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# Large amplitude solitary magnetized plasma waves

C. M. C. NAIRN[1], R. BINGHAM[2] and J. E. ALLEN[3]

[1]Umeå University, Space Physics Department SE-901 87 Umeå, Sweden
[2]Rutherford Appleton Laboratory, Chilton, Didcot, Oxon OX11 OQX, UK
(rbi@vk.rl.ac.uk)
[3]Department of Engineering Science, Parks Road, Oxford OX1 3PJ, UK

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**Abstract.** Waves launched into a magnetized plasma when it is rapidly compressed were studied in the late 1950s by Adlam and Allen. In this paper we show that the equations describing large amplitude magnetized plasma waves, or Adlam–Allen waves, can be reduced to a single nonlinear equation, namely the Korteweg–de Vries equation and that the solutions of this equation are in agreement with the results obtained previously by Adlam and Allen. The solutions of both the Adlam–Allen equations and the Korteweg–de Vries equation take the form of solitary waves and periodic wave trains.

## 1. Introduction

The original Adlam–Allen wave model (Adlam and Allen 1958) applied to large amplitude waves that travel into a plasma containing a magnetic field when it is rapidly compressed. In this paper we describe the particle motion and field compression in the compressed field region from a two-fluid description of the plasma and Maxwell's equations using the Adlam–Allen model. In their earlier paper (Adlam and Allen 1958), Adlam and Allen derived a nonlinear equation describing the time-independent wave amplitude that yielded solitary wave solutions. In their second paper (Adlam and Allen 1960) a set of coupled nonlinear equations were solved numerically to show the space and time behaviour of the wave amplitude. Here, a similar procedure is followed leading to a nonlinear equation describing the space and time dependence of the waves. This equation is the nonlinear Korteweg–de Vries (KdV) equation. Solutions of the KdV equation are found for constant profile waves and compared with the results found by Adlam and Allen.

Section 2 contains a description of the Adlam–Allen wave model and the derivation of the KdV equation from the equations describing the model. In Sec. 3 analytic and numerical solutions of the KdV equation are discussed and compared with the wave solutions of Adlam and Allen, and the numerical solutions of the Adlam–Allen equations are extended beyond those originally published. Section 4 contains a brief summary of the results and possible applications of this theory to explain results obtained by plasma release experiments in space.

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## 2. Adlam–Allen wave model

The Adlam–Allen model describes waves that propagate in a collisionless cold plasma containing a magnetic field when it is rapidly compressed. Only macroscopic electric and magnetic fields are taken into account and displacement currents are neglected. The equations of motion and continuity for the electrons and ions are given by

$$\frac{\partial \mathbf{v_j}}{\partial t} + (\mathbf{v_j} \cdot \nabla)\mathbf{v_j} = \frac{q_j}{m_j}[\mathbf{E} + \mathbf{v_j} \times \mathbf{B}] \tag{1}$$

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \mathbf{v_j}) = 0, \tag{2}$$

where $v$ and $n$ are the velocity and density and the subscripts $j = \mathrm{i}, \mathrm{e}$ refer to ions and electrons respectively. Coordinates are chosen so that the magnetic field lies in the $z$-direction, $\mathbf{B} = (0, 0, B_z)$ and variations are assumed to occur in the $x$-direction only.

Neglecting displacement currents, Maxwell's equations are given by

$$\frac{\partial B_z}{\partial x} = \mu_0 e(n_\mathrm{e} v_{ey} - n_\mathrm{i} v_{iy}) \tag{3}$$

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t} \tag{4}$$

and Poisson's equation

$$\epsilon_0 \frac{\partial E_x}{\partial x} = e(n_\mathrm{i} - n_\mathrm{e}) \tag{5}$$

where we have assumed that the magnitude of the electronic and ionic charges are equal. Since $|n_\mathrm{i} - n_\mathrm{e}| \ll n$, the particle density, the difference between $n_\mathrm{i}$ and $n_\mathrm{e}$ can be ignored except in Poisson's equation. This implies that $v_{ix} = v_{ex}$ from the continuity equation.

For a constant profile wave we can set the time derivatives in (1) and (2) to zero; then adding the $y$-components of (1) for electrons and ions gives the result

$$m_\mathrm{e} \frac{dv_{ey}}{dx} + m_\mathrm{i} \frac{dv_{iy}}{dx} = 0, \tag{6}$$

which may be integrated using the boundary conditions $v_{iy} = v_{ey} = 0$ at $x = -\infty$ to give

$$v_{iy} = -\frac{m_\mathrm{e}}{m_\mathrm{i}} v_{ey}. \tag{7}$$

Clearly $v_{ey} \gg v_{iy}$ and this is the basis of a large transverse drift current. Taking the $x$-components of (1) and subtracting one from the other we arrive at the approximate equation

$$E_x = -v_{ey} B_z, \tag{8}$$

which in combination with (3) and (7) leads to an expression for the electrostatic field $E_x$, produced by charge separation, in terms of a nonlinear force, the ponderomotive force due to the wave with amplitude $B_z$

$$E_x = -\frac{1}{2\mu_0 en(1 + m_\mathrm{e}/m_\mathrm{i})} \frac{d(B_z^2)}{dx}. \tag{9}$$

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$E_x$ may also be written as the divergence of the Maxwell stress tensor. Associated with this $E_x$, there is a cold electron drift velocity in the $y$-direction given by

$$v_y = -\left(\frac{E_x}{B_z}\right)_y . \tag{10}$$

It is this drift velocity which drives the diamagnetic current in the $y$-direction, that is responsible for the diamagnetic cavity.

