REPRESENTING EMBEDDABILITY AS SET INCLUSION

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Abstract

A few steps are made towards a representation theory of embeddability among uncountable graphs. A class of graphs is defined by forbidding some countable configurations which are related to the graph's end-structure. Using uncountable combinatorics, a representation theorem for embeddability in this class is proved, which asserts the existence of a surjective homomorphism from the relation of embeddability over isomorphism types of regular cardinality $\lambda > \aleph_1$ onto set inclusion over all subsets of reals or cardinality λ or less. As corollaries we obtain: (1) the complexity of the class in every regular uncountable $\lambda > \aleph_1$ is at least

$$\lambda^+ + \sup\{\mu^{\aleph_0} : \mu^+ < \lambda\},$$

(2) a characterisation of graphs in the class for which some invariant of the graph has to be inherited by one of fewer than λ subgraphs whose union covers G. Some continuity properties of the homomorphism in the representation theorem are explored and are used to extend the first corollary to all singular cardinals below the first fixed point of second order. The first corollary shows that, contrary to what Shelah has shown for the class of all graphs, the relations of embeddability in the class under discussion is not independent of negations of the generalised continuum hypothesis.

0. *Introduction*

The study of embeddability among infinite structures has a long tradition of invoking combinatorics. One well-known example is Laver's use of Nash-Williams' combinatorial results to show that embeddability among countable order types is well quasi-ordered [**19**]. In the study of embeddability among uncountable structures, the most prominent combinatorial principle has been the generalised continuum hypothesis, which asserts that every infinite set has the least possible number of subsets. Assuming the generalised continuum hypothesis, Hausdorff proved in 1914 that in every infinite power there is a universal linear ordering. Jonsson [**10**] listed six axioms and used the generalised continuum hypothesis to prove that classes of structures which satisfy those axioms have universal structures in all uncountable powers. See [**24**] for the analogous result in graph theory and [**23**] for model theory.

Finer relevant combinatorial principles have come from Jensen's work in Gödel's universe of constructible sets [**9**]. For example, Macintyre [**20**] uses Jensen's diamond – a principle which is stronger than CH – to prove that no Abelian locally finite group of size $\aleph_1$ is embeddable in all universal locally finite groups of size $\aleph_1$. Komjáth and Pach [**17**] use the same principle to prove that there is no universal graph in power $\aleph_1$ among all graphs omitting K_{ω,ω_1}.

A common property of the combinatorial principles which have been mentioned so far is their independence of the usual axioms of set theory. Easton [**4**] showed that the generalised continuum hypothesis can fail for all regular cardinals. Magidor [**21**] showed the generalised continuum hypothesis could fail at $\aleph_\omega$ with the generalised continuum hypothesis below it and Foreman and Woodin [**5**] showed that the

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generalised continuum hypothesis could fail everywhere (both using large cardinals). In spite of this, the folklore in areas of mathematics where infinite cardinals play a significant role – infinite graph theory, infinite Abelian groups, and model theory, to name only a few – remained that the generalised continuum hypothesis was a useful assumption, while its negations were practically useless.

In the context of embeddability this impression was fortified by Shelah's independence results. Shelah showed that universal structures in uncountable powers may or may not exist under negations of the generalised continuum hypothesis. While the generalised continuum hypothesis implies the existence of universal graphs in all infinite λ, the assumption $\lambda < 2^{\aleph_0}$ for regular uncountable λ does not decide the existence of universal graphs in λ (see [**11**, **22**, **28**]).

Shelah's independence results [**26**, **27**, **28**] created the expectation that the existence of universal structures in uncountable cardinalities would always be independent of negations of the generalised continuum hypothesis, unless the existence was trivial (because the class of structures at hand was 'dull').

The understanding of negations of the generalised continuum hypothesis at singular cardinals has changed dramatically in the last decade. The most fascinating developments in this area are Shelah's bounds on exponents of singular cardinals. Shelah proved the following beautiful theorem, formulated here, though, in a way Shelah himself resents.

THEOREM 0.1. *If* $2^{\aleph_n} < \aleph_\omega$ *for all* n *then* $2^{\aleph_\omega} < \aleph_{\omega_4}$.

Since no bound can be put on the exponent of a regular cardinal by Cohen's results, this theorem is exceptionally thrilling. A short proof of it can be found in [**8**].

The formulation Shelah prefers is the following.

THEOREM 0.2. $\mathrm{cf}\,([\aleph_\omega]^{\aleph_0}, \subseteq) < \aleph_{\omega_4}$.

This formulation says that the cofinality of the partial ordering of set inclusion over countable subsets of $\aleph_\omega$ is *always* smaller than $\aleph_{\omega_4}$, no matter how large $\aleph_\omega^{\aleph_0}$ may be. This theorem exposes a robust structure of the partial ordering of set inclusion, which is affected by negations of the generalised continuum hypothesis in a limited way only. The reader will verify that Theorem 0.2 implies Theorem 0.1. A proof of Theorem 0.2 is in Shelah's recent *Cardinal arithmetic* [**25**], in which Shelah reduces the problem of computing the exponent of a singular cardinal to an algebra of reduced products of regular cardinal, and uses a host of new and sophisticated combinatorics to analyse the structure of such reduced products.

A common property of the combinatorial principles Shelah develops in [**25**] and in later works on cardinal arithmetic, is that they are ZFC theorems. This is necessary, since Theorem 0.2 (unlike the conclusion of Theorem 0.1) is a ZFC theorem itself.

In this paper we use some of Shelah's combinatorics to expose a robust connection between the structure of embeddability over a monotone class of infinite graphs and the relation of set inclusion over subsets of reals of bounded cardinality. The connection is established by means of a representation theorem, that asserts the existence of a surjective homomorphism from the former relation onto the latter. One corollary is that the structure of embeddability over the class we shall study is not independent of negations of the CGH, but also information that is not related to cardinal arithmetic is obtained.

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Shelah's ZFC combinatorics on uncountable cardinals was found useful in the study of embeddability in several papers. In [**13**] it was shown that if $\lambda > \aleph_1$ is regular and $\lambda < 2^{\aleph_0}$ then there is no universal linear ordering in λ. Analogous results were proved for models of first order theories [**14**], infinite Abelian groups [**15**, **29**] and metric spaces [**30**], but are impossible for the class of all graphs by Shelah's independence results [**28**].

The theory of universal graphs, which began with Rado's construction of a countable strongly universal graph [**24**], has advanced considerably since Rado's work, especially in the study of universality in monotone classes (see [**3**] for motivation for this). A monotone class of graphs is always of the form Forb (Γ), the class of all graphs which omit as subgraphs some class Γ of 'forbidden' configurations. A good source for the development of this theory is the survey paper [**17**], in which the authors suggest a generalisation of universality, which they name 'complexity': the complexity of a class of graphs is the least number of members in the class needed to embed as induced subgraphs all members in the class. The complexity is 1 exactly when a strongly universal graph exists in the class.

