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MATHEMATICAL INFERENCE AND LOGICAL INFERENCE

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Abstract. The deviation of *mathematical proof*—proof in mathematical practice—from the ideal of *formal proof*—proof in formal logic—has led many philosophers of mathematics to reconsider the commonly accepted view according to which the notion of formal proof provides an accurate descriptive account of mathematical proof. This, in turn, has motivated a search for alternative accounts of mathematical proof purporting to be more faithful to the reality of mathematical practice. Yet, in order to develop and evaluate such alternative accounts, it appears as a necessary prerequisite to first possess a clear picture of what the deviation of mathematical proof from formal proof consists in. The present work aims to contribute building such a picture by investigating the relation between the elementary steps of deduction constituting the two types of proofs—*mathematical inference* and *logical inference*. Many claims have been made in the literature regarding the relation between mathematical inference and logical inference, most of them stating that the former is lacking properties that are constitutive of the latter. Such *differentiating claims* are, however, usually put forward without a clear conception of the properties occurring in them, and are generally considered to be immediately justified by our direct acquaintance, or phenomenological experience, with the two types of inferences. The present study purports to advance our understanding of the relation between mathematical inference and logical inference by developing a detailed philosophical analysis of the differentiating claims, that is, an analysis of the *meaning* of the differentiating claims—through the properties that occur in them—as well as the *reasons* that support them. To this end, we provide at the outset a representative list of the different properties of logical inference that have occurred in the differentiating claims, and we notice that they all boil down to the three properties of *formality*, *generality*, and *mechanicality*. For each one of these properties, our analysis proceeds in two steps: we first provide precise conceptual characterizations of the different ways logical inference has been said to be formal, general, and mechanical, in the philosophical and logical literature on formal proof; we then examine why mathematical inference does not appear to be formal, general, and mechanical, for the different variations of these notions identified. Our study results in a precise conceptual apparatus for expressing and discussing the properties differentiating mathematical inference from logical inference, and provides a first inventory of the various reasons supporting the observations of those differences. The differentiating claims constitute thus a set of data that any philosophical account of mathematical inference and proof purporting to be more faithful to mathematical practice ought to be able to accommodate and explain.

§1. Introduction. In the preface to his book *Dense Sphere Packings: A Blueprint for Formal Proofs*, the mathematician Thomas Hales writes:

> The ultimate standard of proof is a formal proof, which is nothing other than an unbroken chain of logical inferences from an explicit set of axioms. While this may be the mathematical ideal of proof, actual mathematical practice generally deviates significantly from the ideal. (Hales, 2012, p. x)

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The deviation of *mathematical proof*—proof in mathematical practice—from the ideal[1] of *formal proof*—proof in formal logic—has received much attention in the philosophy of mathematics.[2] This originates from the fact that, in contemporary epistemology of mathematics, the notion of formal proof has not only played a *normative* role by specifying what mathematical proofs *should be*, it has also played a *descriptive* role purporting to capture what mathematical proofs *are*.[3] With respect to the latter role, any deviation of mathematical proof from formal proof constitutes an epistemological challenge that demands a philosophical response. In order to be able to develop and evaluate different strategies to address this challenge, and ultimately to provide a philosophical account of mathematical proof that would be more faithful to the reality of mathematical practice, we first need to possess a clear picture of what the deviation of mathematical proof from formal proof consists in. The present work aims to contribute to the philosophical project of building such a picture.

Any attempt to address this issue must begin by recognizing that mathematical proofs and formal proofs are *composite* entities, for both are composed of elementary steps of deduction. An elementary step of deduction in a formal proof is, by definition, a *logical inference*. Analogously, we shall refer to an elementary step of deduction in a mathematical proof as a *mathematical inference*.[4] Insofar as mathematical inference and logical inference are the building blocks of mathematical proof and formal proof, investigating the relation between mathematical proof and formal proof boils down, ultimately, to investigating the relation between mathematical inference and logical inference.

Unsurprisingly, many claims have been made in the philosophical literature on the relation between mathematical inference and logical inference. It is noteworthy that most of these claims are built according to the same scheme, usually stating that mathematical inference and logical inference differ with respect to certain *properties*. In the following, we shall refer to them as the *differentiating claims*, and to the properties that occur in them as the *differentiating properties*.[5] A representative list of the

[1] We shall not discuss in this paper whether, and in which sense, formal proof constitutes *the* mathematical ideal of proof, nor shall we discuss whether there could be a *plurality* of mathematical ideals of proof.

[2] We refer the reader to the following list of references: (Myhill, 1960), (Kreisel, 1967, 1970, 1981), (Corcoran, 1973), (Feferman, 1979, 2012), (Robinson, 1991, 1997, 2000, 2004), (Detlefsen, 1992a,b, 2009), (Thurston, 1994), (Rav, 1999, 2007), (Azzouni, 2004, 2009, 2013), (Bundy, Atiyah, Macintyre, & MacKenzie, 2005), (Bundy, Jamnik, & Fugard, 2005), (Suppes, 2005), (Aberdein, 2006), (Avigad, 2006, 2008), (Cellucci, 2008), (Leitgeb, 2009), (Goethe & Friend, 2010), (Sjögren, 2010), (Larvor, 2012), (Tanswell, 2015), (Burgess, 2015), and (Weir, 2016).

[3] See, for instance, Kitcher: "I do not intend to deny that much mathematical knowledge is gained by constructing or following the sequences of statements contained in mathematics books and labelled "proofs." Nor am I suggesting that the kinds of inferences involved in these proofs are anything other than what logicians and philosophers of mathematics have traditionally taken them to be" (Kitcher, 1984, p. 10).

[4] The distinction between mathematical inference and logical inference is not meant to be interpreted as a dichotomy. According to our terminology, it is thus possible for a mathematical inference to count as a logical inference. This is as it should be, for mathematical proofs do possess many genuine logical inferences as elementary steps of deduction—e.g., steps of deduction consisting of modus ponens, reductio ad absurdum, implication introduction, etc.

[5] As we noted in footnote 4, many mathematical inferences in mathematical proofs are indeed logical inferences. Obviously, the differentiating claims put forward in the literature only concern those mathematical inferences that are not genuine logical inferences, what we might call *proper* mathematical inferences. Thus, we shall assume throughout this paper that the differentiating claims only concern *proper* mathematical inferences.

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differentiating properties which have appeared in differentiating claims is provided in Table 1.[6]

Table 1. *Differentiating properties*

Mathematical inference	**Logical inference**
Local	General, global
Material, informal	Formal
non-Mechanical	Mechanical
Topic-specific	Topic-neutral
Matter dependent	Matter independent
Domain dependent	Domain independent
Content dependent	Content independent
Meaning dependent	Meaning independent
Subject matter dependent	Subject matter independent

Not much effort has been devoted, however, to provide precise conceptual characterizations of the differentiating properties, nor to provide arguments for the differentiating claims. Rather, the differentiating properties are usually taken to be conceptually clear, and the differentiating claims to be self-evident, or at least to be directly justified by our direct acquaintance with mathematical inference and logical inference. Our standpoint is that genuine progress can be made in our understanding of the relation between mathematical inference and logical inference—and thereby of the relation between mathematical proof and formal proof—by putting the differentiating claims on firm conceptual grounds, and by investigating the reasons that support them.

To develop a philosophical analysis of the differentiating claims along these lines is precisely the aim of this study. To this end, we shall proceed in two steps.[7] The first step will consist in providing precise conceptual characterizations of the properties listed in the right side of Table 1, i.e., the properties attributed to logical inference. We shall observe that many of them have been used synonymously, allowing the classification of these properties into three synonymity classes, identified by the following three properties: *formal*, *general*, and *mechanical*. We shall see, however, that these latter properties are themselves susceptible of many variations, since there are different ways in which logical inference has been said to be formal, general, and mechanical. The first step will thus amount to identifying and providing precise conceptual characterizations for the different variations of these three properties. The second step will then consist in examining why mathematical inference does not appear to enjoy the properties attributed to logical

[6] This list has been compiled based on an analysis of the references listed in footnote 2. Notice that, in these references, the differentiating claims might appeal to inference, but might also appeal to proof or reasoning—e.g., comparing mathematical or informal proof to logical or formal proof, and mathematical or informal reasoning to logical or formal reasoning. In this case, a pair of differentiating properties is listed only when it is clear that the differentiating claim in which it appears concerns elementary proof steps or reasoning steps, that is, elementary steps of deduction. Although it is impossible to aim for exhaustivity, we believe that this list is fairly representative of the differentiating claims that have been put forward in the relevant literature.

[7] We only provide here a sketch of the methodology we will adopt. A detailed description of our approach as well as the motivations for adopting it will be presented in more detail in §4.

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inference, i.e., why mathematical inference does not appear to be formal, general, and mechanical. This examination will be carried out for the different variations of these three notions, its starting point being the conceptual characterizations resulting from the analysis conducted in the first step.

