GENERALIZED MATRIX FUNCTION INEQUALITIES ON M-MATRICES

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ABSTRACT

For normalised generalized matrix functions f and g, we say that f dominates g if $f(A) \geqslant g(A)$ for every M-matrix A. We first demonstrate a finite set of test matrices for any such inequality. Then, using results from group representation theory, all comparisons among immanants in certain classes are determined. This work parallels ongoing research into gmf inequalities on positive semidefinite matrices, for which no finite set of test matrices is available. However, the inequalities for the two classes are quite different, and the test matrices permit more rapid progress in the M-matrix case. Just as in the positive semidefinite case, the gmf inequalities we prove may be used to verify previously unknown determinantal inequalities for M-matrices, such as the symmetrized Fischer inequalities recently proved in the positive semidefinite case.

1. *Introduction*

Let d_χ^G denote the *generalized matrix function* (gmf) on n-by-n matrices $A = (a_{ij})$, based upon the character χ on the subgroup G of the full symmetric group S_n, so that

$$d_\chi^G(A) = \sum_{\sigma \in G \subseteq S_n} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

Let e denote the identity element of G, and let

$$\bar{d}_\chi^G = \frac{1}{\chi(e)} d_\chi^G$$

be the *normalization* of d_χ^G.

The determinant ($G = S_n, \chi = \text{sign}$), the permanent ($G = S_n, \chi \equiv 1$), and the product of the main diagonal entries ($G = \{e\}, \chi \equiv 1$) are familiar examples, but there are many others. If $G = S_n$, and χ is irreducible, the gmf is called an *immanant* and $d_\chi^{S_n}$ is abbreviated to d_χ. There is a one-to-one correspondence between immanants and partitions of n, conveniently viewed as Young diagrams, and there is a variety of schemes for calculating immanants on given matrices [**6**].

Recently, there has been much interest in determining all inequalities among normalized immanants that hold at *every* positive semidefinite matrix, and, in particular, the conjecture that the permanent dominates all other normalized immanants on the positive semidefinites [**10**]. Considerable progress has been made in this area, though the complete picture has not yet emerged, nor even has the permanent-on-top conjecture been resolved (see, for example, [**3**, **4**, **11**, **12**, **13**, **14**]). Historically, there has been much interest in parallels between determinantal

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inequalities for M-matrices and for positive definite matrices, but, thus far, there is little literature on gmf inequalities for M-matrices. (See [**9**] for one example.) Our purpose here is to investigate the M-matrix case rather broadly, but focussing primarily upon immanantal inequalities. In particular, we show that for proposed inequalities between functions even more general than immanants or gmfs there is a finite set of test matrices. (The existence of a finite set of test matrices for, say, immanantal inequalities on positive semidefinite matrices is still an open question.) Using our test matrices we then investigate a number of possible inequalities. It appears that there are rather fewer inequalities that are valid for all M-matrices than are valid for all positive semidefinite matrices. In particular, the permanent beats no immanant other than the determinant on all M-matrices. However, there are some valid inequalities, primarily among 'single-hook' immanants and a few among immanants corresponding to diagrams with two rows or two columns. There are other valid inequalities, verified by computer, but they appear to be 'sporadic' and we discern no pattern.

Though identities useful for calculating immanants on particular matrices are employed, for example, [**11**, **8**], the methodology is necessarily rather different from the positive definite case. In particular, several properties of M-matrices are used, and the reader unfamiliar with these may wish to consult a reference, such as [**5**, Chapter 2]. To avoid confusion, we note that we adopt the inclusive definition of 'M-matrix', that is, A is an n-by-n M-matrix if and only if $A = tI - B$, in which B is entry-wise nonnegative and $t \geqslant \rho(B)$, the spectral radius, or Perron root, of B. Some authors distinguish between nonsingular ($t > \rho(B)$) and singular ($t \geqslant \rho(B)$) M-matrices. The set of all M-matrices is the closure of the set of nonsingular ones. The M-matrices are a highly important and heavily studied class (see, for example, [**2**] or [**5**, Chapter 2]), some of whose properties parallel those of the positive definite matrices.

Let P_k denote the k-by-k basic circulant permutation matrix. (The i, j entry of P_k is 1 when $j = i+1$, or $i = n$ and $j = 1$, and is 0 otherwise.)

Let $\mathscr{M}(n)$ denote the set of n-by-n matrices of the form $I - Q$, in which Q is a $(0, 1)$-matrix with at most one 1 in each row and zeros on the main diagonal. There are n^n such matrices.

Let $\mathscr{M}_0(n)$ denote the subset of $\mathscr{M}(n)$ consisting of the n-by-n matrices of the form

$$(I - P_{k_1}) \oplus (I - P_{k_2}) \oplus \cdots \oplus (I - P_{k_t}) \oplus I$$

in which $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_t \geqslant 2$, $k_1 + \cdots + k_t \leqslant n$ and the last I is of size $n - (k_1 + \cdots + k_t)$. There are as many such matrices as there are partitions of n.

1.1. *Test matrices*

The following results show that $\mathscr{M}_0(n)$ is a finite test set for immanantal inequalities on M-matrices.

THEOREM 1.1. *Let $M_n(\mathbb{R})$ be the set of real n-by-n matrices. Suppose that $f: M_n(\mathbb{R}) \to \mathbb{R}$. Assume that*

(i) *f is linear as a function of each row and*

(ii) *f satisfies $f(DA) = (d_1 d_2 \ldots d_n) f(A) = f(AD)$ whenever $D = \operatorname{diag}(d_1, \ldots, d_n) \in M_n(\mathbb{R})$.*

Then f is nonnegative on every M*-matrix if and only if f is nonnegative on each matrix in $\mathscr{M}(n)$.*

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THEOREM 1.2. *Assume that the function f in Theorem* 1.1 *satisfies conditions* (i) *and* (ii) *and also satisfies*

(iii) $f(P^TAP) = f(A)$ *whenever* $P \in M_n(\mathbb{R})$ *is a permutation matrix.*

Then f is nonnegative on every M-*matrix if and only if f is nonnegative on each matrix in* $\mathscr{M}_0(n)$.

Proof. Since $\mathscr{M}_0(n) \subseteq \mathscr{M}(n)$, and both are subsets of the set of M-matrices, one direction of each assertion is immediate.

For the other direction, let A be an arbitrary M-matrix. Since the diagonal entries of A must be nonnegative, A cannot leave the M-matrices via increase in any diagonal entry, and since (i) ensures the continuity of f, we may assume, without loss of generality, that all diagonal entries of A are positive. Then we may choose (see [**5**]) positive diagonal matrices $D = \operatorname{diag}(d_1, \ldots, d_n)$ and $E = \operatorname{diag}(e_1, \ldots, e_n)$ so that $\hat{A} = DAE$ has diagonal entries equal to 1 and is row diagonally dominant. By (ii),

$$f(\hat{A}) = \prod_{i=1}^{n} d_i e_i f(A),$$

and, as $\prod_{i=1}^{n} d_i e_i > 0$, we have $f(A) \geqslant 0$ if and only if $f(\hat{A}) \geqslant 0$. Call a matrix such as $\hat{A}$ a *normalized* M-*matrix*. We next describe a modification process, for normalized M-matrices $\hat{A} = (\hat{a}_{ij})$, that completes the verification of the first assertion. We refer to this as a 'concentration argument', as, typically, a limited amount of weight is concentrated in one position for worst case effect. Consider the first row of $\hat{A}$, and, using (i), write $f(\hat{A})$ as $c_1 + \hat{a}_{12}c_2 + \cdots + \hat{a}_{1n}c_n$, in which $c_1, \ldots, c_n$ are real. Let $c = \max\{c_1, \ldots, c_n\}$. If $c > 0$ and $c_i = c$ for $i \neq 1$, let $\hat{A}_1$ coincide with $\hat{A}$ in all entries except the off-diagonal entries of the first row; these are replaced by zeros except that $\hat{a}_{1i}$ is replaced by -1. Since $c_1 - c_i \leqslant c_1 + \hat{a}_{12}c_2 + \cdots + \hat{a}_{1n}c_n$, we have $f(\hat{A}_1) \leqslant f(\hat{A})$. Observe that $\hat{A}_1$ is row diagonally dominant, and hence an M-matrix. If $c \leqslant 0$, let $\hat{A}_1$ coincide with $\hat{A}$, except that the off-diagonal entries of the first row are all replaced by zeros. Since $c_1 \leqslant c_1 + \hat{a}_{12}c_2 + \cdots + \hat{a}_{1n}c_n$, in this event, we have again that $f(\hat{A}_1) \leqslant f(\hat{A})$. Also observe that $\hat{A}_1$ is row diagonally dominant. Now turn to row 2 and modify the second row of $\hat{A}_1$ similarly to get $\hat{A}_{12}$; it coincides with $\hat{A}_1$, except in row 2. The second row has zeros in the off-diagonal entries except for an occurrence of -1 opposite the entry whose coefficient is largest (unless this coefficient is nonpositive, in which case all off-diagonal entries are 0). Continue analogously to obtain $\hat{A}_{123}$ from $\hat{A}_{12}, \ldots$, and finally $A_0 = \hat{A}_{12\ldots n}$ from $\hat{A}_{12\ldots(n-1)}$. Notice that $A_0 \in \mathscr{M}(n)$ and $f(A_0) \leqslant f(\hat{A}_{12\ldots(n-1)}) \leqslant \cdots \leqslant f(\hat{A}_1) \leqslant f(\hat{A})$. Thus $f(A) \geqslant 0$ for all $A \in \mathscr{M}(n)$ implies that $f(A) \geqslant 0$ for all normalized M-matrices and, thus, for all M-matrices, completing the proof of the first assertion.