0.1. *Organisation of the paper*

The paper is organised as follows. In Section 1 a monotone class of graphs $\mathscr{G}$ is defined by specifying a set of countable configurations which the members of $\mathscr{G}$ must omit. The forbidden configurations are related to the graph's end-structure. It is noted that a generalisation of a theorem by Diestel, Halin and Vogler implies that the complexity $\mathscr{G}_\lambda$ is at least λ^+ for all infinite λ. A surjective homomorphism is now constructed from the relation of (weak) embeddability over $\mathscr{G}_\lambda$ for regular $\lambda > \aleph_1$ onto the relation of set-inclusion over all subsets of reals of cardinality $\leqslant \lambda$. Combining both results, $\max\{\lambda^+, 2^{\aleph_0}\}$ is set as a lower bound for the complexity of $\mathscr{G}_\lambda$ for regular $\lambda > \aleph_1$.

In Section 2 a certain continuity property of the homomorphism from Section 1 is proved. Continuity and combinatorics of singular cardinals are employed to extend the lower bound from Section 1 to all singular cardinals below the first fixed point of second order. In this section the representation theorem is stated in its full generality, generalising Theorem 1.10 to higher cardinals.

In Section 3 a proper subclass $\mathscr{G}^*$ of $\mathscr{G}$ is defined by forbidding some additional countable configurations. A decomposition theorem is proved for $\mathscr{G}^*_\lambda$ for $\lambda > \aleph_1$ regular. The theorem gives a necessary and sufficient condition to when the image of a graph $G \in \mathscr{G}^*_\lambda$ under the homomorphism from Theorem 1.10 is inherited by at least one of fewer than λ subgraphs whose union covers G.

0.2. *Notation*

A graph G is a pair $\langle V, E\rangle$ where V is the set of vertices and $E \subseteq [V]^2$ is the set of edges. By $G[v]$ we denote the *neighbourhood* of $v \in V$ in G, namely $\{u \in V : \{v, u\} \in E\}$.

An *ordinal* is a set which is well ordered by $\in$. A *cardinal* is an initial ordinal number and $\langle \aleph_\alpha : \alpha \text{ ordinal}\rangle$ is the increasing enumeration of all cardinals. The cardinality $|A|$ of a set A is the unique cardinal equinumerous with A. The *cofinality*, $\mathrm{cf}\,\lambda$, of a cardinal λ is the least cardinal κ such that λ can be represented as a union of κ sets, each of cardinality less than λ. A cardinal λ is *singular* if $\mathrm{cf}\,\lambda < \lambda$ and is *regular* if $\mathrm{cf}\,\lambda = \lambda$. A cardinal $\aleph_\alpha$ is a fixed point if and only if $\aleph_\alpha = \alpha$. A cardinal μ

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is a *fixed point of second order* if and only if there are μ fixed points below μ. For cardinals κ, κ', we denote by K_κ the complete graph on κ vertices and by $K_{\kappa,\kappa'}$ the complete bipartite graph with κ vertices in one side and κ' in the other.

If G_1 is isomorphic to a subgraph of G_2 we write $G_1 \leqslant_w G_2$ and we write $G_1 \leqslant G_2$ if G_1 is isomorphic to an *induced* subgraph of G_2. We say that G_1 is *embeddable* in G_2 if $G_1 \leqslant_w G_2$ and that G_1 is *strongly embeddable* in G_2 if $G_1 \leqslant G_2$.

Classes of graphs will be denoted by $\mathscr{G}$ or by Γ and are always assumed to be closed under isomorphism. A class of graphs $\mathscr{G}$ is *monotone* if $G_1 \leqslant_w G_2 \in G \Rightarrow G_1 \in \mathscr{G}$. If Γ is a set of graphs then Forb (Γ) is the class of all graphs without a subgraph in Γ. Let $\mathscr{G}_\lambda$ be the set of all isomorphism types of members of $\mathscr{G}$ whose cardinality is λ. The relations $\leqslant$ and $\leqslant_w$ are reflexive and transitive, and therefore $\langle \mathscr{G}_\lambda, \leqslant \rangle$ and $\langle \mathscr{G}_\lambda, \leqslant_w \rangle$ are quasi-ordered sets for every class $\mathscr{G}$ and cardinal λ.

Let $\operatorname{cp} \mathscr{G}_\lambda$, the *complexity* of $\mathscr{G}_\lambda$, be the least cardinality of a subset $D \subseteq \mathscr{G}_\lambda$ with the property that for every $G \in \mathscr{G}_\lambda$ there exists $G' \in D$ such that $G \leqslant G'$; the *weak complexity* (wcp) is defined by replacing $\leqslant$ by $\leqslant_w$ (see [**17**]). Clearly, $\operatorname{wcp} \mathscr{G} \leqslant \operatorname{cp} \mathscr{G}$ for every class $\mathscr{G}$.

The complexity $\operatorname{cp} \mathscr{G}_\lambda$ is 1 if and only if there is a graph $G^* \in \mathscr{G}_\lambda$ with the property that every member of $\mathscr{G}_\lambda$ is isomorphic to an induced subgraph of G^*. Such a graph G^* is called *universal in* λ (or, sometimes, 'strongly universal') for the class $\mathscr{G}$. The statement $\operatorname{wcp} \mathscr{G}_\lambda = 1$ is equivalent to the existence of a *weakly universal* element in $\mathscr{G}_\lambda$.

Suppose that A is a given infinite set. By $[A]^\lambda$, $[A]^{\leqslant \lambda}$ and $[A]^{< \lambda}$ we denote, respectively, the collection of all subsets of A whose cardinality is λ, $\leqslant \lambda$ and $< \lambda$. Let $\operatorname{cov} A_\lambda$, the covering number of $[A]^\lambda$, be the least cardinality of a subset $D \subseteq [A]^\lambda$ with the property that for every $B \in [A]^\lambda$ there exists $B' \in D$ such that $B \subseteq B'$.

We remark that the least cardinality of a dominating subset is defined for every quasi-ordered set, and bears the name 'cofinality'; we keep here to the existing graph-theoretic and set theoretic terminologies by calling the former 'complexity' and the latter 'covering number'.

DEFINITION 0.3. Let λ be an uncountable regular cardinal. A *club* of λ is a closed (in the order topology) and unbounded subset of λ. A subset of λ is called *stationary* if its intersection with every club of λ is non-empty. The ideal of all subsets of λ which are disjoint to some club of λ is the *non-stationary* ideal. Let S^λ_κ be $\{\alpha : \alpha < \lambda \wedge \operatorname{cf} \alpha = \kappa\}$ and $S^\lambda_0 = \{\alpha : \alpha < \lambda \wedge \operatorname{cf} \alpha = \omega\}$.