We now move to a Lagrangian coordinate system given by

$$\left(\frac{\partial}{\partial t}\right)_h = \left(\frac{\partial}{\partial t}\right)_x + \left(\frac{\partial x}{\partial t}\right)_h \left(\frac{\partial}{\partial x}\right)_t$$

$$\left(\frac{\partial}{\partial x}\right)_t = \left(\frac{\partial h}{\partial x}\right)_t \left(\frac{\partial}{\partial h}\right)_t ,$$

where $h$ is the Lagrangian coordinate and then change to dimensionless variables. Equations (1) to (5) may be reduced to two simultaneous equations in the normalized magnetic field strength $B = B_z/B_0$ and the normalized inverse density $n' = n_0/n$, (Adlam and Allen 1960)

$$\frac{\partial^2 n'}{\partial t^2} = -\frac{\partial}{\partial h}\left[B\frac{\partial B}{\partial h}\right] \tag{11}$$

$$\frac{\partial^3 B}{\partial h^2 \partial t} = \frac{\partial}{\partial t}(n'B) \tag{12}$$

$n_0$ and $B_0$ are the density and magnetic field strength in the undisturbed plasma; $h$ has been normalized by dividing by $d$ where $d$ is given by

$$d = \left[\frac{m_\mathrm{i} m_\mathrm{e}}{n_0 \mu_0 e^2 (m_\mathrm{i} + m_\mathrm{e})}\right]^{\frac{1}{2}} \tag{13}$$

and time has been normalized by dividing by $(m_i m_e/e^2 B_0^2)^{\frac{1}{2}}$.

Initially the plasma is assumed uniform and steady, i.e. $B = 1$, $n' = 1$, at $t = 0$. Integrating (12) using these boundary conditions, the following equation is obtained

$$\frac{\partial^2 B}{\partial h^2} = n'B - 1. \tag{14}$$

Adlam and Allen (1960) solved (11) and (14) numerically to yield solutions representing trains of solitary waves.

Here we use a perturbation technique known as the method of strained parameters (Nayfeh 1973; Dodd et al. 1982) to derive a single nonlinear partial differential equation, known as the KdV equation, the solutions of which are well known (Karpman 1975) and agree with those found by Adlam and Allen.

We start our derivation by making asymptotic expansions for $B$ and $n'$ in terms of a perturbation parameter, $\epsilon$,

$$B = 1 + \epsilon B^{(1)} + \epsilon^2 B^{(2)} \ldots \quad B^{(i)} \to 0 \quad \text{as } |h| \to \infty$$

$$n' = 1 + \epsilon n^{(1)} + \epsilon^2 n^{(2)} \ldots \quad n^{(i)} \to 0 \quad \text{as } |h| \to \infty$$

with the boundary conditions that $B$ and $n' \to 1$ as $|h| \to \infty$. On substituting these expressions for $B$ and $n'$ into (11) and (14) and dropping all terms higher than first

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order we obtain

$$\frac{\partial^2 n^{(1)}}{\partial h^2} - B^{(1)} - n^{(1)} = 0 \tag{15}$$

$$\frac{\partial^2 n^{(1)}}{\partial t^2} + \frac{\partial^2 B^{(1)}}{\partial h^{(2)}} = 0. \tag{16}$$

Substituting for $n^{(1)}$ in (16) from (15) and dropping the small terms gives

$$\frac{\partial^4 B^{(1)}}{\partial t^2 \partial h^2} + \frac{\partial^2 B^{(1)}}{\partial h^2} - \frac{\partial^2 B^{(1)}}{\partial t^2} = 0. \tag{17}$$

Thus, we have reduced our equations to one linear equation for $B$. By assuming that $B$ varies as $e^{i\theta(h,t)}$, where

$$\theta(h,t) = kh - \omega(k)t \tag{18}$$

we obtain the dispersion relation

$$\omega^2 = k^2(1+k^2)^{-1}. \tag{19}$$

For long waves, i.e. waves with small wave number, $k$

$$\omega \approx k - \frac{k^3}{2}. \tag{20}$$

In this long-wave limit we can write $k$ as $k = \epsilon^p \kappa$ where $\kappa$ is a new wave number of $O(1)$ and $p$ is a number to be determined. Equation (18) then takes the form

$$\theta(h,t) = \epsilon^p \kappa h - \omega(\epsilon^p \kappa)t. \tag{21}$$

Let us now take the Taylor expansion of $\omega(\epsilon^p \kappa)$ about $\epsilon^p \kappa - 0$, retaining only the first two terms in the expansion.

