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# Ergodic multiplier properties

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*Abstract*. In this article we will extend 'the weak mixing theorem' for certain locally compact Polish groups (Moore groups and minimally weakly mixing groups). In addition, we will show that the Gaussian action associated with the infinite-dimensional irreducible representation of the continuous Heisenberg group, $H_3(\mathbb{R})$, is weakly mixing but not mildly mixing.

## 1. *Introduction*

We begin with introducing some of the main definitions and the history of known theorems concerning weak mixing. For more details, see [**13**], [**1**] or [**16**].

Let $(X, \mathcal{B}, m)$ be a standard probability space and $T$ a non-singular action.

*Definition 1.1.* $T$ is called an *ergodic action* if every $G$-invariant set is trivial.

PROPOSITION 1.2. *Let $G$ be a group, $T$ a non-singular ergodic action of $G$ on a standard probability space $(X, \mathcal{B}, m)$ and $G_0 \subseteq G$ a dense subgroup. Then the action of $G_0$ on $(X, \mathcal{B}, m)$ is also ergodic.*

*Definition 1.3.* A non-singular action of $G$ on a standard space is called *weakly mixing* if for any probability-preserving ergodic action of $G$, the product action is also ergodic.

One result for probability-preserving $\mathbb{Z}$ actions is 'the weak mixing theorem'. It was originally proved by Koopman and von Neumann in 1932 in their article 'Dynamical systems of continuous spectra' (see [**22**]). A simple proof can also be found in [**28**]. This theorem presents equivalent conditions for weak mixing such as spectral conditions, conditions related to action-invariant $L_2(X, \mathcal{B}, m)$ subspaces and more. The first generalization of this theorem for $\sigma$-finite measure preserving actions was presented by Aaronson, Lin and Weiss in [**3**]. Another generalization of this theorem for probability-preserving actions of locally compact Polish groups will be fully stated soon.

Another result for $\mathbb{Z}$ actions is 'the ergodic multiplier theorem'. This theorem, originally proved by Keane, spectrally characterizes when the product of a non-singular action and a probability-preserving action is ergodic. In this article, we will extend this theorem for general locally compact Polish abelian groups.

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A natural next step was to study similar properties for more general groups. The first progress in this direction, originally proved by Dye in [**10**] and later improved by others, concerned unitary representations of locally compact Polish groups, which means that this method can only be applied to probability-preserving actions.

*Definition 1.4.* A function $f \in C_b(G)$ is called *almost periodic* if the orbit $\{f_g \mid g \in G\}$ is precompact in $C_b(G)$ endowed with the strong topology. A function $f \in C_b(G)$ is called *weakly almost periodic* if the orbit $\{f_g \mid g \in G\}$ is precompact in $C_b(G)$ endowed with the weak topology.

Denote by $WAP(G)$ the algebra of weakly almost periodic functions and by $AP(G)$ the algebra of almost periodic functions.

*Definition 1.5.* Let $\mathfrak{A}$ be a vector space of bounded functions on $G$, which is closed under conjugations and contains the constant functions. A linear functional defined on $\mathfrak{A}$, $\mathfrak{m}$, is called a *mean* if (1) for every $f \in \mathfrak{A}$, $\mathfrak{m}(\overline{f}) = \overline{\mathfrak{m}(f)}$, (2) if $f \geq 0$, then $\mathfrak{m}(f) \geq 0$, (3) $\mathfrak{m}(\mathbf{1}) = 1$.

For every function $f \in \mathfrak{A}$, denote by $f_g$ the function $f_g(h) = f(gh)$. The mean $\mathfrak{m}$ is called a *left $G$-invariant mean* if it is a mean and in addition $m(f_g) = m(f)$ for every $f \in \mathfrak{A}$ and $g \in G$. There is a similar notion for a right $G$-invariant mean.

THEOREM 1.6. *The algebra $WAP(G)$ admits a unique left and right $G$-invariant mean, denoted by* $\mathfrak{m}$.

This theorem was originally proved by Nardzewski in [**32**, Theorem 4]. See also [**13**, Theorem 1.51], and [**11**, Theorem III A.2].

By using the invariant mean on weakly almost periodic functions, similar equivalences to the ones presented for $\mathbb{Z}$ actions can be deduced for representations of locally compact Polish groups.

*Definition 1.7.* A continuous unitary representation of a group $G$ on a Hilbert space $\mathcal{H}$ is called *weakly mixing* if for every $f_1, f_2 \in \mathcal{H}$, $\mathfrak{m}(|\langle \pi(g) f_1, f_2 \rangle|) = 0$.

It is called *ergodic* if for every $f_1, f_2 \in \mathcal{H}$, $\mathfrak{m}(\langle \pi(g) f_1, f_2 \rangle) = 0$.

THEOREM 1.8. (The weak mixing theorem for probability-preserving actions) *Let $(X, \mathcal{B}, m)$ be a standard probability space, $G$ a locally compact Polish group and $T : G \to PPT(X)$. Let $\pi$ be the reduced Koopman representation of $G$ induced by $T$. Then the following are equivalent:*

(1) *$\pi$ is weakly mixing;*
(2) *for every $f_1, f_2 \in L_2(X, \mathcal{B}, m)_0$, $\mathfrak{m}(|\int (\pi(g) f_1) f_2 \, dm|) = 0$;*
(3) *$\pi$ contains no non-zero finite-dimensional subrepresentations;*
(4) *the unitary representation $\pi \times \pi : G \to AUT(\mathcal{H} \times \mathcal{H})$ is ergodic;*
(5) *for every unitary representation $\sigma : G \to AUT(\mathcal{K})$, on a Hilbert space $\mathcal{K}$, the unitary representation $\pi \times \sigma : G \to AUT(\mathcal{H} \times \mathcal{K})$ is ergodic.*

This theorem is a corollary from Bergelson and Rosenblatt's article in 1988 [**6**].

*Remark 1.9.* (Bergelson and Gorodnik [**5**]) Let $G$ be a locally compact Polish group and $T : G \to PPT(X)$ a probability-preserving action of $G$ on a standard probability space $(X, \mathcal{B}, m)$. $T$ is weakly mixing if and only if its reduced Koopman representation is weakly

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mixing if and only if there is no non-trivial finite-dimensional $T$-invariant subspace in $L_2(X, \mathcal{B}, m)_0$.

PROPOSITION 1.10. *Let $G$ be a locally compact Polish group and $T$ a non-singular action on a standard probability space $(X, \mathcal{B}, m)$. Denote by $U$ the Koopman representation, defined by*

$$U_g f = \sqrt{\frac{dm_g}{dm}} \cdot f \circ T_g = \sqrt{\frac{d(m \circ T_g)}{dm}} \cdot f \circ T_g.$$

*If there is no absolutely continuous invariant probability measure, then $U$ is weakly mixing.*

*Proof.* Since $U$ is a continuous unitary representation, if it is not weakly mixing, by Remark 1.9, there exists a finite-dimensional non-trivial subspace in $L_2(X, \mathcal{B}, m)_0$; denote it by $V$. Let $f_1, \ldots, f_d$ be an orthonormal basis for $V$. For every $1 \le j \le d$ and for every $g \in G$, there exist $a_1^j(g), a_2^j(g), \ldots, a_d^j(g) \in \mathbb{R}$ such that $U_g f_j = \sum_{i=1}^d a_i^j(g) f_i$, since $U_g f_j \in V$. Define the matrix

$$A_g := \begin{pmatrix} a_1^1(g) & a_2^1(g) & \ldots & a_n^1(g) \\ a_1^2(g) & a_2^2(g) & \ldots & a_n^2(g) \\ \vdots & \vdots & \ldots & \vdots \\ a_1^n(g) & a_2^n(g) & \ldots & a_n^n(g) \end{pmatrix}.$$

Since $U_g$ is linear, for every $\{a_j\}_{j=1}^d \subset \mathbb{R}$, $\sum_{j=1}^d a_j f_j \in V$. Denote $\psi(\overline{a}) = \sum_{j=1}^d a_j f_j$,

$$(U_g f)(x) = \sum_{j=1}^d a_j \cdot (U_g f_j) = \psi\left( A_g^T \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix} \right).$$

We will show that for every $x \in \mathbb{R}^d$, $\|A_g x\|_2 = \|x\|_2$. Define $\pi : V \to \mathbb{R}^d$ by

$$\pi\left( \sum_{j=1}^d a_j f_j \right) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix}.$$

Then, since $\{f_j\}$ is an orthonormal basis, $\pi$ is an isometry. We have $\pi(U_g f) = A_g^T \cdot \pi(f)$ and, since $\pi$ is an isometry, we conclude that for every $x \in \mathbb{R}^d$, $\|A_g^T x\|_2 = \|x\|_2$ and therefore $\|A_g x\|_2 = \|x\|_2$.

Define $F : X \to \mathbb{R}^d$ by $F(x) = (f_1(x), f_2(x), \ldots, f_d(x))$ and define the function $\phi : X \to \mathbb{R}$ by

$$\phi(x) = \frac{1}{d} \sum_{j=1}^d |f_j(x)|^2 = \frac{1}{d} \|F(x)\|_2^2,$$

where $\|*\|_2$ denotes the Euclidean norm in $\mathbb{R}^d$. Define the measure $d\mu = \phi \cdot dm$; then it is a probability measure and, for every set $B \in \mathcal{B}$, $\mu(T_g B) = \mu(B)$, since $\sum_{j=1}^d |U_g f_j|^2 = \|A_g F\|_2^2 = \|F\|_2^2$. □

For $G = \mathbb{Z}$, this proposition was already known; see [**1**, remark on p. 75].

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Bergelson and Rosenblatt's result is not enough to characterize weak mixing when it comes to non-singular actions. There are examples of non-singular actions which are not weakly mixing, but no invariant probability measure exists; for example see [**1**, pp. 29–31]. By Proposition 1.10, their reduced Koopman representation is weakly mixing, but the actions are not.

