions, i.e. the functions and their first normal derivatives vanish at the clamped edge $X = 0$. They are defined in all the plane X, Y containing the structure.

The coefficients of the series expansion in Equation (2) are Lagrangian degrees of freedom.

2.2 The stiffness and mass matrices

The structural properties have to be specified for evaluating the stiffness and mass matrices. First the thickness h can be expressed as:

$$h(\mathbf{P})=t_o T(\mathbf{P}) \qquad \ldots (5)$$

where t_o is the thickness value of the uniform reference structure, and $T(\mathbf{P})$ is its non-dimensional relative component, which (like $W(\mathbf{P})$) can be expressed by a series expansion over S:

$$T(\mathbf{P})=t_{l_t} g_{l_t}(\mathbf{P}) \quad l_t=1,2....N_t \qquad \ldots (6)$$

The elements $g_{l_t}(\mathbf{P})$ of the series expansion (6) (like the previous ones considered $f_i(\mathbf{P})$) are arbitrary polynomial trial describing functions, belonging to the space $H_2(S)$ of the square summable functions in S, which can be expressed as:

$$g_{l_t}(\mathbf{P})=X^{l_{tx}}Y^{l_{ty}} \qquad \ldots (7)$$

$$l_{tx}=0,1,2....N_{tx}-1 \quad l_{ty}=0,1,2....N_{ty}-1 \qquad \ldots (8)$$

$$l_t=l_{tx}N_{ty}+l_{ty}+1 \quad l_t=1,2....N_t \quad N_t=N_{tx}N_{ty} \qquad \ldots (9)$$

for which there aren't particular boundary conditions to be satisfied. Also these are defined in all the plane X, Y containing the structure.

It is necessary, for forming the stiffness matrix, to compute the cubic power of the thickness $t_o T$, and consequently the following expression has to be considered:

$$T^3(\mathbf{P})=\sum_{l_t m_t n_t} t_{l_t} t_{m_t} t_{n_t} X^{l_{t3x}} Y^{l_{t3y}} \qquad \ldots (10)$$

obtained from Equations (6) and (7), with:

$$l_{t3x}=l_{tx}+m_{tx}+n_{tx} \quad l_{t3y}=l_{ty}+m_{ty}+n_{ty} \qquad \ldots (11)$$

$$l_{t3x}=0,1,2....N_{t3x}-1 \quad l_{t3y}=0,1,2....N_{t3y}-1 \qquad \ldots (12)$$

$$N_{t3x}=3(N_{tx}-1)+1 \quad N_{t3y}=3(N_{ty}-1)+1 \qquad \ldots (13)$$

where m_{tx}, n_{tx} and m_{ty}, n_{ty} vary from 0 to $N_{tx}-1$ and $N_{ty}-1$, respectively, like l_{tx} and l_{ty} in Equation (8), and:

$$m_t=m_{tx}N_{ty}+m_{ty}+1 \quad n_t=n_{tx}N_{ty}+n_{ty}+1 \quad m_t, n_t=1,2....N_t \qquad \ldots (14)$$

as for l_t in Equation (9). The expression (10) can be written also as:

$$T^3(\mathbf{P})=\sum_{l_{t3}} t_{l_{t3}} X^{l_{t3x}} Y^{l_{t3y}} \qquad \ldots (15)$$

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$$l_{t3} = l_{t3x} N_{t3y} + l_{3y} \quad l_{t3} = 1,2....N_{t3} \quad N_{t3} = N_{t3x} N_{t3y} \qquad \ldots (16)$$

where the coefficients $t_{l_{t3}}$ in Equation (15) depend on t_{l_t}, t_{m_t}, t_{n_t} as follows:

$$t_{l_{t3}} = \sum_{l_t m_t n_t} t_{l_t} t_{m_t} t_{n_t} \quad l_{t3} = l_{t3}(l_t, m_t, n_t) \qquad \ldots (17)$$

and the relation between l_{t3} and l_t, m_t, n_t can be obtained by Equations (9), (11), (13), (14) and (16). Consequently the stiffness matrix elements k_{ij}, which are proportional to the integral of the product between the cubic power of the thickness $t_o T$ and some structural operators applied to W[15], taking into account Equation (15), can be expressed as:

$$k_{ij} = t_{l_{t3}} k_{ij}^{(l_{t3})} \quad l_{t3} = 1,2....N_{t3} \qquad \ldots (18)$$

considering that each couple of powers $X^{t3x} Y^{t3y}$ in Equation (15) gives a separate contribution $k_{ij}^{(l_{t3})}$ to the elements k_{ij} as shown in Appendix A.

The mass matrix elements m_{ij} which depend linearly on the thickness $t_o T$, can be derived from the thickness series expansion in Equation (6) and written as:

$$m_{ij} = t_{l_t} m_{ij}^{(l_t)} \quad l_t = 1,2....N_t \qquad \ldots (19)$$

because every component function g_{l_t} of the series expansion gives a separate contribution $m_{ij}^{(l_t)}$ to this matrix elements m_{ij}, as shown in Appendix A.

2.3 The generalised structural-aerodynamic force

The generalised force F_i, acting on the generic Lagrangian degree of freedom (dof) w_i, coefficient of the flexural displacement series expansion in Equation (2), if also the aerodynamic loading is considered, can be written as:

$$F_i = k_{ij}^* w_j \quad j = 1,2....N_w \quad i = 1,2....N_w \qquad \ldots (20)$$

where the matrix elements k_{ij}^* take into account both the structural and the aerodynamic forces[15] (see Appendix B):

$$k_{ij}^* = k_{ij} - \sigma \int_S f_j'(\mathrm{P}) f_i(\mathrm{P}) \mathrm{d}S \quad \sigma = \frac{2q}{\beta} L \quad ( )' = \partial( )/\partial Y \qquad \ldots (21)$$

and k_{ij} are the already introduced stiffness matrix elements, corresponding to the structural forces, and further, within the framework of the quasi-steady linearised 'Piston Theory'[14] of the high supersonic aerodynamics, q is the flow dynamic pressure, $\beta = (M_{\text{ach}}^2 - 1)^{1/2}$ (is a nondimensional parameter, with M_{ach} equal to the Mach number. The aerodynamic and structural damping can be neglected till the multipanel structure critical conditions.