The paper is organized as follows. §2 and §3 circumscribe our two main objects of study, i.e., logical inference and mathematical inference. §4 explains our two-step methodology for analyzing the differentiating claims. §5, §6, and §7 apply this methodology, respectively, to the notions of *formality*, *generality*, and *mechanicality*. §8 concludes this paper by discussing two philosophical strategies that have been deployed in the literature to provide alternative accounts of mathematical proof purporting to be more faithful to the reality of mathematical practice.

§2. Logical inference. In the mathematical ideal of proof, every elementary step of deduction in a proof shall consist of a *logical inference*. The importance of logical inference in mathematical reasoning can be traced back to the emergence of the modern standard of mathematical rigor at the end of the nineteenth century, when rigorous mathematical reasoning came to be conceived as proceeding from axioms to theorems by an exclusive use of logical inferences.[8] It is now commonly held that the ultimate way to implement the modern standard of mathematical rigor is through the *formalization* of mathematical reasoning. As Detlefsen put it:[9]

> [W]ith the vigorous development of techniques of *formalization* that has taken place in this century, demands for rigor have increased to a point where it is now the reigning orthodoxy to require that, to be genuine, a proof must be formalizable. This emphasis on formalization is based on the belief that the only kinds of inferences ultimately to be admitted into mathematical reasoning are *logical* inferences [...]. (Detlefsen, 1992b, p. viii)

What does the formalization of mathematical reasoning consist in? Here is Gödel's answer:

> [T]he so-called "formalization" of mathematics [...] means that a perfectly precise language has been invented, by which it is possible to express any mathematical proposition by a formula. Some of these formulas are taken as axioms, and then certain rules of inference are laid down which allow one to pass from the axioms to new formulas and thus to deduce more and more propositions [...]. (Gödel, 1995b, p. 45)

The formalization of mathematical reasoning in all, or a subfield of, mathematics requires then two elements: a *formal language*, in which any propositions of all, or a subfield of, mathematics can be expressed as formulas; a set of *logical rules of inference*, which specify which steps of deduction are to be admitted, i.e., what counts as a logical inference. When some axioms are specified and expressed in the chosen formal language, we obtain

[8] For a philosophical and historical discussion of the modern standard of mathematical rigor, we refer the reader to (Kitcher, 1981, 1984) and (Detlefsen, 2009, sec. 4).

[9] Such a view on mathematical rigor is expressed, for instance, by Saunders Mac Lane: "A mathematical proof is rigorous when it is (or could be) written out in the first order predicate language $L(\in)$ as a sequence of inferences from the axioms ZFC, each inference made according to one of the stated rules" (Mac Lane, 1986, p. 377).

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a *formal deductive system*. Given a formal deductive system, a *formal proof* is a sequence of formulas such that each formula in the sequence is either an axiom or the result of the application of a logical rule of inference to one or more preceding formulas in the sequence. Insofar as they are considered to be the ultimate way to implement the modern standard of mathematical rigor, formal proofs are taken to constitute the current mathematical ideal of proof.

From this perspective, a *logical inference* is no more and no less than a step of deduction that proceeds from some premisses to some conclusion—both expressed as formulas in the considered formal language—according to one of the logical rules of inference of the considered formal deductive system.[10] It follows that, in the context of the formalization of mathematical reasoning, the notion of logical inference is completely determined by the one of logical rule of inference. Now, what a logical rule of inference is—relatively to a given formal deductive system—is entirely specified by its *inference schema*, that is, a pair composed of a *premiss schema* and a *conclusion schema* consisting, respectively, of a set of schemas for the premisses and a schema for the conclusion. Here, a schema is a template or pattern composed of placeholders and of symbols from the vocabulary of the considered formal language, together with some specifications on how the placeholders are to be filled to generate formulas of the language which are then called *instances* of the schema.[11] The archetypal example of an inference schema is given by the one of the logical rule of inference known as *modus ponens*, a rule that one can find in almost any formal deductive system:[12]

[10] It should be noted that certain formal deductive systems possess rules of inference that are not logical rules of inference—e.g., many formal deductive systems for arithmetic possess mathematical induction as a rule of inference. Strictly speaking, such formal deductive systems would not properly implement the modern standard of mathematical rigor, for they would license inferences that are not properly logical. This is not an issue since, from a technical perspective, the distinction between axioms and rules of inference is usually not crucial, and often results from choices on the part of the designer of the system—e.g., mathematical induction can either be represented as an axiom or as a rule of inference in a formal deductive system for arithmetic. Thus, such formal deductive systems can still be considered to implement the modern standard of mathematical rigor, provided that they can be rewritten so as to only possess logical rules of inference. For the purpose of this paper, we will adopt a more restrictive use of the term formal deductive system, and consider that a formal deductive system is one that only possesses logical rules of inference, keeping in mind that a formal deductive system possessing nonlogical rules of inference, and considered as a legitimate implementation of the modern standard of mathematical rigor, can be rewritten so as to only possess logical rules of inference. Furthermore, we would like to emphasize that, throughout this paper, by a *logical* rule of inference we mean a rule of inference that is properly *logical*, and by a *logical* inference we mean an inference that is properly *logical*. We owe this important point of clarification to an anonymous reviewer.

[11] More generally, Corcoran (2014) defines a schema as consisting of two things: (1) a *template-text* or *schema-template* which is "a syntactic string composed of significant words and/or symbols and also of blanks or other placeholders", and (2) a *side condition* which specifies "how the blanks (placeholders, variables or ellipses) are to be filled to obtain instances". Although we are considering in this section the notion of schema only relatively to a formal language, we shall see in the next section that schemas can be defined as well for natural language (e.g., plain English).

[12] Our notion of inference schema corresponds exactly to what Corcoran (2014) calls an *argument-text schema*. Furthermore, Corcoran says that: "Of the various ways of presenting an argument-text perhaps the one least open to misinterpretation is the premises-line-conclusion format which consists in listing the premises followed by a line followed by the conclusion." In the following, we will present inference schemas using the premises-line-conclusion format.

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$$\frac{\Phi \quad \Phi \rightarrow \Psi}{\Psi},$$

where Φ and Ψ are placeholders to be filled with formulas of the considered formal language.

Thus, we shall consider in this study that the notion of logical inference is entirely determined by the one of logical rule of inference. As a consequence, we shall consider the properties attributed to logical inference in Table 1 as being properties of logical rules of inference. In fact, reflecting a moment on the right side of Table 1 reveals that this is indeed the most natural way to think of these properties. More specifically, we shall see in the following that these properties will appear to apply either to the inference schemas of logical rules of inference, or to their applications. It is important to mention, however, that our analysis of these properties will not be tied to the particularities of logical rules of inference as they belong to one or another formal deductive system. Rather, we shall provide conceptual characterizations of them insofar as they constitute properties that any logical rule of inference, from any formal deductive system, should possess in virtue of implementing the modern standard of mathematical rigor, i.e., in virtue of constituting the mathematical ideal of proof.

§3. Mathematical inference. A mathematical inference is an elementary step of deduction in a mathematical proof. In order to carry out a philosophical analysis of the differentiating claims, we need to possess a clear conception of what we take a mathematical inference to be. In this section, we begin by considering as a case study the mathematical inferences making up an ordinary mathematical proof of the irrationality of $\sqrt{2}$. Following Sundholm (2012), we then distinguish between three dimensions of the notion of mathematical inference—the inference act, the inference process, and the inference trace. For the purpose of this study, we shall see that mathematical inferences shall be approached at the level of their inference processes. This will lead us to introduce the notion of *mathematical rule of inference* as a functional description—i.e., as a description in terms of input-ouput functions—of the inference processes underlying mathematical inferences. Since the properties in Table 1 attributed to logical inference are properties of logical rules of inference, the notion of mathematical rule of inference will provide us with a point of comparison between mathematical inference and logical inference, and thereby with a suitable conceptual setting for analyzing the differentiating claims.

3.1. Mathematical inference in mathematical proof: a case study. There are thousands of mathematical proofs out there—in mathematical textbooks, monographs, and journals—ready for philosophical investigations. In this paper, we will consider as a typical example the mathematical proof of the irrationality of $\sqrt{2}$ taken from the fourth edition of Hardy and Wright's *An Introduction to the Theory of Numbers* (Hardy & Wright, 1975, pp. 39–40):[13]

[13] We are choosing this example for three reasons: (1) this is one of the most standard proofs of one of the most basic theorems of mathematics, (2) the proof comes from a classic textbook on number theory, and so can be considered as fairly representative of the way mathematical proofs are written in mathematical practice, (3) the proof is sufficiently simple and compact such that we can analyze it in its entirety within this paper. Surprisingly, we shall see that this small example already illustrates most of the aspects of mathematical inference that we shall touch upon in our analysis of the differentiating claims.