$$\omega(\epsilon^p \kappa) = \omega'(0)\epsilon^p \kappa + \omega'''(0)\epsilon^{3p}\kappa^3. \tag{22}$$

Only odd terms in $\kappa$ appear, since we are considering a purely dispersive system (i.e. no dissipation) and such a system cannot have a mixture of both odd and even terms. Equation (21) may now be written

$$\theta(h,t) = \epsilon^p \kappa(h - \omega'(0)t) - \epsilon^{3p}\kappa^3 \omega'''(0)t. \tag{23}$$

Using (20) to determine $\omega'(0)$ and $\omega'''(0)$, this becomes

$$\theta(h,t) = \epsilon^p \kappa(h-t) + 3\epsilon^{3p}\kappa^3 t \tag{24}$$

from which we obtain natural scalings for $h$ and $t$:

$$\xi = \epsilon^p(h-t), \quad \tau = 3\epsilon^{3p}t. \tag{25}$$

In the frame of reference with coordinates $\xi$ and $\tau$, the partial derivatives with respect to $h$ and $t$ are transformed as

$$\frac{\partial}{\partial t} \to 3\epsilon^{3p}\frac{\partial}{\partial t} - \epsilon^p \frac{\partial}{\partial \xi} \tag{26}$$

$$\frac{\partial}{\partial h} \to \epsilon^p \frac{\partial}{\partial \xi}. \tag{27}$$

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Making these transformations in (21) and (24), and using our asymptotic expansions for $B$ and $n'$ gives us

$$\left\{9\epsilon^{6p}\frac{\partial^2}{\partial\tau^2} - 6\epsilon^{4p}\frac{\partial^2}{\partial\tau\partial\xi} + \epsilon^{2p}\frac{\partial^2}{\partial\xi^2}\right\}\left[1+\epsilon n^{(1)}+\epsilon^2 n^{(2)}\dots\right] + \frac{\epsilon^{2p}}{2}\frac{\partial^2}{\partial\xi^2}\left[1+\epsilon B^{(1)}+\epsilon^2 B^{(2)}\dots\right]^2 = 0 \tag{28}$$

$$\epsilon^{2p}\frac{\partial^2}{\partial\xi^2}\left[1+\epsilon B^{(1)}+\epsilon^2 B^{(2)}\dots\right] + 1 - \left[1+\epsilon n^{(1)}+\epsilon^2 n^{(2)}\dots\right]\left[1+\epsilon B^{(1)}+\epsilon(2)\dots\right] = 0. \tag{29}$$

Gathering terms and considering the coefficients of powers of $\epsilon$ shows that a suitable choice of value for $p$ is $p=1/2$, which results in the following coefficients for (28)

$$[\epsilon^2] \qquad \frac{\partial^2 n^{(1)}}{\partial\xi^2} + \frac{\partial^2 B^{(1)}}{\partial\xi^2} \tag{30}$$

$$[\epsilon^3] \qquad \frac{\partial^2 n^{(2)}}{\partial\xi^2} + \frac{\partial^2 B^{(2)}}{\partial\xi^2} + \frac{1}{2}\frac{\partial^2[B^{(1)}]^2}{\partial\xi^2} - 6\frac{\partial^2 n^{(1)}}{\partial\tau\partial\xi} \tag{31}$$

and for (29):

$$[\epsilon] \qquad -n^{(1)} - B^{(1)} \tag{32}$$

$$[\epsilon^2] \qquad \frac{\partial^2 B^{(1)}}{\partial\xi^2} - n^{(1)}B^{(1)} - n^{(2)} - B^{(2)} \tag{33}$$

$$[\epsilon^3] \qquad \frac{\partial^2 B^{(2)}}{\partial\xi^2} - n^{(1)}B^{(2)} - n^{(2)}B^{(1)} \tag{34}$$

Since (28) and (29) must hold for all values of $\epsilon$, each coefficient of $\epsilon$ must vanish independently because sequences of $\epsilon$ are linearly independent. Thus, we set each of the coefficients given by (30) to (34) equal to zero. Then, from (32),

$$n^{(1)} = -B^{(1)}, \tag{35}$$

which also follows from (30) if we set all integration constants equal to zero. Using this equation to eliminate $n^{(1)}$ from (33) we obtain an expression for $n^{(2)}$ in terms of $B^{(1)}$ and $B^{(2)}$:

$$n^{(2)} = \frac{\partial^2 B^{(1)}}{\partial\xi^2} + \left[B^{(1)}\right]^2 - B^{(2)}. \tag{36}$$

If we now substitute for $n^{(1)}$ and $n^{(2)}$ in (31) using (35) and (36) we obtain an equation for $B^{(1)}$:

$$\frac{\partial^4 B^{(1)}}{\partial\xi^4} + 3\frac{\partial}{\partial\xi}\left[B^{(1)}\frac{\partial B^{(1)}}{\partial\xi}\right] + 6\frac{\partial}{\partial\xi}\frac{\partial B^{(1)}}{\partial\tau} = 0. \tag{37}$$