In this article, we will use representation theory to find ergodic multiplier properties for non-singular actions of locally compact Polish Moore groups (see §4). We will show that if $G$ is a locally compact Polish Moore group, then a non-singular action of $G$ is weakly mixing if and only if every finite-dimensional $G$-invariant subspace of $L_\infty(X, \mathcal{B}, m)$ is trivial. In addition, we will show some examples of such actions.

We will conclude the article with an action of the Heisenberg group, which is weakly mixing but not mildly mixing (see §5).

## 2. *Spectral properties of non-singular actions of locally compact abelian groups*

### 2.1. *Preliminaries.*

THEOREM 2.1. (The scalar spectral theorem) *Let $G$ be a locally compact abelian group and $T : G \to PPT(X)$ an ergodic action of $G$ defined on a standard probability space $(X, \mathcal{B}, m)$. Denote by $\sigma_0$ the spectral type of the reduced Koopman representation induced by $T$. Then there exists a non-trivial sesquilinear map*

$$h : L_2(X, \mathcal{B}, m)_0 \times L_2(X, \mathcal{B}, m)_0 \to L_1(\sigma_0)$$

*such that for every $\alpha, \beta \in L_2(X, \mathcal{B}, m)_0$ and for every $g \in G$,*

$$(h(U_g\alpha, \beta))(\chi) = \chi(-g) \cdot (h(\alpha, \beta))(\chi), \tag{1}$$

*where this equality is an $L_1(\sigma_0)$ equality.*

*Proof.* By the 'spectral theorem' (see [**18**, **26**, **39**, **37**]), for every $\alpha, \beta \in L_2(X, \mathcal{B}, m)_0$ there exists a measure $m_{\alpha,\beta}$ such that

$$\langle U_g\alpha, \beta\rangle = \int_{\hat{G}} \chi(-g)\, dm_{\alpha,\beta}(\chi).$$

Denote by $\sigma_0$ the spectral type of the representation $\{U_g\}_{g\in G}$; then $m_{\alpha,\beta}$ is absolutely continuous with respect to $\sigma_0$. By the Radon–Nikodym theorem there exists a function $f_{\alpha,\beta} \in L_1(\sigma_0)$ such that $(dm_{\alpha,\beta}/d\sigma_0) = f_{\alpha,\beta} \in L_1(\sigma_0)$. Define the map

$$h(\alpha, \beta) = \frac{dm_{\alpha,\beta}}{d\sigma_0}.$$

It is indeed sesquilinear, non-trivial and equation (1) holds. □

THEOREM 2.2. (The eigenvalue theorem for locally compact abelian groups—originally proved by Schmidt (see [**35**])) *Let $G$ be a locally compact abelian group and $T : G \to NST(X)$ a $G$-action defined on a standard probability space $(X, \mathcal{B}, m)$ such that $T$ is non-singular and ergodic. Then:*

- *the set $e(T)$ is a Borel set;*
- *there exists $\psi : e(T) \times X \to S^1$, a measurable map, such that, for every $\chi \in e(T)$ and for every $g \in G$,*

$$\psi(\chi, T_g x) = \chi(g) \cdot \psi(\chi, x) \text{ for } m\text{-almost every } x \in X. \tag{2}$$

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*Proof.* The proof is an extension of the proof in [**1**, p. 76]. □

*Remark 2.3.* It would be interesting to have a generalized version of the eigenvalue theorem for Moore groups (see §4).

*Definition 2.4.* A non-singular action of $G$ on a standard probability space is called *properly ergodic* if it is ergodic and every orbit has measure zero, i.e. for every $x \in X$, $m(\{T_g x \mid g \in G\}) = 0$. For more information, see [**36**].

2.2. *The ergodic multiplier theorem.* In this section, we will prove the ergodic multiplier theorem for actions of locally compact Polish abelian groups.

THEOREM 2.5. (The ergodic multiplier theorem) *Let $G$ be a locally compact Polish abelian group, let $(X, \mathcal{B}, m)$ be a standard space and let $T : G \to NST(X)$ be a non-singular and properly ergodic $G$-action. Let $(Y, \mathcal{C}, \nu)$ be a standard probability space, $S : G \to PPT(Y)$ an ergodic $G$-action and denote by $\sigma_0$ the spectral type of reduced Koopman representation induced by the action $S$. Then the following are equivalent:* (1) *$T \times S$ is ergodic,* (2) *$\sigma_0(e(T)) = 0$.*

*Proof.* The proof is essentially the same proof as in [**1**], with minor changes. We will mention the changes; for a full proof, see [**16**].

(1) $\Leftarrow$ (2): we will show that if $T \times S$ is not ergodic, then $\sigma_0(e(T)) > 0$. Let $G_0 \subseteq G$ be a countable dense subgroup and denote by $T^0$ the action of $G_0$ on $(X, \mathcal{B}, m)$. According to Proposition 1.2, the action of $T^0$ on $(X, \mathcal{B}, m)$ is also ergodic and $T^0 \times S$ is not ergodic. Following the same proof as in [**1**], we get that $\sigma_0(e(T^0)) > 0$. Finally, $e(T) = e(T^0)$; therefore, $\sigma_0(e(T)) = \sigma_0(e(T^0)) > 0$.

(1) $\Rightarrow$ (2): we will show that if $\sigma_0(e(T)) > 0$, then $T \times S$ is not ergodic. Without loss of generality, there exists $f \in L_2(Y, \mathcal{C}, \nu)_0$ such that $\mu_f(e(T)) = \|f\|^2$. Let $U_g$ denote the reduced Koopman representation of $S$, the action of $G$ in $L_2(Y, \mathcal{C}, \nu)_0$, and $\mathcal{H}_f = \overline{\text{span}\{U_g f \mid g \in G\}}$. Let $V : \mathcal{H}_f \to L_2(\hat{G}, \hat{\mathcal{B}}, \mu_f)$ be the Hilbert space isometry defined for every $g \in G$ as

$$(V(U_g f))(\chi) = \chi(-g) \Rightarrow U_g f = V^{-1}(\chi(-g))$$

and, for other elements of $V$, define it as a linear transformation. Then it is well defined on $\mathcal{H}_f$, it is an isometry and, for every $g \in G$ and $h \in \mathcal{H}_f$,

$$V^{-1}(\chi(-g) \cdot h) = (U_g \circ V^{-1}(h))(\chi). \tag{3}$$

In addition, by 'the eigenvalue theorem' (Theorem 2.2), there exists a Borel function $\psi : e(T) \times X \to S^1$ such that for every $\chi \in e(T)$, every $g \in G$ and $m$-almost every $x \in X$, $\psi(\chi, T_g x) = \chi(g)\psi(\chi, x)$. Now, let us look at the function $\phi : X \to \{\eta;\ \eta : e(T) \to S^1\}$ defined by $(\phi(x))(\chi) = \psi(\chi, x)$. By the properties of $\psi$ (equation (2)), for every $g \in G$ and $\chi \in e(T)$,

$$(\phi(T_g x))(\chi) = \psi(\chi, T_g x) = (g(*) \cdot \phi(x))(\chi)\ m\text{-almost everywhere.}$$

Define the function $F : X \times Y \to \mathbb{C}$ by $F(x, y) = V^{-1}(\phi(x))(y)$; $F$ is well defined almost everywhere and it is $T \times S$ invariant.

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It is left to show that $F$ is not constant. Since $V$ is an isometry, it is enough to show that $\psi$ is not constant with respect to $x$. Let $G_0 \subset G$ be a dense countable subgroup and denote by $T^0$ the action of $G_0$ on $(X, \mathcal{B}, m)$. Assume that $\psi$ is constant with respect to $x$ and fix $\chi \in e(T)$. Then there exists $x \in X$ such that for every $g \in G_0$, we have $\psi(\chi, x) = \psi(\chi, T_g x)$ and therefore

$$\psi(\chi, x) = (\phi(x))(\chi) = (\phi(T_g x))(\chi) = \psi(\chi, T_g x) = \chi(g)\psi(\chi, x).$$

We conclude that $\chi(g) = 1$ for every $g \in G_0$ and, by continuity of the characters, for every $g \in G$. This is true for every $\chi \in e(T)$; therefore, $e(T) = \{\chi_0\}$ and $\sigma_0(\{\chi_0\}) = 0$, which is a contradiction. □

*Remark 2.6.* Note that given an eigenvalue theorem for a larger class of groups (for example Moore groups), one could easily extend the ergodic multiplier theorem for this class of group actions.

## 3. *Banach–Kronecker systems*

3.1. *Definitions and preliminaries.* Let $G$ be a locally compact Polish group and $X$ a second countable compact topological space. Denote by $(X, \pi, G)$ the topological dynamical system, where $\pi : G \times X \to X$.

LEMMA 3.1. *Let $X$ be a compact second countable topological space and let $G$ be a locally compact Polish group. If $(X, \pi, G)$ is an equicontinuous topological dynamical system, then there exists an invariant metric.*

The proof is left as an exercise for the reader; a full one can be found in [**16**].

*Definition 3.2.* A dynamical system, $(K/H, \pi, G)$, is called *a homogeneous system* if $K$ is a compact topological group, $H \leq K$ is a closed subgroup and $G$ is embedded in $K$ as a dense subgroup, acting on $K/H$ by left translations, meaning $\pi(g, k) = \tilde{g}k$, where $\tilde{g}$ is the embedding of $g$ in $K$.

