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Piecewise linear immersions

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0. *Introduction.* An *immersion* f of one space X in another Y is a continuous map which is locally an embedding; that is, for any $x \in X$ there is a neighbourhood $N(x)$ of x such that $f|N(x)$ is an embedding. Thus the image of an immersion may intersect itself, but it can contain no worse singularities. For example, the well-known model of the Klein Bottle is the image of an immersion into ordinary 3-space; the locus of points of self intersection is a circle, where two distinct circles in the Klein Bottle have their images. This paper obtains a sufficient condition for a map of PL manifolds to be deformable into a PL immersion.

1. *Definitions and statement of results.* Let M and Q be connected PL manifolds of dimensions m and q respectively. Let M be compact. We say that Q is *relatively* n*-connected* if $\pi_r(Q, \partial Q) = 0$, $1 \leqslant r \leqslant n$. If $f: M \to Q$ is a map, we write $\pi_r(f)$ for the rth homotopy group of the mapping cone C_f of f. There is then an exact sequence

$$\ldots \to \pi_{r+1}(f) \xrightarrow{\partial} \pi_r(M) \xrightarrow{f_*} \pi_r(Q) \xrightarrow{i_*} \pi_r(f) \to \ldots.$$

We say that f is n*-connected* if $\pi_r(f) = 0$, $1 \leqslant r \leqslant n$. The map f is *proper* if $f(\partial M) \subseteq \partial Q$.

The following three theorems have been proved by geometric arguments (6), (4), (3):

THEOREM (IRWIN). *If M is $(2m-q)$-connected, $Q(2m-q+1)$-connected and $q \leqslant m+3$, then any proper map $f: M \to Q$ which restricts to a PL embedding on ∂M is homotopic* rel ∂M *to a PL embedding.*

THEOREM (HUDSON). *If M is relatively $(2m-q)$-connected, Q relatively $(2m-q+1)$-connected and $q \geqslant m+3$, then any proper map $f: M \to Q$ is homotopic to a PL embedding.*

THEOREM (HUDSON). *If M is $(3m-2q+2)$-connected and $q \geqslant m+3$, then any proper $(2m-q+1)$-connected map $f: M \to Q$ which restricts to a PL embedding on ∂M is homotopic* rel ∂M *to a PL embedding.*

This has been strengthened by Casson and Sullivan, using a homotopy theoretic argument (9), (11), and independently by Haefliger and Wall:

THEOREM (CASSON-SULLIVAN *et al.*). *If $q \geqslant m+3$, then any proper $(2m-q+1)$-connected map $f: M \to Q$, which restricts to a PL embedding on ∂M is homotopic* rel ∂M *to a PL embedding.*

In the case of immersions, Irwin has proved (7) that:

THEOREM (IRWIN). *If M and Q are $(2m-q)$-connected and $q \geqslant \frac{3}{2}m+1$, then any proper map $f: M \to Q$ which restricts to a PL embedding on ∂M is homotopic* rel ∂M *to a PL immersion.*

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Our results are that:

THEOREM 1. *If* $\pi_{2m-q}(M) = 0$ *and* $q \geqslant \frac{3}{2}m+1$, *then any proper* $(2m-q)$*-connected map* $f: M \to Q$ *which restricts to a PL embedding on* ∂M *is homotopic* rel ∂M *to a PL immersion.*

THEOREM 2. *If* M *is relatively* $(2m-q-1)$*-connected,* Q *relatively* $(2m-q)$*-connected and* $q \geqslant m+3$, *then any proper map* $f: M \to Q$ *is homotopic to a PL immersion.*

This residual condition on $\pi_*(M)$ in Theorem 1 is displeasing, and one feels it should be replacable by an obstruction theory; the condition cannot simply be removed, since the simply connected 2^{n+1}-manifold Complex Projective 2^n-Space does not immerse in real $(2^{n+2}-2)$-space; and any map of a simply connected manifold into a contractible one must of necessity itself be 2-connected.

2. *Relative engulfing.* We first recall a few lemmas from Hudson (3). Let K be a simplicial complex and X a polyhedron; a *subdivision* of $K \times X$ is a triangulation such that for each simplex $s \in K$ with its original triangulation, $s \times X$ is a subcomplex of $K \times X$. A K*-isotopy* of some subset A of $K \times X$ is a PL isotopy of A in itself, such that $((s \times X) \cap A) \times I \to (s \times X) \cap A$, for each simplex $s \in K$.

LEMMA 2·1 (Lemma 8 of (3)). *Let* X *be a polyhedron in* M, *and* N *a regular neighbourhood of* X *in* M *which meets the boundary regularly. If* K *is a simplicial complex, and* N' *the second derived neighbourhood of* $K \times X$ *in* $K \times M$ *with respect to any subdivision of* $K \times M$ *which subdivides* $K \times X$ *as a subcomplex, then there is an (ambient)* K*-isotopy of* $K \times M$ *in* $K \times M$, *which is fixed on* $K \times X$, *and which throws* N' *onto* $K \times N$. *Moreover, if* $N' \cap (K \times \partial M) = K \times (N \cap \partial M)$, *we may demand that the isotopy leave* $K \times \partial M$ *fixed.*

For any map $f: M \to Q$, we define: $S'_r(f) = (x \in M : f^{-1}(fx)$ contains at least r distinct points), and $S_r(f) =$ the closure of $S'_r(f)$.

We say that f is *non-degenerate* if $S_\infty(f)$ is empty, and that f is *in general position* if, for all finite r, $\dim(S_r(f)) \leqslant m - r(q-m)$, with the convention that a set of negative dimension is empty.

The map f is said to be *regular at the boundary* if:

(i) $f^{-1}(\partial Q) = \partial M$ and (ii) $S_r(f) \cap \partial M = S_r(f|\partial M)$, for all r.