To verify the second assertion, we use (iii) to show that for each $A \in \mathscr{M}(n)$ there is a $B \in \mathscr{M}_0(n)$ such that $f(B) = f(A)$. Given $A \in \mathscr{M}(n)$, use (iii) to put A in Frobenius normal form $\tilde{A} = (\tilde{A}_{ij})$ with $i, j = 1, \ldots, t$, in which $\tilde{A}_{ij} = 0$ if $i > j$, $\tilde{A}_{ii} \in M_{n_i}(\mathbb{R})$ is irreducible (and thus is either (1) if $n_i = 1$, or is permutation similar to $I - P_{n_i}$ if $n_i > 1$), and $\tilde{A}_{ij}$ is possibly nonzero if $i < j$. Notice that (ii) implies that f is unchanged by diagonal similarity of its argument. Let $\tilde{A}_\varepsilon$ coincide with $\tilde{A}$, except that any -1 entries in the block $\tilde{A}_{ij}$ with $i < j$, are replaced by $-\varepsilon^{j-i}$. Since $\tilde{A}_\varepsilon = D^{-1}AD$ for $D = I_{n_1} \oplus \varepsilon I_{n_2} \oplus \varepsilon^2 I_{n_3} \cdots \oplus \varepsilon^{t-1} I_{n_t}$ when $\varepsilon \neq 0$, it follows that $f(\tilde{A}_\varepsilon) = f(\tilde{A})$. Again, using the continuity of f (which follows from (i)), we conclude that $f(\tilde{A}_0) = f(\tilde{A})$ also. Since $\tilde{A}_0$ is clearly permutation similar to an element of $\mathscr{M}_0(n)$, the proof is complete, where again we use (iii).

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It follows readily that $\mathscr{M}(n)$ is a (finite) set of test matrices for generalized matrix function inequalities on the M-matrices and that $\mathscr{M}_0(n)$ is a (finite) test set for immanantal inequalities on M-matrices.

The proofs of the following two corollaries are obvious.

COROLLARY 1.1. *Let $d_{\chi_1}^{G_1}$ and $d_{\chi_2}^{G_2}$ be generalized matrix functions on $M_n(\mathbb{R})$. If*

$$d_{\chi_1}^{G_1}(A) \leqslant d_{\chi_2}^{G_2}(A)$$

for all $A \in \mathscr{M}(n)$, then $d_{\chi_1}^{G_1}(A) \leqslant d_{\chi_2}^{G_2}(A)$ for all M*-matrices A.*

COROLLARY 1.2. *Let d_{χ_1} and d_{χ_2} be immanants on $M_n(\mathbb{R})$. If $d_{\chi_1}(A) \leqslant d_{\chi_2}(A)$ for all $A \in \mathscr{M}_0(n)$, then*

$$d_{\chi_1}(A) \leqslant d_{\chi_2}(A)$$

for all M*-matrices A.*

REMARK. Suppose that χ_1 and χ_2 are irreducible characters on S_n and that $\bar{d}_{\chi_1}(A) \leqslant \bar{d}_{\chi_2}(A)$ for all M-matrices A. Write $\chi_1 \preccurlyeq \chi_2$. If $\chi_1 \neq \chi_2$, it is easy to find a test matrix A_0 in $\mathscr{M}_0(n)$ such that $\bar{d}_{\chi_1}(A_0) \neq \bar{d}_{\chi_2}(A_0)$. Thus $\preccurlyeq$ is a partial ordering of the irreducible characters of S_n, and also a partial ordering of the corresponding partitions of n.

Corollary 1.1 and Corollary 1.2 enable us to verify inequalities on M-matrices for symmetrized Fischer products as defined in [**1**]. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition of n with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > 0$, and all λ_i integers. Let M be an n-by-n matrix. For α an increasing sequence of integers in $\{1, \ldots, n\}$ of length p, let $M[\alpha]$ be the principal submatrix of M contained in rows and columns α, and $M(\alpha)$ the complementary $(n-p)$-by-$(n-p)$ principal submatrix. Put

$$S_\lambda(M) = \sum_\alpha \prod_{i=1}^{r} \det M[\alpha_i],$$

where α runs over all ordered set partitions $\alpha = (\alpha_i, \ldots, \alpha_r)$ of $\{1, \ldots, n\}$ with $|\alpha_i| = \lambda_i$ for $i = 1, \ldots, r$. Then $S_\lambda(M)$ is a *generalised Fischer product* of M. In the case that $r = 2$, and $\lambda = (\lambda_1, \lambda_2)$, we have

$$S_\lambda(M) = \sum_{|\alpha| = \lambda_1} \det M[\alpha] \det M(\alpha).$$

Put $\bar{S}_\lambda(M) = S_\lambda(M)/S_\lambda(I_n)$. In [**1**] the following theorem was proved.

THEOREM 1.3. *Let λ and μ be partitions of n. Then $\bar{S}_\lambda(M) \geqslant \bar{S}_\mu(M)$ for all positive semidefinite matrices M if and only if λ majorizes μ, that is,*

$$\sum_{i=1}^{k} \lambda_i \geqslant \sum_{i=1}^{k} \mu_i$$

for all $k = 1, \ldots, r$.

In Section 5, we shall prove that Theorem 1.3 holds for M-matrices. The proof will be given only for partitions of length 2; the general result is obtained by use of the 'pinching' process of [**1**].

2. *Characters of S_n*

The notation $\lambda \mapsto n$ means that λ is a partition of n.

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There is an established method of indexing the irreducible characters of S_n by partitions of n, and we denote by χ_λ the irreducible character indexed by λ. Moreover, we adopt the following convention.

DEFINITION 2.1. If $\lambda = (\lambda_1, \ldots, \lambda_k)$, where $\lambda_1, \ldots, \lambda_k$ are integers and $\lambda_i < \lambda_{i+1}$ for some i or $\lambda_j < 0$ for some j, then $\chi_\lambda = 0$. Also, if $\lambda = (\ldots, 3^{r_1}, 2^{r_2}, 1^{r_1})$ and some r_i is negative, then $\chi_\lambda = 0$.

Now, the character values of S_n can be found inductively. To explain the method, we first introduce some notation.

DEFINITION 2.2. Assume that $0 \leqslant m \leqslant n$. If χ is a class function on S_n, then define the class function $\chi^{\hat{m}}$ on S_{n-m} as follows. Let h be the m-cycle $(n-m+1, \ldots, n) \in S_n$. For $g \in S_{n-m}$, let

$$\chi^{\hat{m}}(g) = \chi(gh).$$

The Murnaghan–Nakayama Rule [**6**, Chapter 21] gives a method for calculating $\chi^{\hat{m}}$ when χ is the irreducible character χ_λ of S_n. The statement involves the Young diagram $[\lambda]$ for the partition λ, which is defined to be

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : j \leqslant \lambda_i\}.$$

The Murnaghan–Nakayama Rule states that

$$\chi_\lambda^{\hat{m}} = \sum_\nu (-1)^i \chi_\nu,$$

where the sum is taken over all ν such that $[\lambda] \backslash [\nu]$ is a skew m-hook of leg length i. For the definition of a skew m-hook and its leg length, see [**6**, p. 73].

EXAMPLE 2.1. If $\lambda = (5, 4, 3^2)$ and $m = 3$, then

$$\chi_\lambda^{\hat{m}} = -\chi_{(3^4)} - \chi_{(5421)} + \chi_{(543)}.$$

The various skew 3-hooks which arise in this calculation are illustrated below. The deleted skew 3-hooks are indicated in upper case bold face.

$x\,x\,x\,\mathbf{X}\,\mathbf{X}$	$x\,x\,x\,x\,x$	$x\,x\,x\,x\,x$
$x\,x\,x\,\mathbf{X}$	$x\,x\,x\,x$	$x\,x\,x\,x$
$x\,x\,x$	$x\,x\,\mathbf{X}$	$x\,x\,x$
$x\,x\,x$	$x\,\mathbf{X}\,\mathbf{X}$	$\mathbf{X}\,\mathbf{X}\,\mathbf{X}$

When $m = 1$, we obtain the restriction $\chi \downarrow S_{n-1}$ of χ to S_{n-1}. This special case is called the Branching Theorem.

The Branching Theorem states that

$$\chi_\lambda^{\hat{1}} = \chi_\lambda \downarrow S_{n-1} = \sum_{i \geqslant 1} \chi_{(\lambda_1, \ldots, \lambda_i - 1, \ldots)}.$$

EXAMPLE 2.2. $\chi_{(4,2,2)} \downarrow S_7 = \chi_{(3,2,2)} + \chi_{(4,1,2)} + \chi_{(4,2,1)} = \chi_{(3,2,2)} + \chi_{(4,2,1)}$.

EXAMPLE 2.3. The Branching Theorem applied 3 times enables us to calculate the restriction of $\chi_{(5,4,3^2)}$ from S_{15} to S_{12}. We obtain

$$\chi_{(5,4,3^2)} \downarrow S_{12} = 2\chi_{(3^4)} + 2\chi_{(5,4,2,1)} + \chi_{(5,4,3)} + \text{other characters}.$$

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If χ is a class function on S_n, and $\alpha \mapsto n$, then let $\chi(\alpha)$ denote $\chi(g)$, where g is an element of cycle type α in S_n. We shall be particularly interested in the Murnaghan–Nakayama Rule when λ is a hook-partition, that is, $\lambda = (a+1, 1^b)$ for some $a \geqslant 0, b \geqslant 0$. We gather results we need in the next proposition. First we remark that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

is the binomial coefficient if $0 \leqslant k \leqslant n$, and zero otherwise.

