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NORMAL DERIVABILITY IN CLASSICAL NATURAL DEDUCTION

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Abstract. A normalization procedure is given for classical natural deduction with the standard rule of indirect proof applied to arbitrary formulas. For normal derivability and the subformula property, it is sufficient to permute down instances of indirect proof whenever they have been used for concluding a major premiss of an elimination rule. The result applies even to natural deduction for classical modal logic.

§1. The problem and suggested solutions. In a step of indirect inference in natural deduction, a negative temporary assumption $\neg A$ is closed and A concluded. When assumptions are closed otherwise, they are subformulas of the conclusion, as in implication introduction, or of the major premiss, as in those elimination rules that close assumptions, and the subformula property of normal derivations follows. The situation is different with indirect proof, because the conclusion A can be a major premiss in an elimination, and it can be lost trace of even in the absence of nonnormal instances of the rest of the rules.

Gentzen failed to prove normalization for classical natural deduction and was led to believe that even the subformula property fails to hold (cf. von Plato, 2012). The first solution to the problem, by Prawitz (1965), included the leaving out of disjunction and existence from the language and the restriction of conclusions of indirect proof to atomic formulas, by which they cannot be major premisses of elimination rules. Other solutions have involved restrictions in the way elimination rules can be instantiated (Statman, 1974; Stålmarck, 1991), and yet others contain global proof transformations (Andou, 1995).

In the present work, it is shown that derivations in the full language of predicate logic can be so transformed by standard methods of local permutations that no conclusion of an indirect step of proof is the major premiss of an elimination rule. For the rest, normal form is defined as for intuitionistic derivations, in particular, no *a priori* restrictions are imposed on rule instances: Normal derivations and whatever rule instances they may contain come out purely as results of a normalization procedure. It follows especially that normal derivations have the subformula property. The situation is particularly clear if natural deduction is formulated in terms of general elimination rules (cf. von Plato, 2001) with the definition: A derivation is *normal* if all major premisses of elimination rules are assumptions. This definition can be applied directly to classical natural deduction for predicate logic, and also to natural deduction for classical modal logic.

§2. A natural solution. By Prawitz' (1965) proof strategy, no major premiss of an elimination rule in a normal derivation can have been concluded by indirect proof. It turns out that this property is sufficient for normal derivability with disjunction and existence included. For a uniform treatment, we formulate first intuitionistic natural deduction, here

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system **NI**, with general elimination rules as in von Plato (2001), and then look in the following section at the standard rules that will not present any particular difficulties. The rules that differ from standard natural deduction are:

$$\frac{A\&B \quad \begin{array}{c}[A,B]^{1,1}\\ \vdots\\ C\end{array}}{C}\,\&E,1 \qquad \frac{A\supset B \quad A \quad \begin{array}{c}[B]^{1}\\ \vdots\\ C\end{array}}{C}\,\supset E,1 \qquad \frac{\forall x A \quad \begin{array}{c}[A(t/x)]^{1}\\ \vdots\\ C\end{array}}{C}\,\forall E,1$$

Table 1. *General E-rules for* &, ⊃, ∀

The standard rules are special cases, as in:

$$\frac{A\&B \quad [A]^{1}}{A}\,\&E,1 \qquad \frac{A\&B \quad [B]^{1}}{B}\,\&E,1 \qquad \frac{A \supset B \quad A \quad [B]^{1}}{B}\,\supset E,1 \qquad \frac{\forall x A \quad [A(t/x)]^{1}}{A(t/x)}\,\forall E,1$$

Table 2. *Gentzen's E-rules as special cases of general E-rules*

It is convenient not to write out the degenerate derivations of minor premisses.