It is well known that any map $f: M \to Q$ is homotopic to a PL map in general position, and that the homotopy may be required to be fixed on ∂M if $f|\partial M$ is already a PL map in general position. Furthermore:

LEMMA 2·2 (cf. LEMMA 3 OF (3)). *If* $f: M \to Q$ *is a proper PL map in general position, then* f *is homotopic* rel ∂M *to a PL map in general position which is regular at the boundary.*

Proof. Let Q' be a regular neighbourhood of fM in Q; then Q' is compact. Let $M^* = M \cup (\partial M \times I)$, with x identified with $x \times 0$ for all $x \in \partial M$. By Whitehead's collaring theorem there is a PL homeomorphism $h: M \to M^*$, which is the identity outside a regular neighbourhood of the boundary, homotopic to the inclusion, and such that $h(x) = x \times 1$, for all $x \in \partial M$. Define Q^* and $h': Q' \to Q^*$ similarly.

Define the new map as the composition:

$$M \xrightarrow{h} M^* \xrightarrow{f \cup ((f|\partial M) \times 1)} Q^* \xrightarrow{(h')^{-1}} Q' \subseteq Q.$$

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LEMMA 2·3 (LEMMA 12 OF (3)). *Let $f: B^m \to B^q$ be a PL map of PL balls which is regular at the boundary and in general position. Let K be a k-dimensional cone, and $h: K \times B^t \to fS_2(f)$ a PL homeomorphism, with $h(K \times \partial B) = fS_2(f) \cap \partial B^q$, where $t = 2m-q-k$.*

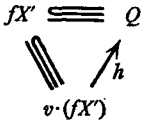
Then, if $k \leqslant q-m-2$ and $q \geqslant m+3$, there is a *PL* map $f': B^m \to B^q$, which is in general position and regular at the boundary, with $f'|\partial B^m = f|\partial B^m$; and a *PL* homeomorphism $h': K \times \partial B^t \times I \to f'S_2(f')$, such that $h'(K \times \partial B \times (0)) = f'S_2(f') \cap \partial B^q$, and $h'|K \times \partial B \times (0) = h|K \times \partial B^t$.

LEMMA 2·4 (RELATIVE ENGULFING). *Let $f: M \to Q$ be a $(k+1)$-connected PL map in general position and regular at the boundary. Let K be a k-dimensional simplicial complex, in $fM \cap \operatorname{int} Q$, which is inessential in Q.*

Then, if $k \leqslant \min(\frac{1}{2}(m-3), q-m-3)$ and $q \geqslant m+3$, there are collapsible polyhedra C and D, such that $K \subseteq D$, $C \subseteq \operatorname{int} M$, $D \subseteq \operatorname{int} Q$, $f^{-1}D = C$, $C \cap S_2(f) = f^{-1}K \cap S_2(f)$, and $D \cap fM = fC$.

Proof. We shall use induction on k.

Write $A \cdot B$ for the join of A and B; v and v' are to be points, so that $v \cdot B$ and $v' \cdot B$ will be cones on B with vertices v and v' respectively. Let $X = f^{-1}K$. By general position, we can isotop X within $\operatorname{int} M$ off $S_2(f)$; this gives a map $j: X \times I \to \operatorname{int} M$, with $j(x, 0) = x$ for all $x \in X$. By general position again, we may take j to be a *PL* embedding, since $(2k+2) < m$, and we may further require that $(S_2(f)) \cap j(X \times I) \subseteq j(X \times (0))$, since $k+1+(2m-q) < m$. Let $X' = j(X \times 1)$, $Z = j(X \times I)$ and $K' = fZ$. Clearly, K' is *PL* homeomorphic to the mapping cylinder of the restriction $f|X: X \to K$. Since $K' \searrow K$, and K is inessential in Q, K' must be inessential in Q. Let, therefore, $h: v \cdot K' \to \operatorname{int} Q$ be a map with $h|K' =$ inclusion. By general position, we may suppose h to be a *PL* embedding, and that $\dim(fM \cap h(v \cdot K' - K')) \leqslant (k+2)+m-q < 0$; that is, that $h(v \cdot K') \cap fM = K'$. There is then a commutative diagram of inclusions and embeddings:



Pick $m \in \operatorname{int} M$, such that $m \notin Z \cup S_2(f)$. Then $fm \notin h(v \cdot K')$. Since Q is path connected, we may choose an embedded path in $\operatorname{int} Q$ from hv to fm, say $p: v \cdot v' \to \operatorname{int} Q$. If necessary, we may move this into general position, so that

$$p(v \cdot v') \cap (fM \cup h(v \cdot K')) = p((v) \cup (v')) = (hv) \cup (fm).$$

Let X_n be the n-skeleton of X', in some triangulation. Let $Y_n = K' \cup (v' \cdot X_n)$, with $x \in X_n \subseteq v' \cdot X_n$ identified with $fx \in K'$. Then Y_k is clearly the mapping cone of the restriction $f|X: X \to K$.

Let $P(n)$ be the assertion:

$P(n)$: There exist *PL* embeddings $\varphi_n: v' \cdot X_n \to \operatorname{int} M$ and $\psi_n: v \cdot Y_n \to \operatorname{int} Q$, such that

(i) $\psi_n|v' \cdot X_n = f\varphi_n$,
(ii) $\psi_n|v \cdot K' = h$,
(iii) $\psi_n|v \cdot v' = p$,
(iv) $\varphi_n|X_n =$ inclusion,
(v) $\varphi_n(v' \cdot X_n) \cap (Z \cup S_2(f)) = \varphi_n(X_n)$,
(vi) $\psi_n(v \cdot Y_n) \cap fM = \psi_n(Y_n)$.