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THEOREM 3.1 (Pythagoras' Theorem). $\sqrt{2}$ *is irrational.*

Proof. The traditional proof ascribed to Pythagoras runs as follows. If $\sqrt{2}$ is rational, then the equation

$$a^2 = 2b^2$$

is soluble in integers a, b with $(a, b) = 1$. Hence a^2 is even, and therefore a is even. If $a = 2c$, then $4c^2 = 2b^2$, $2c^2 = b^2$, and b is also even, contrary to the hypothesis that $(a, b) = 1$. □

What are the mathematical inferences making up this mathematical proof? As a methodological principle, we shall identify mathematical inferences in mathematical proofs by their *conclusions*, and we shall count as the conclusion of a mathematical inference any proposition in a mathematical proof that has been introduced as the result of a step of deduction. In some cases, such conclusions are made readily explicit using signaling terms—e.g., 'therefore', 'hence', 'whence', 'it follows that', 'we conclude that', 'we deduce that', 'we obtain that', etc.—which are usually followed by the conclusions being drawn. For instance, when we see in the above proof statements such as "[h]ence a^2 is even" or "therefore a is even", we can identify two mathematical inferences with conclusions, respectively, "a^2 is even" and "a is even". In other cases, conclusions may not be signaled as such. For instance, it is not made readily explicit that a statement such as "[i]f $a = 2c$" corresponds to the conclusion of a mathematical inference with premiss "a is even". It is nonetheless fairly easy in practice to identify the propositions in a mathematical proof that are conclusions of some mathematical inferences, and to distinguish them from propositions that are introduced either as assumptions, as definitions, or as lemmas or theorems that have been previously established. In the above proof, we can identify 8 propositions as being conclusions of some mathematical inferences—we are listing them in Table 2. Thus, the mathematical proof of our case study comprises 8 mathematical inferences, that we will refer to as the mathematical inferences (I1)–(I8).

Table 2. *Conclusions of the mathematical inferences (I1)–(I8)*

	Assume $\sqrt{2}$ is rational
(I1)	$a^2 = 2b^2$ is soluble in integer a, b with $(a, b) = 1$
(I2)	a^2 is even
(I3)	a is even
(I4)	$a = 2c$
(I5)	$4c^2 = 2b^2$
(I6)	$2c^2 = b^2$
(I7)	b is even
(I8)	false
	$\sqrt{2}$ is irrational

We have identified the conclusions of the mathematical inferences (I1)–(I8), but what are their *premisses*? As it appears, identifying the premisses of a mathematical inference in a mathematical proof turns out to be much more difficult than identifying its conclusion. Again, in some cases, it is true that premisses can be identified easily—this is the case when they occur right before signaling terms for conclusions, or when they are themselves

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signaled by a dedicated term, usually by the term 'from', used either alone or in expressions such as 'it follows from', 'we conclude from', 'we deduce from', 'we obtain from', etc. But, in other cases, identifying premisses can be quite tricky. As an illustration, consider the mathematical inference (I8). To reach the contradiction as a conclusion, one needs as premisses the propositions "b is even" and "$(a, b) = 1$", which can be easily identified from the text, but one also needs the proposition "a is even". Nothing in the text, however, indicates that this latter proposition constitutes a premiss to (I8). Yet, despite these difficulties, the skillful agent should be able in practice to identify which propositions in a mathematical proof are used as premisses to which mathematical inferences.[14] In Table 3, we report the propositions in the text of the above mathematical proof that we have identified as premisses for the mathematical inferences (I1)–(I8) along with their conclusions.

Table 3. *Premisses and conclusions of the mathematical inferences* (*I1*)–(*I8*)

(I1)	$\dfrac{\sqrt{2} \text{ is rational}}{a^2 = 2b^2 \text{ is sol. in int. } a, b \text{ with } (a, b) = 1}$	(I5)	$\dfrac{\begin{array}{l} a^2 = 2b^2 \\ a = 2c \end{array}}{4c^2 = 2b^2}$
(I2)	$\dfrac{a^2 = 2b^2}{a^2 \text{ is even}}$	(I6)	$\dfrac{4c^2 = 2b^2}{2c^2 = b^2}$
(I3)	$\dfrac{a^2 \text{ is even}}{a \text{ is even}}$	(I7)	$\dfrac{2c^2 = b^2}{b \text{ is even}}$
(I4)	$\dfrac{a \text{ is even}}{a = 2c}$	(I8)	$\dfrac{\begin{array}{l} (a, b) = 1 \\ a \text{ is even} \\ b \text{ is even} \end{array}}{\text{false}}$

The premisses and conclusions of the mathematical inferences (I1)–(I8) as reported in Table 3 constitute all the information explicitly available about them in the written text of the above mathematical proof. Are premisses and conclusion all there is to a mathematical inference? The answer is obviously no. To approach the notion of inference in all its complexity, we shall now appeal to a set of distinctions introduced by Sundholm (2012).

3.2. Inference act, inference process, inference trace. Sundholm (2012, p. 948) distinguishes three dimensions of the notion of inference—the *inference act*, the *inference process*, and the *inference trace*. An inference is primarily an *act* that is performed by a given agent on a given occasion, and this is what Sundholm refers to by an *inference act*. When we abstract from the specific performances of an inference act, we obtain an *inference process*.[15] Finally, an *inference trace* is a recipe or blueprint for performing

[14] Notice that this particular skill is likely to play a significant role in the processing of mathematical proofs. We will, however, leave a detailed investigation of it for another occasion.

[15] Prawitz (2012, p. 895) uses the terms 'individual inference act' and 'generic inference act' to designate, respectively, the notions of inference act and inference process.

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an inference act, and so inference traces are precisely what is to be found in written mathematical proofs.[16]

The philosophical analysis of the notion of inference is usually situated at the level of inference processes, since what is philosophically relevant is the nature of inference processes, and not specific performances of inference acts. Sundholm notices, however, that inference processes are *subjective*—i.e., internal to the agent—while their inference traces are *objective*—i.e., external to the agent. Thus, any philosophical analysis of inference processes should be grounded, somehow, in what is objectively accessible, that is, their inference traces. For this reason, we shall demand that our analysis of the inference processes underlying mathematical inferences be accountable to their inference traces.

Now, the notion of inference process can itself be approached at different levels of analysis. In this respect, it will be particularly useful to appeal to David Marr's three levels of analysis of complex information-processing systems—the *functional*, *algorithmic*, and *implementational* levels.[17] According to Marr's trichotomy, describing an information process at the functional, algorithmic, and computational levels consists, respectively, in identifying it as an input-output function, identifying an algorithm to compute that function as well as the representations and processes involved in it, and identifying a physical implementation of the algorithm specified.[18] Insofar as inference processes are information processes *par excellence*, Marr's trichotomy can be readily applied to the inference processes underlying mathematical inferences. Thus, we can study the inference processes underlying mathematical inferences at the functional, algorithmic, and implementational levels.[19] For the purpose of this study, we shall approach the inference processes underlying mathematical inferences at the *functional* level.[20] This means that we should only provide a description of the inference processes underlying mathematical inferences in terms of input-output functions. For reasons that will become clear shortly, we shall refer to such input-output functions as *mathematical rules of inference*.

3.3. Mathematical rule of inference. In order to describe the inference processes underlying mathematical inferences as input-output functions, we first need to identify what they take as input and output. The answer is straightforward: these processes take *premisses* as input and *conclusions* as output. For instance, in the case of mathematical inference (I6), its underlying inference process takes the premiss "$4c^2 = 2b^2$" as input and the conclusion "$2c^2 = b^2$" as output.

Now, it is important to realize that a given inference process is more general, so to speak, than any particular mathematical inference it underlies. The reason for this is that the same

[16] Sundholm (2012, p. 948) mentions that this notion of inference trace is due to Martin-Löf and was introduced in an unpublished lecture given by Martin-Löf in Paris in April, 1992.

[17] This trichotomy has become a standard conceptual tool in fields of inquiry involved with the study or design of information processing systems such as cognitive science, artificial intelligence, and computer science. For the original presentation of the trichotomy, see (Marr, 1982).

[18] Our formulation of Marr's three levels of analysis is directly inspired by the one provided in (Stenning & Van Lambalgen, 2008). Interestingly, Stenning & Van Lambalgen (2008) show the importance of distinguishing between these three levels when it comes to the cognitive analysis of human reasoning.

[19] Ultimately, it might be expected that any philosophical account of mathematical inference should provide a description of those inference processes at these three levels.

[20] We will not argue here that this is the right level of analysis to approach mathematical inferences in order to analyze the differentiating claims. Whether this is so should be judged *a posteriori* from the subsequent analysis of the differentiating claims provided in the following sections.