2.4 The main relations

Thus the governing equations ϕ_i of the multipanel flutter dynamics, can be written as:

$$\phi_i = \left(k_{ij}^* - \omega^2 m_{ij}\right) w_j = 0 \quad j = 1,2....N_w \quad i = 1,2....N_w \qquad \ldots (22)$$

the system of which corresponds to the generalised eigenvalue problem, if the thickness distribution throughout the planform structure surface is known.

Equations (22) are homogeneous and consequently a value of any unknown variable can be arbitrarily chosen. We choose a_1 for w_1, so that we have another governing equation:

$$\phi_{N_w+1} = w_1 - a_1 = 0 \qquad \ldots (23)$$

Since the variation range of the thickness on the multipanel domain has to be limited, particularly with respect to its minimum value, an input control parameter is necessary to avoid instability of the numerical solutions and unrealistic thickness values. This input parameter has been chosen as the mean square value T_{qm} of the thickness distribution over the multipanel surface, and thus the last governing equation is derived:

$$\phi_{N_w+2} = T_{qm} - I^{(T^2)} = 0 \qquad \ldots (24)$$

where:

$$I^{(T^2)} = \frac{\int_S T^2(P) \mathrm{d}S}{S} \qquad \ldots (25)$$

The Lagrangian functional expression reads:

$$\mathcal{L}_a = \mathcal{W} + \lambda_i \phi_i \quad i = 1,2....N_w + 2 \qquad \ldots (26)$$

where W is the relative weight, that is the ratio between the weight of the optimised multipanel structure and the weight of the reference structure with constant thickness t_o, which consequently can be written as:

$$\mathcal{W} = \frac{\int_S T(\mathbf{P}) \mathrm{d}S}{S} \qquad \ldots (27)$$

and, in virtue of the thickness series expansion (6), also as:

$$\mathcal{W} = t_{l_t} \mathcal{W}_{l_t} \quad l_t = 1,2....N_t \qquad \ldots (28)$$

with:

$$\mathcal{W}_{l_t} = \frac{\int_S g_{l_t}(\mathbf{P}) \mathrm{d}S}{S} \qquad \ldots (29)$$

Imposing the stationary conditions of the Lagrangian functional expression (26) with respect to the Lagrange multipliers λ_i leads to the previously introduced governing equations ϕ_i, while by imposing the same conditions with respect to the coefficients w_j of the series expansion (2), the following adjoint equations system is derived:

$$\phi_{N_w+2+j} = \frac{\partial \mathcal{L}_a}{\partial w_j} = \lambda_i \left(k_{ij}^* - \omega^2 m_{ij}\right) + \delta_{j1} \lambda_{N_w+1} = 0 \quad j,i = 1,2....N_w \qquad \ldots (30)$$

where δ_{j1} is a Kronecker's delta. At last the stationary conditions of the Lagrangian functional (26) have to be imposed also with respect to the coefficients of the thickness series expansions (6), which give the relations:

$$\phi_{N_w+2+l_t} = \frac{\partial \mathcal{L}_a}{\partial t_{l_t}} = \mathcal{W}_{l_t} + \lambda_h \left(\frac{\partial t_{l_{t3}}}{\partial t_{l_t}} k_{ij}^{(l_{t3})} - \omega^2 m_{ij}^{(l_t)} \right) w_j \qquad \ldots (31)$$

$$+ \lambda_{N_w+2} \frac{\partial I^{(T^2)}}{\partial t_{l_t}} = 0 \qquad \ldots (32)$$

where the components W_{l_t} of W have been introduced and defined in Equations (28) and (29). In view of Equation (17) one obtains:

$$\frac{\partial t_{l_{t3}}}{\partial t_{l_t}} = 3 \sum_{m_t n_t} t_{m_t} t_{n_t} \quad l_{t3} = l_{t3}(l_t, m_t, n_t) \qquad \ldots (33)$$

with l_{t3} connected with l_t, m_t, n_t, as in the same Equation (17).

The terms W_{l_t} and $\partial I^{(T2)}/\partial t_{l_t}$ are determined in Appendix C.

Equations (31) are the fundamental relations of this optimisation procedure, which have been derived from the Euler-Lagrange optimality criterion[8].

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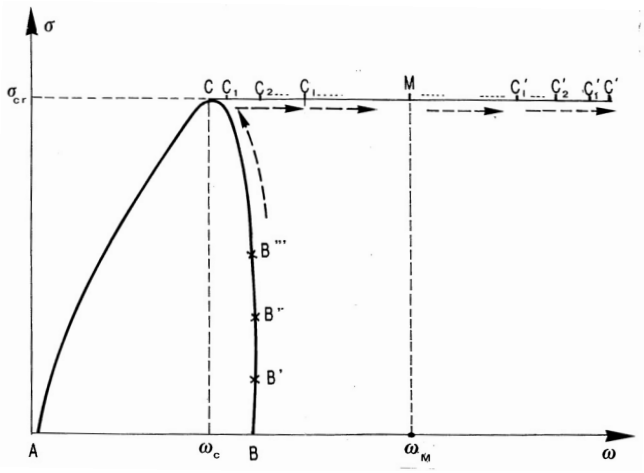


Figure 2. Behaviour σ-ω of a generic uniform multipanel wing structure and the optimisation iterative process.

Finally we have $2N_w + 2 + N_t$ nonlinear algebraic equations:

$$\begin{cases} \phi_i = \dfrac{\partial \mathcal{L}_a}{\partial \lambda_i} = 0 & i = 1,2....N_w + 2 & \text{governing equations} \\ \phi_{N_w+2+j} = \dfrac{\partial \mathcal{L}_a}{\partial w_j} = 0 & j = 1,2....N_w & \text{adjoint equations} \\ \phi_{2N_w+2+l_t} = \dfrac{\partial \mathcal{L}_a}{\partial t_{l_t}} = 0 & l_t = 1,2....N_t & \text{weight stationariness conditions} \end{cases} \quad \ldots (34)$$

and $2N_w + 2 + N_t$ unknown variables:

$$\begin{cases} w_i & i = 1,2....N_w & \text{displacement series expansion coefficients} \\ \lambda_i & i = 1,2....N_w + 2 & \text{Lagrange multipliers} \\ t_{l_t} & l_t = 1,2....N_t & \text{thickness series expansion coefficients} \end{cases} \quad \ldots (35)$$

The solution of the system of the nonlinear algebraic equations can be found easily by the above mentioned algorithm of the NAG utility package, if both the dymamic pressure and the flutter frequency are known, and appropriate values of the same unknown variables are given to the solver algorithm. These values must be chosen quite near to the true solution, otherwise big convergence problems arise for the NAG algorithm. Consequently a researching procedure of these appropriate values is mandatory.