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(*a*) $P(-1)$ *is true*

We have $X_{-1} = \phi$, and $Y_{-1} = K'(v')$. Let $\varphi_{-1}(v') = m$ and $\psi_{-1}|v \cdot K' = h$, $\psi_{-1}|v \cdot v' = p$. Then the conditions of $P(-1)$ are satisfied.

(*b*) $P(n-1) \Rightarrow P(n)$, $0 \leqslant n \leqslant k$

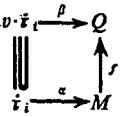
Suppose we have PL embeddings $\varphi_{n-1}: v' \cdot X_{n-1} \to \operatorname{int} M$ and $\psi_{n-1}: v \cdot Y_{n-1} \to \operatorname{int} Q$ satisfying the conditions of $P(n-1)$. Let $(\sigma_1, \ldots, \sigma_r)$ be the n-simplexes of X_n. Let $\tau_i \equiv v' \cdot \sigma_i$, $1 \leqslant i \leqslant r$. Then τ_i is an $(n+1)$-simplex of $v' \cdot X_n$. Clearly, $\dot{\tau}_i = v' \cdot \dot{\sigma}_i \cup \sigma_i$. Define a map $\alpha: \dot{\tau}_i \to \operatorname{int} M$ by

$$\begin{cases} \alpha|v' \cdot \dot{\sigma} = \varphi_{n-1}, \\ \alpha|\sigma_i = \text{inclusion}, \end{cases}$$

and define a map $\beta: v \cdot \dot{\tau}_i \to \operatorname{int} Q$ by

$$\begin{cases} \beta|v \cdot v' \cdot \dot{\sigma}_i = \psi_{n-1}, \\ \beta|v \cdot \sigma_i = h|v \cdot f\sigma_i \quad (\text{with } \sigma_i \text{ and } f\sigma_i \text{ identified under } f). \end{cases}$$

Then there is a commutative diagram:



Such a diagram represents an element of $\pi_{n+1}(f)$, which we know to be trivial. There are thus maps $\varphi^i: \tau_i \to \operatorname{int} M$ and $\psi^i: v \cdot \tau_i \to \operatorname{int} Q$, with

(i) $\varphi^i|\dot{\tau}_i = \alpha$, (iii) $\psi^i|\tau_i = f\varphi^i$,

(ii) $\varphi^i(v') = m$, (iv) $\psi^i|v \cdot \dot{\tau}_i = \beta$.

By general position, we may demand that each φ^i be a PL embedding, and

(i) $\varphi^i\tau_i \cap \varphi^j\tau_j = \alpha(\tau_i \cap \tau_j)$ if $i \neq j$, (ii) $\varphi^i\tau_i \cap (Z \cap S_2(f)) = \varphi^i\sigma_i$;

and then that each ψ^i be a PL embedding, with

(i) $\psi^i(v \cdot \tau_i) \cap \psi^i(v \cdot \tau_j) = \beta(v \cdot \tau_i \cap v \cdot \tau_j)$ if $i \neq j$,

(ii) $\psi^i(v \cdot \tau_i) \cap (fM \cup \psi_{n-1}(v \cdot Y_{n-1})) = \psi^i(v \cdot \dot{\tau}_i)$.

Then we may define φ_n and ψ_n by

$\varphi_n|v' \cdot X_{n-1} = \varphi_{n-1}$, $\varphi_n|\tau_i = \varphi^i$,

$\psi_n|v \cdot Y_{n-1} = \psi_{n-1}$, $\psi_n|v \cdot \tau_i = \psi^i$.

Then φ_n and ψ_n satisfy the conditions of $P(n)$.

(*c*) When we have reached $P(k)$ we are home, for we may set $C = Z \cup \varphi_k(v \cdot X')$, and $D = \psi_k(v \cdot Y_k)$. Then $C \subseteq \operatorname{int} M$, $D \subseteq \operatorname{int} Q$, $K \subseteq D$, $f^{-1}D = C$, $C \searrow \varphi_k(v \cdot X') \searrow 0$, $D \searrow 0$, $C \cap S_2(f) = f^{-1}K \cap S_2(f)$, and $D \cap fM = fC$.

3. *Engulfing the branch points.* We prove in this section the following result:

LEMMA 3·3. *If* $q \geqslant \frac{3}{2}m + 1$, *then any* $(2m-q)$*-connected* PL *map* $f: M \to Q$ *in general position which restricts to a* PL *embedding on* ∂M *and which is regular at the boundary is homotopic* rel ∂M *to a* PL *map* g *in general position and regular at the boundary, with*

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the property that there exists balls $N_2^q \subseteq \operatorname{int} Q$ *and* $N_1^m \subseteq \operatorname{int} M$, *such that (writing* $Q_0 = \overline{Q - N_2}$ *and* $M_0 = \overline{M - N_1}$) $g|M_0$ *is a PL immersion, and the components of* $S_2(g|M_0)$ *are* $(2m-q)$*-balls which meet* ∂N_1 *in their boundary* $(2m-q-1)$*-spheres. Moreover, g embeds them individually, although identifying them in pairs. Each* $(2m-q)$*-ball in* $gS_2(g|M_0)$ *may be supposed to be a face of a* $(2m-q+1)$*-ball in* Q_0, *not meeting* ∂Q_0, *and whose complementary face is in* ∂N_2. *These* $(2m-q+1)$*-balls may be supposed disjoint.*

The techniques used by Irwin in (7) may then be generalized to homotop g to a PL immersion.