To obtain classical natural deduction, system **NK**, we add to **NI** rule *DN*:

$$\frac{\begin{array}{c}[\neg A]^{1}\\ \vdots\\ \bot\end{array}}{A}\,DN,1$$

Table 3. *The rule of indirect proof*

DEFINITION 2.1. *A derivation in* **NK** *is* **normal** *if all major premisses of E-rules are assumptions.*

THEOREM 2.2. **Normalization for classical natural deduction**. *Derivations in NK convert to normal form.*

Proof. We consider first the essential case when the major premiss of an E-rule has been derived by *DN*. Rule $\bot E$ applied to the conclusion of *DN* need not be considered, because the assumption $\neg\bot$ closed by the rule is a theorem.

The proof transformation follows the same pattern in each of the five different E-rules to be treated. It will be helpful to present the transformation schematically through the use of the derivability relation $\vdash$ of natural deduction, to get it right in each of the cases:

We have a major premiss $A \circ B$ and the derivability relation $\neg(A \circ B), \Gamma \vdash \bot$. By rule *DN*, we get $\Gamma \vdash A \circ B$. With the conclusion $A \circ B$ as the major premiss in an E-rule, we have a composition after that rule, as in

$$\frac{\dfrac{\neg(A\circ B),\Gamma\vdash\bot}{\Gamma\vdash A\circ B}\,DN \qquad \dfrac{\vdots}{A\circ B,\Delta\vdash C}\,\circ E}{\Gamma,\Delta\vdash C}\,Comp$$

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The transformation by which *DN* is permuted below the *E*-rule is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{A \circ B, \Delta \vdash C}\circ E \quad \bot \vdash \bot}{A \circ B, \neg C, \Delta \vdash \bot}L\supset}{\neg C, \Delta \vdash \neg(A \circ B)}R\supset \qquad \neg(A \circ B), \Gamma \vdash \bot}{\neg C, \Gamma, \Delta \vdash \bot}Comp}{\Gamma, \Delta \vdash C}DN$$

One just turns the derivation of C from $A \circ B$ into a derivation of $\neg(A \circ B)$ from $\neg C$ and then does the composition, followed by *DN*. The cases are:

1. The major premiss of $\&E$ has been derived by rule *DN*, and the part of derivation and its permutation are:

$$\dfrac{\dfrac{\neg(A \overset{1}{\&} B) \quad \dfrac{\vdots}{\dfrac{\bot}{A \& B}DN,1} \quad \overset{2}{A}, \overset{2}{B}\ \vdots\ C}{C}\&E,2}{} \quad \rightsquigarrow \quad \dfrac{\overset{1}{A}, \overset{1}{B}\ \vdots\ \dfrac{A \overset{2}{\&} B \quad C}{C}\&E,1 \quad \overset{3}{\neg C}}{\dfrac{\dfrac{\bot}{\neg(A \& B)}\supset I,2}{\dfrac{\vdots}{\dfrac{\bot}{C}DN,3}}}\supset E$$

2. With rule $\vee E$, the part of derivation and its permutation are:

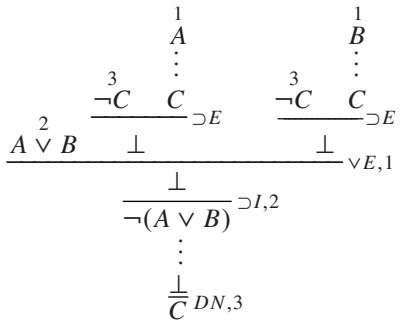
$$\dfrac{\dfrac{\neg(A \overset{1}{\vee} B) \quad \dfrac{\vdots}{\dfrac{\bot}{A \vee B}DN,1} \quad \overset{2}{A}\ \overset{2}{B}\ \vdots\ \vdots\ C\ C}{C}\vee E,2}{} \quad \rightsquigarrow \quad \dfrac{\overset{1}{A}\ \overset{1}{B}\ \vdots\ \vdots\ \dfrac{A \overset{2}{\vee} B \quad C \quad C}{C}\vee E,1 \quad \overset{3}{\neg C}}{\dfrac{\dfrac{\bot}{\neg(A \vee B)}\supset I,2}{\dfrac{\vdots}{\dfrac{\bot}{C}DN,3}}}\supset E$$

3. The transformations for the remaining rules $\supset E$, $\forall E$, and $\exists E$ are similar and need not be detailed out here.