2.5 The iterative process

To this purpose it is necessary to look at Fig. 2, where the σ-ω behaviour, corresponding to the first two flexural vibrating modes of a generic multipanel planform wing structure with uniform thickness, which coalesce with the increasing dynamic pressure, is presented with a thick continuous line.

Let's suppose that we want to optimise the same structure profile in critical flutter conditions, but with a fixed *a priori* minimum value of the thickness. That is the problem at hand consists in searching the lighest multipanel planform structure, able to bear the critical dynamic pressure $q_{cr} = \sigma_{cr}/L$ of the reference uniform structure with the same in-plane geometry, but with the thickness not smaller than a minimum established value throughout its surface. This multipanel structure with optimised thickness profile, researched in the problem, must have its critical dynamic pressure $(q_{cr})_{opt}$ equal to the critical one q_{cr} of the reference structure, which means that for higher values of the dynamic pressure $q = \sigma/L$ there aren't existing flutter solutions for it with this idealised model, like for the uniform multipanel.

The same sequence of operations, used by the author in an above mentioned paper[10], and based on a step-by-step iterative process, will be utilised.

At each step we choose an input value T_{qm} of the mean square relative thickness, so that the minimum thickness on the structure surface is equal to the pre-established one. We start from the point B, corresponding to the free vibration situation of the multipanel structure, where σ = 0 and the natural vibration frequency ω_B is known. It is easy to apply this optimising procedure, and, by utilising the above mentioned NAG solver algorithn, it is possible to determine the $2N_w + 2 + N_t$ unknown variables. Then by moving along the line BC through a series of points B', B'' and B''' and so on..., where we optimise the thickness profile at each point, we reach the critical conditions of the uniform reference structure in C, but with the optimised profile throughout the multipanel surface. The results at each step are the input values of the following step for the NAG solver algorithm, which, considering that each point is close to the subsequent one, can find easily a numerical solution of the same repeated optimisation operations (that is the results in B are the input values of the $2N_w + 2 + N_t$ unknown variables for the optimisation in B', the results in B' are the input values of the same variables for the optimisation in B'', and so on...). Both input parameters σ and ω are known at each step from the σ-ω behaviour of the reference uniform multipanel structure.

Simple considerations about the validity of this iterative process have been performed by the author in the same above mentioned previous work[10]. It is only necessary to remark that in most cases along the way from B to C, the frequency varies much less than the dynamic pressure.

Further it is also necessary to point out that the way from B to C, through B', B'', B'''... and so on, is not obliged and has been chosen by the author only for convenience reasons inherent in the numerical code, but there is possibility also of alternative routes, considering that the structure vibratiion frequencies can vary with continuity for the modified profiles (differently from the structures with uniform thickness profile where there is an infinite sequence of discrete frequency eigenvalues for each chosen dynamic pressure).

The problem has not yet completely solved with the optimised profile in C, because the flutter vibration frequency of the optimised multipanel with minimum weight for $\sigma = \sigma_{cr}$ is surely different from the corresponding one of the uniform reference structure in critical conditions. Consequently it is necessary to continue moving along the straight line CM, characterised by $\sigma = \sigma_{cr}$, where obviously also the flow speed remains constant at its critical value for the uniform structure. Then the iterative process will continue through a series of points C_1 , C_2, ... C_i ... M... C_i '... C_2 ', C_1', following the path CC', where the optimisation operations are repeated at each point. Also in this second part of the process the variables results of each step are the input values of the following step for the NAG algorithm, while the frequency is freely established. Thus it is possible to determine the weight behaviour of the structure with optimised profile vs the frequency, which reach its minimum value in a point M, as shown qualitatively in Fig. 3.

That is in the last part of this iterative process we minimise the optimised profile weight with respect to ω. Also the mean square value of the relative thickness distribution T_{qm} reaches its minimum value in M, as we will see in the two cases considered for application.

In the same point M the critical conditions of the planform structure with optimised profile are reached, where the two vibration frequencies and corresponding fluttering modes coalesce, as it has been sufficiently illustrated in the same previous paper of the author[10]. Consequently the critical dynamic pressure $(q_{cr})_{opt,M}$ of this structure with optimised profile in M is equal to the critical one

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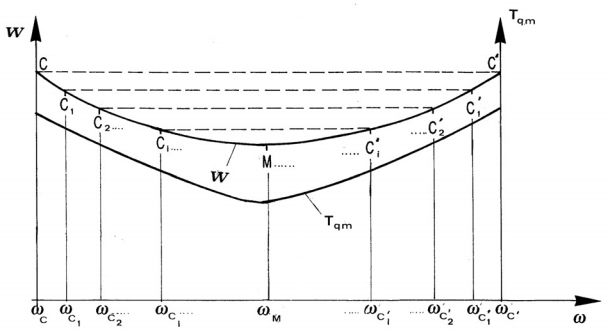


Figure 3. Optimised generic multipanel weight and mean square thickness value *vs* the frequency ω for $\sigma = \sigma_{cr}$.

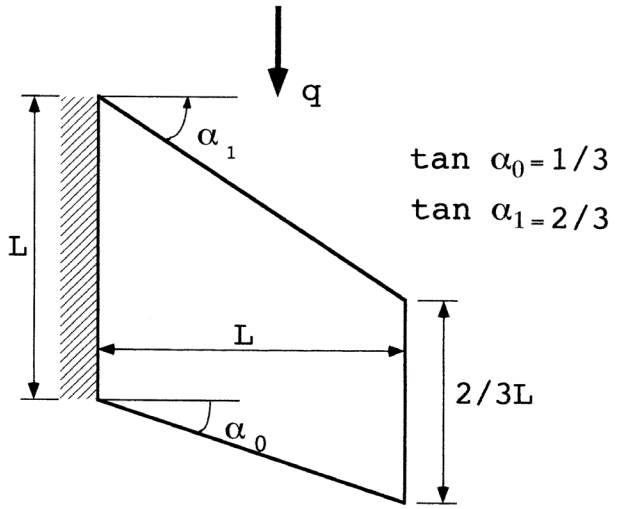


Figure 4. A wing plane structure formed by a single component element and exposed to a high supersonic flow.