Proposition 2.1. *Assume that* $n = a+b+1 = x+y$, *with* $a, b, x, y \geqslant 0$ *and* $x \geqslant y$. *Suppose that* $\lambda = (a+1, 1^b)$, *and* $0 \leqslant m \leqslant n$. *Then*

$$\chi_\lambda^{\hat{m}} = \chi_{(a+1-m, 1^b)} + (-1)^{m-1} \chi_{(a+1, 1^{b-m})}; \tag{2.1}$$

$$\chi_\lambda \downarrow S_{n-m} = \sum_{i=0}^{m} \binom{m}{i} \chi_{(a+1-i, 1^{b-m+i})}; \tag{2.2}$$

$$\chi_\lambda(1) = \binom{a+b}{b}. \tag{2.3}$$

$$\text{If } \alpha = (n), \text{ then } \chi_\lambda(\alpha) = (-1)^b. \tag{2.4}$$

$$\text{If } \alpha = (x, 1^{n-x}), \text{ then } \chi_\lambda(\alpha) = \binom{a+b-x}{a-x} + (-1)^{x-1} \binom{a+b-x}{b-x}. \tag{2.5}$$

$$\text{If } \alpha = (x, y), \text{ then } \chi_\lambda(\alpha) = \begin{cases} (-1)^b & \text{if } a \geqslant b, x, \\ (-1)^{b-1} & \text{if } b > a \text{ and } b \geqslant x, \\ 0 & \text{if } x > \max(a, b). \end{cases} \tag{2.6}$$

Proof. (1) This part is immediate from the Murnaghan–Nakayama Rule.

(2) Apply part (1), with $m = 1$, repeatedly.

(3) Take $m = n-1$ in (2), and note that $\chi_{(a+1-i, 1^{b-n+1+i})}$ is nonzero only if $i = a$.

(4) Take $m = n$ in the Murnaghan–Nakayama Rule, and note that the character χ_0 of S_0 has value 1.

(5) Combine part (1) (with $m = x$) with part (3).

(6) Combine part (1) (with $m = x$) with part (4).

There are useful ways of calculating certain quotients $\chi_\lambda(g)/\chi_\lambda(1)$, and these involve the functions $m_k(\lambda)$ which we define next.

Definition 2.3. Suppose that $\lambda \mapsto n$. We write $(i, j) \in [\lambda]$ if $j \leqslant \lambda_i$. For each integer k, define

$$m_k(\lambda) = \sum_{(i,j) \in [\lambda]} (j-i)^k.$$

Example 2.4. Suppose that $\lambda = (3, 2^2)$. The numbers $j-i$ for $(i, j) \in [\lambda]$ can be displayed as follows:

$$\begin{array}{ccc} 0 & 1 & 2 \\ -1 & 0 & \\ -2 & -1 & \end{array}$$

We have $m_1(\lambda) = 1+2-1-2-1 = -1$, $m_2(\lambda) = 1^2+2^2+1^2+2^2+1^2 = 11$.

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DEFINITION 2.4. Suppose that $\lambda \mapsto n$. Define the function ω_λ on S_n by

$$\omega_\lambda(g) = \frac{n!}{|C(g)|}\frac{\chi_\lambda(g)}{\chi_\lambda(1)} \quad \text{for } g \in S_n, \tag{2.7}$$

where $|C(g)|$ denotes the order of the centralizer of g. (In terms of representation theory, ω_λ is the central character of the group algebra of S_n corresponding to λ.)

PROPOSITION 2.2. *We have*
(1) $\omega_\lambda(g) = m_1(\lambda)$ *if* g *has cycle type* $(2, 1^{n-2})$,
(2) $\omega_\lambda(g) = m_2(\lambda) - n(n-1)/2$ *if* g *has cycle type* $(3, 1^{n-3})$.

Proof. This result was proved by Suzuki [**17**].

Pauncz and Katriel [**15**] give formulae for $\omega_\lambda(g)$ when g has cycle type $(k, 1^{n-k})$ and $2 \leqslant k \leqslant 14$. For example,

$$\omega_\lambda(g) = m_3(\lambda) - (2n-3)\, m_1(\lambda) \quad \text{if } k = 4;$$
$$\omega_\lambda(g) = m_4(\lambda) - (3n-10)\, m_2(\lambda) - 2(m_1(\lambda))^2 + (n-1)\,(5n-19)\, n/6 \quad \text{if } k = 5.$$

We shall use only the cases $k = 2$ and $k = 3$, as given in Proposition 2.2.

3. *Properties of binomial coefficients*

We shall make extensive use of inequalities involving binomial coefficients, and we gather the relevant results in this section.

PROPOSITION 3.1. *Assume that* m, r, *and* s *are integers.*

(1) *We have* $\binom{m}{r} \geqslant \binom{m}{s}$ *if* $r \geqslant s$ *and* $m \geqslant r+s$; *or* $r \leqslant s$ *and* $m \leqslant r+s$.

(2) *If we assume, also, that* $0 \leqslant r \leqslant m$, *then* $\binom{m}{r} > \binom{m}{s}$ *if* $r > s$ *and* $m > r+s$; *or* $r < s$ *and* $m < r+s$.

Proof. These results are entirely routine, and we leave them to the reader.

We shall usually apply the next proposition when every $q_i = 0$, and this special case is easy. Since the Proposition 3.2 is technical, the reader may prefer to postpone checking the details until we come to apply the result in Section 5.

PROPOSITION 3.2. *Assume that* m, n, u *are integers with* $n - m \geqslant 2u > 0$ *and* $m \geqslant 1$. *Let* p *and* q *be functions from* $\mathbb{Z}$ *to* $\mathbb{R}$ *and write* p_i *for* $p(i)$, *and so on. Assume that*

$$p_i \geqslant 0 \quad \text{if } i \leqslant \tfrac{1}{2}n, \quad q_i \geqslant 0 \quad \text{if } i \geqslant \tfrac{1}{2}n+1,$$
$$p_i + p_{n-i} + iq_{n-i} + (n-i)\, q_i = 0, \quad q_{i+1} + q_{n-i} = 0$$

for all $i \in \mathbb{Z}$. *Then* $\sum_{i=1}^{m-1} \binom{m}{i} p_{u+i} \geqslant 0$.

Proof. Let k be the largest integer such that $k < \frac{1}{2}n$. If $u+m-1 \leqslant \frac{1}{2}n$, then $p_{u+i} \geqslant 0$ for $1 \leqslant i \leqslant m-1$, and the required result is immediate. Assume, therefore, that $n-k \leqslant u+m-1$.

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If n is odd, then $n-k=k+1$, so $q_{k+1}=-q_{k+1}=0$. If n is even, then $n=2(k+1)$ and $2p_{n/2}+nq_{n/2}=0$, so $p_{k+1}=-(k+1)\,q_{k+1}$. Hence

$$\begin{aligned}\sum_{i=1}^{m-1}\binom{m}{i}p_{u+i} &= \sum_{i=1}^{k-u}\binom{m}{i}p_{u+i}-(k+1)\binom{m}{k+1-u}q_{k+1}+\sum_{i=n-u-k}^{m-1}\binom{m}{i}p_{u+i}\\ &= \sum_{i=1}^{k-u}\binom{m}{i}p_{u+i}+(k+1)\binom{m}{k+1-u}q_{n-k}\\ &\quad+\sum_{i=n-u-k}^{m-1}\binom{m}{i}(-p_{n-u-i}-(u+i)\,q_{n-u-i}-(n-u-i)\,q_{u+i})=x+y,\end{aligned}$$

where

$$x=\sum_{i=1}^{k-u}\binom{m}{i}p_{u+i}-\sum_{i=n-u-k}^{m-1}\binom{m}{i}p_{n-u-i},$$

$$y=(k+1)\binom{m}{k+1-u}q_{n-k}-\sum_{i=n-u-k}^{m-1}\binom{m}{i}((u+i)\,q_{n-u-i}+(n-u-i)\,q_{u+i}).$$

We shall show that $x\geqslant 0$ and $y\geqslant 0$, since this is sufficient to prove the proposition. First, consider x. By changing the running variables, we obtain

$$x=\sum_{i=u+1}^{k}\binom{m}{i-u}p_i-\sum_{i=n-u-m+1}^{k}\binom{m}{n-u-i}p_i.$$

Now, $m\leqslant n-2u$, so for $i\leqslant k$, Proposition 3.1 shows that

$$\binom{m}{i-u}\geqslant\binom{m}{n-u-i}.$$

Also, $p_i\geqslant 0$ for $i\leqslant k$. Hence, $x\geqslant 0$.

Next, consider y. By splitting the sum which appears in the definition of y into two sums, and changing the running variables, we obtain

$$\begin{aligned}y &= (k+1)\binom{m}{k+1-u}q_{n-k}+\sum_{i=n-k+1}^{u+m}(i-1)\binom{m}{i-1-u}q_i\\ &\quad-\sum_{i=n-k}^{u+m-1}(n-i)\binom{m}{i-u}q_i\\ &= \left\{(k+1)\binom{m}{k+1-u}-k\binom{m}{n-k-u}\right\}q_{n-k}+m(u+m-1)\,q_{u+m}\\ &\quad+\sum_{i=n-k+1}^{u+m-1}\left\{(i-1)\binom{m}{i-1-u}-(n-i)\binom{m}{i-u}\right\}q_i.\end{aligned}$$

We have $k+1-u\leqslant n-k-u$ and $m\leqslant n-2u+1$; also, for $i\geqslant n-k+1$, we have $2i-1-2u\geqslant m$ and $i-1\geqslant n-i$. Hence, by Proposition 3.1

$$(k+1)\binom{m}{k+1-u}-k\binom{m}{n-k-u}\geqslant 0,$$

$$(i-1)\binom{m}{i-1-u}-(n-i)\binom{m}{i-u}\geqslant 0.$$

Finally, $q_i\geqslant 0$ for $i\geqslant n-k$. Hence $y\geqslant 0$, as we wished to show.

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4. *Test matrices*

DEFINITION 4.1. If χ is a real valued function on S_n and $M = (m_{ij})$ is an n-by-n real matrix, then let

$$d_\chi(M) = \sum_{g \in S_n} \chi(g) \prod_{i=1}^{n} m_{i,g(i)}.$$

If $\chi(1) \neq 0$, then define the normalised function $\bar{d}_\chi(M)$ as in the Introduction. If $\chi = \chi_\lambda$, where λ is a partition of n, then the corresponding immanant is labelled d_λ or d_χ.