Integrating with respect to $\xi$ and setting the integration constant equal to zero gives

$$\frac{\partial^3 B^{(1)}}{\partial\xi^3} + 3B^{(1)}\frac{\partial B^{(1)}}{\partial\xi} + 6\frac{\partial B^{(1)}}{\partial\tau} = 0, \tag{38}$$

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which is the KdV equation, returning to the frame of reference with coordinates $h$ and $t$ and setting

$$\epsilon B^{(1)} = b, \tag{39}$$

the fourth-order (17) can be integrated to give a third-order equation:

$$\frac{\partial b}{\partial h} + \frac{\partial b}{\partial t} + \frac{3}{2} b \frac{\partial b}{\partial h} + \frac{1}{2} \frac{\partial^3 b}{\partial h^3} = 0. \tag{40}$$

This is one form of the nonlinear KdV equation. Nonlinear effects arise from the third term in the equation and dispersion effects from the fourth.

## 3. Wave solutions

Solitary wave solutions to the KdV equation may be obtained as in Nicholson (1983). Equation (20) is first changed into a moving frame with coordinates

$$h' = h - v_0 t$$
$$t' = t,$$

where $v_0$ is a constant. The waves are to have constant profiles in this frame so time derivatives are set to zero:

$$(2 - 2v_0 + 3b) \frac{\partial b}{\partial h'} + \frac{\partial^3 b}{\partial h'^3} = 0. \tag{41}$$

This can be immediately integrated and then written in the form of a force equation for particle motion in a potential well:

$$\frac{\partial^2 b}{\partial h'^2} = -\frac{\partial}{\partial b} \left[ (1 - v_0) b^2 + \frac{b^3}{2} + Ab + B \right], \tag{42}$$

where $A$ and $B$ are the integration constants. Particles trapped in the potential well given by the pseudopotential:

$$\phi(b) = (1 - v_0) b^2 + \frac{b^3}{2} + Ab + B \tag{43}$$

would give rise to a localized waveform.

Equation (42) can be integrated to give a solitary wave solution with the following boundary conditions; $b, b', b'' \to 0$ as $h' \to \pm\infty$ which means that $A = B = 0$:

$$b = 2(v_0 - 1) \operatorname{sech}^2 \left[ \frac{(v_0 - 1)^{1/2} h'}{(2)^{1/2}} \right]. \tag{44}$$

This result is in agreement with the solitary wave solutions found by Adlam and Allen (1958) for constant profile waves. However, it should be noted that the solutions obtained here from the KdV equation enter over a smaller range of amplitudes than those obtained by Adlam and Allen. We assumed $b \ll 1$, while Adlam and Allen had a less restrictive range for $b$ of $b \leqslant 2$. For $b > 2$ the solitons undergo wave-breaking and $E_x$ becomes large enough to reflect a large fraction of the incoming upstream ions. Equation (44) is a particular solution of the KdV equation. The general periodic wave solution may be written in terms of Jacobian elliptic functions and referred to as a croidal wave,

$$b = a_2 + (a_3 - a_2) cn^2 \left[ \frac{(a_3 - a_1)^{1/2}}{2} (\xi - \xi_3), m \right], \tag{45}$$

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where

$$m = \frac{a_2 - a_3}{a_1 - a_3}, \tag{46}$$

$a_1$, $a_2$ and $a_3$ are constants (see Appendix); $\xi$ is the spatial coordinate in a frame of reference in which the waves have permanent shape and size and $\xi_3$ is the value of $\xi$ when $b = a_z$. A $\mathrm{sech}^2$ solution is obtained from (45) in the limit $m = 1$.

Adlam and Allen (1960) solved (11) and (14) numerically to yield solutions representing trains of solitary waves. Here the equations are solved again, using the same algorithm as they used, to investigate the form of the waves after a long time and to show that the maximum value to which the amplitude can grow is three times the equilibrium field strength. The details of the algorithm will be repeated here for completeness.

The boundary conditions used are

$$B = 1 \quad \text{when } t = 0 \tag{a}$$

$$n' = 1 \quad \text{when } t = 0 \tag{b}$$

$$\frac{\partial n'}{\partial t} = 0 \quad \text{when } t = 0. \tag{c}$$

The last condition is obtained from the continuity equation by putting the velocity equal to zero when $t = 0$.

At sufficiently large distances into the plasma the magnetic field remains undisturbed, i.e.

$$B \to 1 \quad \text{as } h \to \infty. \tag{d}$$

Adlam and Allen considered two cases: one in which the magnetic field strength increased linearly at the boundary and one in which it increased exponentially. These boundary conditions are expressed as

$$B = 1 + \alpha t \quad \text{at } h = 0 \tag{e}$$

$$B = 1 + C[1 - \exp\{-\alpha t\}] \quad \text{at } h = 0, \tag{f}$$

where $\alpha$ and $C$ are constants.

Equations (11) and (14) are solved alternately in the following manner.

(1) Equation (14) is solved for time $t_1$ to give $B$ as a function of $h$, $n'$ being known.

(2) Equation (11) is solved to give $n'$ and $\partial n'/\partial t$ at time $t_1 + \delta t$ for a range of values of $h$ when $n'$, $\partial n'/\partial t$ and $B$ are known at time $t_1$.