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inference process might underly different mathematical inferences with different, albeit related, premisses and conclusions. To see this, consider again mathematical inference (I6). It seems natural to think that the same inference process also underlies, for instance, the two following mathematical inferences:

$$\frac{4c = 2b}{2c = b}, \qquad \frac{4(a+b)^2 = 2(c+d)^2}{2(a+b)^2 = (c+d)^2}.$$

More generally, it seems natural to think that the inference process underlying mathematical inference (I6) allows to deduce any proposition of the form '$2X = Y$' from one of the form '$4X = 2Y$', as soon as the placeholders X and Y are replaced by expressions denoting concrete or arbitrary natural numbers.

This aspect of inference processes can be captured by introducing the notion of an *inference schema*: the inference schema of an inference process is a pair composed of a *premiss schema* and a *conclusion schema* which consist, respectively, in a set of schemas for the premisses and a schema for the conclusion. As an illustration, the inference schema for the inference process underlying mathematical inference (I6) is given by:

$$\frac{4X = 2Y}{2X = Y},$$

where X and Y are placeholders for expressions denoting concrete or arbitrary natural numbers. In Table 4, we provide the inference schemas associated to the inference processes underlying the mathematical inferences (I1)–(I8).[21,22]

The inference schema of an inference process provides us with all what is needed to describe it as an input-output function. More specifically, given an inference process and its inference schema, the inference process can be described, at the functional level, as a function which takes as input any instance of the premiss schema, and which yields as output the corresponding instance of the conclusion schema. By analogy with logical rules of inference, we shall refer to these input-output functions as *mathematical rules of inference*, and we shall speak of the inference schema *of* a mathematical rule of inference.[23] A mathematical rule of inference is thus a rule which, given any instance of its premiss schema, allows to derive the corresponding instance of its conclusion schema.

[21] How did we obtain the inference schemas listed in Table 4 from the mathematical inferences (I1)–(I8)? We have followed the following methodological principle: for each mathematical inference (I1)–(I8), we have searched for the most general inference schema satisfying the two following conditions: (1) each instance of the inference schema consists of a valid mathematical inference, and (2) the inference schema should be able to generate as instance the initial mathematical inference. This second condition is precisely what allows us to meet a demand expressed previously, namely that our analysis of the inference processes underlying mathematical inferences be accountable to their inference traces.

[22] The side conditions for these inference schemas are as follows: For (I1), X is a placeholder for an expression denoting a concrete or arbitrary natural number, and Y and Z are placeholders for expressions denoting concrete or arbitrary integers, with the specific constraint that, in the arbitrary case, the variables introduced should not have been used previously; for (I2)–(I8), X, Y, and Z are placeholders for expressions denoting concrete or arbitrary natural numbers, with the specific constraint for (I4) that, in the arbitrary case, the variable(s) introduced should not have been used previously.

[23] Thus, the inference schemas of both logical and mathematical rules of inference are instances of what Corcoran (2014) calls *argument-text schemas*, thereby establishing a direct analogy between the two types of rules.

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Table 4. *Inference schemas corresponding to the mathematical inferences (I1)–(I8)*

(I1)	$\dfrac{\sqrt{X} \text{ is rational}}{Y^2 = XZ^2 \text{ is sol. in int. } Y, Z \text{ with } (Y, Z) = 1}$	(I5)	$\dfrac{X^2 = 2Y^2 \quad X = 2Z}{4Z^2 = 2Y^2}$
(I2)	$\dfrac{X = 2Y}{X \text{ is even}}$	(I6)	$\dfrac{4X = 2Y}{2X = Y}$
(I3)	$\dfrac{X^2 \text{ is even}}{X \text{ is even}}$	(I7)	$\dfrac{2X^2 = Y^2}{Y \text{ is even}}$
(I4)	$\dfrac{X \text{ is even}}{X = 2Y}$	(I8)	$\dfrac{(X, Y) = 1 \quad X \text{ is even} \quad Y \text{ is even}}{\text{false}}$

From this perspective, we can see the inference schemas in Table 4 as mathematical rules of inference, and we shall refer to them, respectively, as the mathematical rules of inference (I1)–(I8). As an illustration, the mathematical rule of inference (I6) is then a rule which would allow to derive, for instance, the conclusions "$2c^2 = b^2$", "$2c = b$", and "$2(a + b)^2 = (c + d)^2$", respectively, from the premisses "$4c^2 = 2b^2$", "$4c = 2b$", and "$4(a + b)^2 = 2(c + d)^2$". For later on, it will be useful to introduce yet another example of a mathematical rule of inference, and this is the rule corresponding to what Poincaré (1894, p. 379) considered to be mathematical reasoning *par excellence*: mathematical induction. One possible way to formulate the inference schema of the mathematical rule of inference corresponding to mathematical induction is the following:[24]

$$\frac{H(0) \quad H(X) \rightarrow H(X + 1)}{H(Y)},$$

where H is a placeholder for an expression involving an arbitrary variable ranging over $\mathbb{N}$, and X and Y are placeholders for arbitrary variables ranging over $\mathbb{N}$.

An important remark to make at this is point is that, contrary to logical rules of inference, drawing a mathematical inference does *not* consist in applying a mathematical rule of inference. The only thing we are saying is that drawing a mathematical inference can be described, at the functional level, as applying a mathematical rule of inference. To say that drawing a mathematical inference consists in applying a mathematical rule of inference would be a claim about the nature of the inference processes underlying mathematical inferences at the algorithmic level. In this study, we remain agnostic of what is going on at the algorithmic level, and we only consider mathematical inference at the functional level. In other words, one could say that we are approaching the inference processes underlying

[24] There are other possible ways to formulate the inference schema of mathematical induction, for instance by introducing universal quantifiers instead of arbitrary variables in the second premiss as well as in the conclusion. From the point of view of mathematical practice, these alternative formulations would be, however, equivalent to the one we are providing here.

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mathematical inferences as *black boxes*—to borrow terminology from system analysis in science and engineering—focusing on their 'external' behavior and abstracting from their 'internal' functioning.

To sum up, mathematical rules of inference are descriptions of the inference processes underlying mathematical inferences at the functional level—i.e., as input-output functions—and are fully characterized by their *inference schemas*. As we shall now see, they offer us a point of comparison with logical rules of inference, enabling us to compare mathematical inference and logical inference, and thereby to analyze the differentiating claims.

§4. Analyzing the differentiating claims. The objective of this study is to provide a philosophical analysis of the differentiating claims. As mentioned in the introduction, the differentiating claims are generally formulated according to the same scheme, stating usually that mathematical inference and logical inference differ with respect to certain *properties*. Table 1 reported a representative list of properties that have occurred in differentiating claims put forward in the philosophy of mathematics literature.

Interestingly, Table 1 reveals an important aspect of our understanding of these properties: although we do possess precise and often mathematical characterizations of the properties attributed to logical inference listed in the right column, we do not possess such characterizations for the properties attributed to mathematical inference listed in the left column. This observation leads us to hypothesize that the properties attributed to mathematical inference have been obtained *by opposition* to the properties attributed to logical inference. Or, in other words, the properties from the left column have been obtained by *negating* the properties from the right column. We may suppose that the properties in the left column find their roots in the observation that mathematical inference does not appear to possess the properties in the right column, thus attributing to mathematical inference the opposite or negation of those properties usually attributed to logical inference. These considerations suggest the following scheme for analyzing the differentiating claims:

> "Mathematical inference departs from logical inference since, contrary to logical inference, mathematical inference does not possess property P_R"

where P_R denotes a property from the right column of Table 1.

Now, the properties of Table 1 can be classified under three headings in the way reported in Table 5. It is our contention that the properties occurring under each heading in the right column of Table 5 have been used synonymously in the literature on formal deductive systems.[25] Since, according to our hypothesis, the properties in the left column have been obtained by negating the properties in the right column, those properties can be classified, respectively, under the *negation* of these three headings.[26]

[25] We will not argue for this claim, for the simple reason that this would only amount to list quotes from the literature to attest that certain terms have been used synonymously to others, something which does not appear to be particularly interesting for the present study. It should be noted, however, that these notions do have different historical and philosophical roots, and that they have been used in different ways in various philosophical contexts. It is only in the modern literature on formal deductive systems that, we claim, the notions under each heading have been used synonymously.

[26] Unsurprisingly, the properties occurring under each heading in the left column of Table 5 have often been used synonymously as well in the philosophical literature on mathematical proofs.

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Table 5. *Classification of the differentiating properties under three headings*

Mathematical inference	**Logical inference**
nonFormal	*Formal*
Meaning dependent	Meaning independent
Matter dependent	Matter independent
Content dependent	Content independent
nonGeneral	*General*
Topic-specific	Topic-neutral
Subject matter dependent	Subject matter independent
Domain dependent	Domain independent
nonMechanical	*Mechanical*

Consequently, in our analysis of the differentiating claims, we can restrict our attention to the properties appearing in the three headings of Table 5. Analyzing the differentiating claims using the above scheme amounts then to consider the three cases in which P_R denotes the properties of being *formal*, *general*, and *mechanical*. In other words, our task amounts to analyze the following three differentiating claims:

> "Mathematical inference departs from logical inference since, contrary to logical inference, mathematical inference is not (i) *formal*, (ii) *general*, (iii) *mechanical*."