DEFINITION 4.2. Let M_1 be the 1×1 matrix (1), and for each integer $r \geqslant 2$, let $M_r = I_r - P_r$, where P_r is the basic r-by-r circulant matrix (see Section 1). In addition, if $\alpha = (\alpha_1, \alpha_2, \ldots) \mapsto n$, let $M_\alpha = M_{\alpha_1} \oplus M_{\alpha_2} \oplus \ldots$, so that M_α is an n-by-n matrix.

PROPOSITION 4.1. *Suppose that* $\alpha = (\ldots 3^{k_3}, 2^{k_2}, 1^{k_1}) \mapsto n$. *Then for all class functions* χ *on* S_n, *we have*

$$d_\chi(M_\alpha) = \sum_{\beta = (\ldots, 1^{r_1})} (-1)^{n-r_1} \chi(\beta) \prod_{i \geqslant 2} \binom{k_i}{r_i}.$$

Proof. Let $M_\alpha = (m_{ij})$. If g has cycle type β, where $\beta = (\ldots, 2^{r_2}, 1^{r_1}) \mapsto n$, and $\prod_{i=1}^{n} m_{i,g(i)} \neq 0$, then $r_i \leqslant k_i$ for all $i \geqslant 2$. Assume, therefore, that $r_i \leqslant k_i$ for all $i \geqslant 2$. Then there are $\prod_{i \geqslant 2} \binom{k_i}{r_i}$ elements g of cycle type β for which $\prod_{i=1}^{n} m_{i,g(i)} \neq 0$, and for these elements g,

$$\prod_{i=1}^{n} m_{i,g(i)} = (-1)^{2r_2 + 3r_3 + \cdots} = (-1)^{n-r_1}.$$

The desired expression for $d_\chi(M_\alpha)$ now follows.

The evaluation of $\bar{d}_\chi(M_\alpha)$ when α is a hook partition is particularly straightforward.

If $\alpha = (k, 1^{n-k})$, and χ is a class function on S_n with $\chi(1) \neq 0$, then

$$\bar{d}_\chi(M_\alpha) = 1 + (-1)^k \frac{\chi(\alpha)}{\chi(1)}. \tag{4.1}$$

Now, we are mainly interested in determining for which partitions λ and μ of n it is true that $\bar{d}_\lambda(M) \geqslant \bar{d}_\mu(M)$ for all n-by-n M-matrices M. From Theorem 1.2 we know that this holds if and only if $\bar{d}_\lambda(M_\alpha) \geqslant \bar{d}_\mu(M_\alpha)$ for all $\alpha \mapsto n$. We concentrate first on necessary conditions for the inequality $\bar{d}_\lambda(M) \geqslant \bar{d}_\mu(M)$.

THEOREM 4.1. *If* $\bar{d}_\lambda(M) \geqslant \bar{d}_\mu(M)$ *for all* n*-by-*n M*-matrices* M, *then* $m_1(\lambda) \geqslant m_1(\mu)$ *and* $m_2(\lambda) \leqslant m_2(\mu)$.

Proof. Suppose that $\alpha = (2, 1^{n-2})$, and let g be a transposition. Then

$$\bar{d}_\lambda(M_\alpha) = 1 + \frac{\chi_\lambda(g)}{\chi_\lambda(1)} = 1 + \frac{|C(g)|}{n!} \omega_\lambda(g) = 1 + \frac{|C(g)|}{n!} m_1(\lambda),$$

by (4.1), (2.7), and Proposition 2.2(1). If $\bar{d}_\lambda(M_\alpha) \geqslant \bar{d}_\mu(M_\alpha)$, then $m_1(\lambda) \geqslant m_1(\mu)$.

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If $\beta = (3, 1^{n-3})$, and h has cycle-type β, then

$$\bar{d}_\lambda(M_\beta) = 1 - \frac{\chi_\lambda(\beta)}{\chi_\lambda(1)},$$

by equation (4.1). This time, (2.7) and Proposition 2.2(2) imply that $m_2(\lambda) \leqslant m_2(\mu)$.

COROLLARY 4.1. *Let $\lambda, \mu \vdash n$. If $\bar{d}_\lambda(M) \geqslant \bar{d}_\mu(M)$ for all n-by-n* M*-matrices M, then μ does not majorize λ.*

Proof. From Theorem 4.1, we have $m_1(\lambda) \geqslant m_1(\mu)$. But if μ majorizes λ then it is easy to see that $m_1(\lambda) < m_1(\mu)$ (cf. [7, 1.4.10]). Hence μ does not majorize λ.

COROLLARY 4.2. *If $\lambda = (n-a, a)$ and $\mu = (n-b, b)$ are partitions of n and $\bar{d}_\lambda(M) \geqslant \bar{d}_\mu(M)$ for all n-by-n* M*-matrices M, then $\lambda = \mu$.*

Proof. From Theorem 4.1, we have $m_1(\lambda) \geqslant m_1(\mu)$, and $m_2(\lambda) \leqslant m_2(\mu)$. Since $m_1(\lambda) \geqslant m_1(\mu)$, it follows that $a \leqslant b$. Also, $m_2(\lambda) \leqslant m_2(\mu)$ implies that $a \geqslant b$. Hence, $a = b$ and $\lambda = \mu$.

COROLLARY 4.3 *Suppose that $\mu \vdash n$ and $\bar{d}_{(n)}(M) \geqslant \bar{d}_\mu(M)$ for all n-by-n* M*-matrices M. Then $\mu = (n)$ or $\mu = (1^n)$. (Recall: $\bar{d}_{(n)} = per$.)*

Proof. We have $m_2((n)) = \sum_{i=1}^{n-1} i^2 \leqslant m_2(\mu)$. Hence $\mu = (n)$ or $\mu = (1^n)$.

COROLLARY 4.4. *Suppose that $a+b = c+d = n-1$ and $\lambda = (c+1, 1^d)$ and $\mu = (a+1, 1^b)$ are distinct partitions of n. If $\bar{d}_\lambda(M) \geqslant \bar{d}_\mu(M)$ for all n-by-n* M*-matrices M, then $b \geqslant c > a$.*

Proof. Since $m_1(\lambda) \geqslant m_1(\mu)$, we have $c \geqslant a$ and $b \geqslant d$; indeed, these inequalities are strict, because $\lambda \neq \mu$. Also, $m_2(\lambda) \leqslant m_2(\mu)$, so $\max(c, d) \leqslant \max(a, b)$. Hence $\max(a, b) = b$, and $b \geqslant c$.

If μ has the form $(a+1, 1^b)$ then we call $\bar{d}_\mu$ a *normalized hook immanant*. We shall later classify all the inequalities between normalized hook immanants of M-matrices. The next proposition is one step on the way to this classification.

PROPOSITION 4.2 *Suppose that $a+b = c+d = n-1$ and $\lambda = (c+1, 1^d)$ and $\mu = (a+1, 1^b)$ are partitions of n. We have $\bar{d}_\lambda(M_{(n)}) \geqslant \bar{d}_\mu(M_{(n)})$ if and only if one of the following holds:*

(1) *a is even and c is odd,*
(2) *a and c are both even and $(b-c)(c-a) \geqslant 0$,*
(3) *a and c are both odd and $(b-c)(c-a) \leqslant 0$.*

Proof. Equation (4.1) and Proposition 2.1 give

$$\bar{d}_\lambda(M_{(n)}) = 1 + \frac{(-1)^{c+1}}{\binom{c+d}{c}} \quad \bar{d}_\mu(M_{(n)}) = 1 + \frac{(-1)^{a+1}}{\binom{a+b}{b}}.$$

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Hence $\bar{d}_\lambda(M_{(n)}) \geqslant \bar{d}_\mu(M_{(n)})$ if and only if

$$(-1)^{c+1}\binom{a+b}{b} \geqslant (-1)^{a+1}\binom{c+d}{c}.$$

The stated result now follows, by using Proposition 3.1 and examining the various possibilities.

Information about inequalities between immanants of M-matrices can frequently be obtained by considering the longest rim rook which can be removed from λ. We present some examples of this technique next. If $\lambda = (\lambda_1, \ldots, \lambda_k) \mapsto n$ with $\lambda_k > 0$, then let $h_{11}(\lambda) = \lambda_1 + k - 1$. Thus, $h_{11}(\lambda)$ is the 'hook length for the (1, 1) node of $[\lambda]$'; see [**6**, Chapter 18].

PROPOSITION 4.3. *Assume that* $\lambda, \mu \mapsto n$ *and* $\bar{d}_\lambda(M) \geqslant \bar{d}_\mu(M)$ *for all n-by-n* M-*matrices M*.

(1) *If* μ_1 *is even, then* $h_{11}(\lambda) \geqslant h_{11}(\mu)$.
(2) *If* λ_1 *is odd, then* $h_{11}(\lambda) \leqslant h_{11}(\mu)$.
(3) *If* $h_{11}(\lambda) = h_{11}(\mu)$, *and* μ_1 *is even, then* λ_1 *is even*.

Proof. Suppose that $\nu = (\nu_1, \nu_2, \ldots)$ and $\alpha = (k, 1^{n-k})$ are partitions of n. Then by (4.1) and the Murnaghan–Nakayama Rule,

$$\bar{d}_\nu(M_\alpha) = \begin{cases} 1 & \text{if } k > h_{11}(\nu), \\ 1 + (-1)^{\nu_1}\chi_{\nu'}(1)/\chi_\nu(1) & \text{if } k = h_{11}(\nu), \end{cases} \tag{4.2}$$

where $\nu' = (\nu_2 - 1, \nu_3 - 1, \ldots) \mapsto n - h_{11}(\nu)$.