LEMMA 3.3. (See [**4**]) *Let $X$ be a compact metric space, $G$ a locally compact Polish group and $(X, \pi, G)$ a topological dynamical system, which is minimal and equicontinuous. Then there exists a homogeneous system $(K/H, \tilde{\pi}, G)$ such that $(X, \pi, G)$ is topologically isomorphic to a translation on $(K/H, \tilde{\pi}, G)$. Moreover, there exists an action-invariant ergodic probability measure defined on $X$.*

*Proof.* Define $T : G \to \mathrm{InvHom}(X)$ by $T_g(x) = \pi(g, x)$; then, since the action is equicontinuous, by the Arzelá–Ascoli theorem the set $\{T_g \mid g \in G\}$ is a precompact subset of the set of homeomorphisms of $X$, endowed with the compact open topology. Denote $K := \mathrm{cls}\{T_g \mid g \in G\}$; then $K$ is a compact subgroup of the group of homeomorphisms. Fix $x_0 \in X$ and define $H := \mathrm{cls}\{T \in K;\ Tx_0 = x_0\}$.

Define $\varphi : K/H \to X$ by $\varphi(TH) = Tx_0$. We have that $\varphi$ is a continuous bijection and onto $X$; since $K/H$ is compact, $\varphi$ is an isomorphism.

Define the topological action of $G$ on $K/H$ by left multiplication $\tilde{\pi}(g, TH) = (T_g T)H$. Then

$$\pi(g, \varphi(TH)) = T_g(\varphi(TH)) = T_g(Tx_0) = (T_g T)(x_0) = \varphi(\tilde{\pi}(g, TH)).$$

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We conclude that $(X, \pi, G)$ is isomorphic as a topological dynamical system to the dynamical system $(K/H, \tilde{\pi}, G)$, a homogeneous system. Since $K/H$ is a compact set, the projection of the Haar measure onto $K/H$ is an invariant probability measure for $T$.

The set of invariant measures is a compact convex subset; by the Krein–Milman theorem, it is the convex hull of its extreme points. The action-invariant ergodic measures are the extreme points of this set; specifically, there exists an ergodic $G$-invariant probability measure. □

LEMMA 3.4. *Let $X$ be a compact metric space and $(X, \pi, G)$ a topological dynamical system. Assume that there exists an invariant metric $D$ and that the action is minimal. Then any non-singular measure is globally supported.*

*Proof.* Denote by $T$ the action of $G$ on $X$ ($T_g x = \pi(g, x)$) and let $m$ be a non-singular measure. If $m$ is not globally supported, then there exist $x_0 \in X$ and $\varepsilon_0 > 0$ such that $m(B_D(x_0, \varepsilon_0)) = 0$ and, since the action is minimal, the collection $\{B_D(T_g x_0, \varepsilon_0)\}$ is a cover for $X$. Since $X$ is compact, there exists a finite subcover $\{B_D(T_{g_n} x_0, \varepsilon_0)\}_{n=1}^{N}$. Then

$$\begin{aligned} 1 = m(X) &= m\left(\bigcup_{n=1}^{N} B_D(T_{g_n} x_0, \varepsilon_0)\right) \\ &\leq \sum_{n=1}^{N} m(B_D(T_{g_n} x_0, \varepsilon_0)) = \sum_{n=1}^{N} m(T_{g_n}(B_D(x_0, \varepsilon_0))) = 0, \end{aligned}$$

which is a contradiction. We conclude that the measure is globally supported. □

*Remark 3.5.* Note that this lemma is still true if $X$ is Polish and not compact. In this case the subcover is not finite, but countable, and the same proof holds.

### 3.2. *Banach–Kronecker systems are not weakly mixing.*

*Definition 3.6.* Let $\pi$ be a finite-dimensional representation of a locally compact Polish group $G$ on $\mathbb{R}^d$. If $X \subset \mathbb{R}^d$ is closed, bounded and $\pi$-invariant, and $(X, \pi, G)$ is minimal and equicontinuous, then the topological dynamical system $(X, \pi, G)$ is called a *Banach–Kronecker system*.

*Remark 3.7.* Note that according to Lemma 3.3, there exists an action-invariant measure; specifically, there exist non-singular measures.

THEOREM 3.8. *Let $(X, \pi, G)$ be a Banach–Kronecker system and denote by $T$ the topological action, meaning $T_g(x) = \pi(g, x)$. Then, for any non-singular measure, $m \sim m \circ T$, the action is not weakly mixing.*

*Proof.* Let $m$ be a non-singular measure; if it is not $G$-ergodic, then clearly the system is not weakly mixing. We will show that if $m$ is ergodic, then there exists a probability-invariant measure $P$ such that $(X \times X, \mathcal{B} \times \mathcal{B}, m \times P)$ is not ergodic.

Note that Banach–Kronecker systems are by definition minimal and equicontinuous; by Lemma 3.3, there exists a probability measure $P$ which is action invariant and ergodic. Define the following norm on linear homeomorphisms: $\|A\|_{\mathrm{op}} = \sup_{\|x\|=1} \|Ax\|$, and a

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metric on $X$, $d(x, y) = \|x - y\|$. There exists a constant $C$ such that for every $g \in G$, $\|T_g\|_{\mathrm{op}} < C$. For every $\varepsilon > 0$, define the set

$$D_\varepsilon := \left\{(x, y) \in X \times X;\ d(x, y) < \frac{\varepsilon}{2C}\right\}.$$

This set is an open set and therefore Borel measurable and, for every $g \in G$,

$$d(T_g(x), T_g(y)) \le \|T_g\|_{\mathrm{op}} \cdot d(x, y) < C \cdot d(x, y) < \varepsilon.$$

Let $G_0 \subseteq G$ be a dense countable subgroup in $G$. Define the set

$$O_\varepsilon = \{(T_g(x), T_g(y));\ g \in G_0,\ (x, y) \in D_\varepsilon\}.$$

First, this set is measurable as a countable union of measurable sets. Next, $e \in G_0$ and therefore $D_\varepsilon \subseteq O_\varepsilon$. In addition, it is clearly $G_0$-invariant by definition, and therefore it is also $G$-invariant. Finally, note that since $T_g T_h = T_{gh}$, then, for every $(x, y) \in O_\varepsilon$ and $g \in G$, there exist $x_0,\ y_0 \in D_\varepsilon$ and $h \in G_0$ such that

$$\begin{aligned} d(T_g x, T_g y) &= d(T_g T_h x_0, T_g T_h y_0) \\ &= d(T_{gh} x_0, T_{gh} y_0) \le \|T_{gh}\|_{\mathrm{op}} d(x_0, y_0) < C \cdot \frac{\varepsilon}{2C} < \varepsilon. \end{aligned}$$

This means that $O_\varepsilon \subseteq \{(x, y) \in X \times X;\ d(x, y) < \varepsilon\}$.

Denote by $D$ the action-invariant metric (such a metric exists by Lemma 3.1). The space $X$ is a compact metric space; therefore, all the metrics defined on it are equivalent and there exist $M, m > 0$ such that $m \cdot d(x, y) \le D(x, y) \le M \cdot d(x, y)$, which means that

$$B_d\left(x, \frac{r \cdot m}{M}\right) \subseteq B_D(x, r \cdot m) \subseteq B_d(x, r).$$

Then

$$m \times P(D_\varepsilon) = \int_X P\left(B_d\left(x, \frac{\varepsilon}{2C}\right)\right) dm \ge \int_X P\left(B_D\left(x, m \cdot \frac{\varepsilon}{2C}\right)\right) dm.$$

Now, the action is minimal and probability preserving, $D$ is an invariant metric, by Lemma 3.4, $P$ is globally supported and therefore for every $\varepsilon > 0$, $m \times P(O_\varepsilon) \ge m \times P(D_\varepsilon) > 0$.

We saw that $O_\varepsilon \subseteq \{(x, y);\ d(x, y) < \varepsilon\}$. Evidently there exists $\varepsilon > 0$ such that

$$m \times P(O_\varepsilon) \le m \times P(\{(x, y);\ d(x, y) < \varepsilon\}) < 1.$$

The set $O_\varepsilon$ is a non-trivial $T \times T$ invariant set; therefore, $T$ is not weakly mixing. □

3.3. *Banach–Kronecker factors and non-trivial, finite-dimensional subspaces of* $L_\infty(X, \mathcal{B}, m)$.

THEOREM 3.9. *Let $T$ be a non-singular ergodic $G$-action defined on $(X, \mathcal{B}, m)$, a standard $\sigma$-finite measure space. There exists a $G$-invariant, non-trivial, finite-dimensional subspace of $L_\infty(X, \mathcal{B}, m)$ if and only if there exists a non-trivial Banach–Kronecker factor.*

*Proof.* First assume that there exists a non-trivial Banach–Kronecker factor; then there exists a bounded closed set $Y \subseteq \mathbb{R}^n$, a measure $\nu$, a linear action $S$ and a factor map $\pi : X \to Y$ such that $\nu = m \circ \pi^{-1}$ and $\pi \circ T = S \circ \pi$. Note that $\pi = (\pi_1, \ldots, \pi_n)$ and

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each of these functions $\pi_j : X \to \mathbb{R}$ is bounded, since $Y$ is bounded. In addition,

$$\pi_j(T_g x) = S_g \pi_j(x) = \sum_{i=1}^{n} a_i^j(g)\pi_i(x).$$

Define $L = \text{span}\{\pi_j \mid 1 \le j \le n\}$; then it is a $G$-invariant non-trivial $L_\infty(X, \mathcal{B}, m)$ subspace.

Next, assume that there exists a finite-dimensional non-trivial $G$-invariant subspace of $L_\infty(X, \mathcal{B}, m)$; denote it by $V$ and let $S_\infty := \{f \in L_\infty(X, \mathcal{B}, m);\ \|f\|_\infty = 1\}$. Denote $\dim(V) = n$ and let $\{f_1, f_2, \ldots, f_n\} \subseteq S_\infty$ be a basis for $V$. Define the function $F : X \to \mathbb{R}^n$ by $F(x) = (f_1(x), f_2(x), \ldots, f_n(x))$. The function $F$ is a measurable function; define the measure $\nu(A) = m(F^{-1}(A))$; then it is a well-defined probability measure and $\nu(\text{supp}(m)) = 1$. Define the space $Y = \overline{\text{supp}(\nu)}$.