We shall need the following combinatorial tool.

THEOREM 0.4 (Shelah). *Suppose that $\omega \leqslant \mu$, $\mu^+ < \lambda$, μ is a cardinal and λ is a regular cardinal. Suppose that $S \subseteq \lambda$ is stationary and every $\zeta \in S$ has cofinality* cf μ. *Then there is a stationary $S' \subseteq \lambda$ and a sequence*

$$\bar{C} = \langle c_\delta : \delta \in S' \rangle \tag{0}$$

that satisfies

(a) $\operatorname{otp} c_\delta = \mu$ *and* $\sup c_\delta = \delta$;

(b) *for every closed unbounded $E \subseteq \lambda$ the set $N(E) := \{\delta \in S' : c_\delta \subseteq E\}$ is stationary.*

Proofs of Theorem 0.4 can be found in [**12**, Section 1; **13**, Appendix; **25**].

A sequence $\bar{C}$ as in (0) that satisfies (a) and (b) is called a *club guessing sequence*.

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Suppose $\bar{C}$ is a sequence as in (0) above. We define, relative to $\bar{C}$, two *guessing ideals* over λ, $\mathrm{id}^a(\bar{C})$ and $\mathrm{id}^b(\bar{C})$ as follows.

DEFINITION 0.5. (0) $X \in \mathrm{id}^a(\bar{C})$ if and only if for some club $E \subseteq \lambda$ it holds that $c_\delta \not\subseteq E$ for all $\delta \in S \cap E$.

(1) $X \in \mathrm{id}^b(\bar{C})$ if and only if for some club $E \subseteq \lambda$ it holds that $c_\delta \not\subseteq^* E$ for all $\delta \in S \cap E$.

The relation $c_\delta \subseteq^* E$ means that an end segment of c_δ is contained in E.

A set $X \subseteq \lambda$ is in $\mathrm{id}^a(\bar{C})$ if and only if for some club E, there are no stationarily many $\delta \in X$ such that c_δ is contained in E; $X \subseteq \lambda$ is in $\mathrm{id}^b(\bar{C})$ if and only if for some club E, there are no stationarily many $\delta \in X$ such that C_δ is almost contained in E.

The ideal $\mathrm{id}^a(\bar{C})$ is λ-complete and $\mathrm{id}^b(\bar{C})$ is normal. Both ideals are proper if $\bar{C}$ is a club guessing sequence.

We occasionally call a set which does not belong to an ideal under discussion 'positive' and a set whose complement belongs to the ideal 'a measure 1 set'.

DEFINITION 0.6. Let ω be the set of natural numbers. Let Fin be the set of all finite subsets of ω. Two subsets X, $Y \subseteq \omega$ are *equivalent* mod Fin if and only if the symmetric difference $X \setminus Y \cup Y \setminus X \in \mathrm{Fin}$. By $\mathscr{P}(\omega)$ we denote the *power set* of ω and by $\bar{\mathscr{P}}(\omega)$ we denote $\mathscr{P}(\omega)/\mathrm{Fin}$, the set of all equivalence classes of subsets of ω modulo Fin.

DEFINITION 0.7. Suppose that A is a structure, λ is a cardinal and F is a filter over λ. Let A^λ be the set of all functions from λ to the structure A and let A^λ/F be the reduced power of A modulo F.

1. *Representing embeddability as set inclusion*

In [**11**] the following theorem is proved:

THEOREM 1.1. *If $\mathscr{G}$ is a class of graphs that contains all $K_{\omega,\omega}$-free incidence graphs of $A \subseteq \mathscr{P}(\omega)$, then* $\mathrm{cp}\,\mathscr{G}_\lambda > 2^{\aleph_0}$ *for all λ which satisfies* $\mathrm{cf}\,2^{\aleph_0} \leqslant \lambda < 2^{\aleph_0}$.

In particular, if $\mathrm{cf}\,2^{\aleph_0} = \aleph_1$ there is no universal graph (in the class of all graphs) in all uncountable $\lambda < 2^{\aleph_0}$.

On the other hand, Shelah proved in [**27**]:

THEOREM 1.2. *If λ is regular uncountable, it is consistent that $\lambda < 2^{\aleph_0}$ and that there exists a universal graph in power λ.*

Mekler [**22**] generalised Shelah's result to a broader collection of classes of relational structures. The consistency of non-existence of universal elements in uncountable λ below a regular continuum is easy (see [**13**, Appendix]).

The results above show that embeddability among graphs at a regular uncountable λ is indifferent to the size of the continuum – if the continuum is regular. This behaviour stands in strong contrast to the behaviour of embeddability in, say, linear orderings, where there can be no universal at a regular $\lambda > \aleph_1$ below the continuum, by [**13**].

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It is reasonable to ask whether in some class of graphs embeddability would behave with respect to the continuum similarly to that among linear orderings. If such classes of graphs exist, it is legitimate to ask how far are they from the class of all graphs.

In this section we present a simple set of countable configurations which are related to the end-structure of graphs. It turns out that forbidding those configurations suffices to make way for the method from [**13**], and consequently we are able to prove very tight connections between embeddability over the resulting class and the continuum.

DEFINITION 1.3. A *ray* in a graph G is a 1-way infinite path. A *tail* of a ray $R \subseteq G$ is an infinite connected subgraph of R. Two rays in G are *tail-equivalent* if and only if they share a common tail. Tail-equivalence is an equivalence relation on rays.

We mention in passing that tail-equivalence is a refinement of end-equivalence. For more on both relations see [**2**].

DEFINITION 1.4. Let $\mathscr{G}$ be the class of all graphs G which satisfy that for every $v \in G$ the induced subgraph of G spanned by $G[v]$ contains at most one ray up to tail-equivalence.

CLAIM 1.5. *There is a non-empty set* Γ *of countable graphs, each containing an infinite path, such that* $\mathscr{G} = \mathrm{Forb}(\Gamma)$.

Proof. Let Γ' be the set of all countable graphs that contain (at least) two tail-inequivalent rays and let Γ be all graphs obtained by choosing an element from Γ' and joining a new vertex to all its vertices. If a graph G contains a subgraph in Γ then $G \notin \mathscr{G}$. Conversely, suppose that $G \notin \mathscr{G}$. Let $v \in G$ be a vertex such that there are two rays R_1, $R_2 \subseteq G[v]$ which are not tail-equivalent. Let $G' \subseteq G$ be the induced subgraph spanned by $\{v\} \cup R_1 \cup R_2$. Now $G' \in \Gamma$ and so $G \notin \mathrm{Forb}(\Gamma)$.