To prove (1) and (3), assume that μ_1 is even. We take $\nu = \mu$ and $k = h_{11}(\mu)$ in (4.2), to obtain $\bar{d}_\lambda(M_\alpha) \geqslant \bar{d}_\mu(M_\alpha) > 1$. Now apply (4.2) with $\nu = \lambda$ and $k = h_{11}(\mu)$. Since $\bar{d}_\lambda(M_\alpha) > 1$, we cannot have $h_{11}(\lambda) < h_{11}(\mu)$; therefore, $h_{11}(\lambda) \geqslant h_{11}(\mu)$ and, if $h_{11}(\lambda) = h_{11}(\mu)$, then λ_1 is even.

To verify (2), assume that λ_1 is odd. We take $\nu = \lambda$ and $k = h_{11}(\lambda)$ in (4.2). Then $\bar{d}_\mu(M_\alpha) \leqslant \bar{d}_\lambda(M_\alpha) < 1$. Hence, $h_{11}(\mu) \geqslant h_{11}(\lambda)$.

COROLLARY 4.5. *Assume that* $\lambda = (2^r, 1^{n-2r})$ *and* $\mu = (2^s, 1^{n-2s})$ *are partitions of n with* $s \geqslant 1$. *Assume, also, that* $\bar{d}_\lambda(M) \geqslant \bar{d}_\mu(M)$ *for all n-by-n* M-*matrices M. Then* $\lambda = \mu$.

Proof. By Corollary 4.1, $r \geqslant s$, and by Proposition 4.3(1), $n - r + 1 \geqslant n - s + 1$. Therefore, $r = s$ and $\lambda = \mu$.

COROLLARY 4.6. *Assume that* $\mu = (2, 1^{n-2})$ *and* λ *are partitions of n, and* $\lambda \neq \mu$. *If* $\bar{d}_\lambda(M) \geqslant \bar{d}_\mu(M)$ *for all n-by-n* M-*matrices M, then n is odd and* $\lambda = (n-1, 1)$.

Proof. By Proposition 4.3, $h_{11}(\lambda) \geqslant h_{11}(\mu)$, so λ has the form $(c+1, 1^d)$ for some c and d with $c + d = n - 1$. Now Proposition 4.2 implies that c is odd and $c \geqslant n - 2$. Finally, Corollary 4.4 gives $n - 2 \geqslant c$; hence $c = n - 2$, n is odd and $\lambda = (n-1, 1)$.

In Corollary 4.2, we showed that normalized immanants for distinct 2-rowed partitions are incomparable, and in Corollary 4.5 we proved a similar result for 2-column partitions. We now show that it is rare that a 2-rowed partition can be compared with a 2-column partition.

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COROLLARY 4.7. *Assume that* $\lambda = (n-a, a)$ *and* $\mu = (2^b, 1^{n-2b})$ *are partitions of* n, *with* $b \geqslant 1$, *and that* $\bar{d}_\lambda(M) \geqslant \bar{d}_\mu(M)$ *for all* n*-by-*n M*-matrices* M. *Then* $\lambda = (n-1, 1)$ *and* $\mu = (2, 1^{n-2})$ *and* n *is odd.* (*In particular*, λ *and* μ *are dual partitions.*)

Proof. We have $m_2(\lambda) \leqslant m_2(\mu)$ by Theorem 4.1, so $a \geqslant b$. Also, $h_{11}(\lambda) \geqslant h_{11}(\mu)$ by 4.3(1), so $a \leqslant b$. Therefore, $a = b \geqslant 1$. If $a = b = 1$, then n is odd, by Corollary 4.6, so assume that $a = b \geqslant 2$. Let $\alpha = (n-a, 1^a)$ and $\beta = (n-a+1, 1^{a-1})$. Then it is easy to see, using equation (4.1) and the Murnaghan–Nakayama Rule, that

$$\bar{d}_\mu(M_\alpha) = 1 + \frac{\chi_{(2,1^{a-2})}(1)}{\chi_\mu(1)} \quad \text{and} \quad \bar{d}_\mu(M_\beta) = 1 + \frac{1}{\chi_\mu(1)}.$$

Thus, $\bar{d}_\mu(M_\alpha) > 1$ and $\bar{d}_\mu(M_\beta) > 1$. On the other hand, a similar calculation reveals that one of $\bar{d}_\lambda(M_\alpha)$ and $\bar{d}_\lambda(M_\beta)$ is greater than 1 and the other is less than 1. (In fact $\bar{d}_\lambda(M_\alpha) > 1$ if and only if $n-a$ is odd.) Hence, it is not true, in the case where $a \geqslant 2$, that $\bar{d}_\lambda(M) \geqslant \bar{d}_\mu(M)$ for all M-matrices M.

Now recall, for $\alpha \mapsto n$, that $M_\alpha = M_{\alpha_1} \oplus M_{\alpha_2} \oplus \cdots$. We next give a definition which, in the case where some part of α is equal to m, removes one of the m-by-m submatrices M_m from M_α.

DEFINITION 4.3. Assume that $\alpha = (\ldots 2^{k_2}, 1^{k_1}) \mapsto n$ and $k_m \geqslant 1$. Define the $(n-m)$-by-$(n-m)$ matrix $M_\alpha^{\hat{m}}$ by

$$M_\alpha^{\hat{m}} = M_\beta, \quad \text{where } \beta = (\ldots, m^{k_m - 1}, \ldots, 2^{k_2}, 1^{k_1}).$$

We shall make extensive use of the following theorem, which reduces the calculation of $d_\chi(M_\alpha)$ to a calculation involving a smaller matrix.

THEOREM 4.2. *Assume that* $\alpha \mapsto n$ *and* $m \geqslant 2$, *and that some part of* α *is equal to* m. *Suppose that* χ *is a class function on* S_n, *and let* ϕ *be the class function on* S_{n-m} *which is given by*

$$\phi = \chi \downarrow S_{n-m} + (-1)^m \chi^{\hat{m}}.$$

Then

$$d_\chi(M_\alpha) = d_\phi(M_\alpha^{\hat{m}}).$$

Proof. Assume that $\alpha = (\ldots 2^{k_2}, 1^{k_1})$. For each partition $\beta = (\ldots 2^{r_2}, 1^{r_1})$ of n, let $y_\beta = \prod_{i \geqslant 2, i \neq m} \binom{k_i}{r_i}$ Let

$$A = \{\beta = (\ldots 2^{r_2}, 1^{r_1}) \mapsto n : k_i \geqslant r_i \text{ for all } i \geqslant 2 \text{ and } k_m > r_m\}.$$

If $\beta = (\ldots 2^{r_2}, 1^{r_1}) \in A$, then note that $r_1 \geqslant m$; let $\bar{\beta} = (\ldots, 3^{r_3}, 2^{r_2}, 1^{r_1 - m}) \mapsto n-m$. Now

$$\sum_{\beta \mapsto n} (-1)^{n-r_1} \chi(\beta) \binom{k_m - 1}{r_m} y_\beta = \sum_{\beta \in A} (-1)^{n-r_1} \chi(\beta) \binom{k_m - 1}{r_m} y_\beta$$

$$= \sum_{\beta \in A} (-1)^{n-r_1} (\chi \downarrow S_{n-m})(\bar{\beta}) \binom{k_m - 1}{r_m} y_\beta = d_{\chi \downarrow S_{n-m}}(M_\alpha^{\hat{m}}),$$

by Proposition 4.1.

Next let

$$B = \{\beta = (\ldots 2^{r_2}, 1^{r_1}) \mapsto n : k_i \geqslant r_i \text{ for all } i \geqslant 2 \text{ and } r_m \geqslant 1\}.$$

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If $\beta = (\dots, 2^{r_2}, 1^{r_1}) \in B$ then let $\tilde{\beta} = (\dots, m^{r_m - 1}, \dots, 1^{r_1}) \mapsto n-m$. Now,

$$\sum_{\beta \mapsto n} (-1)^{n-r_1} \chi(\beta) \binom{k_m - 1}{r_m - 1} y_\beta = \sum_{\beta \in B} (-1)^{n-r_1} \chi(\beta) \binom{k_m - 1}{r_m - 1} y_\beta$$
$$= \sum_{\beta \in B} (-1)^{n-r_1} \chi^{\hat{m}}(\tilde{\beta}) \binom{k_m - 1}{r_m - 1} y_\beta.$$

Now, by Proposition 4.1, $d_{\chi^{\hat{m}}}(M_\alpha^{\hat{m}}) = \sum_{\tilde{\beta}} (-1)^{n-m-r_1} \chi(\tilde{\beta}) \binom{k_m - 1}{r_m - 1} y_\beta$, so

$$\sum_{\beta \mapsto n} (-1)^{n-r_1} \chi(\beta) \binom{k_m - 1}{r_m - 1} y_\beta = (-1)^m d_{\chi^{\hat{m}}}(M_\alpha^{\hat{m}}).$$

Finally, using Proposition 4.1 once more, we obtain

$$d_\chi(M_\alpha) = \sum_{\beta \mapsto n} (-1)^{n-r_1} \chi(\beta) \binom{k_m}{r_m} y_\beta = \sum_{\beta \mapsto n} (-1)^{n-r_1} \chi(\beta) \left(\binom{k_m - 1}{r_m} + \binom{k_m - 1}{r_m - 1} \right) y_\beta$$
$$= (d_{\chi \downarrow S_{n-m}} + (-1)^m d_{\chi^{\hat{m}}})(M_\alpha^{\hat{m}}),$$

the last equality following from the two results which appear earlier in the proof of Theorem 4.2. This completes the proof.