Let there be $g \in G$; then for every $1 \le j \le n$ we know that $f_j \circ T_g \in V$; therefore, there are $\{a_i^j(g)\}_{i=1}^n$ such that $\sum_{i=1}^n a_i^j(g) f_i = f_j \circ T_g$. Define the matrix $S_g$ by

$$S_g = \begin{pmatrix} a_1^1(g) & a_2^1(g) & \ldots & a_n^1(g) \\ a_1^2(g) & a_2^2(g) & \ldots & a_n^2(g) \\ \vdots & \vdots & \ldots & \vdots \\ a_1^n(g) & a_2^n(g) & \ldots & a_n^n(g) \end{pmatrix}.$$

Then, for every $g \in G$, $F \circ T_g = S_g \cdot F$. Define the action of $G$ on $Y$, $S : G \to NST(Y)$ by $S_g(y) = S_g \cdot y$; then $S_g(y) \in \text{Im}(F)$ for $y \in \text{Im}(F)$ and therefore for every $g \in G$, $S_g(Y) \subseteq Y$ and the action is well defined. Next, by the definition of the measure $\nu$, the transformation $F$ is well defined on a set of full measure, and $F(X) \subseteq Y$ $\nu$-almost everywhere. Overall, $F : X \to Y$ is a factor map.

Moreover, $Y$ is compact and specifically Borel measurable.

It is left to show that $(Y, \mathcal{C}, \nu)$ is a Banach–Kronecker system. First, $\mathbb{R}^n$ is a finite-dimensional Banach space, $Y$ is a closed bounded subspace and $G$ is a locally compact Polish group by hypothesis. In addition, the action is linear and, since there exists a constant $C \in \mathbb{R}_+$ such that for every $g \in G$, $\|S_g\| < C$, then the action is also equicontinuous. We will show that it is minimal.

Denote by $D$ the action-invariant metric created in Lemma 3.1. By the definition of $\nu$, there exists $y_0 \in Y$ such that for every $\varepsilon > 0$, $\nu(B_D(y_0, \varepsilon)) > 0$. Let $G_0 \subset G$ be a dense countable subgroup. If $y_0$ is not a transitive point, then there exist $\delta > 0$ and $y \in Y$ such that for every $g \in G_0$, the intersection $B_D(y, \delta/2) \cap B_D(S_g(y_0), \delta/2) = \emptyset$. Define the set $B_0 = \bigcup_{g \in G_0} B_D(S_g(y_0), \delta/2)$; it is measurable and $G_0$-invariant. By Proposition 1.2, the action of $G_0$, denoted by $S^0$, is also ergodic; therefore, this set is trivial. Now $B_D(y_0, \delta) \subset B_0$ and therefore $\nu(B_0) = 1$; then $\nu(B_D(y, \delta/2)) = 0$, which is a contradiction to the definition of $Y$. Then $y_0$ is a transitive point. Now every equicontinuous action has an invariant metric (according to Lemma 3.1) and, if a transitive action has an invariant metric, then it is minimal. We conclude that the action of $S$ on $(Y, \mathcal{C}, \nu)$ is minimal.

Finally, the system is not trivial, since the original subspace was not a trivial one. We conclude that this system has a non-trivial Banach–Kronecker factor. □

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4. *The weak mixing theorem for non-commutative groups*

In this section, we will show that a non-singular action of a locally compact Polish Moore group (see definition below) is weakly mixing if and only if it has a non-trivial finite-dimensional $L_\infty$ invariant subspace. A recent article by Glasner and Weiss (see [**15**]) shows that a non-singular action is weakly mixing if and only if it is ergodic with unitary coefficients. They also showed that if the group is a locally compact Polish Moore group, then a non-singular action is ergodic with unitary coefficients if and only if it has a non-trivial finite-dimensional $L_\infty$-invariant subspace, using the main theorem below.

*Definition 4.1.* [**38**, Definition 4.1.3] A locally compact group is called a *Moore group* if all irreducible unitary representations are finite dimensional.

*Example 4.2.* According to the spectral theorem, every abelian group is a Moore group.

We will now state and prove the main theorem.

THEOREM 4.3. *Let $G$ be a locally compact Polish Moore group and let $T : G \to NST(X)$ be an ergodic action defined on a standard measure space $(X, \mathcal{B}, m)$. Then the action is weakly mixing if and only if there is no non-trivial finite-dimensional $G$-invariant subspace of $L_\infty(X, \mathcal{B}, m)$.*

This theorem gives necessary and sufficient conditions for weak mixing.

4.1. *The proof.* First, we will show that every action of a locally compact Polish group that has a non-trivial finite-dimensional action-invariant $L_\infty(X, \mathcal{B}, m)$ subspace is not weakly mixing.

THEOREM 4.4. *Let $G$ be a locally compact Polish group, $(X, \mathcal{B}, m)$ a standard space and $T : G \to NST(X)$ an ergodic action. If there exists a $G$-invariant, non-trivial, finite-dimensional subspace of $L_\infty(X, \mathcal{B}, m)$, then $T$ is not weakly mixing.*

*Proof.* The proof is derived from Theorems 3.8 and 3.9 and the fact that if a system has a non-weakly mixing factor, then it is not weakly mixing as well. □

Next, we will show that every non-weakly mixing action of a locally compact Polish Moore group has a non-trivial finite-dimensional action-invariant $L_\infty(X, \mathcal{B}, m)$ subspace.

4.1.1. *Definitions and preliminaries: the direct integral.* The following definitions and theorems are quoted from [**9**].

*Definition 4.5.* Let $Z$ be a Borel space, $\mu$ a positive measure and $\{\mathcal{H}(\zeta)\}_{\zeta \in Z}$ a family of Hilbert spaces. A *vector field* is an assignment to each point of an element of the associated Hilbert space.

*Definition 4.6.* Let $(Z, \mu)$ be a standard space. A *$\mu$-measurable field of Hilbert spaces over $Z$* is a pair $(\{H(\zeta)\}_{\zeta \in Z}, \Gamma)$, where $\{H(\zeta)\}_{\zeta \in Z}$ is a family of Hilbert spaces indexed by $Z$ and $\Gamma$ is a set of vector fields satisfying the following conditions:

(1) $\Gamma$ is a vector subspace of $\prod_{\zeta \in Z} \mathcal{H}(\zeta)$;
(2) there exists a sequence $\gamma_1, \gamma_2 \ldots$ of elements in $\Gamma$ such that for every $\zeta \in Z$, the sequence $\{\gamma_n(\zeta)\}$ forms a total sequence in $\mathcal{H}(\zeta)$;

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(3) for every $\gamma \in \Gamma$, the function $\zeta \mapsto \|\gamma(\zeta)\|^2$ is $\mu$-measurable;
(4) let $X$ be a vector field. If, for every $\gamma \in \Gamma$, the function $\langle X(\zeta), \gamma(\zeta)\rangle$ is $\mu$-measurable, then $X \in \Gamma$.

Under these conditions, the elements of $\Gamma$ are called *the measurable vector fields of* $(\{H(\zeta)\}_{\zeta\in Z}, \Gamma)$. If $\gamma_1, \gamma_2 \in \Gamma$, then the function $\zeta \mapsto \langle\gamma_1(\zeta), \gamma_2(\zeta)\rangle$ is measurable.

*Definition 4.7.* Let $Z$ be a Borel space, $\mu$ a positive measure and $(\{H(\zeta)\}_{\zeta\in Z}, \Gamma)$ a $\mu$-measurable field of Hilbert spaces over $Z$. An *operator field* is an assignment to each point $\zeta \in Z$ of an element $T(\zeta) \in \mathfrak{L}(\mathcal{H}(\zeta))$, $\mathfrak{L}(\mathcal{H}(\zeta))$, the collection of all the linear operators of $\mathcal{H}(\zeta)$. We say that $\zeta \mapsto T(\zeta)$ is a *$\mu$-measurable field of operators* if, for every $\gamma \in \Gamma$, the field $\zeta \mapsto T(\zeta)\gamma(\zeta)$ is measurable. If this is so, the function $\zeta \mapsto \|T(\zeta)\|$ is measurable. Suppose further that this function is essentially bounded, in which case the field is said to be essentially bounded, and $T = \int^{\oplus} T(\zeta)d\mu(\zeta)$ is well defined. The operators of the form $\int^{\oplus} T(\zeta)\, d\mu(\zeta)$ on $\mathcal{H}$ are said to be *diagonalizable*.

*Definition 4.8.* Let $Z$ be a Borel space, $\mu$ a positive measure and $(\{H(\zeta)\}_{\zeta\in Z}, \Gamma)$ a $\mu$- measurable field of Hilbert spaces over $Z$. For each $\zeta \in Z$, let $\pi(\zeta)$ be a representation of a group $G$ in $\mathcal{H}(\zeta)$. Then we call $\zeta \mapsto \pi(\zeta)$ a *field of representations* of $G$. This field is called a *measurable field of representations* if for every $g \in G$, the field of operators $\zeta \mapsto \pi(\zeta)(g)$ is measurable.

*Definition 4.9.* Given a measurable field of representations, one could construct for every $g \in G$ the continuous operator $\pi(g) = \int^{\oplus} \pi(\zeta)(g)\, d\mu(\zeta)$ on the Hilbert space $\mathcal{H} = \int^{\oplus} \mathcal{H}(\zeta)\, d\mu(\zeta)$. The operator $\pi$ is said to be *the direct integral* of $\{\pi(\zeta)\}$ and we write $\pi = \int^{\oplus} \pi(\zeta)\, d\mu(\zeta)$.