$q_{cr} = \sigma_{cr}/L$ of the uniform reference structure, which means that with this idealised model it can't bear higher values of q, whereas the critical dynamic pressure $(q_{cr})_{opt,C}$, $(q_{cr})_{opt,C_1}$, $(q_{cr})_{opt,C_2}$ of the optimised profiles in C, C_1 , C_2 ... and so on, is higher than the one q_{cr} of the reference multipanel structure. The optimised thickness behaviour throughout the multipanel surface at the frequency ω_M and for $\sigma = \sigma_{cr}$ is the final purpose of this proposed method.

3.0 APPLICATIONS

First a wing plane structure formed by a single element, exposed to an idealised high supersonic flow from the leading edge towards the trailing edge with dynamic pressure q, is considered in Fig. 4. We want to optimise it in critical flutter conditions, and with a minimum of the nondimensional relative thickness equal to 0·6.

In the second case a bipanel wing plane structure, in presence of a high supersonic flow parallel to the clamped edge is considered, as in Fig. 5. We search also for it the oprimised profile in critical flutter conditions throughout its surface, with the minimum relative thickness equal to 0·65.

The series expansion for the buckling displacement W in Equation (2) is applied, for which, in the light also of Equations (3) and (4), we have $N_w = N_a(N_b + 1)$ coefficients to be determined, which are unknown Lagrangian variables. From Equations (6-9) we have

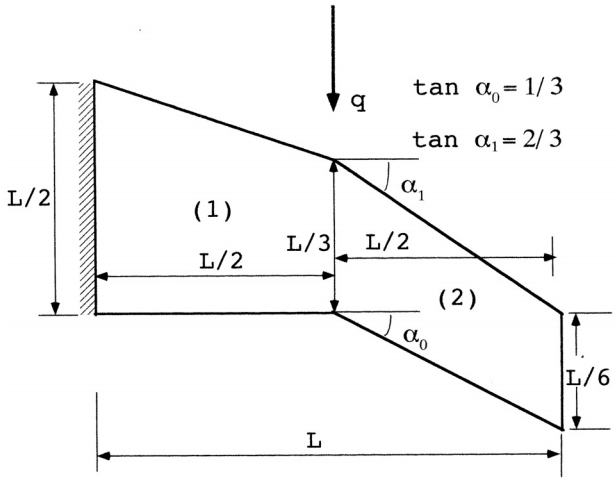


Figure 5. A bipanel wing plane structure exposed to a high supersonic flow.

Table 1
Lagrangian component variables of the optimisation problem

N	N_a	N_b	N_w	N_{tw}	N_{ty}	N_t
58	4	4	20	4	4	16

$N_t = N_{tx}N_{ty}$ unknown coefficients of the thickness T series expansion, likewise employed in the computational code, which in the same way are Lagrangian variables. Further there are $N_w + 2$ Lagrange multipliers from Equations (22)-(26), which are further Lagrangian variables. Thus the whole number of unknowns to be determined is:

$$N = 2N_w + 2 + N_t \qquad \ldots (36)$$

as in Equation (35). The numbers N_a and N_b of the function elements of the displacement W series expansions *vs* X and Y, respectively, as in Equations (3) and (4), are shown in Table 1, together with the number N_{tx} and N_{ty} of the function elements of the thickness T series expansions *vs* X and Y, respectively, as in Equations (7)-(9)

With these input values of N_a, N_b and N_{tx}, N_{ty}, the whole number N of the unknown Lagrangian variables written in the table is obtained from Equation (36).

4.0 RESULTS

For the analysis of the results it is better to refer to both input parameters, σ and ω, reformulated in nondimensional form. Consequently the following dimensionless dynamic pressure parameter σ_d is introduced:

$$\sigma_d = \frac{(1-\nu^2)\sigma L^2}{EI_o} \qquad \ldots (37)$$

where $I_o = m_o^3/12$ is the second moment of the inertial area of the reference wing plate structure with constant thickness t_o per unit length, together with the dimensionless frequency parameter ω_d, which is connected with the true frequency ω via the following relation:

$$\omega_d = \omega^2 \frac{(1-\nu^2)\rho t_o L^4}{EI_o} \qquad \ldots (38)$$

In the first case of a unipanel planform structure in idealised flutter

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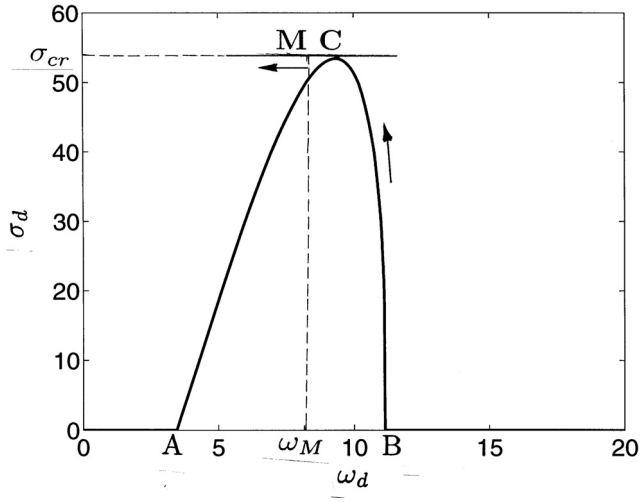


Figure 6. The $\sigma_d - \omega_d$ behaviour of the unipanel planform structure with uniform thickness profile.

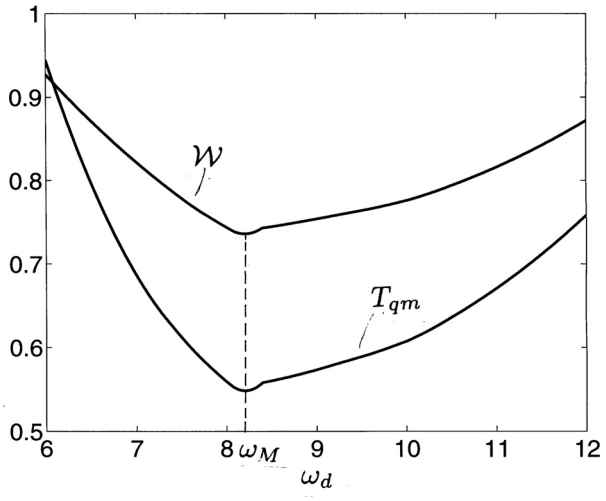


Figure 7. Behaviour of the relative weight and of the thickness mean square value *vs* the frequency of the optimised profile for $\sigma_d - \sigma_{cr}$, in the unipanel planform structure case.