COROLLARY 4.8. *If $\lambda \mapsto n$, then $\bar{d}_\lambda(M) \geqslant \det(M) \geqslant 0$ for all $n \times n$ M-matrices M.*

Proof. If $\alpha = (1^n)$, then $\bar{d}_\lambda(M_\alpha) = \det(M_\alpha) = 1$. Assume, therefore, that $\alpha \mapsto n$ and $\alpha \neq (1^n)$. Let $m = \alpha_1$. Then $m \geqslant 2$. When we write $\chi_\lambda \downarrow S_{n-m}$ as a linear combination of the irreducible characters $\chi_\mu (\mu \mapsto n-m)$ of S_{n-m}, all partitions μ of $n-m$ such that $[\mu]$ is a subset of $[\lambda]$ occur with coefficient at least 1, by the Branching Theorem applied m times. In particular, all those ν such that $[\lambda] \backslash [\nu]$ is a skew m-hook have coefficient at least 1 in $\chi_\lambda \downarrow S_{n-m}$. Therefore, the Murnaghan–Nakayama Rule shows that the function ϕ, given by

$$\phi = \chi_\lambda \downarrow S_{n-m} + (-1)^n \chi_\lambda^{\hat{m}},$$

is a character of S_{n-m} (or is zero). Compare Examples 2.1 and 2.3. Hence, by induction, $d_\phi(M_\alpha^{\hat{m}}) \geqslant 0$. Theorem 4.2 now shows that $d_\lambda(M_\alpha) \geqslant 0$. Since $\det(M_\alpha) = 0$, we have $\bar{d}_\lambda(M_\alpha) \geqslant \det(M_\alpha)$, and the result follows.

5. *A Fischer-type inequality*

In this section, we verify inequalities between special class functions on M-matrices. Let u be an integer between 0 and n, let $v = n-u$, and let $Q_{u,n}$ be the $\binom{n+1}{u}$ increasing integer sequences of length u chosen from $\{0, \dots, n\}$. For $\alpha \in Q_{u,n}$, and M an $n \times n$ matrix, let $M[\alpha]$ be the $n \times n$ principal submatrix of M determined by rows and columns α (see Section 1). Let $M(\alpha)$ be the complementary principal submatrix. Put

$$\Delta_{u,v} = \sum_{\alpha \in Q_{u,n}} \det M[\alpha] \det M(\alpha).$$

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Now if $1 \leqslant u \leqslant \frac{1}{2}n$, then $\Delta_{u,v} - \Delta_{u-1,v+1}$ is the (unnormalized) immanant corresponding to the partition $(2^u, 1^{v-u})$ [**11**] ($\Delta_{0,n} = \det$). Thus, if M is positive semidefinite, since immanants of positive semidefinite matrices are nonnegative [**16**], we have

$$\Delta_{0,n} \leqslant \Delta_{1,n-1} \leqslant \cdots \leqslant \Delta_{k,n-k},$$

where $k = [\frac{1}{2}n]$. Notice that the correspondence here is between unnormalized $\Delta_{i,n-i}$. It is more natural to normalize them first so that all take the value 1 on the identity matrix. In fact, for entirely different reasons, the normalized $\bar{\Delta}_{i,n-i}$ still line up. Let

$$\bar{\Delta}_{u,v} = \frac{\Delta_{u,v}}{\binom{n}{u}},$$

so that $\bar{\Delta}_{u,v}(I_n) = 1$. Then if M is positive semidefinite, we have [**1**, p. 71] $\bar{\Delta}_{0,n} \leqslant \bar{\Delta}_{1,n-1} \leqslant \cdots \leqslant \bar{\Delta}_{k,n-k}$, where $k = [\frac{1}{2}n]$. The purpose of this section is to prove analogous inequalities for M-matrices. The exact statement is Theorem 5.1.

DEFINITION 5.1. Write $n = u+v$, where u, v are arbitrary integers. If $u, v \geqslant 0$, then let $\chi_{u,v}$ be the character of S_n given by inducing the sign character of $S_u \times S_v$. If u or $v < 0$, then let $\chi_{u,v} = 0$.

We have (see [**6**, 14.4, and 6.6]):

$$\chi_{u,v} = \chi_{v,u}, \quad \chi_{u,v}(1) = \binom{n}{u}, \quad \text{and} \quad \chi_{u,v} = \sum_{i=0}^{\min(u,v)} \chi_{(2^i, 1^{n-2i})}.$$

Moreover, for $0 \leqslant m \leqslant n$ we have

$$\chi_{u,v}^{\hat{m}} = (-1)^{m-1}(\chi_{u,v-m} + \chi_{u-m,v}), \tag{5.1}$$

as can be proved by the method of [**6**, 21.12]. In particular, $\chi_{u,v}\downarrow S_{n-1} = \chi_{u,v-1} + \chi_{u-1,v}$. By iterating this result, we obtain

$$\chi_{u,v}\downarrow S_{n-m} = \sum_{i=0}^{m} \binom{m}{i} \chi_{u-i,v-m+i}. \tag{5.2}$$

If M is an $n \times n$ M-matrix, then let $d_{u,v}(M) = d_{\chi_{u,v}}(M)$ and $\bar{d}_{u,v}(M) = d_{u,v}(M)/\chi_{u,v}(I)$ if $u, v \geqslant 0$. The theorem which we prove in this section shows that

$$\bar{d}_{n,0}(M) \leqslant \bar{d}_{n-1,1}(M) \leqslant \cdots \leqslant \bar{d}_{n-k,k}(M)$$

for all $n \times n$ M-matrices M, where k is the largest integer less than or equal to $n/2$.

THEOREM 5.1. *Assume that s, t, u, v are integers with $s+t = u+v = n$ and $\min(u,v) \leqslant \min(s,t)$. Suppose that M is an $n \times n$ M-matrix. Then*

(1) $d_{u,v}(M) \leqslant d_{s,t}(M)$,

(2) *if, in addition, $u, v \geqslant 0$, then $\bar{d}_{u,v}(M) \leqslant \bar{d}_{s,t}(M)$.*

Proof. If $\min(u,v) < 0$, then $d_{u,v}(M) = 0$, and part (1) holds. If $\min(u,v) \geqslant 0$, then $0 \leqslant \binom{n}{u} \leqslant \binom{n}{s}$ by Proposition 3.1, so that part (1) would follow from part (2). Hence, it is sufficient to prove the following.

$$\text{If } s \geqslant t \text{ and } \alpha \mapsto n, \text{ then } (s+1)\, d_{s+1,t-1}(M_\alpha) \leqslant t d_{s,t}(M_\alpha) \tag{5.3}$$

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(In other words, $\bar{d}_{s+1,t-1}(M_\alpha) \leqslant \bar{d}_{s,t}(M_\alpha)$.)

In the case where $\alpha = (1^n)$, the inequality (5.3) reduces to equality. Assume, therefore, that $s \geqslant t$ and $\alpha = (\alpha_1, \alpha_2, \ldots)$ where $\alpha_1 \geqslant 2$. Let $m = \alpha_1$, and for all integers i, let

$$p_i = i d_{n-m-i,i}(M_\alpha^{\hat{m}}) - (n-m-i+1)\, d_{n-m-i+1,i-1}(M_\alpha^{\hat{m}}).$$

Both parts of Theorem 5.1 are true for 0×0 (and for 1×1) M-matrices. Therefore, we may assume, by induction, that

$$d_{s-m+1,t-1}(M_\alpha^{\hat{m}}) - d_{s,t-m}(M_\alpha^{\hat{m}}) \geqslant 0, \tag{5.4}$$

and that $p_i \geqslant 0$ for all i with $i \leqslant \frac{1}{2}(n-m+1)$.

Note that $p_i + p_{n-m-i+1} = 0$ (since $\chi_{u,v} = \chi_{v,u}$), and $n-2m+1 \geqslant 2(t-m)$. Hence, by applying Proposition 3.2 (with each $q_i = 0$), we have $\sum_{i=1}^{m-1} \binom{m}{i} p_{t-m+i} \geqslant 0$. That is,

$$\sum_{i=1}^{m-1} \binom{m}{i} \{(t-m+i)\, d_{n-t-i,t-m+i}(M_\alpha^{\hat{m}}) - (n-t-i+1)\, d_{n-t-i+1,t-m+i-1}(M_\alpha^{\hat{m}})\} \geqslant 0.$$

By substituting s for $n-t$, we may rewrite this inequality as $x_1 + x_2 + x_3 + x_4 \geqslant 0$, where

$$x_1 = t \sum_{i=1}^{m-1} \binom{m}{i} d_{s-i,t-m+i}(M_\alpha^{\hat{m}}),$$

$$x_2 = -\sum_{i=1}^{m-1} (m-i) \binom{m}{i} d_{s-i,t-m+i}(M_\alpha^{\hat{m}}),$$

$$x_3 = -(s+1) \sum_{i=1}^{m-1} \binom{m}{i} d_{s-i+1,t-m+i-1}(M_\alpha^{\hat{m}}),$$

$$x_4 = \sum_{i=1}^{m-1} i \binom{m}{i} d_{s-i+1,t-m+i-1}(M_\alpha^{\hat{m}}).$$

Now,

$$d_{s,t}(M_\alpha) = \sum_{i=0}^{m} \binom{m}{i} d_{s-i,t-m+i}(M_\alpha^{\hat{m}}) - d_{s,t-m}(M_\alpha^{\hat{m}}) - d_{s-m,t}(M_\alpha^{\hat{m}}),$$

by Theorem 4.2, (5.2), and (5.1). Hence, $x_1 = t d_{s,t}(M_\alpha)$. Similarly,

$$x_3 = -(s+1)\, d_{s+1,t-1}(M_\alpha).$$

Next,

$$\begin{aligned} x_4 + x_2 &= \sum_{i=0}^{m-2} (i+1) \binom{m}{i+1} d_{s-i,t-m+i}(M_\alpha^{\hat{m}}) - \sum_{i=1}^{m-1} (m-i) \binom{m}{i} d_{s-i,t-m+i}(M_\alpha^{\hat{m}}) \\ &= m d_{s,t-m}(M_\alpha^{\hat{m}}) - m d_{s-m+1,t-1}(M_\alpha^{\hat{m}}), \end{aligned}$$

since $(i+1)\binom{m}{i+1} = (m-i)\binom{m}{i}$.