THEOREM 4.10. (Mautner and Dixmier) *Let $Z$ be a Borel space, $\mu$ a positive measure on $Z$, $(\{H(\zeta)\}_{\zeta\in Z}, \Gamma)$ a $\mu$-measurable field of Hilbert spaces over $Z$ and $\zeta \mapsto \pi(\zeta)$ a measurable field of representations of $G$ in $\mathcal{H}(\zeta)$. Denote*

$$\mathcal{H} = \int_Z^{\oplus} \mathcal{H}(\zeta)\, d\mu(\zeta); \quad \pi = \int_Z^{\oplus} \pi(\zeta)\, d\mu(\zeta)$$

*and denote by $\mathfrak{Z}$ the algebra of diagonalizable operators. Then the following are equivalent:*

(1) *$\mathfrak{Z}$ is a maximal commutative von Neumann subalgebra of $\pi(G)'$, where $\pi(G)' := \{B;$ for all $g \in G$, $B\pi(g) = \pi(g)B$, $B$ bounded$\}$;*
(2) *$\pi(\zeta)$ is irreducible for $\mu$-almost every $\zeta \in Z$.*

The theorem above has a rich history. It was originally stated by Mautner and later proved by many other mathematicians. References can be found in [**9**].

THEOREM 4.11. *Let $\mathcal{H}$ be a separable Hilbert space, $\pi$ a representation of $G$ in $\mathcal{H}$ and $\mathcal{A}$ a maximal commutative von Neumann subalgebra of $\pi(G)'$. Then there exist a standard Borel space $Z$, a bounded positive measure $\mu$ on $Z$, a $\mu$-measurable field $(\{H(\zeta)\}_{\zeta\in Z}, \Gamma)$ of Hilbert spaces over $Z$, a measurable field $\zeta \mapsto \pi(\zeta)$ of irreducible representations of $\mathcal{A}$ in $\mathcal{H}(\zeta)$ and an isomorphism of $\mathcal{H}$ onto $\int_Z^{\oplus} \mathcal{H}(\zeta)\, d\mu(\zeta)$ (an isomorphism of Hilbert spaces) which transforms $\mathcal{A}$ into the algebra of diagonalizable operators and $\pi$ into $\int_Z^{\oplus} \pi(\zeta)\, d\mu(\zeta)$.*

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*Remark 4.12.* This decomposition is not always unique. It is known that if the group is postliminal (see definition below), then the decomposition is unique, but there are examples where it is not unique. For more details, see [**9**].

*Definition 4.13.* A group is called *postliminal* [**9**] or *GCR* or *type I* [**33**] if for every irreducible representation $\pi$, there exists $g \in G$ such that $\pi(g)$ is compact.

The definition in [**9**] is a different one, but in [**33**] they proved that the definition above and the definition in [**9**] are equivalent.

*Remark 4.14.* Every Moore group is a type I group.

LEMMA 4.15. (Uniqueness) *Let $f \in \mathcal{H}$ be such that $f = 0$. Let $(\{H(\zeta)\}_{\zeta \in Z}, \Gamma)$ be a $\mu$-measurable field of Hilbert spaces over $Z$, $\mathcal{H} = \int_Z^{\oplus} \mathcal{H}(t)\, d\lambda(t)$ a direct integral and $\bar{f} : Z \to (\{H(\zeta)\}_{\zeta \in Z}, \Gamma)$ be such that $f = \int_Z^{\oplus} \bar{f}(t)\, d\lambda(t)$. Then, for $\lambda$-almost every $t \in Z$, $\bar{f}(t) = 0$.*

*Proof.* By hypothesis, $0 = \int_Z^{\oplus} \bar{f}(t) d\mu(t)$. If there exists a measurable set $Z_0 \subseteq Z$ such that $\lambda(Z_0) > 0$, and for every $t \in Z_0$, $\bar{f}(t) \neq 0$, then $\|\bar{f}(t)\| > 0$ and

$$0 = \langle f, f \rangle = \left\langle \int_Z^{\oplus} \bar{f}(t)\, d\mu(t), \int_Z^{\oplus} \bar{f}(t)\, d\mu(t) \right\rangle = \int_Z \langle \bar{f}(t), \bar{f}(t) \rangle\, d\mu(t)$$
$$\geq \int_{Z_0} \|\bar{f}(t)\|_0^2\, d\lambda(t) > 0,$$

which is a contradiction. □

4.1.2. *No Banach–Kronecker factor implies weakly mixing.*

THEOREM 4.16. *Let $G$ be a locally compact Polish Moore group. Let $T : G \to NST(X)$ be an ergodic action of $G$ defined on $(X, \mathcal{B}, m)$, a standard space. If $T$ is not weakly mixing, then there exists a finite-dimensional non-trivial $G$-invariant subspace of $L_\infty(X, \mathcal{B}, m)$.*

*Proof.* If $T$ is not weakly mixing, then there exist $(Y, \mathcal{C}, \nu)$, a standard probability space, and $S : G \to PPT(Y)$ ergodic such that $T \times S$ is not ergodic. Let $F : X \times Y \to \{0, 1\}$ be a non-constant invariant function. Define the function $\phi : X \to L_2(Y, \mathcal{C}, \nu)_0$ by

$$(\phi(x))(y) = F(x, y) - \int_Y F(x, y)\, d\nu(y).$$

Denote by $Isom(L_2(Y, \mathcal{C}, \nu)_0)$ the set of invertible isomorphisms of $L_2(Y, \mathcal{C}, \nu)_0$ and by $U : G \to Isom(L_2(Y, \mathcal{C}, \nu))_0$ the operators defined by $U_g h = h \circ S_{g^{-1}}$. Then, since $F$ is $T \times S$ invariant, it follows that for every $g \in G$ for $m$-almost every $x \in X$,

$$\phi(T_g x) = \phi(x) \circ S_{-g} = U_g(\phi(x)). \tag{4}$$

Let $G_0$ be a countable dense subgroup. For every $g \in G_0$, there is a measurable set of full measure $X_g \subset X$ such that for every $x \in X_g$, equality (4) holds. Denote $X_0 = \bigcap_{g \in G_0} X_g$; then it is measurable as a countable intersection of measurable sets, of full measure and, for every $x \in X_0$ and $g \in G_0$, equation (4) holds.

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Using Theorem 4.11, there exists a $\lambda$-measurable field of Hilbert spaces over $\mathbb{T}$ $(\{H(t)\}_{t\in\mathbb{T}}, \Gamma)$ such that

$$L_2(Y, \mathcal{C}, \nu)_0 \simeq \int_{\mathbb{T}}^{\oplus} \mathcal{H}(t)\, d\lambda(t) = \mathcal{H},$$

where $\lambda$ is a positive probability measure defined on $\mathbb{T}$. Denote by $\psi : L_2(Y, \mathcal{C}, \nu)_0 \to \mathcal{H}$ the isomorphism between the two Hilbert spaces and define $\tilde{U} : G \to Isom(\mathcal{H})$ by $\tilde{U}_g(h) = \psi(U_g(\psi^{-1}(h)))$. For every $h \in \mathcal{H}$, there exists $\bar{h} : \mathbb{T} \to (\{H(t)\}_{t\in\mathbb{T}}, \Gamma)$ such that, for every $t \in \mathbb{T}$, $\bar{h}(t) \in \mathcal{H}(t)$ and

$$h = \int_{\mathbb{T}}^{\oplus} \bar{h}(t)\, d\lambda(t).$$

In addition, by the same theorem (4.11), for every $g \in G$, the operator $\tilde{U}_g$ is diagonalizable. For every $x \in X_0$, by Lemma 4.15, there exists a set of full measure $\mathbb{T}_x \subset \mathbb{T}$ such that for every $t \in \mathbb{T}_x$ and $g \in G_0$,

$$(\tilde{\phi}(T_g x))(t) := (\psi(\phi(T_g x)))(t) = (\psi(U_g \phi(x)))(t) = \tilde{U}_g(t)(\tilde{\phi}(x))(t). \tag{5}$$

By Fubini's theorem, for $\lambda$-almost every $t \in \mathbb{T}$, there exists a set of full measure $X_t \subseteq X_0$ such that equation (5) holds for $t$ and for every $x \in X_t$. In addition, there exists a set of positive measure $\mathbb{T}_0 \subseteq \mathbb{T}$ such that for every $t \in \mathbb{T}_0$, the function $(\tilde{\phi}(\cdot))(t)$ is not constant as a function of $X$. If not, for almost every $t \in \mathbb{T}$, fix $x_0(t) \in X_0$ such that $(\tilde{\phi}(x_0(t)))(t) = (\tilde{\phi}(x))(t)$ for $m$ almost every $x \in X$ and specifically for every $g \in G_0$, $(\tilde{\phi}(T_g x_0(t)))(t) = (\tilde{\phi}(x_0(t)))$. Define the function

$$E = \int_{\mathbb{T}}^{\oplus} (\tilde{\phi}(x_0(t)))(t)\, d\lambda(t).$$

By definition, $E \in \mathcal{H}$, and $\psi^{-1}(E) \in L_2(Y, \mathcal{C}, \nu)_0$. In addition, for every $g \in G_0$,

$$\begin{aligned}\tilde{U}_g E &= \int_{\mathbb{T}}^{\oplus} \tilde{U}_g(t)(\tilde{\phi}(x_0(t)))(t)\, d\lambda(t) \overset{(\star)}{=} \int_{\mathbb{T}}^{\oplus} (\tilde{\phi}(T_g x_0(t)))(t)\, d\lambda(t) \\ &= \int_{\mathbb{T}}^{\oplus} (\tilde{\phi}(x_0(t))(t))\, d\lambda(t) = E,\end{aligned}$$

where $(\star)$ is because $x_0 \in X_0$; then there is a set of full measure $\mathbb{T}_{x_0}$ such that equation (5) holds for every $t \in \mathbb{T}_{x_0}$ and $g \in G_0$.