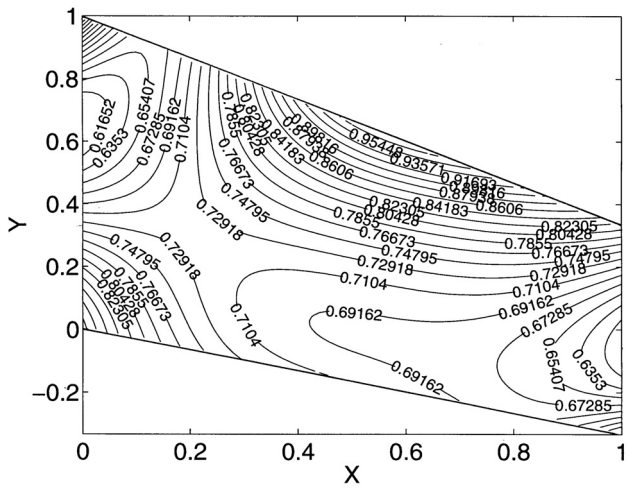


Figure 8. Plane mapping of the nondimensional thickness optimised profile in minimum weight conditions, obtained *vs* the frequency in the unipanel planform structure case.

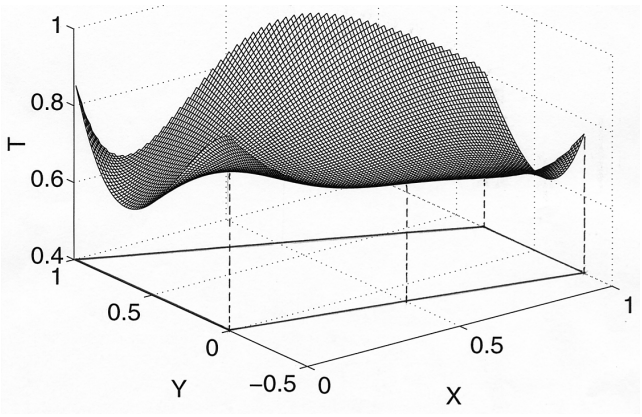


Figure 9. Trimetric view of the nondimenional thickness optimised profile in minimum weight conditions, obtained *vs* the frequency in the unipanel planform structure case.

conditions and with a uniform thickness t_o distribution, the behaviour of σ_d *vs* ω_d is shown in Fig. 6.

It easy to optimise the structure thickness profile in free vibration conditions, corresponding to the point B of Fig. 6. Then, as previously stated in the explanation of the iterative process, by a step-by-step sequence of repeated optimisation operations along the line BB'B"B"'...C, it is possible to arrive at the critical flutter conditions of the uniform reference panel in C, but with the optimised profile. At each step an input value T_{qm} of the mean square relative thickness is properly chosen so that the minimum of T on the surface is 0·6. The results at each step are the input values of the following step for the NAG solver algorithm, which allow to determine easily the $2N_w + 2 + N_t$ unknown variables, while both input parameters, σ_d and ω_d, are known at each step.

It is then necessary to continue moving along the straight line CM, characterised by the condition $\sigma_d = \sigma_{cr}$. Following the path CM and over, the behaviour of the optimised profile weight W and of the relative thickness mean square value T_{qm} *vs* ω_d is obtained, as plotted in Fig. 7. We have in M the minimum of the relative weight W and of T_{qm}, with approximately a frequency $\omega_M = 8{\cdot}08$.

The plane mapping of the nondimensional relative thickness T behaviour throughout the whole wing surface of the optimised panel in M, is sketched in Fig. 8, while Fig. 9 shows the corresponding trimetric drawing.

The case of a bipanel in idealised flutter conditions, displayed in Fig. 4, now has to be considered.

The $\sigma_d - \omega_d$ behaviour of the reference bipanel with uniform thickness is drawn in Fig. 10, where, as in the previous case, the point C corresponds to the critical situation, and B to the free vibration conditions.

Also in this case, starting from the point B, we arrive at C with an optimised profile, by a step-by-step iterative sequence of optimisation operations. In this second case at each step an input value of the mean square relative thickness T_{qm} is chosen so that the minimum of T is 0·65.

Then following the path CM and over, characterised by $\sigma_d - \sigma_{cr}$, the behaviour of the optimised profile weight W and of the thickness mean square value T_{qm} *vs* ω_d is derived, as plotted in Fig. 11. We have in M the conditions of minimum weight W and T_{qm}, where the

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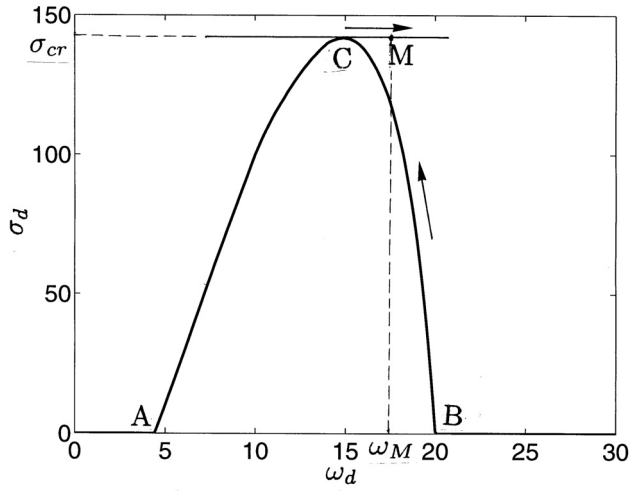


Figure 10. The σ_d — ω_d behaviour of the bipanel planform structure with uniform thickness profile.