Finally, using (5.4), we obtain inequality (5.3), as follows:

$$\begin{aligned} t d_{s,t}(M_\alpha) - (s+1)\, d_{s+1,t-1}(M_\alpha) &= x_1 + x_3 \\ &\geqslant -x_2 - x_4 = m(d_{s-m+1,t-1}(M_\alpha^{\hat{m}}) - d_{s,t-m}(M_\alpha^{\hat{m}})) \geqslant 0. \end{aligned}$$

6. *The inequalities between normalized hook immanants*

In this section, we determine precisely which inequalities $\bar{d}_\lambda(M) \geqslant \bar{d}_\mu(M)$ hold for all M-matrices M, in the case where λ and μ are hook partitions.

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LEMMA 6.1. *Assume that* n, a, b, x, y *are integers, with* $n = a+b+1 = x+y$, $b > a \geqslant 0$ *and* $x \geqslant y > 0$. *Then*

$$d_{(b+1,1^a)}(M_{(x,y)}) \geqslant d_{(a+1,1^b)}(M_{(x,y)}).$$

Proof. If $y = 1$, then by Proposition 2.1, parts (2.3) and (2.5), we have

$$d_{(b+1,1^a)}(M_{(x,y)}) = \binom{a+b}{a} + (-1)^{n-1}\binom{0}{a} \geqslant \binom{a+b}{a} - \binom{0}{a} = d_{(a+1,1^b)}(M_{(x,y)}).$$

Assume, therefore, that $y \geqslant 2$. Let $\varepsilon = 0$ if $x > b$ and $\varepsilon = 1$ if $b \geqslant x$. Using Proposition 2.1, parts (2.3), (2.5), and (2.6), we obtain

$$\begin{aligned} d_{(a+1,1^b)}(M_{(x,y)}) = &\binom{a+b}{b} + (-1)^x\binom{a+b-x}{b} - \binom{a+b-x}{a} \\ &+ (-1)^y\binom{a+b-y}{b} - \binom{a+b-y}{a} + (-1)^{b-1+x+y}\varepsilon. \end{aligned}$$

Now, $(-1)^{b-1+x+y} = -(-1)^{b+n}$. The corresponding term for $d_{(b+1,1^a)}(M_{(x,y)})$ is $(-1)^{a+x+y} = -(-1)^b$. We find that $d_{(b+1,1^a)}(M_{(x,y)}) - d_{(a+1,1^b)}(M_{(x,y)}) = x_1 + x_2 + x_3$, where

$$\begin{aligned} x_1 &= ((-1)^x + 1)\left\{\binom{a+b-x}{a} - \binom{a+b-x}{b}\right\}, \\ x_2 &= ((-1)^y + 1)\left\{\binom{a+b-y}{a} - \binom{a+b-y}{b}\right\}, \\ x_3 &= (-1)^b((-1)^n - 1)\varepsilon. \end{aligned}$$

Using Proposition 3.1 (1), we see that $x_1 \geqslant 0$ and $x_2 \geqslant 0$. Hence $x_1 + x_2 + x_3 \geqslant 0$ if $\varepsilon = 0$. Assume, therefore, that $\varepsilon = 1$; then $b \geqslant x \geqslant y > 0$. Hence, by Proposition 3.1 (2),

$$\binom{a+b-x}{a} > \binom{a+b-x}{b} \quad \text{and} \quad \binom{a+b-y}{a} > \binom{a+b-y}{b}.$$

Since $x_3 \geqslant -2$, we deduce that $x_1 + x_2 + x_3 \geqslant 0$ unless both x and y are odd. But if both x and y are odd, then $n = x+y$ is even, and $x_3 = 0$. Thus, in every case, $x_1 + x_2 + x_3 \geqslant 0$, which proves the lemma.

LEMMA 6.2. *Assume that* n, a, b, x, y *are integers, with* $n = a+b+1 = x+y$, $b-1 \geqslant a \geqslant 0$ *and* $x \geqslant y \geqslant 0$. *Then*

$$\bar{d}_{(a+2,1^{b-1})}(M_{(x,y)}) \geqslant \bar{d}_{(a+1,1^b)}(M_{(x,y)}).$$

Proof. Let g be an element of S_n of cycle type $(x, 1^y)$. Then by Proposition 2.1 (2.5),

$$\begin{aligned} (a+1)\chi_{(a+2,1^{b-1})}(g) - b\chi_{(a+1,1^b)}(g) = &(a+1)\left\{\binom{a+b-x}{b-1} + (-1)^{x-1}\binom{a+b-x}{a+1}\right\} \\ &- b\left\{\binom{a+b-x}{b} + (-1)^{x-1}\binom{a+b-x}{a}\right\}. \end{aligned}$$

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Now,

$$b\binom{a+b-x}{b} = (a+1-x)\binom{a+b-x}{b-1}$$

and

$$(a+1)\binom{a+b-x}{a+1} = (b-x)\binom{a+b-x}{a}.$$

Hence,

$$(a+1)\chi_{(a+2,1^{b-1})}(g) - b\chi_{(a+1,1^b)}(g) = \left\{\binom{a+b-x}{b-1} + (-1)^x\binom{a+b-x}{a}\right\}x.$$

The inequality we wish to prove is $(a+1)\,d_{(a+2,1^{b-1})}(M_{(x,y)}) \geqslant b d_{(a+1,1^b)}(M_{(x,y)})$. If $b-1=a$, then the required inequality is given by Lemma 6.1. Assume, therefore, that $b-1>a$. We find (using Proposition 2.1 (2.6)), that

$$(a+1)\,d_{(a+2,1^{b-1})}(M_{(x,y)}) - b d_{(a+1,1^b)}(M_{(x,y)}) = y_1+y_2+y_3,$$

where

$$y_1 = \left\{\binom{a+b-x}{a} + (-1)^x\binom{a+b-x}{b-1}\right\}x, \quad y_2 = \left\{\binom{a+b-y}{a} + (-1)^y\binom{a+b-y}{b-1}\right\}y,$$

$$y_3 = \pm(a+1)\,\varepsilon_1 \pm b\varepsilon_2,$$

and $\varepsilon_1 = 0$ or 1, $\varepsilon_2 = 0$ or 1, and $\varepsilon_1 = \varepsilon_2 = 0$ if $x > b$ or $y = 1$.

Using Proposition 3.1 (1), we see that $y_1, y_2 \geqslant 0$. Hence $y_1+y_2+y_3 \geqslant 0$ if $y_3 = 0$. Assume, therefore that $b \geqslant x$ and $y > 1$. By Proposition 3.1 (2), $y_1 \geqslant x$, and $y_2 \geqslant y$. Therefore, $y_1+y_2+y_3 \geqslant x+y-(a+1+b) = 0$. The lemma now follows.

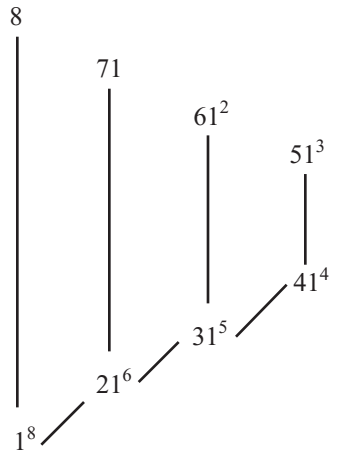


FIG. 1.

Figure 1 illustrates the next lemma by indicating the inequalities between normalized hook immanants for $n = 8$, evaluated at M_α for $\alpha \neq (8)$.

Lemma 6.3. *Assume that n, a, b are integers with $a+b = n-1 \geqslant 0$ and $b \geqslant a+1$. Suppose that $\alpha \mapsto n$ and $\alpha \neq (n)$. Then*

(1) $(a+1)\,d_{(a+2,1^{b-1})}(M_\alpha) \geqslant b d_{(a+1,1^b)}(M_\alpha)$,

(2) $d_{(b+1,1^a)}(M_\alpha) \geqslant d_{(a+1,1^b)}(M_\alpha)$.

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Proof. If $\alpha = (1^n)$, then both the required inequalities are immediate. If $\alpha = (x, y)$ with $x \geqslant y > 0$, then Lemmas 6.1 and 6.2 give the inequalities. Assume, therefore, that $\alpha = (\alpha_1, \alpha_2, \ldots) \mapsto n$, where $\alpha_1 \geqslant 2$ and $\alpha_3 \geqslant 1$. Let $m = \alpha_1$.

Now, $M_\alpha^{\hat{m}} = M_{(\alpha_2, \alpha_3, \ldots)}$ and $\alpha_3 \geqslant 1$, so we may assume, by induction, that the results corresponding to the ones we want to prove hold for the matrix $M_\alpha^{\hat{m}}$.