Omission of countable configurations each of which contains an infinite path was considered by Diestel, Halin and Vogler [**3**] for $\lambda = \aleph_0$. They prove [**3**, Theorem 4.1] the following theorem.

THEOREM 1.6 (Diestel–Halin–Vogler). *Let* Γ *be a non-empty set of countable graphs each containing an infinite path. Then* $\mathrm{Forb}_{\aleph_0}(\Gamma)$ *does not contain a universal element.*

Theorem 1.6 applies to our class $\mathscr{G}$ by Claim 1.5, and because every forbidden configuration in Γ contains an infinite path. The following is a straightforward generalisation of Theorem 1.6 to higher cardinals. We included its proof for completeness.

THEOREM 1.7. *Let* Γ *be a non-empty set of countable graphs each containing an infinite path. Then* $\mathrm{wcp}\,\mathrm{Forb}_\lambda(\Gamma) \geqslant \lambda^+$ *for all infinite cardinals* λ.

Proof. By induction on $\alpha < \lambda^+$ define graphs G_α as follows: G_α is obtained by joining a vertex w_α to all vertices in a disjoint union of G_β for all $\beta < \alpha$. For all $\lambda \leqslant \alpha < \lambda^+$ the graph G_α contains no infinite path and therefore belongs to $\mathrm{Forb}_\lambda(\Gamma)$.

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Suppose that $\mathscr{F}$ is a collection of λ graphs and $f_\alpha: G_\alpha \longrightarrow (\alpha)$ is an embedding of G_α into some $G(\alpha) \in \mathscr{F}$. By the pigeon-hole principle there is a fixed graph $G \in \mathscr{F}$ such that $G = G(\alpha)$ for λ^+ many $a < \lambda^+$. A second use of the pigeon-hole principle gives a vertex $w(0) \in G$ and an unbounded set $X \subseteq \lambda^+$ such that $w(0) = f_\alpha(w_\alpha)$ for all $\alpha \in X$. This implies, by the construction of the G_α, that $G[w(0)]$ contains as subgraphs copies of G_α for unboundedly many $\alpha < \lambda^+$ and therefore for all $\alpha < \lambda^+$.

Repeating this argument inductively, a set $\{w(n): n < \omega\} \subseteq G$ is found so that $w_{n+1} \in :\bigcap_{m<n} G[w_n]$ and $\bigcap_{m \leqslant n+1} G[w_m]$ contains copies of G_α for all $\alpha < \lambda^+$. Thus the set $\{w(n): n < \omega\} \subseteq G$ spans in G a copy of K_ω. Therefore G contains all countable configurations and therefore does not belong to $\mathrm{Forb}_\lambda(\Gamma)$.

We conclude that for every infinite λ, $\mathrm{wcp}\,\mathscr{G}_\lambda \geqslant \lambda^+$. Also Theorem 1.1 applies to $\mathscr{G}$, because $\mathscr{G}$ contains all bipartite graphs.

The virtue of $\mathscr{G}$ is, however, that much more than this can be said about embeddability over it. Embeddability over $\mathscr{G}_\lambda$ admits representation as set inclusion over sets of reals, as stated in the following.

THEOREM 1.8. *If $\lambda > \aleph_1$ is regular then there is a surjective homomorphism $\Phi: \langle \mathscr{G}_\lambda, \leqslant_w \rangle \longrightarrow \langle [\mathbb{R}]^{\leqslant\lambda}, \subseteq \rangle$.*

COROLLARY 1.9. *If $\lambda > \aleph_1$ is regular then* $\mathrm{wcp}\,\mathscr{G}_\lambda \geqslant \max\{\lambda^+, 2^{\aleph_0}\}$.

Proof. By Theorem 1.7, $\mathrm{wcp}\,\mathscr{G}_\lambda \geqslant \lambda^+$. Assume, then, that $2^{\aleph_0} > \lambda^+$. By Theorem 1.8, $\langle [2^{\aleph_0}]^{\leqslant\lambda}, \subseteq \rangle$ is a homomorphic image of $\langle \mathscr{G}_\lambda, \leqslant_w \rangle$ and therefore

$$\mathrm{cf}\langle [2^{\aleph_0}]^{\leqslant\lambda}, \subseteq \rangle \leqslant \mathrm{cf}\langle \mathscr{G}_\lambda, \leqslant_w \rangle \overset{\mathrm{def}}{=} \mathrm{cp}\,\mathscr{G}_\lambda.$$

The first cofinality is at least $2^{\aleph_0}$ trivially, as $2^{\aleph_0} > \lambda^+$.

In Corollary 1.9, the inequality $\mathrm{wcp}\,\mathscr{G}_\lambda \geqslant \max 2^{\aleph_0}$ holds regardless of the cardinality or cofinality of the continuum for all regular $\lambda > \aleph_1$. This corollary will be extended to some singular values of λ in the next section.

We turn now to the proof of Theorem 1.8. The homomorphism Φ will be factored through a reduced product of the inclusion relation over subsets of reals. We will prove the following stronger formulation.

THEOREM 1.10. *Suppose that $\lambda > \aleph_1$ is regular. Then*

(0) *there is a surjective homomorphism $\Phi: \langle \mathscr{G}_\lambda, \leqslant_w \rangle \longrightarrow \langle [\mathbb{R}]^{\leqslant\lambda}, \subseteq \rangle$;*

(1) *Φ from (0) can be chosen to be a composition $\psi\phi$ where $\phi: \langle \mathscr{G}_\lambda, \leqslant_w \rangle \longrightarrow ([\mathbb{R}]^{\leqslant\lambda}, \subseteq)^\lambda/I$ is a surjective homomorphism and I is a normal ideal over λ.*

Proof. First let us notice that $\langle [\mathbb{R}]^{\leqslant\lambda}, \subseteq \rangle$ is a homomorphic image of $\langle [\mathbb{R}]^{\leqslant\lambda}, \subseteq \rangle^\lambda/I$ for every ideal I. Suppose that $\bar{A}$ is a representative of an equivalence class of $([\mathbb{R}]^{\leqslant\lambda})^\lambda/I$. Define $\psi([\bar{A}]) := \{x \in \mathbb{R}: \{\delta < \lambda: x \in A(\delta)\} \notin I\}$. It is routine to check that the definition of ψ does not depend on the choice of a representative and that ψ is a homomorphism.

Thus it suffices to prove that there is a surjective homomorphism $\phi: \langle \mathscr{G}_\gamma, \leqslant_w \rangle \longrightarrow \langle [\mathbb{R}]^{\leqslant\lambda}, \subseteq \rangle^\lambda/I$ for some normal ideal I over λ. This is in fact more than is needed for (0). The set $\mathbb{R}$ can be replaced here by any set of equal cardinality. It is convenient for us to work with $\bar{\mathscr{P}}(\omega) = \mathscr{P}(\omega)/\mathrm{Fin}$.