LEMMA 3·1 (cf. PROP. 1 OF (3)). *If* $q \geqslant \frac{3}{2}m+1$, *then any* $(2m-q)$*-connected PL map* $f: M \to Q$ *in general position which restricts to a PL embedding on* ∂M *and which is regular at the boundary is homotopic* rel ∂M *to a PL map g in general position and regular at the boundary, such that* $gS_2(g)$ *consists of disjoint collapsible* $(2m-q)$*-complexes.*

Proof. (*a*) $q \geqslant m+3$. Let $k: K \to fS_2(f)$ be a triangulation. Let $\sigma_1, \ldots, \sigma_R$ be the simplexes of K of dimension $\geqslant 1$, in order of decreasing dimension. Let $\hat{\sigma}_i$ be the barycentre of σ_i, and U_i the closed simplicial neighbourhood of $\hat{\sigma}_i$ in the barycentric second derived K'' of K. Let X_i be the closure of $\left(|K| - \bigcup_{j=1}^{i-1} U_j\right)$.

We shall define, inductively, sequences (f_i) and (k_i) of PL maps and homeomorphisms respectively, such that:

$k_i: X_i \to f_i S_2(f_i)$, and

$f_i: M \to Q$, $f_i \sim f$ rel ∂M, and f_i is in general position and regular at the boundary.

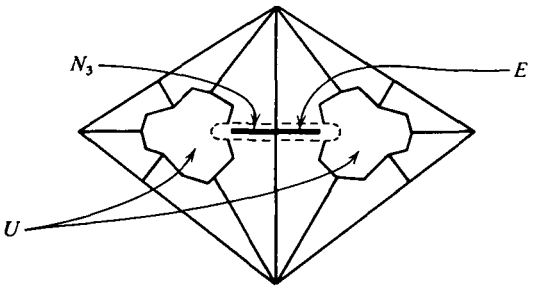


Fig. 1

When $i = R+1$ we shall have established our lemma, since X_{R+1} consists only of the stars in K'' of the vertices of K.

(*a*) Let $f_1 = f$, and $k_1 = k$.

(*b*) Suppose f_i and k_i already defined. Write $U = \bigcup_{j=1}^{i-1} U_j$. Let $s = \dim \sigma_i$. Then $s \leqslant 2m-q$. Let $E_1 = \bigcup (\sigma \in K' : \sigma = \hat{\sigma}_{i_1} \ldots \hat{\sigma}_{i_p} : i_n \leqslant i$ for all $n)$. Let $E = \overline{E_1 - E_1 \cap U}$; see fig. (i). E is a cone of dimension $r \leqslant 2m-q-s \leqslant 2m-q-1$, and $E \subseteq \operatorname{int} Q$. By Lemma 2·4, there are collapsible sets $C \subseteq \operatorname{int} M$, $D \subseteq \operatorname{int} Q$, such that $f_i^{-1}D = C$, $E \subseteq D$, $C \cap S_2(f_i) = f_i^{-1}E$ and $D \cap f_i M = f_i C$. Subdivide X_i, M and Q until k_i and f_i are simplicial, and C and D subcomplexes. Let N_2 be the second derived neighbourhood of D in Q. Let $N_1 = f_i^{-1}N_2$ and $N_3 = k_i^{-1}N_2$. Then N_1 and N_2 are balls in $\operatorname{int} M$ and $\operatorname{int} Q$ respectively.

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Now $U_i \cong E \times B^s$, and so by Lemma 1·1, there is a PL homeomorphism $\beta: N_3 \to E \times B^s$ sending $N_3^{\cdot} \to E \times \partial B^s$, where $N_3^{\cdot} = \partial N_3 \cap X_i$. But $f_i|N_1: N_1 \to N_2$ and

$$k_i\beta^{-1}: E \times B^s \to f_i S_2(f_i) \cap N_2$$

are such that $k_i\beta^{-1}(E \times \partial B^s) = f_i S_2(f_i) \cap \partial N_2$. Moreover, $r = 2m-q-2 \leqslant q-m-2$, and $q \geqslant m+3$. Thus, by Lemma 2·3, there is a PL map $f': N_1 \to N_2$ in general position, and a PL homeomorphism $\psi_1: E \times \partial B^s \times I \to f' S_2(f')$, with $f'|\partial N_1 = f|\partial N_1$,

$$\psi_1(x, 0) = k_i\beta^{-1}x \quad \text{on} \quad E \times \partial B^s.$$

Let $f_{i+1}: M \to Q$ be f' on N_1 and f_i elsewhere. Then $f_{i+1} \sim f \operatorname{rel} \partial M$, is in general position and regular at the boundary. Let $Y_i = \overline{X_i - N_3} \cup (N_3^{\cdot} \times I)$, with $x \times 0 \in N_3^{\cdot} \times I$ identified with $x \in N_3^{\cdot} \subseteq \overline{X_i - N_3}$, for all $x \in N_3^{\cdot}$. Then Y_i is PL homeomorphic to $f_{i+1}S_2(f_{i+1})$, say by ψ_2, where

$$\psi_2|\overline{X_i - N_3} = k_i|\overline{X_i - N_3},$$
$$\psi_2|N_3^{\cdot} \times I = \psi_1(\beta \times 1)|N_3^{\cdot} \times I.$$

It remains to verify that $Y_i \cong X_{i+1}$. Let V be the second derived neighbourhood of U_i in X_i. Then, since star $(\hat{\sigma}_i; K') \cong \hat{\sigma}_i \cdot \operatorname{link}(\sigma_i; K) \cong \Sigma^{s-1}E$, (the $(s-1)$-fold suspension of E) there is a PL homeomorphism $\varphi: V \to E \times B^s$, with $\varphi(E) = E \times (0)$, and $\varphi(U_i) = E \times B_0^s$ for some ball $B_0^s \subset \operatorname{int} B^s$. By Lemma 2·1, there is an ambient isotopy of V rel $V^{\cdot}$, taking N_3 onto U_i, where $V^{\cdot} = \partial V \cap X_i$. Thus

$$\overline{V - N_3} \cup (N_3^{\cdot} \times I) \cong \overline{V - U_i} \cup (U_i^{\cdot} \times I) \cong (V^{\cdot} \times I) \cong \overline{V - U_i} \operatorname{rel} V^{\cdot}.$$

Thus $Y_i \cong X_{i+1}$.

