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Some bilinear generating functions for Jacobi polynomials

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1. *Introduction.* In a previous paper (3) the writer has proved

$$\sum_{n=0}^{\infty}\binom{m+n}{m}P_{m+n}^{(\alpha-n,\beta-n)}(x)\,t^n = [1+\tfrac{1}{2}(x+1)t]^{\alpha}[1+\tfrac{1}{2}(x-1)t]^{\beta}P_m^{(\alpha,\beta)}[x+\tfrac{1}{2}(x^2-1)t], \quad (1\cdot1)$$

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{(\mu)_n}(y-x)^n P_n^{(\alpha-n,\beta-n)}\left(\frac{x+y}{x-y}\right) = F_1(\lambda;\,-\alpha,-\beta;\,\mu;\,x,y), \quad (1\cdot2)$$

where $P_n^{(\alpha,\beta)}(x)$ is a Jacobi polynomial defined as ((4) p. 255)

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha+\beta)_{2n}}{n!(1+\alpha+\beta)_n}\left(\frac{x+1}{2}\right)^n {}_2F_1\left[\begin{matrix}-n,\,-\beta-n;\\ -\alpha-\beta-2n;\end{matrix}\;\frac{2}{x+1}\right]. \quad (1\cdot3)$$

Replacing x and y by $-\frac{1}{2}(x+1)t$ and $-\frac{1}{2}(x-1)t$ respectively we write (1·2) as

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{(\mu)_n}P_n^{(\alpha-n,\beta-n)}(x)\,t^n = F_1[\lambda;\,-\alpha,-\beta;\,\mu;\,-\tfrac{1}{2}(x+1)t,\,-\tfrac{1}{2}(x-1)t]. \quad (1\cdot4)$$

Putting $\mu = -\alpha-\beta$ and using ((1), p. 238)

$$F_1(\alpha;\,\beta;\,\beta';\,\beta+\beta';\,x,y) = (1-y)^{-\alpha}\,{}_2F_1\left(\alpha,\beta;\,\beta+\beta';\,\frac{x-y}{1-y}\right),$$

(1·4) yields

$$\sum_{n=0}^{\infty}\frac{(\lambda_n)}{(-\alpha-\beta)_n}P_n^{(\alpha-n,\beta-n)}(x)\,t^n = [1-\tfrac{1}{2}(1-x)t]^{-\lambda}\,{}_2F_1\left[\begin{matrix}\lambda,\,-\alpha;\\ -\alpha-\beta;\end{matrix}\;\frac{-t}{1-\frac{1}{2}(1-x)t}\right]. \quad (1\cdot5)$$

In this paper, by means of the above formulae, we propose to obtain some results, believed to be new, which, besides Jacobi polynomials, will involve Lauricella's functions F_A and F_D and Appell's functions F_1 and F_2.

For Lauricella's functions see (2), while for Appell's functions see ((1), p. 224).

2. In this section we shall prove as follows:

$$\sum_{n=0}^{\infty}P_n^{(\alpha-n,\beta-n)}(x)\,F_1(\lambda;\,-n,-\gamma;\,\mu;\,y,z)\,t^n = [1+\tfrac{1}{2}(x+1)t]^{\alpha}[1+\tfrac{1}{2}(x-1)t]^{\beta} \quad (2\cdot1)$$

$$F_D\left[\lambda;\,-\alpha,\,-\beta,-\gamma;\,\mu;\,\frac{(x+1)yt}{2+(x+1)t},\,\frac{(x-1)yt}{2+(x-1)t},\,z\right]$$

and

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{(-\alpha-\beta)_n}P_n^{(\alpha-n,\,\beta-n)}(x)\,F_2(\lambda+n;\gamma,\gamma';\delta,\delta';y,z)\,t^n$$

$$= [1-\tfrac{1}{2}(1-x)t]^{-\lambda}F_A\left[\lambda;\,\gamma,\gamma',\,-\alpha;\delta,\delta',\,-\alpha-\beta;\right.$$

$$\left.\frac{y}{1-\frac{1}{2}(1-x)t},\,\frac{z}{1-\frac{1}{2}(1-x)t},\,\frac{-t}{1-\frac{1}{2}(1-x)t}\right]. \quad (2\cdot2)$$

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To prove (2·1) we start with its left-hand side

$$\sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta-n)}(x)\, F_1(\lambda;\ -n,\ -\gamma;\ \mu;\ y,\ z)\, t^n$$

$$= \sum_{n,k=0}^{\infty} \sum_{r=0}^{n} \frac{(\lambda)_{r+k}(-n)_r(-\gamma)_k}{r!\,k!\,(\mu)_{r+k}} P_n^{(\alpha-n,\beta-n)}(x)\, y^r z^k t^n$$

$$= \sum_{r,k=0}^{\infty} \frac{(\lambda)_{r+k}(-\gamma)_k}{k!\,(\mu)_{r+k}} (-yt)^r z^k \sum_{n=0}^{\infty} \binom{n+r}{r} P_{n+r}^{(\alpha-n-r,\beta-n-r)}(x)\, t^n$$

$$= [1+\tfrac{1}{2}(x+1)t]^{\alpha}[1+\tfrac{1}{2}(x-1)t]^{\beta} \sum_{r,k=0}^{\infty} \frac{(\lambda)_{r+k}(-\gamma)_k}{k!(\mu)_{r+k}} P_r^{(\alpha-r,\beta-r)}[x+\tfrac{1}{2}(x^2-1)t]$$

$$\times \left[\frac{-yt}{\{1+\tfrac{1}{2}(x+1)t\}\{1+\tfrac{1}{2}(x-1)t\}}\right]^r z^k \quad \text{by (1·1)}$$