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We shall define a mapping $\phi: \langle \mathscr{G}_\lambda, \leqslant \rangle \longrightarrow \langle [\bar{\mathscr{P}}(\omega)]^{\leqslant \lambda}, \subseteq \rangle^\lambda / I$ after specifying I and proceed to show that ϕ is well defined, is a homomorphism and is surjective. For the definition of ϕ we fix a club guessing sequence $\bar{C} = \langle c_\delta : \delta \in S \rangle$, $S \subseteq \lambda$ stationary and otp $c_\delta = \omega$ for c_δ in $\bar{C}$. Now let $I = \mathrm{id}^b(\bar{C})$. For each $\delta \in S = S^\lambda_0$ let $\langle \alpha^\delta_n : n < \omega \rangle$ be the increasing enumeration of c_δ.

Given $G \in \mathscr{G}_\lambda$ we define $\phi(G)$ after choosing two auxiliary parameters on G. First, fix a bijection h between G and λ, and second, fix a function r so that $r(v) = \varnothing$ for $v \in G$ in case there are no rays in $G[v]$ and $r(v)$ is a ray in $G[v]$ otherwise.

Let $G^{\alpha, h} = \{v \in G : h(v) < \alpha\}$ for $\alpha < \lambda$. When h is unambiguous we write G^α for $G^{\alpha, h}$. For a vertex $v \in G$ and an ordinal $\delta \in S$ we define:

$$\phi_{h,r}(v, \delta) = [\{n < \omega : r(v) \cap G^{\alpha_n} \neq r(v) \cap G^{\alpha_{n+1}}\}]_{\mathrm{Fin}}. \tag{1}$$

Thus $\phi_{h,r}(v, \delta)$ belongs to $\bar{\mathscr{P}}(\omega)$. The definition in (1) depends strongly on the choice of the bijection h. In fact, for a vertex v whose neighbourhood $G[v]$ does contain a ray, $\phi_{h,r}(v, \delta)$ can be made any prescribed element of $\bar{\mathscr{P}}(\omega)$ by a suitable choice of h. But replacing $r(v)$ by a tail-equivalent $r'(v)$ produces at most a finite change in $\{n < \omega : r(v) \cap G^{\alpha_n} \neq r(v) \cap G^{\alpha_{n+1}}\}$ and therefore does not change the definition (1).

Now let

$$\phi_{h,r}(G, \delta) = \{\phi_{h,r}(v, \delta) : v \in G\}. \tag{2}$$

Since $|G| = \lambda$, we conclude that

$$\phi_{h,r}(G, \delta) \in [\bar{\mathscr{P}}(\omega)]^{\leqslant \lambda}. \tag{3}$$

Finally, let

$$\phi(G) = \phi_{h,r}(G) = [\langle \phi_{h,r}(G, \delta) : \delta \in S \rangle]_I. \tag{4}$$

The sequence $\langle \phi_{h,r}(G, \delta) : \delta \in S \rangle$ belongs to $([\bar{\mathscr{P}}(\omega)]^{\leqslant \lambda})^\lambda$, and we let $\phi_{h,r}(G)$ be the equivalence class of this sequence in the reduced power mod I.

PROPOSITION 1.11. (a) *The definition of $\phi(G)$ does not depend on the choice of h and r.*

(b) *ϕ is a homomorphism: if $G_1 \leqslant G_2$ then $\phi(G_1) \subseteq_I \phi(G_2)$.*

(c) *ϕ is surjective.*

Proof. We shall prove (a) and (b) simultaneously by proving (d) if $G_1 \leqslant G_2$ are in $\mathscr{G}_\lambda$ and h_1, h_2, r_1, r_2 are parameters as in (1) for G_1 and for G_2 respectively, then $\phi_{h_1, r_1}(G_1) \subseteq_I \phi_{h_2, r_2}(G_2)$.

Then (a) follows from (d) by putting $G_1 = G_2 = G$ and (b) follows from (a) and (d).

Let h_1, h_2, r_1, r_2 be given. We need to show that $\phi_{h_1, r_1}(G_1) \subseteq_I \phi_{h_2, r_2}(G_2)$, namely, that:

$$S \setminus \{\delta \in S : \phi_{h_1, r_1}(G_1, \delta) \subseteq \phi_{h_2, r_2}(G_2, \delta)\} \in I. \tag{5}$$

Fixing an embedding from G_1 to G_2 we assume, without loss of generality, that G_1 is a subgraph of G_2. The following set is closed unbounded in λ by a standard back-and-forth argument:

$$E = \{\alpha < \lambda : G_2^{\alpha, h_2} \cap G_1 = G_1^{\alpha, h_1}\}. \tag{6}$$

Therefore, by the definition of I, the set $N(E) = \{\delta \in S : c_\delta \subseteq^* E\}$ satisfies

$$S \setminus N(E) \in I. \tag{7}$$

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We shall show that for every $\delta \in N(E)$ we have for all $v \in G_1$ such that $r_1(v) \neq \varnothing$:

$$\phi_{h_1, r_1}(v, \delta) = \phi_{h_2, r_2}(v, \delta). \tag{8}$$

Let $v \in G_1$ be given. Clearly, $G_1[v] \subseteq G_2[v]$. Thus $r_1(v)$ and $r_2(v)$ are both rays in $G_2(x)$. Since $G_2 \in \mathscr{G}$, the rays $r_1(v), r_2(v)$ are tail-equivalent. Fix a common tail $r(v)$ of $r_1(v), r_2(v)$. Using $r(v)$ instead of either $r_1(v)$ or $r_2(v)$ in the definition (1) for $\phi_{h_i, r_i}(v, \delta)$ $(i = 1, 2)$ makes no difference. So we may assume without loss of generality that $r_1(v) = r_2(v) = r(v)$ for all $v \in G_1$.

Suppose that $v \in G_1$ and that $c_\delta \subseteq^* E$. Then the sets $\{n < \omega : r(v) \cap G_1^{\alpha_n} \neq r(v) \cap G_1^{\alpha_{n+1}}\}$ and $\{n < \omega : r(v) \cap G_2^{\alpha_n} \neq r(v) \cap G_2^{\alpha_{n+1}}\}$ are equivalent modulo finite, because $G_1^{\alpha_n} = G_2^{\alpha_n} \cap G_1$ for an end segment of ω. Thus, definition (2) gives for every $\delta \in N(E)$:

$$\begin{aligned} \phi_{h_1, r_1}(G_1, \delta) &= \{\phi_{h_1, r_1}(v, \delta) : v \in G_1\} = \{\phi_{h_2, r_2}(v, \delta) : v \in G_1\} \\ &\subseteq \{\phi_{h_2, r_2}(v, \delta) : v \in G_2\} = \phi_{h_2, r_2}(G_2, \delta). \end{aligned} \tag{9}$$

And now (5) follows by (7).