(*b*) $q < m+3$. This consists only of $(m, q) = (0, 1)$, $(0, 2)$, $(1, 3)$ and $(2, 4)$, all of which can be dealt with by general position arguments.

LEMMA 3·2. *Let $q \geqslant \frac{3}{2}m + 1$ and $f: M \to Q$ be a $(2m-q)$-connected PL map in general position which restricts to a PL embedding on ∂M and which is regular at the boundary, and such that $fS_2(f)$ consists of disjoint contractible $(2m-q)$-complexes.*

Then we can find balls $N_2^q \subseteq \operatorname{int} Q$ and $N^m = f^{-1}N_2 \subseteq \operatorname{int} M$, such that (writing $M_0 = \overline{M - N_1}$ and $Q_0 = \overline{Q - N_2}$) $f|M_0: M_0 \to Q_0$ is a PL immersion, with the components of $S_2(f|M_0)$ disjoint $(2m-q)$-balls, meeting ∂N_2 in their boundary spheres. Moreover f embeds them individually, but identifies them in pairs; we may suppose further that each $(2m-q)$-ball in $fS_2(f|M_0)$ is a face of a $(2m-q+1)$-ball in Q_0 whose complementary face is in ∂N_2, and that these $(2m-q+1)$-balls are disjoint.

Proof. (*a*) $q > m+3$. It does not matter whether $fS_2(f)$ is connected or not. Let $k: K \to fS_2(f)$ be a triangulation. Since K is inessential in Q, there is a map $k': v \cdot K \to Q$ extending k. By the general position theorems, we may suppose that:

$$\dim(S_2(k'|(v \cdot K - K))) \leqslant 2(2m-q+1) - q = 4m - 3q + 2 < 0,$$

and

$$\dim(k'(v \cdot K - K) \cap fM) \leqslant (2m-q+1) + m - q = 3m - 2q + 1 < 0.$$

Then k' is an embedding, and $k'(v \cdot K) \cap fM = k(K) = fS_2(f)$. Let L be the $(2m-q-1)$-skeleton of K. Although L is not necessarily contractible, kL is inessential in Q, $k'|v \cdot L$

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giving a contraction. Lemma 2·4 can be applied to give collapsible polyhedra $C \subseteq \operatorname{int} M$ and $D \subseteq \operatorname{int} Q$, with $kL \subseteq D$, $f^{-1}D = C$, $C \cap S_2(f) = f^{-1}kL$, and $D \cap fM = fC$. We may clearly also strengthen 2·4 slightly to give $D \cap k'(v \cdot K) = k'(v \cdot L)$. Triangulate so that k' and f are simplicial, and C and D are subcomplexes. Let N_1 be the second derived neighbourhood of C in M, and N_2 that of D in Q, and N_3 that of $v \cdot L$ in $v \cdot K$. Then:

$$f^{-1}N_2 = N_1, \quad k'^{-1}N_2 = N_3, \quad N_1 \subseteq \operatorname{int} M \quad \text{and} \quad N_2 \subseteq \operatorname{int} Q.$$

Moreover, the $(2m-q-1)$-skeleton of $S_2(f)$ is in N_1, and so in particular the branch points of f must be in N_2. The singular set of $f|\overline{M-N_1}$ consists of the centres of the $(2m-q)$-simplexes of $S_2(f)$, and so must be of the required form; the image under f of one is killed to N_2 by a $(2m-q+1)$-ball in $k'(\overline{v \cdot K - N_3})$. In fact if σ is a $(2m-q)$-simplex of K, $\overline{k\sigma - N_2}$ is killed by $k'(v \cdot \sigma - N_3)$. These $(2m-q+1)$-balls are clearly disjoint.

(*b*) $q < m+3$. As in the previous lemma, this consists only of cases which can be dealt with by a general position argument.

Combining Lemmas 3·1 and 3·2 gives Lemma 3·3.

4. *Removal of the branch points*

LEMMA 4·1. *Let $q \geqslant \frac{3}{2}m+1$, $q \geqslant m+3$, and $f: M \to Q$ be a PL map in general position. Let $\pi_{2m-q}(M) = 0$. Let $N_1 \subseteq \operatorname{int}(M)$ and $N_2 \subseteq \operatorname{int}(Q)$ be PL balls with $f^{-1}N_2 = N_1$. Write $M_0 = \overline{M-N_1}$ and $Q_0 = \overline{Q-N_2}$. Suppose that two components, B_1 and B_2, of $S_2(f|M_0)$ meet ∂N_1, that each is a $(2m-q)$-ball in $\operatorname{int}(M)$ meeting ∂N_1 in its boundary $(2m-q-1)$-sphere, and that f embeds them individually in $\operatorname{int}(Q)$ but identifies them.*

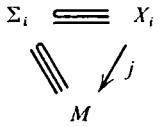
Then the B_i can be extended by disjoint $(2m-q)$-balls in N_1 to $(2m-q)$-spheres in $\operatorname{int}(M)$, with trivial normal block bundle, and meeting ∂N_1 in ∂B_1.