To this end, we shall proceed by addressing the two following questions: (Q_1) What does it mean to say that logical inference is (i) *formal*, (ii) *general*, (iii) *mechanical*? and (Q_2) Why does mathematical inference not appear to be (i) *formal*, (ii) *general*, (iii) *mechanical*? The three following sections develop an analysis of the three differentiating claims, respectively, by addressing these two questions in turn. Before turning to this analysis *per se*, we shall say a few words on the methodology we will adopt to address questions Q_1 and Q_2.

Question Q_1 turns out to be a typical issue in the philosophy of logic. Interestingly, substantial works have recently been done in this field to analyze the notions of formality, generality, and mechanicality from a philosophical and historical perspectives. Of particular interest are the two analyses of the formality of logic developed by MacFarlane (2000) and Dutilh Novaes (2011).[27] The main lesson to be drawn from these two studies is methodological: insofar as the notion of formality has been subject to many conceptual variations in the history and philosophy of logic, any rigorous discussion of it must begin by identifying the different ways in which the notion has been used—a lesson which applies as well to the notions of generality and mechanicality. Similarly, we shall indeed see that the notions of formality, generality, and mechanicality, when applied to logical inference, have been used in different ways in the context of formal deductive systems. Thus, our approach to Q_1 will consist in identifying the different ways in which logical inference has been said to be formal, general, and mechanical in the context of formal deductive systems, and in

[27] These two analyses also address, albeit to a lesser extent, the notions of generality and mechanicality.

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providing precise conceptual characterizations of the different variations of these three notions.[28]

To figure out how to approach question Q_2, we shall first recall that the notions of formality, generality, and mechanicality apply primarily to logical rules of inference—they apply to logical inferences only insofar as logical inferences are governed by logical rules of inference. As we shall see when we will come to the different characterizations of these notions, when we say that logical rules of inference are formal, general, or mechanical this can refer either to a property of their *inference schemas*, or to a property of their *application*. Thanks to the notion of mathematical rules of inference introduced in the previous section, we can compare like with like. Thus, our approach to question Q_2 will consist in investigating why mathematical inference does not appear to be formal, general, and mechanical, by examining why mathematical rules of inference are not formal, general, and mechanical for each one of the variations identified and characterized in Q_1, i.e., by examining why these different properties do not hold either of the inference schema or of the application of mathematical rules of inference.[29]

In the three following sections, we will address questions Q_1 and Q_2, respectively, for the notions of *formality*, *generality*, and *mechanicality*.

§5. Formality. It has been claimed that mathematical inference departs from logical inference with respect to its *formality*—contrary to logical inference, mathematical inference would not be formal, meaning independent, matter independent, or content independent. For instance, Kleene writes:

> [T]he ordinary practice in mathematics illustrates only a partial symbolization and formalization, since part of the statements remain expressed in words, and part of the deductions are performed in terms of the meanings of the words rather than by formal rules. (Kleene, 1952, p. 61)

In this section, we first identify two different senses in which logical inference is said to be formal in the context of formal deductive systems. We then examine why mathematical inference does not appear to be formal for each one of these two senses.

5.1. The formality of logical inference. It is often said that logic is *formal*—i.e., that it is only concerned with the form of propositions, and not with their meaning, matter, or content. Although the distinction between form and matter is usually traced back to Aristotle, MacFarlane (2000) has shown that it is only with Kant that formality became the

[28] Although we shall make important uses of MacFarlane's and Dutilh Novaes's analyses, our approach to Q_1 will differ in aim and scope from these two studies: while MacFarlane is interested in the notion of formality in its capacity to provide a criterion for the demarcation of logic, and Dutilh Novaes is interested in the different ways in which logic has been said to be formal in the history and philosophy of logic, and that both consider a wide temporal window going back to Kant and Aristotle, our aim is rather to analyze the notions of formality, generality, and mechanicality in order to analyze the differentiating claims, and so we should focus on the uses of these notions in the restricted context of formal deductive systems. In the following, we will point out the correspondences, when there are some, between the conceptual characterizations we introduce and the notions identified and investigated in MacFarlane's and Dutilh Novaes's taxonomies.

[29] In doing so, we shall often appeal to the mathematical inferences of our case study, either to examine the issue at hand on concrete cases, or to illustrate the discussion with concrete examples.

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hallmark of logic.[30] It has remained so in our contemporary conception of logic, as Church put it:

> (formal) logic is concerned with the analysis of sentences or of propositions and of proof with attention to the *form* in abstraction from the *matter*. (Church, 1956, p. 1)

The term 'formal' is applied here to logic as a field, but what does it mean to apply it more specifically to logical inference? The most natural suggestion seems to be the following:

Formality: To say that logical inference is *formal* is to say that it is governed by rules of inference which only depend on the logical form of premisses and conclusion, and not on their meaning, matter, or content.[31,32]

This characterization provides a highly general conception of the formality of logical inference. In order to specify it further, we need to identify the precise senses in which logical rules of inference are said to only depend on the logical form of premisses and conclusion, and not on their meaning. To this end, we proceed in two steps: we first make explicit the notions of *logical form* and *meaning* at work in the context of formal deductive systems; we then spell out what it means for a logical rule of inference to *only depend on* the logical form of premisses and conclusion, and not on their meaning.

Recall, first of all, that a formal deductive system is composed of a formal language and a set of logical rules of inference, and notice that the notions of logical form and meaning pertain to the *formulas* of a formal language.[33] In the following passage, Goldfarb states in a clear way how the notion of logical form is used when applied to formulas (called 'schemata' in Goldfarb's terminology):

> Logical forms [. . .] are simply schemata: representations of the composition of the sentences, constructed from the logical signs (quantifiers and truth-functional connectives, in the standard case) using schematic letters of various sorts (predicate, sentence, and function letters). Schemata do not state anything and so are neither true nor false, but they can be interpreted: a universe of discourse is assigned to the quantifiers, predicate letters are replaced by predicates or assigned extensions (of the appropriate r-ities) over the universe, sentence letters can be replaced by sentences or assigned truth-values. (Goldfarb, 2001, p. 26)

According to this conception, the logical form of a formula is given by its *logico-syntactic structure* which reflects how the formula has been built from its nonlogical terms using the

[30] For detailed philosophical and historical analyses of the formality of logic, we refer the reader to (MacFarlane, 2000, 2002) and (Dutilh Novaes, 2011).

[31] The terms 'meaning', 'matter', and 'content' have been used interchangeably in the context of formal deductive systems, although they do have different historical and philosophical origins. In the following, instead of repeating every time the three terms, we will only use the term 'meaning'.

[32] Insofar as MacFarlane's and Dutilh Novaes's analyses have for main object of study the notion of formality, this characterization encompasses, at this level of generality, the three notions of formality—1-formality, 2-formality, and 3-formality—investigated by MacFarlane (2000) as well as the five notions of formality investigated by Dutilh Novaes (2011) under the heading of 'formal as pertaining to forms'.

[33] In the following, when we use the term 'formula', we always mean 'well-formed formula', i.e., formulas built according to the production rules of the considered formal language.

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logical terms of the considered formal language; a formula is then attributed a meaning when it is given an *interpretation*, i.e., a domain of discourse together with an assignment for its nonlogical terms. Such a conception is fairly representative, we believe, of the use of these notions in the contemporary practice of mathematical logic and formal deductive systems.[34]

This gives us precise conceptions of the notions of logical form and meaning for the formulas of a formal language, but it does not tell us yet what it means for a logical rule of inference to only depend on the logical form of premisses and conclusion, and not on their meaning. This can be made precise in a semantic way and in a syntactic way, leading to two different notions of formality: *semantic formality* and *syntactic formality*.

Semantic formality. The semantic formality of logical inference has its roots in the common view that understands the validity of logical inference via the notion of logical consequence, as Prawitz put it:

> In logic, as well as in philosophy in general, the dominant view concerning the notions of logical consequence and valid inference has been for long that the essential questions concerning the former notion were settled by Alfred Tarski in his paper "On the Concept of Logical Consequence", and that the latter notion is accounted for by saying that an inference is valid if and only if its conclusion is a logical consequence of its premisses in the sense defined by Tarski. (Prawitz, 2013, p. 182)

According to this view, to say that a logical inference is valid is to say that its conclusion is a logical consequence of its premisses, where logical consequence is understood in the Tarskian sense, that is, as truth-preservation under all possible interpretations of the nonlogical terms occurring in the premisses and conclusion (see (Tarski, 2002)). This definition of logical consequence was precisely designed by Tarski to capture the pretheoretical intuition that the relation of logical consequence only depends on the logical form of the propositions involved:

> [I]t is a question here of the relation of following logically, i.e., formally, and therefore of a relation which has to be completely determined by the form of the sentences among which it obtains [...]. (Tarski, 2002, p. 183)

This is achieved by Tarski's definition insofar as the requirement of truth-preservation under all possible interpretations of the nonlogical terms occurring in the premisses and conclusion is such that:

> [T]he relation of following logically is completely independent of the sense of the extra-logical constants occurring in the sentences among which this relation obtains [...]. (Tarski, 2002, p. 187)

Thus, by reducing the validity of inference to the holding of a logical consequence between premisses and conclusion, the formality of logical consequence inherited from Tarski's definition transfers, so to speak, to logical inference. We thereby obtain a precise sense in which logical rules of inference only depend on the logical form of premisses and

[34] Indeed, Goldfarb's quote is extracted from a passage in which he aims to describe what he calls "the contemporary conception of logic" (Goldfarb, 2001, sec. 1).