For all integers i, let

$$r_i = id_{(i+1,\,1^{n-m-i-1})}(M_\alpha^{\hat{m}}) - (n-m-i)\, d_{(i,\,1^{n-m-i})}(M_\alpha^{\hat{m}}),$$
$$s_i = d_{(i,\,1^{n-m-i})}(M_\alpha^{\hat{m}}) - d_{(n-m-i+1,\,1^{i-1})}(M_\alpha^{\hat{m}}).$$

Our induction hypothesis shows that $r_i \geqslant 0$ if $i \leqslant \frac{1}{2}(n-m)$ and $s_i \geqslant 0$ if $i \geqslant \frac{1}{2}(n-m)+1$. Note, also, that

$$r_i + r_{n-m-i} + is_{n-m-i} + (n-m-i)\, s_i = 0, \quad \text{and} \quad s_{i+1} + s_{n-m-i} = 0$$

for all integers i. Apply Proposition 3.2 with $p_i = r_i$, $q_i = s_i$ and $u = a+1-m$. We obtain

$$\begin{aligned}
0 &\leqslant \sum_{i=1}^{m-1} \binom{m}{i} r_{a+1-m+i} \\
&= \sum_{i=1}^{m-1} \binom{m}{i} (a+1-m+i)\, d_{(a+2-m+i,\,1^{b-1-i})}(M_\alpha^{\hat{m}}) - \sum_{i=1}^{m-1} \binom{m}{i} (b-i)\, d_{(a+1-m+i,\,1^{b-i})}(M_\alpha^{\hat{m}}) \\
&= \sum_{i=2}^{m} \binom{m}{i-1} (a-m+i)\, d_{(a+1-m+i,\,1^{b-i})}(M_\alpha^{\hat{m}}) - \sum_{i=1}^{m-1} \binom{m}{i} (b-i)\, d_{(a+1-m+i,\,1^{b-i})}(M_\alpha^{\hat{m}}) \\
&= \sum_{i=2}^{m-1} \left(\binom{m}{i-1} (a+1) - \binom{m}{i} b \right) d_{(a+1-m+i,\,1^{b-i})}(M_\alpha^{\hat{m}}) \\
&\quad + (ma)\, d_{(a+1,\,1^{b-m})}(M_\alpha^{\hat{m}}) - m(b-1)\, d_{(a+2-m,\,1^{b-1})}(M_\alpha^{\hat{m}})
\end{aligned}$$

since $(-m+i-1)\binom{m}{i-1} + i\binom{m}{i} = 0$. Therefore,

$$\begin{aligned}
&\sum_{i=1}^{m-1} \binom{m}{i} (a+1)\, d_{(a+2-m+i,\,1^{b-1-i})}(M_\alpha^{\hat{m}}) - \sum_{i=1}^{m-1} \binom{m}{i} b d_{(a+1-m+i,\,1^{b-i})}(M_\alpha^{\hat{m}}) \\
&\quad - m d_{(a+1,\,1^{b-m})}(M_\alpha^{\hat{m}}) + m d_{(a+2-m,\,1^{b-1})}(M_\alpha^{\hat{m}}) \geqslant 0. \qquad (6.1)
\end{aligned}$$

Since $r_{a+1-m} \geqslant 0$, it follows that

$$((-1)^m + 1)\{(a+1)\, d_{(a+2-m,\,1^{b-1})}(M_\alpha^{\hat{m}}) - b d_{(a+1-m,\,1^b)}(M_\alpha^{\hat{m}})\} \geqslant 0. \qquad (6.2)$$

Also, for $i \leqslant (n-m)/2$, we have $r_i \geqslant 0$, and so

$$d_{(i+1,\,1^{n-m-i-1})}(M_\alpha^{\hat{m}}) - d_{(i,\,1^{n-m-i})}(M_\alpha^{\hat{m}}) \geqslant 0.$$

These inequalities, together with the inequalities $s_i \geqslant 0$ for $i \geqslant (n-m)/2+1$ give us the following.

If t, u, v, w are integers with $t+u = v+w = n-m-1$ and $u \geqslant v \geqslant t$, then

$$d_{(v+1,\,1^w)}(M_\alpha^{\hat{m}}) \geqslant d_{(t+1,\,1^u)}(M_\alpha^{\hat{m}}). \qquad (6.3)$$

Therefore,

$$m d_{(a+1,\,1^{b-m})}(M_\alpha^{\hat{m}}) - m d_{(a+2-m,\,1^{b-1})}(M_\alpha^{\hat{m}}) \geqslant 0. \qquad (6.4)$$

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Add the inequalities (6.1), (6.2), and (6.4) to obtain

$$(a+1)\left\{\sum_{i=1}^{m-1}\binom{m}{i}d_{(a+2-m+i,1^{b-1-i})}(M_\alpha^{\hat{m}})+((-1)^m+1)\,d_{(a+2-m,1^{b-1})}(M_\alpha^{\hat{m}})\right\}$$
$$-b\left\{\sum_{i=1}^{m-1}\binom{m}{i}d_{(a+1-m+i,1^{b-i})}(M_\alpha^{\hat{m}})+((-1)^m+1)\,d_{(a+1-m,1^b)}(M_\alpha^{\hat{m}})\right\}\geqslant 0.$$

By Theorem 4.2 and Proposition 2.1 (2.1), (2.2), we obtain

$$(a+1)\,d_{(a+2,1^{b-1})}(M_\alpha)-bd_{(a+1,1^b)}(M_\alpha)\geqslant 0.$$

This completes the proof of part (1) of the lemma.

Next, apply Proposition 3.2 with

$$p_i=d_{(n-m-i,1^i)}(M_\alpha^{\hat{m}})-d_{(i+1,1^{n-m-i-1})}(M_\alpha^{\hat{m}}),\quad q_i=0,\quad u=a-m.$$

(Note that $p_i+p_{n-m-1-i}=0$ and $p_i\geqslant 0$ if $i\leqslant\frac{1}{2}(n-m-1)$). We obtain

$$0\leqslant\sum_{i=1}^{m-1}\binom{m}{i}p_{a-m+i}$$
$$=\sum_{i=1}^{m-1}\binom{m}{i}(d_{(b+1-i,1^{a-m+i})}(M_\alpha^{\hat{m}})-d_{(a+1-m+i,1^{b-i})}(M_\alpha^{\hat{m}})).$$

Also, by (6.3) $d_{(b+1-m,1^a)}(M_\alpha^{\hat{m}})-d_{(a+1-m,1^b)}(M_\alpha^{\hat{m}})\geqslant 0$. Hence,

$$\sum_{i=1}^{m-1}\binom{m}{i}d_{(b+1-i,1^{a-m+i})}(M_\alpha^{\hat{m}})+((-1)^m+1)\,d_{(b+1-m,1^a)}(M_\alpha^{\hat{m}})$$
$$-\sum_{i=1}^{m-1}\binom{m}{i}d_{(a+1-m+i,1^{b-i})}(M_\alpha^{\hat{m}})+((-1)^m+1)\,d_{(a+1-m,1^b)}(M_\alpha^{\hat{m}})\geqslant 0.$$

By Theorem 4.2 and Proposition 2.1 (2.1), (2.2), we get

$$d_{(b+1,1^a)}(M_\alpha)-d_{(a+1,1^b)}(M_\alpha)\geqslant 0.$$

This completes the proof of part (2) of the lemma.

THEOREM 6.1. *Assume that* $a+b=c+d=n-1$ *with* $b\geqslant c\geqslant a$, *and* $\lambda=(c+1,1^d)$ *and* $\mu=(a+1,1^b)$ *are partitions of* n. *Suppose that* $\alpha\mapsto n$ *and* $\alpha\neq(n)$. *Then* $\bar{d}_\lambda(M_\alpha)\geqslant\bar{d}_\mu(M_\alpha)$.

Proof. The cases where $\lambda=(a+2,1^{b-1})$ and $\lambda=(b+1,1^a)$ are given by parts (1) and (2) of Lemma 6.3, respectively. The general case follows from the transitivity of $\geqslant$. (Compare the example of $n=8$ given before Lemma 6.3.)

THEOREM 6.2. *Assume that* $\lambda=(c+1,1^d)$ *and* $\mu=(a+1,1^b)$ *are distinct hook partitions of* n. *Then* $\bar{d}_\lambda(M)\geqslant\bar{d}_\mu(M)$ *for all* M*-matrices* M *if and only if one of the following holds:*

(1) a *is even and* $b\geqslant c>a$,

(2) n *and* a *are odd and* $b=c>a$.

Proof. That the stated conditions are necessary comes from Corollary 4.4 and Proposition 4.2. That they are sufficient is implied by Proposition 4.2 and Theorem 6.1.

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7. *Other inequalities*

In the previous sections, we have determined all possible inequalities that hold for normalized immanants of M-matrices, provided that the immanants being compared correspond to diagrams with at most two rows or at most two columns. The foregoing results by no means complete the list of all possible inequalities. Since we have a finite set of test matrices to check possible comparisons of two given immanants on all M-matrices, we can determine all inequalities for small values of n. At this point, we have run such tests for all $n \leqslant 11$. We produce here, for $n = 11$, a diagram for all immanantal inequalities on M-matrices (Figure 2). We have omitted

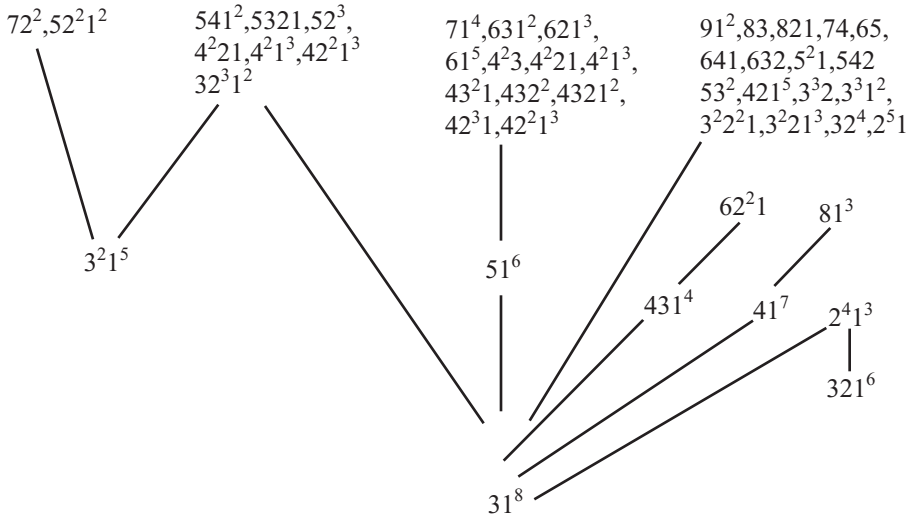


FIG. 2.

the determinant from the diagram, since it sits at the bottom of the partial ordering. While we see from this diagram no particular conjectures worth making, there are some observations for the reader to consider.

(1) There are many cases for which $\bar{d}_\lambda(M) \geqslant \bar{d}_\mu(M)$ for all M-matrices M, but λ does not majorize μ.

(2) The immanant $\bar{d}_{(3,1^{n-3})}$ 'loses' to many immanants. There do not appear to be any immanants which are 'big winners'.

(3) The immanants which we have investigated, namely those with two rows or two columns, compare with other immanants not of these classes.

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