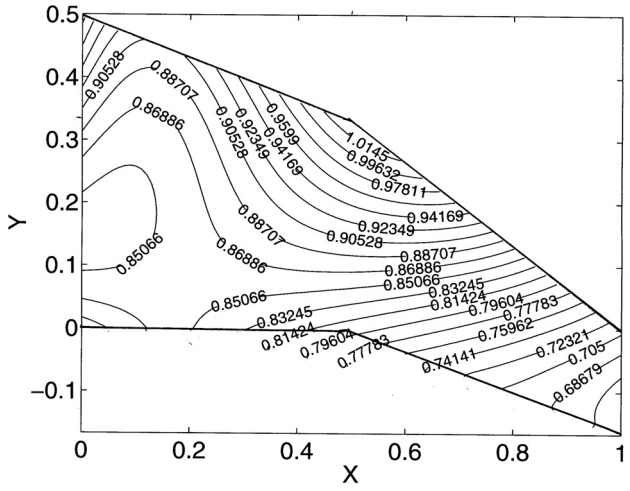


Figure 12. Plane mapping of the nondimensional thickness optimised profile in minimum weight conditions, obtained *vs* the frequency in the bipanel planform structure case.

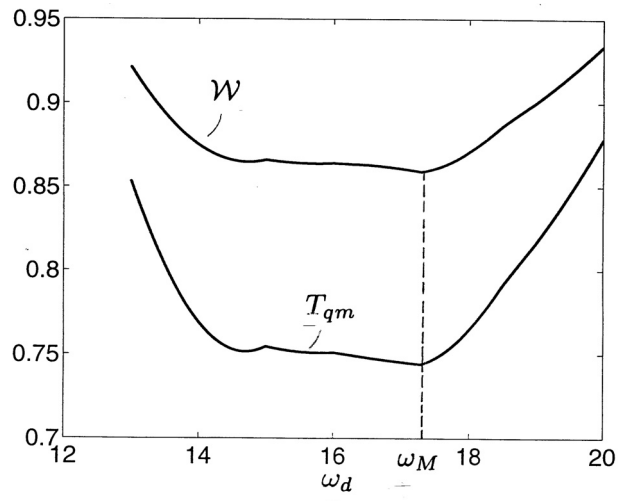


Figure 11. Behaviour of the relative weight and of the thickness mean square value *vs* the frequency of the optimised profile for σ_d — σ_{cr}, in the bipanel planform structure case.

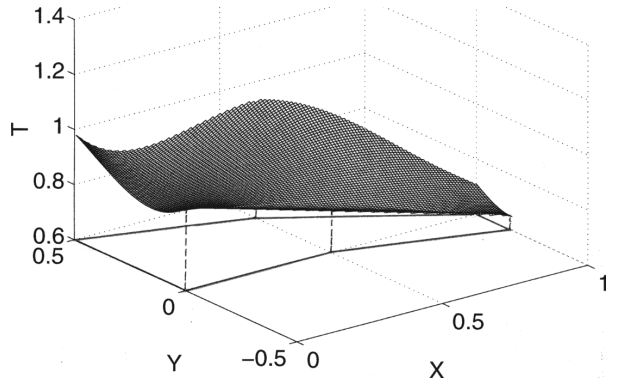


Figure 13. Trimetric view of the nondimenional thickness profile in minimum weight conditions, obtained *vs* the frequency in the bipanel panform structure case.

frequency ω_M is approximately equal to 17·28. Figure 12 shows the plane mapping of the optimised profile in M, while the corresponding trimetric view is sketched in Fig. 13.

5.0 DISCUSSION

Close examination of Figs 8, 9 and Figs 12, 13 reveals that in both cases we have the highest values of the relative thickness in the central part of the leading edge. The minimum is reached in both cases (0·6 and 0·65, respectively) at the free tip of the plane cantilever structure, but in the first case there exists such a minimum 0·6 also on the clamped edge.

In the first case, with a minimum value of the relative thickness equal to 0·6, it is possible to save 26% of the weight, while in the second case with a minimum value equal to 0·65, it is possible to save about 14% of the weight.

Some likeness of the obtained results, as in Figs 8 and 12, with those of a particular case of optimised wing structure to enhance flutter speeed, obtained by Khot *et al*[24] (as in Fig. 10 of the same paper) can be evinced, also taking into account the limited number of the grid points in the mesh chosen by the paper authors. Not very different results have been obtained by Klimmek *et al*[25], who were interested to build wing models for design and structutal optimisation.

6.0 CONCLUSIONS

The above mentioned results are only indicative of an optimised thickness profile throughout a plane wings surface, but sufficiently illustrative for an approximate idea of a possible thickness distribution which allows to save weight in critical flutter conditions.

More sophisticated algorithms to find solutions of non-linear

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algebraic equations systems, will enable the use of a higher number of trial describing functions, both for the flexural displacement and for the thickness behaviour, and more accurate results will be obtained.

The possibility of static aeroelastic divergence of the structure with optimised profile, has to be considered. It must be emphasised that with backward arrow wings, this phenomenon is reduced and in most cases the critical dynamic pressure of the two fluttering modes, which coalesce at the critical conditions, is insufficient to bring the multipanel structure to aeroelastic static instability. With forward arrow wings there exists this possibility and the danger of that must be seriously taken into account. This Ritz procedure could be applied also to determine the static aeroelastic instability conditions, but this is not the case herein considered.

One could object that this procedure, which utilises isotropic plate models is not very useful for practical problems, considering that the current way of construction of wing structures is the semi-monocoque type with stiffened shell elements and with the external skins in multi-layered composite materials. By appropriate modifications in the utilised code, new applications of this numerical approach to more realistic orthotropic plate models, such as the ones proposed by Giles[26,27] and Livne[28,29] for modeling with the Ritz analysis generic multipanel planform wing box structures, can be envisaged. In such a way the optimisation of the skins thickness profile throughout the wing surface for true cases of aircraft in a high supersonic flow, and in critical flutter conditions, can be actually carried out.

Since this paper, like an above mentioned previous work of the author[15], shows the possibilità of applying Ritz procedures also in cases of structural-aerodynamic forces interaction, typical of aeroelastic problems, and also to optimise the thickness profile of plane multipanel structures with this particular procedure, it can be considered as a reference starting point for following-up researchers, who can implement and improve this approaching model for applications to true cases of wing box structures in critical flutter conditions, and offer large opportunities to aircraft designers.