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conclusion and not on their meaning—which we will call *semantic formality*—and which can be rendered by the following characterization:

Semantic formality: To say that logical inference is *semantically formal* is to say that it is governed by rules of inference whose inference schemas generate instances that are truth-preserving under all possible interpretations of the nonlogical terms occurring in the premisses and conclusion.[35]

In the case of semantic formality, independence from meaning is thus obtained by requiring truth-preservation under *all* possible interpretations of the nonlogical terms. This contrasts drastically with the notion of syntactic formality we now turn to in which independence from meaning is achieved by *not* appealing *at all* to the notion of interpretation.

Syntactic formality. The syntactic formality of logical inference lies in the purely syntactic character of the logical rules of inference of formal deductive systems. This conception of the formality of logical inference is expressed very clearly by Gödel when he says:

> [T]he outstanding feature of the rules of inference being that they are purely formal, i.e., refer only to the outward structure of the formulas, not to their meaning, so that they could be applied by someone who knew nothing about mathematics, or by a machine. (Gödel, 1995b, p. 45)

When Gödel speaks of the "outward structure of the formulas", he speaks of what we have called the logico-syntactic structure of formulas, i.e., their logical forms according to the conception of logical form inherent to formal deductive systems. Thus, by saying that logical rules of inference only refer to the logico-syntactic structures of premisses and conclusion, and not to their meaning, we obtain another precise sense in which logical rules of inference only depend on the logical form of premisses and conclusion, and not on their meaning:

Syntactic formality: To say that logical inference is *syntactically formal* is to say that it is governed by rules of inference whose application only refer to the logico-syntactic structure of the premisses and conclusion representing their logical form, and not to their meaning.[36]

In the case of syntactic formality, independence from meaning is thus achieved by considering only the logico-syntactic structure of the premisses and conclusion, and by not appealing *at all* to any interpretation of them. As mentioned above, this contrasts with the notion of semantic formality where independence from meaning is achieved by appealing to all possible interpretations of the premisses and conclusion.

5.2. Why does mathematical inference not appear to be formal? It has often been claimed that, contrary to logical inference, mathematical inference is not formal. Thanks to the two conceptual characterizations of the formality of logical inference just identified, we are now in a position to examine precisely why mathematical inference does not appear

[35] This notion is discussed by MacFarlane (2000, sec. 2.2) under the heading of 'schematic formality' and by Dutilh Novaes (2011, sec. 2.1.2) under the heading of 'formal as indifference to particulars'.

[36] This notion is discussed by MacFarlane (2000, sec. 2.1) under the heading of 'syntactic formality' and by Dutilh Novaes (2011, sec. 2.2.3) under the heading of 'formal as de-semantification'.

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to be formal. In the following, we examine in turn why mathematical inference does not appear to be semantically formal and syntactically formal.[37]

Mathematical inference and semantic formality. Why does mathematical inference not appear to be semantically formal? To claim that mathematical inference is not semantically formal amounts to claim that it is governed by rules of inference whose inference schemas can generate instances that are not truth-preserving under all possible interpretations of the nonlogical terms occurring in the premisses and conclusion. In other words, this would amount to say that, for some mathematical inferences, there could be some interpretations of the nonlogical terms occurring in the premisses and the conclusion which would make the premisses true and the conclusion false. In order to examine this claim, we will investigate it in the particular case of the mathematical inferences of our case study. We shall see that it is possible for these mathematical inferences to exhibit an interpretation of the premisses and the conclusion which would make the premisses true and the conclusion false. We now do so, more specifically, for the mathematical inferences (I3) and (I6) of our case study.[38]

Recall that mathematical inference (I3) has for single premiss the proposition "a^2 is even" and for conclusion the proposition "a is even", where "a" denotes an arbitrary natural number. These two propositions can be formalized in a first-order language. The most direct way to do so is to represent "a" as a free variable, "^{2}" as a function, and "even" as a predicate. If we denote the free variable by 'x', the function square by 'sq' and the predicate even by 'E', we obtain the following formalization:

$$\frac{E(sq(x))}{E(x)}.$$

Now, an interpretation of the premiss and the conclusion consists of a domain and an assignment for the nonlogical terms occurring in the formulas '$E(sq(x))$' and '$E(x)$'. Consider the following interpretation: take as domain the integers $\mathbb{Z}$, assign to 'sq' the multiplication function $x \mapsto x \times x$ on $\mathbb{Z}$, to 'E' the predicate $\{x \in \mathbb{Z} \mid x > 0\}$, and to the free variable 'x' the value -1. Under this interpretation, the premiss of (I3) is true while its conclusion is false. Hence, the mathematical inference (I3) is not semantically formal.

We can proceed in the exact same way to show that the mathematical inference (I6) is not semantically formal. To this end, we begin by formalizing (I6) as follows:

$$\frac{4 \times x = 2 \times y}{2 \times x = y}.$$

It suffices then to consider the following interpretation: take as domain the integers $\mathbb{Z}$, assign to '$\times$' the minus function $x, y \mapsto x - y$ on $\mathbb{Z}$, to '2' and '4' the numbers 2 and 4,

[37] Before being able to do so, we still need to say a word on one underdetermination present in our two conceptual characterizations, namely the fact that both notions of formality, insofar as they appeal to the notion of logical form, require a distinction between logical and nonlogical terms. For the sake of the discussion, we will simply assume the usual distinction between logical and nonlogical terms in the case of first-order logic, and so we will take as logical terms the usual logical connectives (negation, conjunction, disjunction, implication) and the first-order quantifiers (existential quantifier, universal quantifier).

[38] The same methodology can be applied in a similar way to the other mathematical inferences of our case study.

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and to 'x' and 'y' the values 0 and -2. We then get that the premiss is true since $4 - x = 4 - 0 = 2 - (-2) = 2 - y$, but the conclusion false, since $2 - x = 2 - 0 = 2 \neq -2 = y$. Hence, the mathematical inference (I6) is not semantically formal.

Thus, mathematical inference does not appear to be semantically formal because it can be nontruth-preserving for some interpretations of the nonlogical terms occurring in its premisses and conclusion. There is an obvious explanation for this phenomenon, and it lies in the fact that substantial information is often left implicit in the context of mathematical proofs. In particular, it is often the case that (logically necessary) additional premisses to mathematical inferences are left implicit.[39] The absence of these additional premisses is what makes it possible to find interpretations that make the premisses of these inferences true and their conclusions false. Given that almost any mathematical proof leaves a certain amount of information implicit, it is then quite likely that a given mathematical inference in a mathematical proof will lack some (logically necessary) additional premisses, and so will not appear to be semantically formal.

Mathematical inference and syntactic formality. Why does mathematical inference not appear to be syntactically formal? To claim that mathematical inference is not syntactically formal amounts to claim that it is governed by rules of inference whose application does not only refer to the logico-syntactic structure of the premisses and conclusion representing their logical form, but might refer to their meaning as well. To understand why mathematical inference does not appear to be syntactically formal, we shall see that the nonsyntactic formality of mathematical inference follows from its nonsemantic formality plus the requirement that the application of mathematical rules of inference be truth-preserving.

To establish this, it suffices to realize that, if the application of mathematical rules of inference was 'blind', so to speak, to the meaning of the propositions they are applied to, then, due to the nonsemantic formality of mathematical inference, they would run the risk of leading from true premisses to false conclusions. This would happen, more specifically, if a mathematical rule of inference would be applied to some premisses and conclusion whose intended interpretation would make the premisses true and the conclusion false. As an illustration of this point, consider the following toy example of a mathematical rule of inference:

$$\frac{X + Y = 0}{X = Y = 0},$$

where 'X' and 'Y' are placeholders for expressions denoting natural numbers. This mathematical rule of inference is truth-preserving as soon as the instances of its inference schema have for intended interpretation the natural numbers as domain and the sign "+" as the associated addition operation. Suppose now that one encounters the (admitted true) proposition "$x + y = 0$" in a context where its intended interpretation has for domain the integers. Then, if one would apply the above mathematical rule of inference to "$x + y = 0$", one would happen to deduce a false conclusion, namely "$x = y = 0$", from a(n admittedly) true premiss.