In the solution of type (1) the boundary conditions $(d)$, $(e)$ and $(f)$ are used, and in the solution of type (2) the boundary conditions $(b)$ and $(c)$ are used when $t = 0$. The boundary condition $(d)$ was replaced by the condition $B = 1$ when $h = H$ and $H$ was made as large as practicable.

Solutions for a linear increase in $B$ at the boundary are shown in Fig. 1, which were obtained using $\alpha = 1$. The program used to solve the equations stopped running when the inverse density $n'$ reached zero. An inverse density of zero corresponds to infinite density, which is physically unrealistic. The last time step for which solutions were obtained was $t = 1.65$, corresponding to $B_{\max} = 2.65$ and $n_{\min} = 0.025$. As $B_{\max} \to 3$, $n_{\min} \to 0$. These solutions were obtained by Adlam and Allen (1960).

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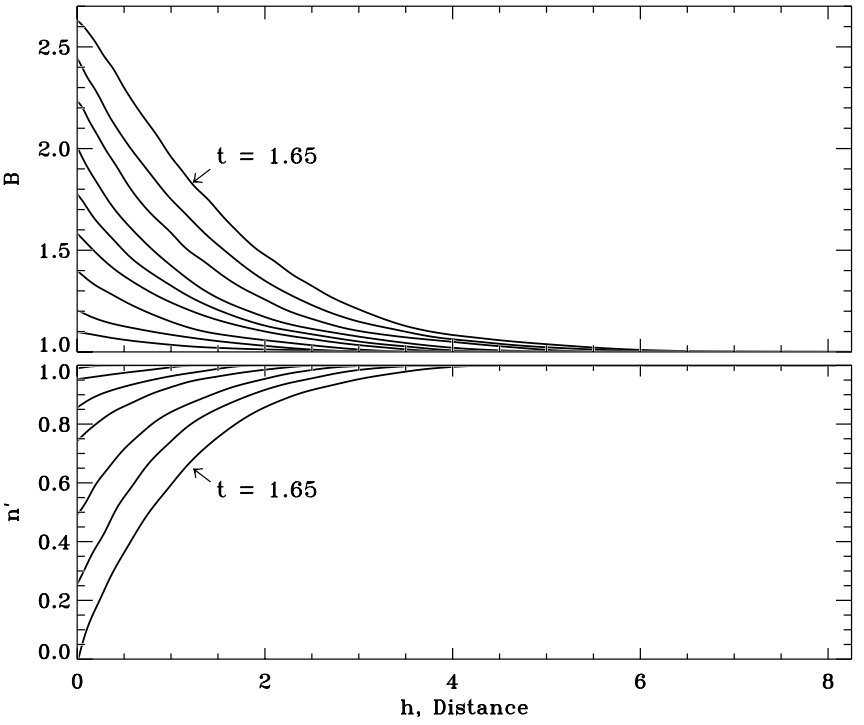



**Figure 1.** Magnetic field strength and inverse density as a function of the coordinate $h$ for a linear increase in $B$ at the boundary $h=0$. The lowest $B$ curve and highest $n'$ curve correspond to $t=0.1$. The curves between these and the ones labelled $t=1.65$ correspond to the times $t=0.2, 0.4, 0.6, 0.8, 1.0, 1.25$ and $1.45$.

Figures 2–5 show the numerical results obtained for an exponential increase of the magnetic field strength at the boundary for times $t=8.2$, 16.2, 24.0, 32.0. The values $C=\alpha=1$ have been used. It can be seen that as the waves propagate into the plasma they grow in amplitude. The leading oscillation seems to be developing into a solitary pulse. The solutions obtained by Adlam and Allen (1960) for these boundary conditions extended only to $t=9$.

In Fig. 6 numerical results are shown for the boundary conditions $C=0.5$, $\alpha=1$, at times $t=25.2$, 50.0, 75.2. These results look very similar to those found by Gurevich and Pitaevskii (1974) when they solved the KdV equation for the decay of an initial discontinuity. Their solutions extended further than those shown here. After a length of time the leading oscillation in the wave train developed into a solitary wave moving at constant velocity. After a longer time a string of equi-spaced solitary pulses developed. Since, as will be shown in the following section, the Adlam–Allen equations can be reduced to the KdV equation, it seems likely that if the solutions of the Adlam–Allen equations were extended further, they too would develop into a train of solitary pulses.

The Adlam–Allen waves propagate perpendicular to the magnetic field at approximately the lower-hybrid frequency, $(\omega_{ce}\omega_{ci})^{1/2}$.