Now, by definition, $\psi^{-1}(\tilde{U}_g E) = U_g \psi^{-1}(E) = \psi^{-1}(E) \circ S_{g^{-1}}$; therefore, we get that $\psi^{-1}(E)$ is $G_0$-invariant. Now, $S$ is ergodic; by Proposition 1.2, the action of $G_0$ on $(Y, \mathcal{C}, \nu)$ is also ergodic. The function $\psi^{-1}(E)$ is a $G_0$-invariant function; therefore, it is constant, but then the original function, $F$, is also constant, which is a contradiction.

We conclude there exists a set of positive measure $\mathbb{T}_0 \subseteq \mathbb{T}$ such that $(\tilde{\phi}(\cdot))(t)$ is not constant for every $t \in \mathbb{T}_0$. Finally, for almost every $t \in \mathbb{T}$, $\dim(\mathcal{H}(t)) < \infty$ and $\tilde{U}_g(t)$ is unitary (according to Lemma 4.15); specifically, there exists $t_0 \in \mathbb{T}_0$ such that $\dim(\mathcal{H}(t_0)) < \infty$, $\tilde{U}_g(t_0)$ is a unitary operator and $(\tilde{\phi}(\cdot))(t_0)$ is not constant.

Now, $\|\tilde{\phi}(\cdot)(t_0)\|$ is bounded $m$-almost everywhere, since the function $\|(\tilde{\phi}(\cdot))(t_0)\|$ is $G_0$ invariant and the action of $G_0$ on $(X, \mathcal{B}, m)$ is ergodic; then it is constant and specifically bounded.

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Define the function $\varphi : X \to \mathcal{H}(t_0)$ by $\varphi(x) = (\tilde{\phi}(x))(t_0)$ and the space $\tilde{V} := \text{span}\{\tilde{U}_g(t_0)(\varphi(\cdot));\ g \in G_0\}$; then it is non-trivial and $G_0$-invariant. We will show that it is finite dimensional. Note that since $\dim(\mathcal{H}(t_0)) = d < \infty$, the operators $U_g$ are actually defined uniquely by matrices based on an orthonormal basis, $\{e_j\}_{i=1}^d$. Denote $\varphi_{j,k}(x) = \langle e_j, \varphi(x)\rangle e_k$; then $\tilde{V} \subseteq \text{span}\{\varphi_{j,k};\ 1 \le j, k \le d\}$ and specifically $\tilde{V}$ is finite dimensional.

Denote by $\{\varphi_1(x), \ldots, \varphi_n(x)\}$ an orthonormal basis for $\tilde{V}$. Since $\mathcal{H}(t_0)$ is a finite-dimensional Hilbert space, there exists an isomorphism between $\mathcal{H}(t_0)$ and $\mathbb{R}^d$; denote it by $\tilde{\pi}$. In addition, since $\tilde{V}$ is not trivial, there exist $1 \le j \le d$ and $1 \le k \le n$ such that the function $\tilde{\pi}(\varphi_k(x))_j$ is not constant. Define $\pi : \mathcal{H}(t_0) \to \mathbb{R}$ by $\pi(h) = (\tilde{\pi}(h))_j$ and define the space $V = \text{span}\{\pi(f(\cdot));\ f \in \tilde{V}\}$. One could verify that $V$ is not trivial, finite-dimensional, $G_0$-invariant, and $V \subseteq L_\infty(X, \mathcal{B}, m)$.

We will show that it is also $G$-invariant. For every $g \in G$, we want to show that $\pi(\tilde{\phi}(T_g x)(t_0)) \in V$, for which it is enough to show that $\tilde{\phi}(T_g x)(t_0) \in \tilde{V}$. Let there be $g \in G$ and let $\{g_n\} \subseteq G_0$ be a sequence such that $g_n \underset{n\to\infty}{\longrightarrow} g$; then $\|\tilde{\phi} \circ T_{g_n}\|$ converges in measure to $\|\tilde{\phi} \circ T_g\|$ and there exists a subsequence $\{n_k\}$ such that $\|\tilde{\phi} \circ T_{g_{n_k}}\| \overset{\text{a.e.}}{\to} \|\tilde{\phi} \circ T_g\|$. Since $\tilde{V}$ is a closed subset in the set of bounded functions from $X$ to $\mathcal{H}(t_0)$, the fact that $\tilde{\phi}(T_g x)(t_0)$ is bounded gives us $\tilde{\phi}(T_g x)(t_0) \in \tilde{V}$. □

*Remark 4.17.* While proving Theorem 4.3, we actually proved that an action is weakly mixing if and only if it has no non-trivial Banach–Kronecker factor. Moreover, we proved that if an action is not weakly mixing, then it has a special non-trivial Banach–Kronecker factor, one that is a unitary action on a Euclidean sphere.

## 4.2. *Corollaries.*

### 4.2.1. *Either finite dimensional or mildly mixing.*

*Definition 4.18.* A non-singular action of $G$ on a standard space is called *mildly mixing* if for any non-singular properly ergodic action of $G$, the product action is also ergodic.

*Definition 4.19.* A *rigid set* is a set $B \in \mathcal{B}$ such that there exists a sequence $g_n \to \infty$ such that $m(B \triangle g_n B) \underset{n\to\infty}{\longrightarrow} 0$.

THEOREM 4.20. (Schmidt and Walters [**36**]) *Let $G$ be a locally compact second countable group. Let $T : G \to NST(X)$ be a properly ergodic action on a standard probability space. This action is mildly mixing if and only if it has no rigid sets.*

*Definition 4.21.* A representation of $G$ on a Hilbert space, $\mathcal{H}$, is called *mildly mixing* if it has no rigid factor, meaning that for every $0 \neq h \in \mathcal{H}$ and $g_n \to \infty$, $\liminf \|\pi(g_n)h - h\| > 0$.

*Definition 4.22.* Let $G$ be a locally compact Polish group. A representation $\pi$ is called *mixing* (or *strongly mixing*) if for every $h_1, h_2 \in \mathcal{H}$, the function $g \mapsto \langle \pi(g)h_1, h_2\rangle$ is in $C_0(G)$, meaning that it is continuous and vanishes at infinity.

THEOREM 4.23. *Let $G$ be a locally compact Polish group such that any irreducible unitary representation of $G$ is either finite dimensional or mildly mixing. Let $T : G \to NST(X)$ be*

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*a properly ergodic action of $G$ defined on $(X, \mathcal{B}, m)$, a standard probability space. Then the action is weakly mixing if and only if every finite-dimensional, $G$-invariant subspace of $L_\infty(X, \mathcal{B}, m)$ is trivial.*

*Proof.* Note that by Theorem 4.4, if there exists a finite-dimensional invariant $L_\infty(X, \mathcal{B}, m)$ subspace, then the action is not weakly mixing. We will show that if the action is not weakly mixing, then there exists a finite-dimensional invariant $L_\infty(X, \mathcal{B}, m)$ subspace. Following the proof of Theorem 4.16 (and using the same notation), there exists a function $\phi : X \to L_2(Y, \mathcal{C}, \nu)_0$ such that for every $g \in G$ for $m$ almost every $x \in X$,

$$\phi(T_g x) = \phi(x) \circ S_{-g} = U_g(\phi(x)).$$

In addition, there exists a set of positive measure $\mathbb{T}_0 \subseteq \mathbb{T}$ such that for every $t \in \mathbb{T}_0$, the function $(\tilde{\phi}(\cdot))(t)$ is not constant as a function of $X$. If, for every $t \in \mathbb{T}_0$, we have $\dim(\mathcal{H}(t)) = \infty$, then for every such $t \in \mathbb{T}_0$, by hypothesis, the representation of $G$ on $\mathcal{H}(t)$ is mildly mixing, which means that every rigid factor is trivial. We will show that there exists a rigid factor to get a contradiction.

Since $T^0 := T|_{G_0}$ is an ergodic action, it is recurrent and therefore, for almost every $x \in X$, there exists a sequence $\{g_n\} = \{g_n(x)\} \subset G_0$ such that $T_{g_n} x \to x$. Specifically, there exists $x$ such that $\tilde{\phi}(x)(t_0) \neq \tilde{\phi}(T_{g_n} x)(t_0)$ and $\|\tilde{U}_{g_n}\tilde{\phi}(x)(t_0) - \tilde{\phi}(x)(t_0)\| = \|\tilde{\phi}(T_{g_n} x)(t_0) - \tilde{\phi}(x)(t_0)\| \to 0$, but, since the representation is mildly mixing, this implies that $\tilde{\phi}(x)(t_0) = 0$, which is a contradiction. We conclude that necessarily there exists $t_0 \in \mathbb{T}_0$ such that $\dim(\mathcal{H}(t_0)) < \infty$ and, using the last part of the proof of Theorem 4.16, there exists a finite-dimensional non-trivial $G$-invariant $L_\infty(X, \mathcal{B}, m)$ subspace. □

PROPOSITION 4.24. *Let $G$ be a locally compact Polish group. If every weakly mixing representation is mildly mixing, then every irreducible representation of $G$ is either finite dimensional or mildly mixing.*

*Proof.* Let $\pi$ be an irreducible representation. If $\pi$ is not weakly mixing, then, according to Theorem 1.8, there exists a finite-dimensional subrepresentation, but this representation is irreducible; we conclude that it is finite dimensional. Otherwise, $\pi$ is weakly mixing and by hypothesis mildly mixing. □

*Definition 4.25.* A group $G$ is called *minimally weakly mixing* (m.w.m.) if $\overline{B(G)} = AP(G) \oplus C_0(G)$, where

$$B(G) := \operatorname{span}\{\langle \pi(*) f, h\rangle;\ \ f, h \in \mathcal{H}, \pi \text{ a continuous irreducible representation}\}.$$

It is called *minimally weakly almost periodic* if $\overline{B(G)} = C \oplus C_0(G)$, where $C$ denotes the constant functions.