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APPENDIX A: DETERMINATION OF THE CONTRIBUTIONS TO THE STIFFNESS AND MASS MATRICES DUE TO THE THICKNESS SERIES EXPANSIONS ELEMENTS

The contributions to the stiffness and mass matrices of a generic plane multipanel structure, as in Fig. 1, due to the component elements of the thickness series expansion, can be evaluated.

The flexural-torsionl strain energy expression *vs* the buckling displacement W reads[15]:

$$\mathcal{U}_{fl} = \frac{EI_o}{2\left(1-\nu^2\right)}\int_{S_d} T^3 \left[\left(\frac{\partial^2 W}{\partial X^2}\right)^2 + 2\left(1-\nu\right)\left(\frac{\partial^2 W}{\partial X \partial Y}\right)^2 + \left(\frac{\partial^2 W}{\partial Y^2}\right)^2 + 2\nu\left(\frac{\partial^2 W}{\partial X^2}\right)\left(\frac{\partial^2 W}{\partial Y^2}\right)\right] d\vec{S}_d \qquad \ldots (A.1)$$

where $S_d = S/L^2$ is the whole structure dimensionless surface, $I_o = t_o^3/12$ is the second moment of inertial area per unit length of the reference structure with constant thickness t_o on the multipanel surface.

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It is necessary to compute the integral of the generic term $X^l Y^m$ over the whole structure dimensionless surface S_d, to evaluate the stiffness matrix elements k_{ij} and for other forthcoming computations:

$$\mathcal{J}(l+1,m+1)=\int_{S_d} X^l Y^m \mathrm{d}S_d \qquad \ldots (A.2)$$

It can be determined by utilising the natural reference system X,η, where $0 \le \eta \le 1$[16,17]. The coordinate Y is connected with X and η by a nonlinear algebraic relation[16].

If the series expansions (2) and (15) are substituted into the strain energy expression (A.1), which can be also written in the classical form:

$$\mathcal{U}_{fl}=\frac{1}{2}k_{ij}w_i w_j \qquad \ldots (A.3)$$

and in view of Equations (3), (18) and (A.2), we can derive the contribution to the stiffness matrix elements due to the powers product $X^{l_{t3x}} Y^{l_{t3y}}$:

$$k_{ij}^{(l_{t3})}=\frac{EI_o}{1-\nu^2}\left[(I)+(II)+(III)+(IV)+(V)\right] \quad l_{t3}=1,2....N_{t3} \qquad \ldots (A.4)$$

where:

$$(I)=i_a(i_a+1)j_a(j_a+1)\mathcal{J}(i_a+j_a-1+l_{t3x},i_b+j_b+1+l_{t3y}) \qquad \ldots (A.5)$$

$$(II)=i_b(i_b-1)j_b(j_b-1)\mathcal{J}(i_a+j_a+3+l_{t3x},i_b+j_b-3+l_{t3y}) \quad i_b,j_b \ge 2 \qquad \ldots (A.6)$$

$$(III)=\nu i_b(i_b-1)j_a(j_a+1)\mathcal{J}(i_a+j_a+1+l_{t3x},i_b+j_b-1+l_{t3y}) \quad i_b \ge 2 \qquad \ldots (A.7)$$

$$(IV)=\nu i_a(i_a+1)j_b(j_b-1)\mathcal{J}(i_a+j_a+1+l_{t3x},i_b+j_b-1+l_{t3y}) \quad j_b \ge 2 \qquad \ldots (A.8)$$

$$(V)=2(1-\nu)(i_a+1)(j_a+1)i_b j_b \mathcal{J}(i_a+j_a+1+l_{t3x},i_b+j_b-1+l_{t3y}) \quad i_b,j_b>0 \qquad \ldots (A.9)$$

and:

$$i_a,j_a=1,2....N_a \quad i_b,j_b=1,2....N_b \quad i=i_b N_a+i_a \quad j=j_b N_a+j_a \qquad \ldots (A.9a)$$

$$l_{t3x}=0,1....N_{t3x}-1 \quad N_{t3x}=3(N_{tx}-1)+1$$
$$l_{t3y}=0,1....N_{t3y}-1 \quad N_{t3y}=3(N_{ty}-1)+1 \qquad \ldots (A.9b)$$

$$l_{t3}=l_{t3x}N_{t3y}+l_{t3y}+1 \quad l_{t3}=1,2....N_{t3} \quad N_{t3}=N_{t3x}N_{t3y} \qquad \ldots (A.9c)$$

like in Equations (4), (12), (13) and (16) the kinetic energy expression of an oscillating structure with frequency can be written as:

$$\mathcal{T}=\frac{1}{2}\omega^2 \rho L^4 t_o \int_{S_d} TW^2 \mathrm{d}S_d \qquad \ldots (A.10)$$

If the series expansions (2) and (6) are substituted into Equation (A.10), in the light of the kinetic energy classical expression:

$$\mathcal{T}=\frac{1}{2}\omega^2 m_{ij} w_i w_j \qquad \ldots (A.11)$$

along with Equations (3), (7), (19) and (A.2), we have the contribution to the mass matrix elements due to the function element $\mathcal{G}_{lt}\mathbf{P}$:

$$m_{ij}^{(l_t)}=\rho t_o L^4 \mathcal{J}(i_a+j_a+3+l_{tx},i_b+j_b+1+l_{ty}) \qquad \ldots (A.12)$$

where:

$$l_{tx}=0,1....N_{tx}-1 \quad l_{ty}=0,1....N_{ty}-1 \qquad \ldots (A.12a)$$

$$l_t=l_{tx}N_{ty}+l_{ty}+1 \quad l_t=1,2....N_t \quad N_t=N_{tx}N_{ty} \qquad \ldots (A.12b)$$

as in Equations (8) and (9).

APPENDIX B: EXPRESSION OF THE GENERALISED STRUCTURAL-AERODYNAMIC FORCE

The expression of the generalised force acting on the generic Lagrangian degree of freedom w_i reads:

$$F_i=\frac{\partial \mathcal{U}^{(s)}}{\partial w_i}+\frac{\partial \mathcal{U}^{(a)}}{\partial w_i} \qquad \ldots (B.1)$$

where $\mathrm{d}U^{(s)}$ and $\mathrm{d}U^{(a)}$ are the elementary works performed by the structural and aerodynamic forces on the same dof w_i, respectively.