The frequency of magnetosonic waves increases with angle of propagation for a given wavenumber; the exact frequencies for any angle are obtained from the Appleton–Hartree equation. The scale length of the Adlam–Allen waves is given by

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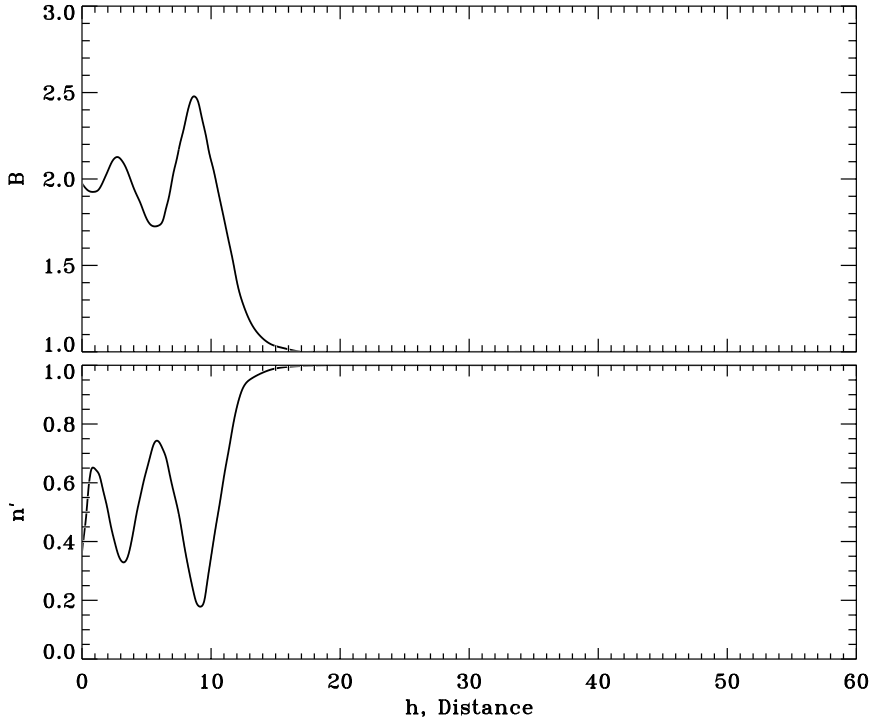



**Figure 2.** Magnetic field and inverse density profile as a function of $h$ for $t = 8.2$, for an exponential increase in $B$ at the boundary $h = 0$.

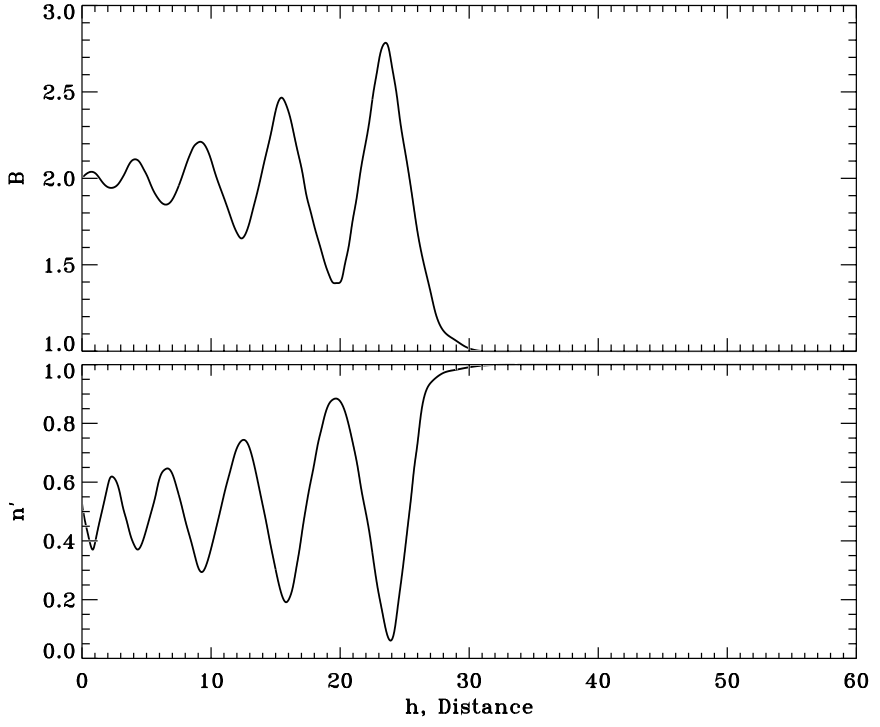



**Figure 3.** Magnetic field and inverse density profile as a function of $h$ for $t = 16.2$, for an exponential increase in $B$ at the boundary $h = 0$.