Proof. Write $\Sigma_i = \partial B_i$, $i = 1, 2$. Then the Σ_i are disjoint $(2m-q-1)$-spheres in the $(m-1)$-sphere ∂N_1, and $2m-q-1 \leqslant m-4$. Thus the Σ_i can be spanned by embedded $(2m-q)$-balls A_i in ∂N_1. By general position, we may require that:

$$\dim(A_1 \cup A_2) \leqslant 2(2m-q)-(m-1) = 3m-2q+1 < 0,$$

and $$\dim((A_i-\Sigma_i) \cup S_2(f|M_0)) \leqslant (2m-q)+(2m-q)-(m-1) < 0 \quad (i = 1, 2).$$

That is, A_1 and A_2 can be supposed disjoint, and to meet $S_2(f|M_0)$ in their boundary spheres Σ_{1i} and Σ_2 respectively. Consider the $(2m-q)$-spheres $(A_i \cup B_i)$, $i = 1, 2$. Since $\pi_{2m-q}(M) = 0$, there are balls X_i in M with $\partial X_i = A_i \cup B_i$. By using a collar of ∂N_1 (if necessary), we may suppose that $X_i \cap N_1 = A_i$, $i = 1, 2$. By using a collar of ∂M (if necessary), we may suppose that $X_i \cap \partial M = \phi$, $i = 1, 2$. By general position, we may suppose that the X_i are immersed in M, with singularities in $\operatorname{int} X_i$; indeed they may be embedded, unless $q = \frac{3}{2}m+1$. Let ξ_i denote the normal block bundle to Σ_i in M, η_i that to X_i in M and ζ_i that to Σ_i in X_i. We have a commutative diagram of inclusions and immersions:



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There is induced an equivalence of normal bundles:

$$\xi_i \simeq \eta_i|\Sigma_i \oplus \zeta_i.$$

Now X_i is a disk, so that η_i must be trivial. Furthermore, ζ_i is trivial, by the collaring theorems. Thus ξ_i is trivial. Thus $A_i \cup B_i$ has trivial normal block bundle in M, $i = 1, 2$. Now pushing A_i into $\operatorname{int} N_1$, (by using a collar of ∂N_1) does not affect the normal bundle structure, so we may contruct the required balls by so doing.

LEMMA 4·2. *Let $q \geqslant \frac{3}{2}m + 1$, $q \geqslant m + 3$, and $f: M \to Q$ be a PL map in general position. Let $N_2^q \subseteq \operatorname{int} Q$ and $N_1^m \subseteq \operatorname{int} M$ be PL balls, with $f^{-1}N_2 = N_1$. Write $M_0 = \overline{M - N_1}$ and $Q_0 = \overline{Q - N_2}$. Suppose that precisely $2k$ components $B_{j,i}$, $(1 \leqslant j \leqslant k, i = 1, 2)$, of $S_2(f|M_0)$ meet ∂N_1, that each $B_{j,i}$ is a $(2m-q)$-ball in $\operatorname{int} M$ meeting ∂N_1 in $\partial B_{j,i}$, and that f embeds them individually but identifies $B_{j,1}$ with $B_{j,2}$ for all j. Suppose that $B_j = fB_{j,1} = fB_{j,2}$ is a component of $fS_2(f|M_0)$, and is a face of a $(2m-q+1)$-ball C_j in Q_0, not meeting ∂Q, and with $\overline{\partial C_j - B_j} \subseteq \partial N_2$. Suppose further that the $B_{j,i}$ can be extended by disjoint $(2m-q)$-balls $G_{j,i}$ in N_1 to $(2m-q)$-spheres $G_{j,i} \cup B_{j,i}$ meeting ∂N_1 in $\partial G_{j,i} = \partial B_{j,i}$, and with trivial normal block bundles.*

Then $f \sim g \operatorname{rel} M_0$, where g is a PL immersion, and only $2k$ components of $S_2(g)$ meet ∂N_1, namely the $G_{j,i} \cup B_{j,i}$, which g embeds individually, but identifies $G_{j,i} \cup B_{j,i}$ with $G_{j,2} \cup B_{j,2}$ for all j.

Proof. Triangulate M and Q so that f is simplicial, and $N_1, N_2, G_{j,i}, B_{j,i}, C_j$ and $C_j \cup \partial N_2$ are subcomplexes, for all j and i. Let $F_{j,i}$ be the second derived neighbourhood of $B_{j,i}$ in M_0, and $H_{j,i}$ that of $G_{j,i}$ in N_1, for all (j, i). Then $F_{j,i} \cap \partial N_1 = H_{j,i} \cap \partial N_1 = J_{j,i}$, say· and $F_{j,i} \cap F_{l,n} = H_{j,i} \cap H_{l,n} = \phi$ if $(j, i) \neq (l, n)$. There are block bundle equivalences:

$$\psi_{j,i}: F_{j,i} \cong B_{j,i} \times I^{q-m} \quad \text{and} \quad \varphi_{j,i}: H_{j,i} \cong G_{j,i} \times I^{q-m},$$

such that

$$\psi_{j,i}|J_{j,i} = \varphi_{j,i}|J_{j,i}: J_{j,i} \cong \partial B_{j,i} \times I^{q-m}.$$

Let $\rho_{j,i}: B_{j,i} \to G_{j,i}$ be the cone on the identity map $1: \partial B_{j,i} = \partial G_{j,i}$. Then

$$r_{j,i} = \varphi_{j,i}^{-1}(\rho_{j,i} \times 1)\psi_{j,i}: F_{j,i} \to H_{j,i}$$

is a PL homeomorphism, with $r_{j,i}|J_{j,i} = 1$. Similarly, let F_j be the second derived neighbourhood of C_j in Q_0, and H_j that of $C_j \cap \partial N_2$ in N_2. Then $F_j \cap \partial N_2 = H_j \cap \partial N_2$ is a $(q-1)$-ball, which is a face of the q-balls F_j and H_j. Extending the identity on $F_j \cap \partial N_2$ to a map $\partial F_j \cong \partial H_j$, and then to a map $r_j: F_j \cong H_j$, we may define a map $g: M_0 \cup \bigcup_{(j,i)} H_{j,i} \to Q_0 \cup \bigcup_j H_j$ by:

$$g|M_0 = f|M_0 \quad \text{and} \quad g|H_{j,i} = r_j f r_{j,i}^{-1} \quad \text{for all} \quad (j, i).$$