For homogeneous boundary conditions (as in the cases herein considered), the work performed by the structural forces is equal to the internal plate structure strain energy, and consequently from Equation (A.3), it can be also written in the classical form:

$$\mathcal{U}^{(s)}=\frac{1}{2}k_{ij}w_i w_j \qquad \ldots (B.2)$$

Since the interest of the paper is limited to the critical conditions of the fluttering structures, the aerodynamic force expression lies within the framework of the quasi-steady linearised 'Piston-Theory' of the high supersonic aerodynamics[14], where the aerodynamic damping is not considered. Then the distributed aerodynamic force acting on the generic point $\mathbf{P}(x, y)$ of the multipanel surface is given by:

$$f^{(a)}(\mathbf{P})=-\sigma\frac{\partial W(\mathbf{P})}{\partial Y} \qquad \ldots (B.3)$$

considering that the flow is oriented along the negative direction of the axis Y. In view of the series expansion of $W(\mathbf{P})$ in Equation (2), the expression of $f^{(a)}$ in Equation (B.3) becomes:

$$f^{(a)}(\mathbf{P})=-\sigma w_j \frac{\partial f_j(\mathbf{P})}{\partial Y} \quad j=1,2....N_w \qquad \ldots (B.4)$$

The resultant elementary work $\mathrm{d}U^{(a)}$, performed for an increment $\mathrm{d}W^{(a)}$ of the flexural displacement, is then equal to:

$$\mathrm{d}\mathcal{U}^{(a)}=L^2\int_{S_d} f^{(a)}(\mathbf{P})\mathrm{d}W(\mathbf{P})\mathrm{d}S_d$$
$$=-L^2\sigma w_j \mathrm{d}w_i \int_{S_d} f_j'(\mathbf{P})f_i(\mathbf{P})\mathrm{d}S_d \quad (\,)'=\frac{\partial(\,)}{\partial Y} \qquad \ldots (B.5)$$

$$j,i=1,2....N_w \qquad \ldots (B.5a)$$

taking into account the aerodynamic distributed force expression in Equation (B.4), and that from the series expansion in Equation (2) we have $\mathrm{d}W(\mathbf{P})=f_i(\mathbf{P})\mathrm{d}w_i$. Thus it is true that:

$$\frac{\partial \mathcal{U}^{(a)}}{\partial w_i}=-L^2\sigma w_j \int_{S_d} f_j'(\mathbf{P})f_i(\mathbf{P})\mathrm{d}S_d \quad j=1,2....N_w \qquad \ldots (B.6)$$

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If the expression of $\partial U^{(a)}/\partial w_i$ in Equation (B.6) is substituted into Equation (B.1), and in the light of the expression of $U^{(s)}$ in Equation (B.2), it is possible to derive the generalised force F_i:

$$F_i = k_{ij} w_j - \sigma w_j L^2 \int_{S_d} f_j'(\mathbf{P}) f_i(\mathbf{P}) \mathrm{d}S_d \quad j = 1, 2 \ldots . N_w \qquad \ldots (B.7)$$

If it is introduced the matrix with elements k_{ij}^*, which take into account both structural and aerodynamic forces[15], as in Equation (20), and also the expression of the generalised force in Equation (B.7) is considered, it is possible to prove the validity of the expression of k_{ij}^* in Equation (21).

Thus the same above mentioned elements k_{ij}^* can be determined from the stiffness matrix elements k_{ij}, and in the light of the polynomial components expression (in Equation (3)) of the displacement W series expansion (in Equation (2)), and also taking into account the expression of the generic integral in Equation (A.2)

$$k_{ij}^* = k_{ij} - \sigma j_b L^2 \mathcal{J}(i_a + j_a + 3, i_b + j_b) \quad j_b > 0 \qquad \ldots (B.8)$$

where:

$$i_a, j_a = 1, 2 \ldots . N_a \quad i_b, j_b = 0, 1, 2 \ldots . N_b$$
$$i = i_b N_a + i_a \quad j = j_b N_a + j_a \qquad \ldots (B.8a)$$

as in Equation (4).

APPENDIX C: DETERMINATION OF SOME PARAMETERS IN THE FUNDAMENTAL OPTIMALITY CRITERION RELATIONS

C.1 The mean square thickness expression and its derivative

The expression of the relative thickness square from Equations (6) and (7) reads:

$$T^2(\mathbf{P}) = \sum_{l_t m_t} t_{l_t} t_{m_t} X^{l_{tx} + m_{tx}} Y^{l_{ty} + m_{ty}} \qquad \ldots (C.1)$$

where:

$$l_{tx}, m_{tx} = 0, 1, 2 \ldots . N_{tx} - 1 \quad l_{ty}, m_{ty} = 0, 1, 2 \ldots . N_{ty} - 1 \qquad \ldots (C.1a)$$

$$l_t = l_{tx} N_{ty} + l_{ty} + 1 \quad m_t = m_{tx} N_{ty} + m_{ty} + 1 \quad l_t, m_t = 1, 2 \ldots . N_t \qquad \ldots (C.1b)$$

as in Equations (8), (9) and (14). The expression (25) of the mean square relative thickness $I^{(T^2)}$, if Equations (C.1) and (A.2) are considered, becomes:

$$I^{(T^2)} = \sum_{l_t m_t} \frac{t_{l_t} t_{m_t}}{S_d} \mathcal{J}(l_{tx} + m_{tx} + 1, l_{ty} + m_{ty} + 1) \quad l_t, m_t = 1, 2 \ldots . N_t \qquad \ldots (C.2)$$

and its partial derivative expression with respect to the generic coefficient of the thickness series expansion reads:

$$\frac{\partial I^{(T^2)}}{\partial t_{l_t}} = 2 \sum_{m_t} \frac{t_{m_t}}{S_d} \mathcal{J}(l_{tx} + m_{tx} + 1, l_{ty} + m_{ty} + 1) \qquad \ldots (C.3)$$

C.2 The expression of the weight series expansion component W_{l_t}

The component W_{l_t} in Equation (29), of the relative weight series expansion (28), if Equations (7) and (A.2) are considered, is equal to

$$\mathcal{W}_{l_t} = \frac{J(l_{tx} + 1, l_{ty} + 1)}{S_d} \qquad \ldots (C.4)$$

where l_t, l_{tx}, l_{ty} are the same of Equations (C.1a) and (C.1b).

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