Thus, mathematical inference does not appear to be syntactically formal because mathematical rules of inference are such that their application *must* refer, on pain of being

[39] For instance, both mathematical inferences (I3) and (I6) would require at least as additional premisses propositions expressing the properties of the multiplication operation.

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nontruth-preserving, to the meaning of the premisses and conclusion they are applied to. This is a direct consequence, as we have just seen, of the nonsemantic formality of mathematical inference. Consequently, since it is likely that a given mathematical inference in a mathematical proof will not appear to be semantically formal, it is then likely that a given mathematical inference in a mathematical proof will not appear to be syntactically formal.

§6. Generality. It has been claimed that mathematical inference departs from logical inference with respect to its *generality*—contrary to logical inference, mathematical inference would not be general, topic-neutral, subject matter independent, or domain independent. For instance, Detlefsen writes:[40]

> The mathematician's inferences stem from and reflect a knowledge of the local "architecture" (Poincaré's term) of the particular subject with which they are concerned, while those of the logician represent only a globally valid, topic-neutral (and, therefore, locally insensitive!) form of knowledge. (Detlefsen, 1992a, p. 350)

In this section, we first identify three different senses in which logical inference is said to be general in the context of formal deductive systems. We then examine why mathematical inference does not appear to be general for each of these senses.

6.1. The generality of logical inference. It is often said that logic is *general*—i.e., that it is concerned with principles of reasoning that are valid whatever the topic, subject matter, or domain of what is reasoned about. This appears to be one of the essential characteristics of logic spanning the whole historical development of the discipline from Aristotle[41] to Frege,[42] a characteristic still central to our contemporary conception of logic, witness for instance the following passage in the opening paragraph of one of our contemporary logic textbooks:

> [T]he principles of logic are general: they do not govern reasoning in one specific subject matter or another, but [. . .] reasoning as it applies to any and all areas of study. (Goldfarb, 2003, p. xiii)

The term 'general' is here applied to logic or the principles thereof, but what does it mean to apply it more specifically to logical inference? The central notion in play seems to be the one of *applicability*: logical inference is said to be general because it is governed by logical rules of inference that are applicable whatever the topic, subject matter, or domain of what is reasoned about. This idea can be rendered by the following characterization:

Generality: To say that logical inference is *general* is to say that it is governed by rules of inference that are *generally applicable*, i.e., that are applicable to propositions—

[40] This quote is extracted from the article (Detlefsen, 1992a) in which Detlefsen proposes a reconstruction of Poincaré's epistemology of mathematical inference. The view the quote expresses reflects thus Detlefsen's own interpretation of Poincaré's view.

[41] See, e.g., Barnes: "It is true—and Aristotle says—that dialectic or syllogistic is 'topic neutral', that it can be applied in any and every field of inquiry" (Barnes, 2007, p. 362).

[42] Frege writes: "the task we assign logic is only that of saying what holds with the utmost generality for all thinking, whatever its subject matter" (Frege, 1979, p. 128). On the role of generality in Frege's conception of logic see (MacFarlane, 2000, 2002).

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premisses and conclusions—belonging to any and every topic, subject matter, or domain.[43,44]

To specify this characterization further, we need to spell out what it means for a rule of inference to be generally applicable. We now identify three different ways to do so, leading to three different notions of generality: *schematic generality*, *syntactic generality*, and *semantic generality*.

Schematic generality. The notion of *schematic generality* locates the generality of logical inference in the generality or topic-neutrality of the expressions occurring in the inference schemas of logical rules of inference. The main idea is the following: for a rule of inference to be generally applicable, its inference schema should be able to generate instances—i.e., pairs of premisses and conclusion—whose propositions can potentially belong to any and every possible subject matter. This can be achieved by requiring that in the inference schemas of logical rules of inference only appear expressions that are *general* or *topic-neutral*, i.e., expressions that can occur in discourse about any and every possible subject matter.[45] Interestingly, MacFarlane (2000) perfectly captures the notion of schematic generality we are interested in, a conception of the generality of logic that he calls 'Gen-4' and which he formulates as follows:[46]

> A natural suggestion is to spell out the generality or topic-neutrality of logic in terms of the generality or topic-neutrality of the expressions that occur essentially in logical truths or inference rules—the "logical constants."
>
> **(Gen-4)** To say that logic is general or topic-neutral is to say that its truths or inference rules contain only general or topic-neutral expressions essentially. (MacFarlane, 2000, p. 72)

This leads to the following characterization of schematic generality:

Schematic generality: To say that logical inference is *schematically general* is to say that it is governed by rules of inference whose inference schemas only contain general or topic-neutral expressions.[47]

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[43] The terms 'topic', 'subject matter', and 'domain' have been used interchangeably in the contemporary practice of formal deductive systems. In the following, instead of repeating every time the three terms, we will only use the term 'subject matter'.

[44] This notion is discussed by MacFarlane (2000, sec. 3.1) under the heading of '1-formality' and by Dutilh Novaes (2011, sec. 2.2.1) under the heading of 'formal as topic-neutral'.

[45] MacFarlane (2014) relates the general applicability of logic to the notion of topic-neutrality as follows: "Logic, it seems, is not about anything in particular; relatedly, it is applicable everywhere, no matter what we are reasoning about. So it is natural to suppose that the logical constants can be marked out as the "topic-neutral" expressions [. . .]. We have reason to care about the topic-neutral expressions, and to treat them differently from others, because we are interested in logic as a *universal* canon for reasoning, one that is applicable not just to reasoning about this or that domain, but to all reasoning" (MacFarlane, 2014, sec. 4).

[46] MacFarlane points out that this is the view of the generality of logic held by Russell in the period 1903–1911 (e.g., in Russell, 1913).

[47] This notion is discussed by MacFarlane (2000, sec. 3.2, 3.5) under the headings of '2-formality' and 'Gen-4' and by Dutilh Novaes (2011, sec. 2.1.2) under the heading of 'formal as indifference to particulars'.

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To make this characterization precise, we still need to specify what a general or topic-neutral expression is. Below, in §6.2, we shall present three accounts of the generality or topic-neutrality of expressions and consider the resulting notions of schematic generality.

Semantic generality. The *semantic generality* of logical inference is a direct consequence of its semantic formality. The view that the generality of logic follows from its semantic formality is expressed for instance by Dutilh Novaes in the following passage:

> [I]t would seem that logic is topic-neutral *because* it is formal, i.e., *because* it deals with forms and patterns that are valid irrespective of the specific subject-matter being dealt with. A valid argument-form will be valid for all substitutional instances of its material terms [. . .], regardless of the discipline or topic dealt with. (Dutilh Novaes, 2011, p. 315)

The reasoning behind this view runs as follows: since logical inference is semantically formal, i.e., truth-preserving for all possible interpretations of the nonlogical terms occurring in the premisses and the conclusion, it is truth-preserving whatever the meaning of the premisses and the conclusion is, and *a fortiori* whatever the subject matter of the premisses and the conclusion is. In this conception of the generality of logical inference, the general applicability of logical rules of inference is understood as truth-preservation whatever the subject matter of the propositions they are applied to. This leads to the following characterization:

Semantic generality: To say that logical inference is *semantically general* is to say that it is governed by rules of inference whose applications are truth-preserving whatever the subject matter of the premisses and conclusion they are applied to.

It should be noted that the *subject matter of* a proposition is an intuitive epistemological notion which should be made precise in order to specify completely the characterization of semantic generality. Below, in §6.2, we will identify different senses in which a proposition can be said to belong to a given subject matter, and we will consider the resulting notions of semantic generality.

Syntactic generality. The *syntactic generality* of logical inference is a direct consequence of its syntactic formality. The view that the generality of logic originates from the syntactic treatment of the logical form of propositions is expressed, for instance, by Goldfarb in the following passage:

> The pivotal role [. . .] of uninterpreted formulas that represent logical forms, gives a specific cast to the generality of logic. Logic deals with logical forms, which schematize away the particular subject matter of sentences. Thus logic is tied to no particular subject matter because it deals with these "empty" forms rather than with particular contents. (Goldfarb, 2001, p. 27)

The reasoning behind the view that the syntactic generality of logical inference is directly inherited from its syntactic formality runs as follows: since logical inference is syntactically formal, i.e., governed by rules of inference that operate only on the logico-syntactic structures of propositions, it is governed by rules of inference whose application is insensitive to the meaning of the propositions they are applied to, and *a fortiori* is insensitive to their subject matter. In this conception of the generality of logical inference,

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the notion of general applicability is understood as the insensitivity of the logical rules of inference to the subject matter of the propositions they are applied to. This leads to the following characterization:

Syntactic generality: To say that logical inference is *syntactically general* is to say that it is governed by rules of inference whose application is insensitive to the subject matter of the propositions they are applied to.