*Remark 4.26.* Examples of such groups are discussed in [**7**].

THEOREM 4.27. (Bergelson and Rosenblatt [**6**]) *If $G$ is minimally weakly mixing, then any weakly mixing representation is strongly mixing.*

*Example 4.28.* According to Theorem 4.27 and Proposition 4.24, every minimally weakly mixing group $G$ will satisfy the conditions of Theorem 4.23. We conclude that Theorem 4.23 can be applied to every minimally weakly mixing group $G$.

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4.2.2. *Poisson actions.* The definition of a Poisson action can be found in [**2**].

THEOREM 4.29. (Aaronson and Lemańczyk [**2**]) *Let $G$ be a locally compact Polish group. Denote by $\rho(T,\ f)$ a Poisson action of $G$. If $p \in \mathcal{P}(G)$ is globally supported, then $\rho(T,\ f)$ is weakly mixing.*

COROLLARY 4.30. *Let $G$ be a locally compact Polish group. Then its Poisson action has no finite-dimensional $L_\infty$-invariant subspaces.*

*Proof.* Let $\rho(G,\ p)$ be a Poisson action of a locally compact group $G$. By Theorem 4.29, it is weakly mixing and, according to Theorem 4.3, it has no finite-dimensional $L_\infty$ subspaces. □

4.3. *Examples.* In this section, we will present some characterizations of Moore groups. In addition, we will see some examples.

*Definition 4.31.* A group $G$ is called a *$Z$ group* if the group $G/Z(G)$ is compact, where $Z(G)$ is the centre of $G$.

THEOREM 4.32. (Grosser and Moskowitz [**17**, Theorem 2.1]) *Let $G$ be a $Z$-group; then any irreducible unitary representation is finite dimensional.*

*Definition 4.33.* Let $G$ be a connected Lie group; then $G$ is called a *Lie group by definition*. Let $G$ be a group which is not connected. We will call $G$ a *Lie group by definition* if its component of identity $G_0$ is open in $G$, and if $G_0$ is a Lie group by definition.

THEOREM 4.34. (Moore [**25**]) *Let $G$ be a Lie group; then every irreducible unitary representation of $G$ is finite dimensional if and only if there exists an open subgroup of finite index $H$, which is a $Z$-group.*

*Definition 4.35.* We say that $G$ is a *projective limit* and denote it $G = projlim(G_\alpha)$ if there is a family of normal subgroups $\{H_\alpha\}_{\alpha\in I}$ directed by inclusion such that $G_\alpha = G/H_\alpha$ and $\bigcap_{\alpha\in I} H_\alpha = \{e\}$, where $H_\alpha$ is compact for each $\alpha \in I$.

THEOREM 4.36. (Moore [**25**]) *For a locally compact group $G$, every irreducible unitary representation is finite dimensional if and only if $G = projlim G_\alpha$, where each $G_\alpha$ is a Lie group, which satisfies that every irreducible unitary representation is finite dimensional.*

*Example 4.37.* Every finite group is a Moore group.

*Example 4.38.* Every abelian group is a Moore group.

THEOREM 4.39. (Peter–Weyl theorem) *Let $G$ be a compact group; then, for every unitary representation $\pi$, there exists a decomposition into finite-dimensional irreducible representations.*

*Example 4.40.* By the Peter–Weyl theorem, every compact group is a Moore group.

*Example 4.41.* (The infinite dihedral group) One of many representations of this group is the group generated by a reflection and a rotation. The reflection of $\mathbb{R}^2$ is a multiplication

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by the matrix $r$ and the rotation is a multiplication by the matrix $s$, where

$$r = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \quad s = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

First, we will show that this group is a Moore group. Let $H = \langle s \rangle$; then it is an abelian group, and it is of index 2, since $s^k \cdot r = r \cdot s^{-k}$. In addition, if the infinite dihedral group is endowed with the discrete topology, then $H$ is an open subgroup. By Theorem 4.34, $D_\infty$ is a Moore group.

Let us look at the action of $D_\infty$ on $(\mathbb{R}^2, \mathcal{B}, Leb)$. It can be defined as matrix multiplications, which is a non-singular action on $(\mathbb{R}^2, \mathcal{B}, m)$, for some finite measure $m \sim Leb$. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} y, & |y| < 1, \\ 1, & \text{otherwise.} \end{cases}$$

Then it is $D_\infty$-invariant and, in addition, $f \in L_\infty(X, \mathcal{B}, m)$; therefore, the space $L = \text{span}\{f\}$ is finite dimensional, $D_\infty$-invariant and non-trivial. By Theorem 4.3, the action in not weakly mixing.

*Example 4.42.* (The quaternion group) Denote the following vectors in $\mathbb{R}^4$:

$$\mathbf{1} := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad i := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad j := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad k := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Define the quaternion multiplication by

$$i^2 = j^2 = k^2 = ijk = -1.$$

This multiplication can be extended to $H := \text{span}\{\mathbf{1}, i, j, k\}$.

Denote $H^\star := \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in H;\, a^2 + b^2 + c^2 + d^2 \neq 0 \right\}$. Then $H^\star$ is a Lie group with quaternion multiplication. In addition, one of its representations is as complex $2 \times 2$ matrices:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \mapsto \begin{pmatrix} a + ib & c + id \\ -(c - id) & a - ib \end{pmatrix},$$

where $i$ is the imaginary number, $i = \sqrt{-1}$. We shall use this representation to show that this group is a Moore group. Calculations show that

$$Z(H^\star) = \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix};\, r \in \mathbb{R} \right\},$$

$$H^\star / Z(H^\star) = \left\{ \begin{pmatrix} a + ib & c + id \\ -(c - id) & a - ib \end{pmatrix};\; a^2 + b^2 + c^2 + d^2 = 1 \right\}.$$

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Here $H^\star/Z(H^\star)$ is a continuous image of a closed bounded set in $\mathbb{R}^4$ and therefore a compact set. We conclude by Theorem 4.34 that $H^\star$ is a Moore group.

5. *Mild mixing versus weak mixing*

Every mildly mixing action is also weakly mixing. For a long time it was not known whether these two concepts have different meanings for non-commutative groups.

For $\mathbb{Z}$ actions, there exists a rigid action which is weakly mixing but not mildly mixing; see [**12**]. For non-commutative groups this is not so simple. On one hand there is an example of an amenable group action which is rigid and weakly mixing (see [**34**]). On the other hand, Howe and Moore showed in their paper [**19**] that every action of an exponential solvable Lie group is mixing. The question whether these two concepts are different for other non-commutative groups first arose in Samet's paper and a partial answer has been given there (see [**34**]).

In this section, we will present an example of a probability-preserving action of the Heisenberg group which is not mildly mixing but weakly mixing.

*Definition 5.1. The Heisenberg group* $H_3(\mathbb{R})$ is the group of $3 \times 3$ upper triangular matrices of the form

$$M(a, b, c) := \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

It is known that every unitary irreducible representation is precisely one of the following representations:

(1) a one-dimensional representation on $\mathbb{C}$ of the form

$$\pi_{\alpha,\beta}(M(a, b, c))z := e^{i(\alpha a+\beta b)}z;$$

(2) an infinite-dimensional representation on $L_2(\mathbb{R})$ of the form

$$[\pi_\gamma(M(a, b, c))f](t) := e^{i\gamma(c+bt)}f(t + a), \quad \gamma \in \mathbb{R} \setminus \{0\}.$$

For more details, see [**21**].

The first example of an action of $H_3(\mathbb{R})$ which is rigid and weakly mixing is a rank-one action. This example was found by Danilenko in [**8**].

We will construct the Gaussian action (for definition, see [**13**, p. 90]) associated with the infinite-dimensional unitary irreducible representation of the Heisenberg group (see Theorem 5.6) and show that it is weakly mixing but not mildly mixing.

Let us look at the infinite-dimensional irreducible representation on $L_2(\mathbb{R})$. It is irreducible, as we mentioned in Definition 5.1, and therefore it has no invariant subspaces, specifically no finite-dimensional non-trivial subrepresentations, and therefore this representation is weakly mixing.

Next, we will show that the representation has a rigid factor: define the sequence

$$g_n := M\left(0, 0, 2\pi n + \frac{1}{n}\right) = \begin{pmatrix} 1 & 0 & 2\pi n + \frac{1}{n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

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This sequence is well defined and $g_n \underset{|n|\to\infty}{\longrightarrow} \infty$. In addition, this is a rigid factor in the sense that for every non-constant function $f \in L_2(\mathbb{R})$ and $n \in \mathbb{Z}$,

$$f(t) \neq \left[\pi\left(M\left(0, 0, 2\pi n + \frac{1}{n}\right)\right) f\right](t) = e^{i(2\pi n + (1/n)) + i0t} \cdot f(t+0) \to f(t).$$

*Remark 5.2.* According to Theorem 4.27, the Heisenberg group is not minimally weakly mixing, since the above representation is weakly mixing but not strongly mixing.

The rest of this paper is dedicated to show that the Gaussian action associated with $\pi$ is weakly mixing but not mildly mixing.

THEOREM 5.3. (Peterson and Sinclair [**29**]) *Let $G$ be a countable discrete group and let $\pi : G \to \mathcal{H}$ be a unitary representation of $G$ on a Hilbert space $\mathcal{H}$. Then there exist a Gaussian dynamical system $(X, \mathcal{B}, m)$ and an action $T : G \to PPT(X)$ such that $L_2(X, \mathcal{B}, m)_0 \simeq \bigoplus_{n=1}^{\infty} \mathcal{H}^{\odot n}$ and $U_T|_{\mathcal{H}} = \pi$. This action is called the Gaussian process associated with $\pi$.*

This construction was also mentioned in [**13**, **20**, **27**, **29**].