Then g is a proper map, and the components of $S_2(g)$ meeting ∂N_1 are the $2k$ spheres $B_{j,i} \cup G_{j,i}$ which are individually embedded, but identified by g in pairs as required. Write $M_1 = \overline{N_1 - \bigcup_{(j,i)} H_{j,i}}$ and $Q_1 = \overline{N_2 - \bigcup_j H_j}$. Then $g|\partial M_1$ is a PL embedding of ∂M_1 in ∂Q_1. Since Q_1 is a ball, this extends to a continuous map from M_1 to Q_1. This may be made PL and put into general position by a homotopy rel ∂M_1. But M_1 is $(q-m-2)$-connected, since a $(q-m-1)$-complex in general position does not meet any $G_{j,i}$.

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Moreover $q-m-2 \geqslant 2m-q$. Thus Irwin's embedding theorem (6) gives us an embedding, which we name $g|M_1$, of M_1 in Q_1. We have now defined g throughout M, and since g and f differ only on $\operatorname{int} N_1$, they must be homotopic rel $M_0 = \overline{M-N_1}$.

THEOREM 1. *If $\pi_{2m-q}(M) = 0$ and $q \geqslant \frac{3}{2}m+1$, then any proper $(2m-q)$-connected map $f: M \to Q$ which restricts to a PL embedding on ∂M is homotopic* rel ∂M *to a PL immersion.*

Proof (*a*) $q \geqslant m+3$. By Lemma 2·2 we may alter f if necessary by a homotopy rel ∂M, to a map which is PL, regular at the boundary and in general position. By Lemma 3·3, we may alter this map by a further homotopy rel ∂M to a map g for which there exist balls $N_2 \subseteq \operatorname{int} Q$ and $N_1 = g^{-1}N_2 \subseteq \operatorname{int} M$, for which $S_2(g|\overline{M-N_1})$ consists of $(2m-q)$-balls with their boundaries in ∂N_1, which g embeds individually but identifies in pairs, and which have image balls that are faces of disjoint $(2m-q+1)$-balls with complementary faces in ∂N_2. We may now apply Lemma 4·1 k times to g, and then Lemma 4·2 to give the immersion.

(*b*) $q < m+3$. This case can be dealt with trivially, using a general position argument.

Remark. In fact our immersion has no triple points, and (unless (m, q) is $(2, 4)$) the singular set consists of $2k$ disjoint $(2m-q)$-spheres which g embeds individually but identifies in pairs.

5. *The counter-example.* In this section we establish a non-immersion result, which shows that the condition in Theorem 1 on $\pi_*(M)$ is necessary.

LEMMA 5·1. *If $n \geqslant 2m+1$, then any PL immersion of M in R^n is regularly homotopic to a PL embedding.*

Proof. Shifting the immersion to general position provides the embedding; moreover the shift is clearly a regular homotopy.

LEMMA 5·2 (Corollary to Theorem 6 of (8)). *If f and g are regularly homotopic PL immersions of M in Q, and $q \geqslant m+3$, then they have isomorphic normal block bundles.*

COROLLARY. *If f is a PL immersion with a normal k-frame, and g is regularly homotopic to f, then g has a normal k-frame.*

LEMMA 5·3 (cf. (10)). *If M PL immerses in R^s, Q PL embeds in R^t, and $s+t \geqslant 2m+1$, then $M \times Q$ PL embeds in R^{s+t}.*

Proof. Since M immerses in R^s, M must immerse in R^{s+t} with a normal t-frame. Dimensions are such that we may regularly homotop this immersion to an embedding; this preserves the normal structure. Thus $M \times R^t$ embeds in R^{s+t}. It is now clear that $M \times Q$ embeds in $M \times R^t$, and so in R^{s+t}.

LEMMA 5·4 (1). *If M and Q are differentiable manifolds, and $q \geqslant \frac{3}{2}(m+1)$, then if M embeds in Q it also smoothly embeds in Q.*

LEMMA 5·5 (10). *If $p = 2^r$, and $s \geqslant p+1$ is the sum of two distinct powers of two, then $CP_p \times CP_{2s}$ does not smoothly immerse in $R^{4(p+2s-1)}$.*

COROLLARY. *$CP_{(2^r)}$ does not PL immerse in $R^{(2^{r+2}-2)}$.*

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Proof. Suppose it did. For s as in the hypotheses of Lemma 5·5, CP_{2s} embeds in R^{8s-2}. Then Lemma 5·3 implies $CP_{(2^r)} \times CP_{2s}$ embeds in $R^{4(2^r+2s-1)}$. For large enough s, Lemma 5·4 may now be applied to give a smooth embedding, contradicting Lemma 5·5. Thus our supposition was false, and $CP_{(2^r)}$ cannot be PL immersed in $(2^{r+2}-2)$-space.

6. *Pushing branch points through the boundary.* The technique used by Hudson in (4) of engulfing the singular set from the boundary and then bodily removing it can equally well be applied with slightly weaker connectivity conditions to produce an immersion.

LEMMA 6·1 (engulfing from the boundary) (2). *Let M be relatively k-connected, and X^k and C be (compact) polyhedra in M, such that $C \searrow C \cap \partial M$.*

Then, if $k \leqslant m-3$, there is a (compact) polyhedron C' in M, such that $C \cup X \subseteq C' \searrow C' \cap \partial M$, and $\dim((C'-C) \cap \mathrm{int}(M)) \leqslant k+1$. If $X \cap \partial M \subseteq C \cap \partial M$, we may further require that $C' \cap \partial M = C \cap \partial M$.