This proves (d) and hence (a) and (b).

To prove (c) fix a member $\bar{A} = \langle A_\alpha : \alpha < \lambda \rangle \in ([\bar{\mathscr{P}}(\omega)]^{\leqslant \lambda})^\lambda$. We shall construct a graph $G \in \mathscr{G}_\lambda$ so that $\phi(G) = [\bar{A}]_I$.

Let $\langle X_{\delta, \alpha} : \alpha < \lambda \rangle$ list all members of A_δ for $\delta \in S$.

Let T be the rooted λ-regular tree, with root e. For every $v \in T$ there exists a unique path in T from v to e. The *height* $\mathrm{hg}(v)$ of v is the length of this unique path. An *immediate successor* of a vertex $v \in T$ is a vertex $u \in T[v]$ with $\mathrm{hg}(u) = \mathrm{hg}(v) + 1$.

Fix an increasing continuous union $T = \bigcup_{\alpha < \lambda} T^\alpha$ with $e \in T^0$. By passing to a closed unbounded set of indices we may assume that for every $\alpha < \lambda$ and $v \in T^\alpha$ there is at least one immediate successor of v in $T^{\alpha+1} \setminus T^\alpha$.

Define a graph G on the vertex set $T \cup \lambda \times \lambda$, where this union is disjoint. Let all edges of T be edges of G as well. For every $\delta \in S$ and $\gamma < \lambda$ find a ray $\langle v_n : n < \omega \rangle$ in T^δ so that $\mathrm{hg}(v_n) = n$ and so that $\min\{\alpha < \delta : v_n \in T^\alpha\} = \alpha^\delta_m$ where m is the nth member of $X_{\delta, \gamma}$. This is possible by our assumption on the union $T = \bigcup_{\alpha < \lambda} T^\alpha$. Connect each v_n by an edge to $\langle d, \gamma \rangle$.

By this definition, $\lambda \times \lambda$ is an independent set in G; for every $v \in \lambda \times \lambda$ the neighbourhood $G[v]$ is either empty or is a branch in T; and for every $v \in T$, the set $G[v] \cap T = T[v]$ is independent.

CLAIM 1.12. *If $v \in T$ then $G[v]$ contains no rays.*

Proof. Suppose that $v \in T$ and let $m = \mathrm{hg}(v)$. Suppose to the contrary that $\langle u_n : n < \omega \rangle \subseteq G[v]$ is a ray. By omitting an initial segment of the ray we assume, without loss of generality, that the predecessor of v in T is not among the u_n. Since $G[v] \cap T$ and $G[v] \setminus T \subseteq \lambda \times \lambda$ are independent sets, $u_n \in T$ if and only if $u_{n+1} \in \lambda \times \lambda$. Without loss of generality, $u_0 \in \lambda \times \lambda$. Thus $u_n \in \lambda \times \lambda$ if and only if n is even and $u_n \in T$ if and only if n is odd, and therefore $\mathrm{hg}(v_n) = m + 1$ for all odd n. Let $n > 0$ be even. Then $u_n \in \lambda \times \lambda$ and u_n is connected by edges to u_{n-1} and u_{n+1}. Since $\mathrm{hg}(u_{n-1}) = \mathrm{hg}(u_{n+1}) = m + 1$ the neighbourhood $G[u_n]$ is neither empty nor a branch – a contradiction.

Let us verify that $G \in \mathscr{G}_\lambda$. If $v \in \lambda \times \lambda$ then $G[v]$ is either empty or is a ray and if $v \in T$ then by the Claim 1.12 $G[v]$ contains no rays. Thus $G[v]$ contains at most one ray up to tail equivalence for all $v \in G$ and therefore $G \in \mathscr{G}_\lambda$.

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Let $G^\alpha := T^\alpha \cup \alpha \times \alpha$ for all $a \in \lambda$. Fix a bijection $h: G \longrightarrow \lambda$. It is standard to check that for a closed unbounded set $E \subseteq \lambda$ it holds that $G_\alpha = G^{\alpha,h}$ for all $\alpha \in E$. We may restrict attention to this set of indices. Suppose now that $\delta \in N(E)$ and let $\gamma < \lambda$ be given. Let $v = \langle \delta, \gamma \rangle \in G$. By the definition of ϕ above, $\phi_{h,r}(v) = X_{\delta,\gamma}$ for all $\gamma < \delta$, where r is any function as in (1) above. This shows that $A \subseteq_I \phi_{<,r}(G)$. The converse inclusion is obvious. This proves (c).

2. *Continuity, singulars and r-subgraphs*

In this section we study the homomorphism Φ from Theorem 1.10 and show that it has a certain continuity property.

As a result we are able to prove that Corollary 1.9 holds also for many singular cardinals μ.

DEFINITION 2.1. For $G_1, G_2 \in \mathscr{G}_\lambda$ say that G_1 is an *r*-subgraph of G_2 if and only if G_1 is a subgraph of G_2 and for all $v \in G_1$ if $G_2[v]$ contains a ray then $G_1[v]$ contains a ray. Equivalently, if r is a function on G_2 such that $r(x)$ is a ray in $G_2[x]$ if such a ray exists and $\varnothing$ otherwise, then for all $x \in G_1$, $r(x) \subseteq^* G_1$.

CLAIM 2.2 (Continuity). *If $G \in \mathscr{G}_\lambda$ and $G = \bigcup_{\alpha<\beta} G_\alpha$ for some $\beta < \lambda$ so that G_α is an r-subgraph of G for every $\alpha < \beta$, then*

$$\Phi(G) = \bigcup_{\alpha<\beta} \Phi(G_\alpha).$$

Proof. For every $\alpha < \beta$ the relation $\Phi(G_\alpha) \subseteq_I \Phi(G)$ holds because Φ is a homomorphism and $G_\alpha \leqslant G$.

On the other hand, suppose that $A \in \Phi(G)$, and we will show that for some $\alpha < \beta$ we have $A \in \Phi(G_\alpha)$.

Let h, r and h_α, r_α be chosen auxiliary parameters for G and for every G_α for $\alpha < \beta$ respectively, as in the definition of ϕ. For every $\alpha < \beta$ there is a set $X_\alpha \in I$ such that for every $\delta \in \lambda \setminus X_\alpha$ and every $v \in G_\alpha$ for which $r(v) \neq \varnothing$ condition (8) in the proof of Proposition 1.11 holds:

$$\phi_{h_\alpha, r_\alpha}(v, \delta) = \phi_{h,r}(v, \delta). \tag{10}$$

Let $X = \bigcup_{\alpha<\beta} X_\alpha$. By λ-completeness of I we deduce that $X \in I$. Suppose that $\delta \in \lambda \setminus X$ and let $v \in G$.