Denote by $\pi^{\mathfrak{C}}$ the Koopman representation on $\bigoplus_{n=1}^{\infty} \mathcal{H}^{\odot n}$.

*Definition 5.4.* Let $G$ be a Polish group and $(X, \mathcal{B}, m)$ a standard probability space. As in [**14**], define a *near action* (or a *Boolean action*) of $G$ on $(X, \mathcal{B}, m)$ to be a Borel map $T : G \to PPT(X)$ such that:

(1) if we denote by $e$ the identity element of $G$, then, for $m$-almost every $x \in X$, $T_e x = x$;
(2) for every $g, h \in G$, $T_g(T_h x) = T_{gh}(x)$ for $m$-almost every $x \in X$.

THEOREM 5.5. (Mackey [**23**] and Ramsay [**30**]) *Let $G$ be a locally compact Polish group. Then every near action is isomorphic to an action.*

For more details, see [**14**].

THEOREM 5.6. *Let $G$ be a locally compact Polish group and let $\pi : G \to \mathcal{H}$ be a unitary representation of $G$ on a Hilbert space $\mathcal{H}$. Then there exist a Gaussian dynamical system $(X, \mathcal{B}, m)$ and an action $T : G \to PPT(X)$ such that $L_2(X, \mathcal{B}, m)_0 \simeq \bigoplus_{n=1}^{\infty} \mathcal{H}^{\odot n}$ and $U_T|_{\mathcal{H}} = \pi$. This $G$-action is called the Gaussian process associated with $\pi$.*

*Proof.* Let $G_0 \subset G$ be a countable dense subgroup. Endow $G_0$ with the discrete topology; then, by Theorem 5.3, there exists a Gaussian action $T^0 : G_0 \to PPT(X)$ on a standard probability space $(X, \mathcal{B}, m)$ such that $U_{T^0}|_{\mathcal{H}} = \pi|_{G_0}$ ($U_{T^0}$ is the Koopman representation induced by $T^0$) and $L_2(X, \mathcal{B}, m)_0 \simeq \bigoplus_{n=1}^{\infty} \mathcal{H}^{\odot n}$. We will show that there exists a near action of $G$ on the same space, $T$, such that $T|_{G_0} = T^0$ and then use Theorem 5.5 to conclude that we have an action isomorphic to this near action.

Let $\{g_k\}$ be a Cauchy sequence; then, by definition of a representation, $\{\pi(g_k)\}$ is also a Cauchy sequence. Let there be $n \in \mathbb{N}$; then, if one denotes by $\pi^{\odot n}(g)$ the representation defined on $\mathcal{H}^{\odot n}$, the sequence $\{\pi^{\odot n}(g_k)\}$ is also a Cauchy sequence, since, for every vector of the form $e_1 \otimes e_2 \otimes \cdots \otimes e_n \in \mathcal{H}^{\otimes n}$, calculations show that $\|\pi^{\otimes n}(g_m)(e_1 \otimes e_2 \otimes \cdots \otimes e_n) - \pi^{\otimes n}(g_k)(e_1 \otimes e_2 \otimes \cdots \otimes e_n)\| \to 0$ (a detailed

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calculation can be found in [**16**]). This is also true for linear combinations of such vectors; we conclude that the induced Koopman representation $U_{T^0}$ is continuous on $G_0$.

Let $g \in G$; then there exists a Cauchy sequence $\{g_n\} \subseteq G_0$ such that $g_n \to g$. The sequence $\{U_{T_{g_n}}\}$ is a Cauchy sequence in the topology of positive isometries; denote by $U_g$ its limit, which is positive, unitary and does not depend on the choice of the sequence $\{g_n\}$. According to Lamperti's theorem (see [**31**]), there exists a non-singular transformation $S$ such that $U_g f = \sqrt{(d(m \circ S)/dm)}\, f \circ S$. Denote $T_g = S = S(g)$. Since the mapping between positive isometries and non-singular actions is a bijection, if $g \in G_0$, then $T_g^0 = T_g$. In addition, since $U_{g_n} \to U_g$ (in the weak operator topology), then, for every $k$, $U_g|_{\mathcal{H}^{\odot k}} = U_g^{\odot k}$, $(d(m \circ T_g)/dm) = 1$ and $f \circ T_{g_n} \xrightarrow{m} f \circ T$.

We will show that $T_{gh} \overset{\text{a.e.}}{=} T_g T_h$. Let there be $g, h \in G$; then there exist $\{g_n\}, \{h_n\} \subseteq G_0$ such that $g_n \to g$, $h_n \to h$. Since $T^0$ is an action, there exists a measurable set of full measure, $X_0$, such that, for every $x \in X_0$ and every $n, m \in \mathbb{N}$, we have $T_{g_n} T_{h_m} x = T_{g_n h_m} x$. In addition, as we showed above, if $g_n \to g$, then $T_{g_n} \xrightarrow{m} T_g$ and specifically there exists a subsequence $\{n_k\}$ such that $T_{g_{n_k}} \xrightarrow{\text{a.e.}} T_g$. Define the set $X_1 := \{x;\, T_{g_{n_k}} x \not\to T_g x \text{ or } T_{h_{m_j}} x \not\to T_h x\}$; then it is of measure zero and, for every $x \in X_0 \setminus X_1$,

$$T_g T_h x = \lim_{k\to\infty} \lim_{j\to\infty} T_{g_{n_k}} T_{h_{m_j}} x = \lim_{k\to\infty} \lim_{j\to\infty} T_{g_{n_k} h_{m_j}} x = T_{\lim_{k\to\infty} \lim_{j\to\infty} g_{n_k} h_{m_j}} x = T_{gh} x.$$

We conclude that $T$ is indeed a near action and, by Theorem 5.5, there exists an action which is isomorphic to this action. □

THEOREM 5.7. *Let $G$ be a locally compact Polish group, $\mathcal{H}$ a separable Hilbert space, $\pi : G \to Isom(\mathcal{H})$ a unitary representation and $(X, \mathcal{B}, m, T)$ the Gaussian process associated with $\pi$. Then the action $T$ is ergodic if and only if it is weakly mixing if and only if $\pi$ is weakly mixing.*

*Proof.* Remember that the action described is measure preserving and therefore ergodicity and weak mixing properties of the action are equivalent to the same properties of the induced Koopman representation. Note that since $\pi^{\mathfrak{C}} \otimes \pi^{\mathfrak{C}} = \pi^{\mathfrak{C}}$, the action is ergodic if and only if it is weakly mixing. We will show that the action is ergodic if and only if $\pi$ is weakly mixing.

If $\pi$ is not weakly mixing, then $\pi \otimes \pi$ is not ergodic and therefore there exists $\xi \in H^{\otimes 2}$ which is $G$-invariant. Denote by $B_2(\mathcal{H})$ the collection of Hilbert–Schmidt operators defined on $\mathcal{H}$ (see [**9**, p. 314]). There exists an isometric isomorphism $\phi : \mathcal{H}^* \otimes \mathcal{H} \to B_2(\mathcal{H})$ defined by $\phi(f \otimes e) = \langle *, f\rangle e$. It is indeed an isometry and $h \in \mathcal{H}^* \otimes \mathcal{H}$ is symmetric if and only if $\phi(h)$ is self adjoint. Denote $\tilde{\xi} = \phi(\xi)$ and let us look at $((\tilde{\xi} + \tilde{\xi}^*)/2)$. It is a self-adjoint operator and indeed a Hilbert–Schmidt operator. In addition, $\phi^{-1}(((\tilde{\xi} + \tilde{\xi}^*)/2))$ is both symmetric and $G$-invariant. We conclude that $\pi^{\mathfrak{C}}$ has an invariant vector and therefore it is not ergodic.

Conversely, assume that $\pi^{\mathfrak{C}}$ is not ergodic; then there exists $\xi \in \bigoplus_{n=1}^{\infty} \mathcal{H}^{\odot n}$ which is $G$-invariant. Since $\pi^{\mathfrak{C}}$ is a subrepresentation of $\bigoplus_{n=1}^{\infty} \pi^{\otimes n}$, this representation also has an invariant vector. Note that

$$\bigoplus_{n=1}^{\infty} \pi^{\otimes n} = \pi \oplus \bigoplus_{n=2}^{\infty} \pi^{\otimes n} = \pi \otimes \left(\mathbf{1} \oplus \bigoplus_{n=1}^{\infty} \pi^{\otimes n}\right).$$

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Then the product representation $\pi \otimes (\mathbf{1} \oplus \bigoplus_{n=1}^{\infty} \pi^{\otimes n})$ has an invariant vector, where $\mathbf{1} \oplus \bigoplus_{n=1}^{\infty} \pi^{\otimes n}$ is also a unitary representation. We conclude that $\pi \otimes \pi$ has an invariant vector, which means that it is not ergodic and $\pi$ is not weakly mixing. □

The first proof of this theorem was done by Maruyama for $G = \mathbb{Z}$ in 1949 (see [**24**]). The proof above was taken from notes created by Peterson. For more details, see Peterson's homepage.

According to Theorem 5.6 applied to the unitary representation of $H_3(\mathbb{R})$ described above, the action $T$ is weakly mixing. If we show that it is also not mildly mixing, then we will have the desired example.

We will show that there that exists a rigid sequence and, by Schmidt–Walter's result (Theorem 4.20), the action is not mildly mixing. But, since $\pi$ is contained in the Koopman representation of $H_3(\mathbb{R})$ on $L_2(X, \mathcal{B}, m)_0$, then we already saw that there is a rigid factor. We conclude that this action is weakly mixing and not mildly mixing.

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