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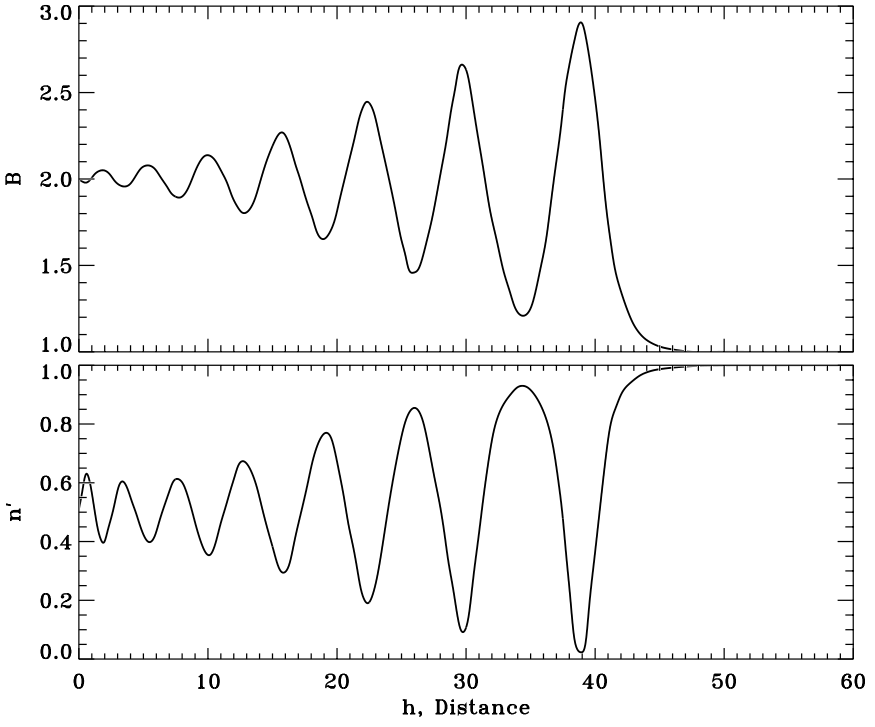



**Figure 4.** Magnetic field and inverse density profile as a function of $h$ for $t = 24.0$, for an exponential increase in $B$ at the boundary $h = 0$.

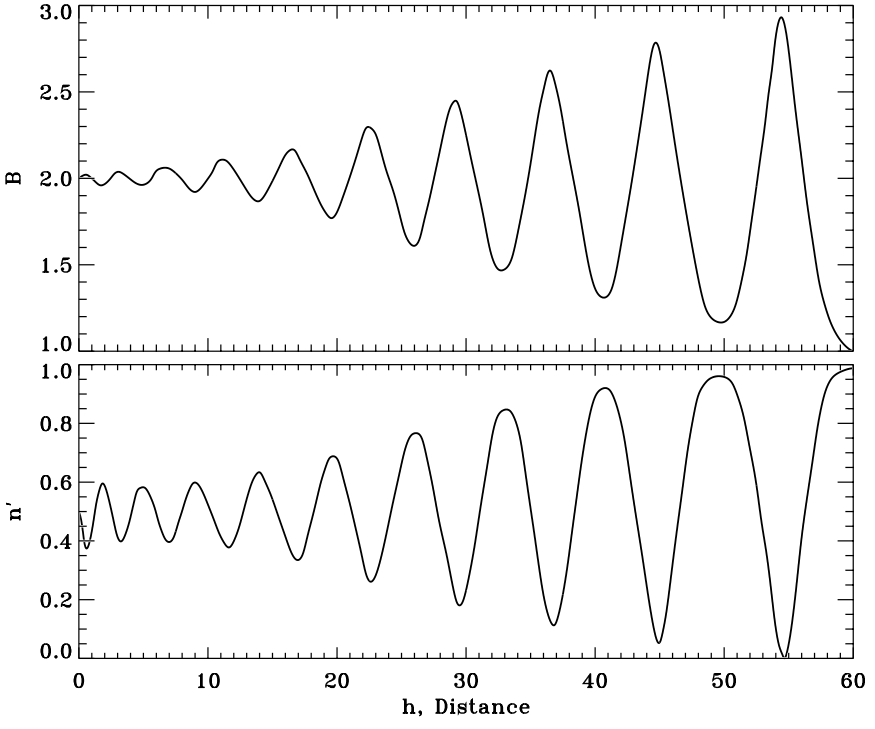



**Figure 5.** Magnetic field and inverse density profile as a function of $h$ for $t = 32.0$, for an exponential increase in $B$ at the boundary $h = 0$.

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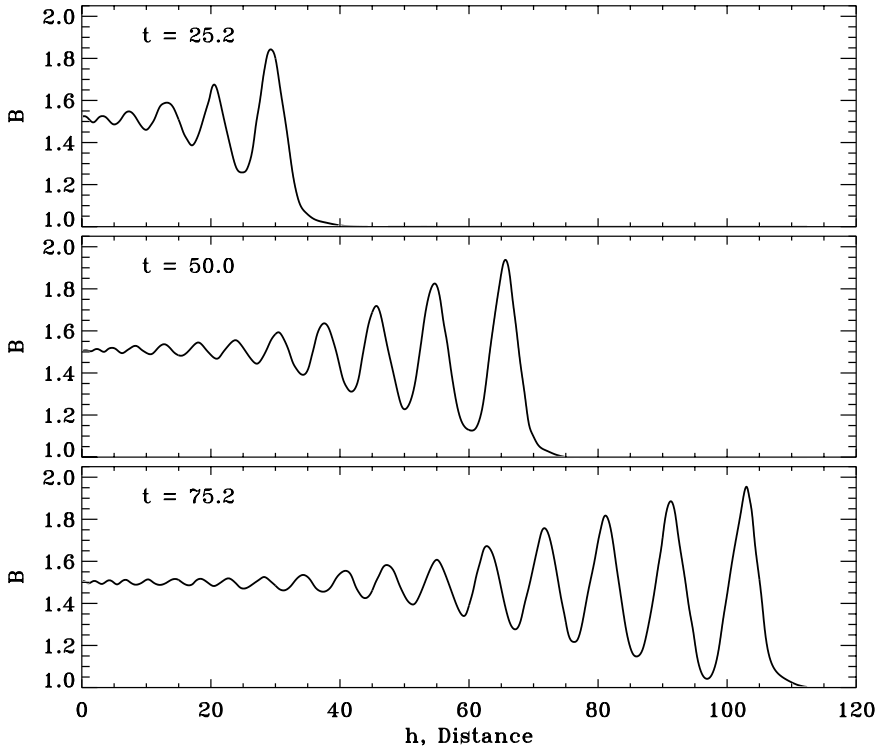