Since $A \in \Phi(G)$, the set $Y := \{\delta < \lambda : A \in \phi_{h,r}(G, \delta)\}$ is positive. For every $\delta \in Y$ pick $v_\delta \in G$ so that $\phi_{h,r}(v_\delta, \delta) = A$.

For every $\delta \in Y$ there is some $\alpha(\delta) < \beta$ for which $v_\delta \in G_{\alpha(\delta)}$. Since I is λ-complete, there is a fixed $\alpha < \beta$ and a positive set $Y' \subseteq Y$ such that $\alpha(\delta) = \alpha$ for all $\delta \in Y'$.

Since $X \in I$, we may, without loss of generality, subtract this union from Y', and Y' would still be positive. But now it follows from (10) that for every $\delta \in Y'$ it holds that

$$\phi_{h_\alpha, r_\alpha}(v_\delta, \delta) = \phi_{h,r}(v_\delta, r) = A.$$

Since Y' is positive this shows that $A \in \Phi(G_\alpha)$ and thus completes the proof.

COROLLARY 2.3. *If μ is singular below the first fixed point of second order, then* $\mathrm{cp}\, \mathscr{G}_\mu \geqslant \max\{\mu^+, 2^{\aleph_0}\}$.

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Proof. We address only the term $2^{\aleph_0}$, as the other follows from Theorem 1.7. Assume that μ is singular, smaller than the first fixed point of second order, and that $\mu < 2^{\aleph_0}$. To prove that $\mathrm{wcp}\,\mathscr{G}_\mu \geqslant 2^{\aleph_0}$ suppose that $\kappa < 2^{\aleph_0}$ and that for every $\alpha < \kappa$ we are given $G_\alpha \in \mathscr{G}_\mu$, and we will exhibit a graph $G \in \mathscr{G}_\lambda$ for some regular $\lambda < \mu$ which is not embedded in any G_α for $\alpha < \kappa$.

By the assumption on μ, and since $|G_\alpha| = \mu$, there exists a regular $\aleph_2 \leqslant \lambda < \mu$ and, for every $\alpha < \kappa$, there is a family $\mathscr{F}_\alpha \subseteq [G_\alpha]^\lambda$ with $|\mathscr{F}_\alpha| = \mu$ such that for all $X \in [G_\alpha]^\lambda$ there is $F \in [\mathscr{F}_\alpha]^{<\lambda}$ so that $X \subseteq \bigcup F$ (see [**13**, 4.5]). So while $\mathscr{F}_\alpha$ itself may not be dominating in $\langle [G_\alpha]^\lambda, \subseteq \rangle$, the set of all unions of $< \lambda$ members of G_α is dominating (but has cardinality larger than μ in this case). We may assume, by increasing each $A \in \mathscr{F}_\alpha$ to a larger set of the same cardinality, that each $A \in \mathscr{F}_\alpha$ spans an r-subgraph of G_α, and, abusing notation, we shall not distinguish between A and the subgraph it spans.

Fix a homomorphism Φ as in Theorem 1.10 for λ.

The cardinality of the following set is $\kappa \times \mu$:

$$D := \{\Phi(A) \colon A \in \bigcup_{\alpha < \kappa} \mathscr{F}_\alpha\}.$$

As $\kappa \times \mu < 2^{\aleph_0}$ and $|\Phi(A)| = \lambda$ for $\Phi(A) \in D$, we have $|\bigcup D| < 2^{\aleph_0}$. Thus we can find a set $B \in [R]^\lambda$ such that $B \notin \bigcup D$.

Using the surjectivity of Φ, fix a graph $G \in \mathscr{G}_\lambda$ with $\Phi(G) = B$.

Suppose to the contrary that G is isomorphic to a subgraph of G_α for some $\alpha < 2^{\aleph_0}$. Without loss of generality G is a subgraph of, say, G_0. By the covering property of $\mathscr{F}_0$ we can find a subset $F \in [\mathscr{F}_0]^{<\lambda}$ such that $G \leqslant \bigcup F$. Because every member of F is an r-subgraph of G_0, every member of F is also an r-subgraph of (the induced subgraph of G_0 spanned by) $\bigcup F$. Therefore $\Phi(\bigcup F) = \bigcup_{A \in F} \Phi(A)$ by continuity Claim 2.2 . Thus

$$B = \Phi(G) \subseteq \Phi(\bigcup F) = \bigcup_{A \in F} \Phi(A) \subseteq \bigcup D,$$

a contradiction to the choice of B.

Corollary 2.3 has two improvements over Corollary 1.9. The first is that it extends the result to singular cardinals. The second is that also it says something stronger than setting a lower bound for complexity: it says that for every set $\mathscr{A}$ of fewer than $2^{\aleph_0}$ members in $\mathscr{G}_\mu$ there is a graph in $\mathscr{G}$ of some smaller cardinality λ which is not weakly embeddable in any member of $\mathscr{A}$. The latter statement holds for all regular $\lambda > \aleph_2$ below the first fixed point of second order as well, by the same proof.

2.1. *Generalisation to higher cardinals*

We state now the most general form of representation we know for $\mathscr{G}$. This involves replacing club guessing using c_δ of order type ω with club guessing using c_δ of order type μ, for some μ of cofinality $\aleph_0$.

Replacing ω and $\mathscr{P}(\omega)$ with μ and $[\mu]^{\aleph_0}$ respectively in the proof of Theorem 1.10, there is only one difficulty: in establishing condition (d) in the proof, the argument that $c_\delta \subseteq^* E$ implies that $\phi_{h_1, r}(v, \delta) = \delta_{h_2, r}(v, d)$ breaks down, because a proper initial segment of c_δ may contain all the members of the subset of c_δ chosen by v in equation (1) in the proof.

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This difficulty vanishes if we demand that $c_\delta \subseteq E$, namely work with $I = \mathrm{id}^a(\bar{C})$ rather than $I = \mathrm{id}^b(\bar{C})$.

Since normality of I was not used so far (only λ-completeness), all the results so far hold when replacing $I = \mathrm{id}^b(\bar{C})$ by the λ-complete $I = \mathrm{id}^a(\bar{C})$.

Thus we have proved the following.

