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Prime characteristic functions on compact separable groups

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Existence is established on some groups for characteristic functions which are prime in the sense of being indecomposable for product factorization.

Let G denote a topological group. A continuous normalized positive-definite complex-valued function on G is called a *characteristic function*. It is called a *character* if it is a homomorphism into the group of complex numbers of unit modulus. For a characteristic function ϕ on G, denote by $|\phi|$ the characteristic function defined by $|\phi|(g) = |\phi(g)|$ for all g in G and denote by G_ϕ the subgroup of G generated by $\{g: \phi(g) \neq 0\}$. Multiplication for characteristic functions will be the pointwise product, i.e. $\phi = \phi_1 \phi_2$ if $\phi(g) = \phi_1(g)\phi_2(g)$ for all g in G. Corresponding to the definitions in [3] we define a characteristic function ϕ to be *idempotent* if $\phi = \phi^2$, *infinitely divisible* if for each $n \in \mathbb{N}$ it can be factorized as $(\phi_n)^n \chi_n$ for some characteristic function ϕ_n and character χ_n, and *prime* if it cannot be factorized as a product of two characteristic functions neither of which is a character. In [5] the question of existence of primes was ignored. Existence of primes is established below for non-trivial Abelian groups and compact separable groups, with the exception of C_2, the cyclic group of order 2, which has none. It is also shown that connected compact groups have no idempotent characteristic functions and that certain characteristic functions, induced from characteristic functions on subgroups, do not have idempotent factors. The presence of idempotent factors was, in [5], found to be an obstacle to factorization.

Let ρ be a probability measure on $\hat{G}$, the dual group of an Abelian group G. Denote the support of ρ by $\operatorname{supp}\rho$. The *measure* ρ is called *decomposable* if $\rho = \mu * \nu$ where the support of each of the measures μ and ν contains at least two points. A *subset* A of a (multiplicative) group is said to be *decomposable* if there exist two sets A_1, A_2 each containing at least two elements and such that $A_1 A_2 = A$. If ρ is decomposable as $\mu * \nu$ and either $\operatorname{supp}\mu$ or $\operatorname{supp}\nu$ is compact then $\operatorname{supp}\rho$ is decomposable as $\operatorname{supp}\mu . \operatorname{supp}\nu$. A characteristic function on G is prime if and only if the corresponding probability measure on $\hat{G}$ is indecomposable. Thus, to prove existence of a prime characteristic function on an Abelian group G it is sufficient to find a finite indecomposable set in $\hat{G}$; any measure having this set as support is indecomposable.

Proposition 1. *Infinite Abelian groups have prime characteristic functions.*

Proof. Since the dual group is infinite it contains finite indecomposable sets by [4] Lemma 5·3.

Proposition 2. *Every cyclic group C_n of finite order $n \neq 2$ has prime characteristic functions.*

Proof. Denote the generator of $\hat{C}_n \cong C_n$ by χ. Let i, j_t, p, q be integers and all equations in the integers be modulo n. If $j_1 \neq j_2$ then $\chi^{i+j_1} \neq \chi^{i+j_2}$ so the only possible non-trivial

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decompositions of a set (χ^p, χ^q) would be of the form $(\chi^{j_1}, \chi^{j_2})(\chi^{j_3}, \chi^{j_4}), j_1 \neq j_2, j_3 \neq j_4$. We may order the j_i in such a way as to obtain the matrix equation

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \end{pmatrix} = \begin{pmatrix} p \\ q \\ q \\ p \end{pmatrix}.$$

The equation is inconsistent for simultaneous equations whenever $2(q-p) \neq n$, so for $n \neq 2$ there are indecomposable sets (χ^p, χ^q).

PROPOSITION 3. *Any finite Abelian group with two or more generators has prime characteristic functions.*

Proof. Denote the identity element and two of the generators of the dual group by e and a, b respectively. Assume a has order m and b order n. Let $X = ((a^{i_p} b^{j_q}))_{1 \leqslant \{p,q\} \leqslant r}$ and $Y = ((a^{i_p} b^{j_q}))_{r < \{p,q\} \leqslant s}$ where i_p, j_q, p, q, r, s are integers. Equations in the i will be modulo m and equations in the j will be modulo n. Assume that there is no repetition of elements in X, i.e. $j_p \neq j_q$ or $i_p \neq i_q$ whenever $p \neq q, 1 \leqslant \{p,q\} \leqslant r$, nor repetition in Y, i.e. $j_p \neq j_q$ or $i_p \neq i_q$ whenever $p \neq q, r < \{p,q\} \leqslant s$. We may order the i's and j's so that

$$\begin{aligned} i_1 + i_r &= 0 & j_1 + j_r &= 0 \\ i_1 + i_{r+1} &= 0 & j_1 + j_{r+1} &= 0 \\ i_2 + i_r &= 1 & j_2 + j_r &= 0, \end{aligned}$$

so $i_2 + i_{r+1} = 1$, $j_2 + j_{r+1} = 1$. Thus (e, a, b) cannot be decomposed as XY since an element of XY has to be ab. No other decomposition of (e, a, b) is possible since any elements of the factors in generators other than a and b cancel out on multiplication. Similarly (e, a, ab) and (e, b, ab) are indecomposable.