All the other cases of nonnormalities are *E*-rules with a major premiss derived by an *I*-rule or another *E*-rule. These are covered by the normalization theorem for intuitionistic natural deduction, as in Negri & von Plato (2001, pp. 199–201) and, slightly corrected, Negri & von Plato (2011, pp. 27–28). QED.

It is possible to permute in the second transformed derivation the instance of $\supset E$ above $\vee E$, with the result

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In Statman (1974), the restriction is put on $\vee E$ that it should always appear in this form, with the formula $\bot$ as the conclusion, and analogously for rule $\exists E$.

If we consider a normal derivation in **NK**, from the endformula upward, there will be a sequence of I-rules and their nested premisses, two in case of $\&I$, until a conclusion of rule *DN* is reached. Its premiss is $\bot$. Looking from the other direction, from topformulas downward, we find a nested sequence of major premisses of E-rules. It is now clear how the presence of rule *DN* can force conclusions of E-rules in normal derivations to be equal to the premiss $\bot$ of *DN*, without any *a priori* requirement that this should be so.

The subformula structure of normal derivations is captured by the notion of a **thread** in a derivation tree, as above. Threads are the equivalents of derivation branches in the $\vee, \exists$-free fragment, obtained by starting from the endformula of a derivation and moving upward to premisses until a conclusion of an E-rule is encountered. Threads now continue with the minor premisses, until assumptions closed by the E-rule are reached. From there they jump into the major premiss of which the closed assumptions are subformulas, and continue towards uppermost formulas. With rule $\supset E$, a new thread begins from its minor premiss (cf. Negri & von Plato, 2001, p. 196, or Negri & von Plato, 2011, p. 26 for formal definitions).

There is at most one instance of rule *DN* along any thread of a normal derivation, for otherwise there would be an instance of *DN* with the conclusion $\bot$. The assumption closed by *DN*, however, would then be the provable formula $\bot \supset \bot$. Therefore, as in Prawitz (1965), steps of indirect proof separate a part with E-rules from a successive part with I-rules, save in threads instead of derivation branches.

COROLLARY 2.3. **Subformula property.** *All formulas in normal derivations of A from the open assumptions* Γ *are subformulas of* $\neg C$*, with C a subformula of* A, Γ.

Proof. Consider the threads of a normal derivation. As noted, there is at most one instance of rule *DN* along any thread of a normal derivation, and let its conclusion be B. The derivation of its premiss $\bot$ is intuitionistic, so that all formulas in such a normal derivation are subformulas of open assumptions or of $\neg B$, or equal to $\bot$. QED.

The subformula property of normal derivations becomes particularly clear through a translation to sequent calculus, because in sequent calculus derivations, that property is carried along branches of derivation trees instead of threads. With general elimination rules, the translation of derivations in intuitionistic natural deduction into derivations in sequent calculus was given in von Plato (2001), and with the standard rules, in von Plato (2011). Rule *DN* is translated as in

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$$\dfrac{\dfrac{\begin{array}{c}\neg^{1}A\\ \vdots\\ \bot\end{array}}{A}\,DN,1}{} \quad\rightsquigarrow\quad \dfrac{\neg A, \Gamma \to \bot}{\Gamma \to A}\,DN$$

If $\neg A$ is vacuously discharged in the natural rule, *DN* is in fact an instance of $\bot E$, if it is multiply discharged, contractions on $\neg A$ are made in the sequent derivation before *DN* is applied. The subformula property is immediate.

It is possible to translate also nonnormal classical derivations to sequent calculus, followed by cut elimination, along the lines of von Plato (2001) for the general rules, and von Plato (2011) for the standard ones.