THEOREM 2.4. *Suppose that λ is regular. Then for every μ satisfying $\mu^+ < \lambda$ there is a surjective homomorphism $\Phi_\mu: \langle \mathscr{G}_\lambda, \leqslant_w \rangle \longrightarrow \langle [\mu^{\aleph_0}]^{\leqslant \lambda}, \subseteq \rangle$.*

COROLLARY 2.5. *For every regular uncountable $\lambda > \aleph_1$ we have $\mathrm{cp}\,\mathscr{G}_\lambda \geqslant \sup(\{\lambda^+\} \cup \mu^{\aleph_0}: \mu^+ < \lambda\})$.*

Thus, for example, one can have a model of set theory V in which the generalised continuum hypothesis holds up to $\aleph_\omega$ and $\aleph_\omega^{\aleph_0}$ large, say $\aleph_{\omega+\omega+1}$ (see [**21**]). In this model the complexity of $\mathscr{G}_{\aleph_{\omega+2}}$ is larger than $\aleph_{\omega+3}$.

3. *A decomposition theorem*

Forbidding a few more countable configurations enables a decomposition theorem: we prove that such a graph is r-indecomposable if and only if its image is a singleton.

DEFINITION 3.1. Suppose that $\lambda > \aleph_1$ is regular and that $\Phi: \mathscr{G}_\lambda \longrightarrow [\mathbb{R}]^{\leqslant \lambda}$ is the homomorphism from Theorem 1.10. Say that $G \in \mathscr{G}_\lambda$ is (r, λ)-indecomposable if and only if for every $\beta < \lambda$ and r-subgraphs $\langle G_\alpha: \alpha < \beta \rangle$ so that $G = \bigcup_{\alpha<\beta} G_\alpha$, there is some $\alpha < \beta$ such that $\Phi(G) = \Phi(G_\alpha)$.

A graph $G \in \mathscr{G}_\lambda$ with $\Phi(G) = \varnothing$ is clearly (r, λ)-indecomposable. The next claim proves the existence of other examples.

CLAIM 3.2. *Suppose that $\lambda > \aleph_1$ is regular and that $\Phi: \mathscr{G}_\lambda \longrightarrow [\mathbb{R}]^{\leqslant \lambda}$ is the homomorphism from Theorem 1.10. If $G \in \mathscr{G}_\lambda$ and $|\Phi(G)| \leqslant 1$ then G is (r, λ)-indecomposable.*

Proof. Suppose $\Phi(G) = \{a\}$. Fix a positive set $Y \subseteq \lambda$ and vertices $v_\delta \in G$ for $\delta \in Y$ so that $\phi(v, \delta) = a$. By completeness of I we can find a fixed $\alpha < \beta$ and a positive $Y' \subseteq Y$ such that $v_\delta \in G_\alpha$ for all $\delta \in Y'$. Since G_α is an r-subgraph, $r(v_\delta) \subseteq^* G_\alpha$ for every $\delta \in Y'$. On a measure 1 set equation (8) above holds and tells us that $\phi(v_\delta, \delta)$ can be computed in either G_α or in G. Intersecting Y' with this measure 1 set yields a positive measure $Y'' \subseteq Y$ so that $\phi(v_\delta, \delta) = \{a\}$ for $\delta \in Y''$. This shows that $\Phi(G) \subseteq \Phi(G_\alpha)$. The other inclusion holds because $G_\alpha \leqslant G$.

We ask if the converse to Claim 3.2 is true, namely whether $\Phi(G)$ can be represented as a union of fewer than λ r-subgraphs G_α, each with $\phi(G_\alpha)$ different from, hence properly included in, $\Phi(g)$, whenever $|\Phi(G)| > 1$. While this is not true for $\mathscr{G}$, forbidding an additional set of countable configurations gives a large subclass of $\mathscr{G}$ for which the converse to Claim 3.2 holds.

DEFINITION 3.3. (0) Let Γ^* be the set of all countable graphs G in which there is a vertex $v \in G$ with a ray $R \subseteq G[v]$ and a vertex $u \in R$ with a ray $S \subseteq G[u]$.

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(1) Let $\mathscr{G}^*$ be the subclass of $\mathscr{G}$ resulting from forbidding all graphs in Γ^*.

CLAIM 3.4. *If λ and Φ are as above, then $\Phi^{-}\mathscr{G}^*_\lambda$ is surjective. Hence the theorems and corollaries proved above for $\mathscr{G}$ hold also for $\mathscr{G}^*$.*

Proof. The graphs constructed in the proof of Theorem 1.10 to demonstrate surjectivity belong to $\mathscr{G}^*$ by Claim 1.12.

CLAIM 3.5. *Suppose that λ and Φ are as above. Suppose that $G \in \mathscr{G}^*$ and that $|\Phi(G)| > 1$. Then there are two r-subgraphs G_1, G_2 of G such that $\Phi(G) \not\subseteq \Phi(G_i)$ for $i = 1, 2$.*

Proof. let $a, b \in \Phi(G)$ be two distinct members of $\Phi(G)$. For every $\delta < \lambda$ let $\langle v_{\delta,i} : i < j(\delta, a)\rangle$ be an enumeration of all vertices $v \in G$ with $\phi(v, \delta) = a$. Similarly, let $\langle u_{\delta,i} : i < j(b, \delta)\rangle$ enumerate all vertices in G with $\phi(u, \delta) = b$. Let r be a function on G such that $r(v)$ is a ray in $G[v]$ if such a ray exists and is $\varnothing$ otherwise. Let $A := \{v_{\delta,i} : \delta < \lambda, i < j(\delta, \alpha)\}$ and let $G_1 = A \cup \{r(v) : v \in A\}$. Since $r(u) = \varnothing$ for every vertex $u \in r(v)$, for $v \in A$ (because Γ^* is omitted) we conclude that G_1 is an r-subgraph of G. Also, since $r(u) \neq \varnothing$ for every $u \in B := \{v_{\delta,i} : \delta < \lambda, i < j(\delta, b)\}$, it follows that $G_1 \cap B = \varnothing$.

Let $C := G \setminus G_1$ and let $G_2 = C \cup \{r(v) : v \in C\}$. Again, $r(u) \cap A = \varnothing$ for $u \in C$.

By restricting attention to a measure 1 subset of λ we may assume that $\phi(v, \delta)$ for a vertex $v \in G_1$ (or in G_2) remains unchanged when computed in G. Thus we see that $\Phi(G_1) = \{a\}$. On the other hand, $\phi(v, \delta) \neq a$ for all $u \in G_2$, because all vertices $v \in G$ for which $\phi(v, \delta) = a$ for some $\delta < \lambda$ are in A, and $A \cap G_2 = \varnothing$. This proves the claim.

THEOREM 3.6. *If $\lambda > \aleph_1$ is regular and $G \in \mathscr{G}^*_\lambda$ then G is (r, λ)-indecomposable if and only if $|\Phi(G)| \leqslant 1$.*

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