PROPOSITION 4. *The cyclic group C_2 has no prime characteristic functions; all the characteristic functions are infinitely divisible.*

Proof. Every characteristic function is of the form $\chi_{x,y} = x\chi_0 + y\chi$ where χ_0, χ are the characters of C_2, $x, y \in \mathbb{R}_+$ and $x + y = 1$. For each $n \in \mathbb{N}$ one has

$$\chi_{x,y} = (\chi_{x^{1/n}, y^{1/n}})^n.$$

From the propositions above we have the following:

THEOREM 1. *All Abelian groups have prime characteristic functions except for the cyclic group C_2 which has none.*

We shall use the term 'representation' (of a topological group) to mean a continuous unitary representation of the group in a complex Hilbert space. A characteristic function ϕ is canonically defined, up to isomorphism, by a representation π and cyclic vector ξ in a Hilbert space H, by $\phi(g) = (\pi(g)\xi|\xi)$ for g in G. We call the dimension of H the *degree* of ϕ. A characteristic function is a character if and only if it is of degree one. The function ϕ, defined for g in G by $\phi(g) = (\pi(g)\xi|\xi)$ where π is a representation and ξ an arbitrary vector in a Hilbert space H, is also a characteristic function; the Hilbert subspace H_ϕ of H spanned by $\pi(G)\xi$ is such that the projection ξ_ϕ of ξ on H_ϕ is cyclic for the restriction π_ϕ of π to H_ϕ.

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PROPOSITION 5. *Let ϕ be a characteristic function on a compact group G and suppose $G_\phi = G$. If ϕ is not a character it is not infinitely divisible.*

Proof. By [1] Remark 4·5 and Proposition 4·5, if ϕ is infinitely divisible it is the product of a character and a characteristic function defined by a representation and cyclic vector in an infinite-dimensional Fock space.

PROPOSITION 6. *Let G be a compact separable group. A characteristic function ϕ, for which $G_\phi = G$ and which is not infinitely divisible, has a prime factor.*

Proof. This is immediate from [5] Proposition 6.

COROLLARY 1. *Let G be a compact separable group. A characteristic function ϕ which is of finite degree but not a character, and such that $G_\phi = G$, has a prime factor.*

COROLLARY 2. *For any characteristic function ϕ on a connected compact separable group G one has $G_\phi = G$ so there are no idempotent characteristic functions. If G is non-Abelian all irreducible representations in spaces of dimension greater than one define characteristic functions with prime factors, so G has prime factors.*

Let G' be a subgroup of G such that G is a semi-direct product WG' where W is a transversal with a finite number of elements $(w_1 = e, w_2, \ldots, w_n)$. Any $g \in G$ can be uniquely expressed as $g_1 g_2$ where $g_1 \in W$, $g_2 \in G'$. Let ϕ be a characteristic function of G' defined by a representation π of G' in a Hilbert space H with cyclic vector ξ. Denote the representation of G induced from π by ρ^π. Its Hilbert space $\mathbf{H} = nH$ may be considered to be a space of H-valued functions on W. One has

$$\rho^\pi(g) f(w) = \pi((gw)_2) f((gw)_1)$$

for $g \in G$, $w \in W$. Choose f such that $f(w) = \xi$ for all $w \in W$ and denote by $\mathrm{Ind}\phi$ the characteristic function $g \mapsto (\rho^\pi(g) f \mid f)_{\mathbf{H}}$. If $g \in W$ then $\mathrm{Ind}\phi(g) = 1$.

LEMMA 1. *Let G be a compact separable group which is a semi-direct product WG' with finite transversal W. If ϕ is a characteristic function on G' such that $G'_\phi = G'$ then $G_{\mathrm{Ind}\phi} = G$ and so $\mathrm{Ind}\phi$ has no idempotent factors.*

Proof. Let $S = \{g \in G' : \phi(g) \neq 0\}$. If $g \in S$ then $\mathrm{Ind}\phi(g) = \phi(g)$. By [2] § 32·7, for any characteristic function ψ if $\psi(h) = 1$ then $\psi(hg) = \psi(g)$ for any $g \in G$. Since G is generated by WS one has $G_{\mathrm{Ind}\phi} = G$. Now use [5] Corollary 1.