§3. Normal derivability in standard natural deduction In standard intuitionistic natural deduction, normal derivations can contain major premisses of elimination rules that have been derived instead of assumed. The possible cases are rules $\&E$, $\supset E$, and $\forall E$. In each of these, the inference is to a subformula of the major premiss, and the subformula property of normal derivations is maintained.

The notion of normal derivability in the case of indirect inferences, namely that no conclusion of rule *DN* is a premiss of an *E*-rule, works also when rule *DN* is added to standard intuitionistic natural deduction.

THEOREM 3.1. **Normalization for classical standard natural deduction.** *Derivations in standard natural deduction extended by rule DN convert to a form in which no major premiss of an E-rule has been concluded by rule DN.*

Proof. Whenever the conclusion of rule *DN* is a major premiss of one of $\&E$, $\supset E$, or $\forall E$, the indirect inference can be permuted down, as in

$$\dfrac{\dfrac{\neg(A \,\&\, B)^{1}\\ \vdots\\ \bot}{A \,\&\, B}\,DN,1}{A}\,\&E \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\dfrac{A \,\&\, B}{A}\,\&E \quad \neg A^{2}}{\bot}\,\supset E}{\neg(A \,\&\, B)}\,\supset I,1}{\begin{array}{c}\vdots\\ \bot\end{array}}\,DN,2}{A}$$

The permutations are similar with rules $\supset E$ and $\forall E$. QED.

These permutations convert the indirect proof of a formula $A \,\&\, B$, $A \supset B$, or $\forall x\, A(x)$ into one on a subformula and are used to this purpose in Prawitz (1965). It has been found recently that they were known already by Gentzen in January 1933 (cf. von Plato, 2012).

It remains to check that if a derivation has none of the standard convertibilities to normal form of intuitionistic natural deduction, and if no conclusion of rule *DN* is a major premiss of any *E*-rule, the derivation has the subformula property: As in Section 2, there will be at most one instance of *DN* along any derivation thread, so that conclusions of *DN* are subformulas of the following *I*-rules and therefore in the end subformulas of the endformula of the whole derivation.

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§4. Normal derivability in classical modal logic Indirect proof can be permuted down with respect to all of the propositional rules, and several such rules contract into just one. Therefore one indirect inference as a last rule suffices for classical propositional logic. The situation is different in predicate logic, because the variable condition in rule $\forall I$ makes it impossible to permute down *DN*. A similar thing happens in modal logic: There is a condition in rule $\Box I$ that is entirely analogous to the eigenvariable condition of rule $\forall I$. The solution to the problem for predicate logic, as in the present note, applies also to natural deduction for classical modal logic, the latter obtained by adding to the intuitionistic propositional rules and *DN* the two modal rules:

$$\frac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}}{\Box A}\Box I \qquad \frac{\Box A \quad \begin{array}{c}1\\ A\\ \vdots\\ C\end{array}}{C}\Box E, 1$$

Table 4. *Natural modal rules*

Rule $\Box I$ has the restriction that the open assumptions Γ on which the premiss A depends must be modal formulas, and the effect is the same as with rule $\forall I$, namely, that indirect proof cannot be permuted below $\Box I$.

Normal derivability in modal logic with general elimination rules is defined as above in Definition 2.1: major premisses of elimination rules have to be assumptions. The normalization theorem for intuitionistic natural deduction extended by the two modal rules is proved in von Plato (2005) and carries over to the classical case:

THEOREM 4.1. **Normalization for classical modal logic.** *Derivations in* **NK** *plus* $\Box I, \Box E$ *convert to normal form.*

Proof. The only new case is rule $\Box E$. If its major premiss has been concluded by rule *DN*, the latter is permuted down exactly as indicated in the proof of Theorem 2.2. QED.

The subformula property follows as in the proof for **NK**.

The normalization of derivations holds equally well if in place of rule $\Box E$ of Table 4 the special case with $C \equiv A$ is used.

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