LEMMA 6·2 (LEMMA 4 of (5)). *If N is a compact $(m-1)$-submanifold of ∂M, then there is a PL embedding $c: N \times I \to M$, such that $c(x, 0) = x$ for all $x \in N$, and*

$$c^{-1}\partial M = (N \times (0)) \cup (\partial N \times I).$$

Such an embedding is called a collar of N in M.

THEOREM 2. *If M is relatively $(2m-q-1)$-connected, Q relatively $(2m-q)$-connected and $q \geqslant m+3$, then any proper map $f: M \to Q$ is homotopic to a PL immersion.*

Proof. We may put f and $f|\partial M$ into general position by a homotopy, and by Lemma 2·2 we may suppose that f is regular at the boundary as well. Triangulate so that $S_2(f)$ is a subcomplex. Suppose inductively that there are (compact) polyhedra C_i, $K_i \subseteq M$, and $D_i \subseteq Q$, such that:

$$C_{i-1} \cup K_i \subseteq C_i \searrow C_i \cap \partial M, \quad fC_i \cup D_{i-1} \subseteq D_i \cap \partial Q,$$
$$f^{-1}D_i = C_i \cup K_{i+1}, \quad \dim(K_i) \leqslant 2m-q-i-1 \quad (i \geqslant 0).$$

(*a*) $i = 0$. Let $C_{-1} = D_{-1} = \phi$, and K_0 be the $(2m-q-1)$-skeleton of $S_2(f)$.

(*b*) *Suppose D_{i-1}, C_{i-1} and K_i have been constructed, $i \geqslant 0$.*

Lemma 6·1 gives us a polyhedron $C_i \subseteq M$ such that $K_i \cup C_i \subseteq C_i \searrow \partial M$, and $\dim(C_i - K_i) \leqslant 2m-q-i$. By Lemma 6·1 again, there is a polyhedron D_i in Q for which $fC_i \cup D_{i-1} \subseteq D_i \searrow D_i \cap \partial M$, and $\dim(D_i - fC_i) \leqslant 2m-q-i+1$. Shift D_i if necessary until $D_i - fC_i - (D_i \cap \partial Q)$ is in general position with respect to fM. Then $f^{-1}D_i = C_i \cup K_{i+1}$ say, where K_{i+1} is a (compact) polyhedron in M of dimension at most $(2m-q-i+1)+m-q \leqslant 2m-q-i-2$.

(*c*) When $i = k$, $K_i = \phi$. Set $C = C_k$ and $D = D_k$. Then

$$K_0 \subseteq C \searrow C \cap \partial M, \quad fC \subseteq D \searrow D \cap \partial Q \quad \text{and} \quad f^{-1}D = C.$$

Let Q^* be a regular neighbourhood of $fM \cup D$ in Q which meets the boundary regularly. Then Q^* is compact. Triangulate so that f is simplicial and C and D subcomplexes.

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Let B be the second derived neighbourhood of $D \cup \partial Q^*$ in ∂Q^*, and $A = f^{-1}B$ that of $C \cap \partial M$ in ∂M. Subdivide twice so that A and B become subcomplexes, and let N_2 be the second derived neighbourhood of $B \cup D$ in Q^* and $N_1 = f^{-1}N_2$ be that of $A \cup C$ in M. Observe that $f|\overline{N - N_1}: \overline{M - N_1} \to \overline{Q^* - N_2}$ must be a PL immersion, since the branch points of f lie in N_2. When we reached this stage in Theorem 1, we extended f by an immersion of N_1 in N_2; here we may proceed more simply by pulling the boundaries in across N_1 and N_2; more precisely: Let $c: A \times I \to M$ be a collar of A in M. Since N_1 is a regular neighbourhood of A in M meeting the boundary regularly, the uniqueness of regular neighbourhoods allows us to apply an ambient isotopy, if necessary, to ensure that $c(A \times [0, \frac{1}{2}]) = N_1$. Define maps $F_t; M \to M$, $0 \leqslant t \leqslant 1$, by

$$F_t c(x, s) = c(x, s + \tfrac{1}{2}t(1 - s)) \quad \text{for} \quad (x, s) \in A \times I,$$

and $F =$ the identity outside $c(A \times I)$. Then F_t gives a homotopy of $(M, \overline{\partial M - A}, A)$ in $(M, \overline{\partial M - A}, N_1)$, F_0 is the identity and F_1 a PL homeomorphism $M \cong \overline{M - N_1}$. Let $d: B \times I \to Q^*$ be a collar of B in Q^*. As before, we may assume that $d(B \times [0, \frac{1}{2}]) = N_2$. Define maps $G_t: Q^* \to Q^*$. $0 \leqslant t \leqslant 1$, by

$$G_t d(x, s) = d(x, s(1 - t)) \qquad 0 \leqslant s \leqslant \tfrac{1}{2},$$
$$G_t d(x, s) = d(x, s + t(s - 1)) \quad \tfrac{1}{2} \leqslant s \leqslant 1,$$

and $G_t =$ the identity outside $d(B \times I)$. Then G_t gives a homotopy of $(Q^*, \partial Q^*, N_2)$ in itself, G_0 is the identity, $G_1(N_2) \subseteq B$, and $G_1|\overline{Q^* - N_2}$ is a PL homeomorphism $\overline{Q^* - N_2} \cong Q^*$. Define g to be the composition:

$$M \xrightarrow{F_1} \overline{M - N_1} \xrightarrow{f|\overline{M - N_1}} \overline{Q^* - N_2} \xrightarrow{G_1} Q^* \subseteq Q.$$

Then $g = G_1 f F_1 \sim G_0 f F_0 = f$ and is the required immersion.

Remark. Observe that g has no triple points, and t that the singular set consists of $2k$ $(2m - q)$-balls meeting ∂M in their boundaries, which g identifies in pairs but embeds individually.

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