$$= [1+\tfrac{1}{2}(x+1)t]^{\alpha}[1+\tfrac{1}{2}(x-1)t]^{\beta} \sum_{k=0}^{\infty} \frac{(-\gamma)_k(\lambda)_k}{k!(\mu)_k} z^k,$$

$$\times \sum_{r=0}^{\infty} \frac{(\gamma+k)_r}{(\mu+k)_r} P_r^{(\alpha-r,\beta-r)}[x+\tfrac{1}{2}(x^2-1)t] \left[\frac{-yt}{\{1+\tfrac{1}{2}(x+1)t\}\{1+\tfrac{1}{2}(x-1)t\}}\right]^r$$

$$= [1+\tfrac{1}{2}(x+1)t]^{\alpha}[1+\tfrac{1}{2}(x-1)t]^{\beta}$$

$$\times F_D\left[\lambda;\ -\alpha,\ -\beta,\ -\gamma;\ \mu;\ \frac{(x+1)yt}{2+(x+1)t},\ \frac{(x-1)yt}{2+(x-1)t},\ z\right], \quad \text{by} \quad (1\cdot4)$$

Putting $z = y, \gamma = m$ and using ((1), p. 239)

$$F_1(\alpha;\ \beta,\beta';\ \gamma;\ x,x) = {}_2F_1(\alpha,\beta+\beta';\ \gamma;\ x),$$

(2·1) becomes

$$\sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta-n)}(x)\, {}_2F_1(-m-n,\lambda;\ \mu;\ y)\, t^n = [1+\tfrac{1}{2}(x+1)t]^{\alpha}[1+\tfrac{1}{2}(x-1)t]^{\beta}$$

$$\times F_D\left[\lambda;\ -\alpha,\ -\beta,\ -m;\ \mu;\ \frac{(x+1)yt}{2+(x+1)t},\ \frac{(x-1)yt}{2+(x-1)t},\ y\right]. \qquad (2\cdot3)$$

In (2·3) we replace λ, μ, y and t by $-\mu-m$, $-\lambda-\mu-2m$, $2/(y+1)$ and $-\frac{1}{2}(y+1)t$, respectively. Then with the help of (1·3) we arrive at

$$\sum_{n=0}^{\infty} \frac{(m+n)!}{(-\lambda-\mu-m)_n} P_n^{(\alpha-n,\beta-n)}(x)\, P_{m+n}^{(\lambda-n,\mu-n)}(y)\, t^n$$

$$= (1+\lambda+\mu+m)_m [\tfrac{1}{2}(y+1)]^m [1-\tfrac{1}{4}(x+1)(y+1)t]^{\alpha}[1-\tfrac{1}{4}(x-1)(y+1)t]^{\beta}$$

$$\times F_D\left[-\mu-m;\ -\alpha,\ -\beta,\ -m;\ -\lambda-\mu-2m;\right.$$

$$\left.\frac{2(x+1)t}{(x+1)(y+1)t-4},\ \frac{2(x-1)t}{(x-1)(y+1)t-4},\ \frac{2}{y+1}\right]. \qquad (2\cdot4)$$

For $m = 0$, (2·4) reduces to the known result due to Manocha and Sharma (3).

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Next, for proving (2·2) we have

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(-\alpha-\beta)_n} P_n^{(\alpha-n,\beta-n)}(x)\, F_2(\lambda+n;\gamma,\gamma';\delta,\delta';y,z)\, t^n$$

$$= \sum_{n,r,k=0}^{\infty} \frac{(\lambda)_{n+r+k}(\gamma)_r(\gamma')_k}{r!\,k!\,(-\alpha-\beta)_n(\delta)_r(\delta')_k} P_n^{(\alpha-n,\beta-n)}(x)\, t^n y^r z^k$$

$$= \sum_{r,k=0}^{\infty} \frac{(\lambda)_{r+k}(\gamma)_r(\gamma')_k}{r!\,k!\,(\delta)_r(\delta')_k} y^r z^k \sum_{n=0}^{\infty} \frac{(\lambda+r+k)_n}{(-\alpha-\beta)_n} P_n^{(\alpha-n,\beta-n)}(x)\, t^n$$

$$= [1-\tfrac{1}{2}(1-x)t]^{-\lambda} F_A\left[\lambda;\ \gamma,\gamma',\ -\alpha;\ \delta,\delta',\ -\alpha-\beta;\right.$$

$$\left. \frac{y}{1-\frac{1}{2}(1-x)t},\ \frac{z}{1-\frac{1}{2}(1-x)t},\ \frac{-t}{1-\frac{1}{2}(1-x)t}\right] \quad \text{by} \quad (1\cdot5).$$

In (2·2) we put $\lambda = -m$, $z = 0$ and replace γ, δ, y and t by $-\delta-m$, $-\gamma-\delta-2m$, $2/(1+y)$ and $-2t/(y+1)$, respectively. This combined with (1·3) yields

$$\sum_{n=0}^{m} \frac{(1+\gamma+\delta+m)_n}{(-\alpha-\beta)_n} P_n^{(\alpha-n,\beta-n)}(x)\, P_{m-n}^{(\gamma+n,\delta+n)}(y)\, t^n = \frac{(1+\gamma+\delta+m)_m}{m!\,2^m}[1+y+(1-x)t]^m$$

$$\times F_2\left[-m;\ -\delta-m,\ -\alpha;\ -\gamma-\delta-2m,\ -\alpha-\beta;\ \frac{2}{1+y+(1-x)t},\ \frac{2t}{1+y+(1-x)t}\right]. \tag{2·5}$$

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