LEMMA 2. *Let G be a semi-direct product WG' where W is a finite transversal. If ϕ is a characteristic function on G' which is not a character, then $\mathrm{Ind}\phi$ is not a character of G.*

Proof. Suppose ϕ is defined by π and cyclic ξ. There exist $g, h \in G'$ such that $\pi(g)\xi$ and $\pi(h)\xi$ are linearly independent. One has

$$\rho^\pi(g) f = \pi(g)\xi \oplus \pi((gw_2)_2)\xi \oplus \ldots \oplus \pi((gw_n)_2)\xi$$

so $\rho^\pi(g) f \neq \lambda \rho^\pi(h) f$ for any $\lambda \in \mathbb{C}$. Thus $\dim \mathbf{H}_{\mathrm{Ind}\phi} > 1$ and so $\mathrm{Ind}\phi$ is not a character.

LEMMA 3. *Let G be a semi-direct product WG' with finite transversal W and let ϕ be a characteristic function on G' of finite degree. If G' is a normal subgroup then $\mathrm{Ind}\phi$ is not a character.*

Proof. For a finite-dimensional representation π of G one has

$$\mathrm{Tr}\, \rho^\pi(g) = \sum_{w \in W} \mathrm{Tr}\, \pi(w^{-1} g w)$$

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where $\mathrm{Tr}\,\pi$ is extended to G by defining it to be zero off G'. Thus if G' is normal $\mathrm{Tr}\,\rho^\pi$ vanishes off G'. Since $\mathrm{Ind}\phi$ does not vanish off G' it cannot be a character.

LEMMA 4. *If ϕ is a character of a group G then G_ϕ is a normal subgroup.*

Proof. Since $\phi(g^{-1}hg) = \phi(h) \neq 0$, if $g \in G$ and $h \in G'$, also $g^{-1}hg \in G'$.

PROPOSITION 7. *If G is a finite non-Abelian group it has prime characteristic functions.*

Proof. For any characteristic function ϕ of finite degree take G' to be G_ϕ and consider the restriction ϕ' of ϕ to G'. If ϕ' is not a character $\mathrm{Ind}\phi'$ has a prime factor by Lemmas 1, 2 and Corollary 1. If ϕ' is a character then $\mathrm{Ind}\phi'$ has a prime factor by Lemmas 3, 4 and Corollary 1.

PROPOSITION 8. *If G is a locally connected compact separable group it has prime characteristic functions.*

Proof. Take G' to be the connected component of the identity and consider any characteristic function ϕ on G' of finite degree. Then $\mathrm{Ind}\phi$ has a prime factor by Corollary 2, Lemmas 1, 3 and Corollary 1.

PROPOSITION 9. *There exist prime characteristic functions on any infinite totally disconnected compact group.*

Proof. A totally disconnected compact group G can be realised as the projective limit of groups G_α, $\alpha \in B$, which are direct products of finite groups. Denote $\{g\colon \phi(g) = 1\}$ by N_ϕ. Then $G/N_\phi = G_\alpha$ for some $\alpha \in B$. When considering ϕ as a characteristic function of G_α, write $\phi = \phi_\alpha$. Suppose that ϕ_α is prime in G_α but that ϕ can be factorised as $\psi_1\psi_2$. Since $|\phi|^2 = |\psi_1|^2|\psi_2|^2$ and $|\psi| \leqslant 1$ for any characteristic function ψ, $\{g\colon |\psi_i(g)| = 1\} \supset N_\phi$ for $i = 1, 2$. Thus both $|\psi_1|$ and $|\psi_2|$ are characteristic functions of G_α. Thus $\phi = |\psi_1|\chi_1|\psi_2|\chi_2$ where χ_1 and χ_2 are characters of G, i.e. characters of G_β, G_γ respectively for some $\beta, \gamma \in B$. Since G_β and G_γ are direct products we may write $\chi_1 = \chi_3\chi_\beta$, $\chi_2 = \chi_4\chi_\gamma$ where χ_3 and χ_4 are characters of G_α, and χ_β and χ_γ are characters of G such that $\chi_\beta = \chi_\gamma = 1$ on G_α. Since $\phi(g) = 1$ on N_ϕ so $\chi_\beta\chi_\gamma \equiv 1$. Thus $\phi = \psi_3\psi_4$ where $\psi_3 = |\psi_1|\chi_3$ and $\psi_4 = |\psi_2|\chi_4$ are characteristic functions of G_α, which contradicts the primeness of ϕ_α.

THEOREM 2. *Every compact separable non-Abelian group has prime characteristic functions.*

Proof. We have proved the theorem for connected, for locally connected and for totally disconnected compact non-Abelian groups, which constitute all non-Abelian compact groups.

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