**Figure 6.** Magnetic field profiles as a function of $h$ for $t = 25.2, 50.0, 75.2$, and an exponential increase in $B$ at the boundary $h = 0$. The value $C = 0.5$ has been used to obtain the solutions.

$d$, which is $c/\omega_{pe}$, where $\omega_{pe}$ is the electron plasma frequency. If finite-temperature effects are added to the model the frequencies of the waves are unchanged, but the scale length is then given by $c/\omega_{pe}$ multiplied by a temperature dependent factor. When the waves have a small amplitude, the width is very large and the ion gyroradius and gyration time are much smaller than the width and duration of the pulse, then the appropriate multiplication factor is

$$\left[1 + \frac{1}{16}\frac{m_{\mathrm{i}}}{m_{\mathrm{e}}}\beta\right]^{1/2},$$

where $\beta$ is the ratio of kinetic pressure to magnetic pressure in the plasma. For waves having amplitudes satisfying the condition

$$1 > \frac{b_m - b_0}{b_0} > \frac{m_{\mathrm{e}}}{m_{\mathrm{i}}} + \frac{\beta_+}{16},$$

where $b_m$ is the maximum value of $b$, $b_0$ is the equilibrium value and $\beta_+$ is the ion plasma beta, then the multiplication factor is

$$(1 - 3\beta_-/4)^{1/2},$$

where $\beta_-$ is the electron plasma beta.

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## 4. Summary

It has been shown that starting from the cold plasma model of Adlam and Adlam it is possible to derive the KdV equation to describe nonlinear magnetosonic waves propagating perpendicular to the magnetic field at frequencies around the lower-hybrid frequency, which propagate into a magnetized plasma when it is rapidly compressed. The solutions of the KdV equation have been shown to have the same form as those obtained by Adlam and Allen.

A more extensive treatment elsewhere will deal with finite-geometry effects, magnetosonic waves propagating at an angle to the magnetic field and plasma heating due to the Adlam–Allen waves. The analysis presented here can be applied to the study of large amplitude waves at the Earth's bow shock, coronal mass ejections (CMEs), ejected plasmoids and plasma release experiments.

## Appendix. Derivation of constants $a_1, a_2$ and $a_3$

The constants $a_1, a_2$ and $a_3$ are given by the three distinct zeros of

$$b^3 - 2Wb^2 + 2Ab + 2B = 0, \tag{A 1}$$

where we have replaced $(v_0 - 1)$ in (43) by $W$. Using the substitution $b = y + 2W/3$ brings the cubic equation (A 1) to reduced form:

$$y^3 + \left(2A - \tfrac{4}{3}W^2\right)y - \tfrac{16}{27}W^3 + \tfrac{4}{3}AW + 2B = 0, \tag{A 2}$$

which may be written

$$y^3 + \alpha y + \beta = 0, \tag{A 3}$$

with $\alpha$ and $\beta$ identified as

$$\alpha = 2A - \tfrac{4}{3}W^2 \quad \beta = -\tfrac{16}{27}W^3 + \tfrac{4}{3}AW + 2B. \tag{A 4}$$

Cardan's formula gives the solution of (A 3) as

$$y = \sqrt[3]{-\frac{\beta}{2} + \left[\frac{\beta^2}{4} + \frac{\alpha^3}{27}\right]^{1/2}} + \sqrt[3]{-\frac{\beta}{2} - \left[\frac{\beta^2}{4} + \frac{\alpha^3}{27}\right]^{1/2}}. \tag{A 5}$$

Thus, the solutions of (A 1) are given by

$$b = \sqrt[3]{-\frac{\beta}{2} + \left[\frac{\beta^2}{4} + \frac{\alpha^3}{27}\right]^{1/2}} + \sqrt[3]{-\frac{\beta}{2} - \left[\frac{\beta^2}{4} + \frac{\alpha^3}{27}\right]^{1/2}} + \frac{2W}{3}. \tag{A 6}$$

In the special case $A = B = 0$, from (A 5),

$$y = \frac{2W}{3} + \frac{2W}{3} = \frac{4W}{3} \tag{A 7}$$

and

$$b = 2W. \tag{A 8}$$

From (A 1) the other two roots are equal and given by $b = 0$ in this special case.

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