In order to specify this characterization, we still need to make explicit what it means for the application of a rule of inference to be *insensitive* to the subject matter of the propositions it is applied to. We will return to this point below when we will examine why mathematical inference does not appear to be syntactically general.

6.2. Why does mathematical inference not appear to be general? It has often been claimed that, contrary to logical inference, mathematical inference is not general. Thanks to the three conceptual characterizations of the generality of logical inference just identified, we are now in a position to examine precisely why mathematical inference does not appear to be general. In the following, we examine in turn why mathematical inference does not appear to be schematically general, semantically general, and syntactically general.

Mathematical inference and schematic generality. Why does mathematical inference not appear to be schematically general? To claim that mathematical inference is not schematically general amounts to claim that it is governed by rules of inference whose inference schemas contain expressions that are not general or topic-neutral. For instance, in the case of the mathematical inferences (I2)–(I8), this would amount to claim that the expressions in Table 6 are not general or topic-neutral:[48]

Table 6. *Expressions in the inference schemas of mathematical inferences (I2)–(I8)*

Constants	Predicates/Relations	Functions
2, 4	even(.)	$x, y \mapsto x \times y,\ x \mapsto x^2$ $x, y \mapsto (x, y)$

Such claims make sense, however, only with respect to a given account of the generality or topic-neutrality of expressions. To examine why mathematical inference does not appear to be schematically general, we now consider three different accounts of what a general or topic-neutral expression is.

A first possible account is obtained by simply following the ordinary practice of mathematical logic, and by identifying the general or topic-neutral expressions with the so-called *logical constants*. In the paradigmatic case of first-order logic, the logical constants are usually taken to consist of the logical connectives (negation, conjunction, disjunction, implication) and the first-order quantifiers (existential quantifier, universal quantifier). It should be noted that this particular choice reflects some of our pre-intuitions of what a

[48] To simplify the discussion, we do not consider the expressions occurring in the inference schema of the mathematical inference (I1), since the expressions occurring in the inference schemas of the mathematical inferences (I2)–(I8) already provide us with enough examples for the analysis to be developed here.

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general or topic-neutral expression is since there are few discussions on whether these expressions should count as logical constants, as MacFarlane put it:

> [I]t is generally agreed that signs for negation, conjunction, disjunction, conditionality, and the first-order quantifiers should count as logical constants, and that words like "red", "boy", "taller", and "Clinton" should not [. . .]. (MacFarlane, 2014, introduction)

Given that the propositions occurring in our case study can be formalized using the resources of a first-order language, we can easily determine whether the expressions occurring in the mathematical rules of inference of our case study constitute logical constants. The answer is straightforward: the expressions listed above as appearing in the inference schemas of those mathematical rules of inference are neither logical connectives nor first-order quantifiers, and so would not count as logical constants from the paradigmatic point of view of first-order logic. Thus, one reason why mathematical inference does not appear to be schematically general is simply because the inference schemas of mathematical rules of inference contain expressions that are not part of the sets of logical constants—i.e., not part of the set of general or topic-neutral expressions—which are usually admitted as such in the context of formal deductive systems.[49]

A second possible account is obtained by using a criterion proposed by Ryle which purports to determine whether an expression is topic-neutral or not. The criterion is the following:

> We may call English expressions "topic-neutral" if a foreigner who understood them, but only them, could get no clue at all from an English paragraph containing them what that paragraph was about. (Ryle, 1954, p. 116)

The problem with this criterion is that it suffers from what MacFarlane (2014, sec. 4) calls the *antinomy of topic-neutrality*. To see this, suppose that a foreigner is faced with an English paragraph containing the expressions "2" and "4", and that these two expressions are the only ones that the foreigner understands in this paragraph. On the one hand, one might say that, since "2" and "4" can be applied to any object that can be counted, the foreigner would still have no clue on what the paragraph is about, and so "2" and "4" should be counted as topic-neutral. On the other hand, one might say that the foreigner already knows that the paragraph is about numbers, or at least about discrete objects that can be counted, and so "2" and "4" should not be counted as topic-neutral.[50] It follows from the antinomy that Ryle's criterion cannot determine decisively whether the expressions occurring in mathematical rules of inference are topic-neutral. For this reason, Ryle's criterion does not appear to be particularly helpful for explaining why mathematical inference does not appear to be schematically general.

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[49] It should be noted that these remarks are not necessarily tied up to the specific case of first-order logic, and would hold as well in the cases of second-order or higher-order logic. More generally, we are not aware of formal deductive systems in which the expressions occurring in the mathematical rules of inference of our case study would count as logical constants in the ordinary practice of mathematical logic.

[50] Such observations might lead to think of the notion of topic-neutrality not as a dichotomous one according to which any expression could be categorized as topic-neutral or not, but rather as one that admits *degrees* (see (Lycan, 1989)).

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A third possible account is obtained by considering as general or topic-neutral the expressions that are invariant under all permutations of the domain of objects. Such a proposal has originally been made by Tarski (1986), and has been explored further by several logicians and philosophers, most notably by Sher (1991).[51] At the basis of this approach is the notion of a *permutation* π_D of a domain of objects D which is defined as a bijection from D onto itself. An expression is then said to be *permutation-invariant* just in case its extension on each domain of objects D is invariant under all permutations π_D of that domain.[52] To fully specify the notion of permutation-invariance, one still needs to spell out for each type of expression what it means for its extension on a domain D to be invariant under all permutations π_D of D. This can be done in a precise way by considering a hierarchy of types for interpreted symbols that can be defined recursively (see (Van Benthem, 1989)). Are the expressions occurring in the mathematical rules of inference of our case study permutation-invariant? Thanks to observations already made by Tarski (1986), we can answer this question in a straightforward way. Tarski notices that for any domain with at least two objects, no constant is permutation-invariant, and so there is no general or topic-neutral expressions on this level. It follows that the constants "2" and "4" are not permutation-invariant, and so do not count as general or topic-neutral. Furthermore, Tarski notices that the only predicates that are permutation-invariant are the ones whose extensions on every domain are either the empty class or the universal class. It follows that the predicate "even(.)" is not permutation-invariant, and so does not count as general or topic-neutral. Finally, the only binary relations that are permutation-invariant are the ones whose extensions on every domain are either the empty relation, the universal relation, the identity relation, or its complement. It follows that the function "$x \mapsto x^2$" is not permutation-invariant since its extension on $\mathbb{N}$ does not coincide with one of these four cases, and so this function does not count as general or topic-neutral. Similarly, it can be shown as well that the functions "$x, y \mapsto x \times y$" and "$x, y \mapsto (x, y)$" are not general or topic-neutral by considering the permutation-invariance criterion for ternary relations. In short, all the expressions occurring in the mathematical rules of inference of our case study do not categorize as general or topic-neutral from the perspective of the permutation-invariance criterion.

Thus, mathematical inference does not appear to be schematically general because the expressions that occur in the inference schemas of mathematical rules of inference do not classify as general or topic-neutral expressions, when the general or topic-neutral expressions are determined either by our pre-intuition as exhibited in the ordinary practice of mathematical logic, or by Tarski's permutation-invariance criterion for generality. Ryle's criterion for topic-neutrality cannot help to explain why mathematical inference does not appear to be schematically general because it does not allow to decide decisively whether the expressions occurring in the inference schemas of mathematical rules of inference should count as topic-neutral or not.

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[51] For an overview of the literature on the permutation-invariance criterion, we refer the reader to (MacFarlane, 2014, sec. 5) and (Bonnay, 2006). For various proposals to modify the original permutation-invariance criterion, see (Sher, 1991), (McGee, 1996), (Feferman, 1999), and (Bonnay, 2006, 2008).

[52] It should be noted that Tarski's version of the permutation-invariance criterion applies to operations and not to symbols or expressions. The development of the permutation-invariance criterion for symbols and expressions is due to Sher (1991). On this point see (Bonnay, 2006, sec. 3.1.1) and (Bonnay, 2008, sec. 1.2).

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Mathematical inference and semantic generality. Why does mathematical inference not appear to be semantically general? To claim that mathematical inference is not semantically general amounts to claim that it is governed by rules of inference that are not truth-preserving whatever the subject matter of the premisses and conclusion they are applied to. In other words, this would amount to claim that mathematical rules of inference might allow to deduce false conclusions from true premisses, when the premisses and conclusions are taken to belong to some particular subject matters. To examine these claims, we shall first notice that our characterization of the semantic generality of logical inference appeals to a *variation* of the subject matter of propositions.[53] Making this idea precise requires, in turn, an account of what it means for a proposition *to belong to*—or equivalently *to be about*—a particular subject matter. To provide such an account turns out to be a difficult philosophical issue which has triggered some literature in the philosophy of language and metaphysics.[54] We shall not attempt to provide here an exhaustive analysis of how the notion of the subject matter(s) of a proposition has been used in the context of mathematical discourse, as this would require a dedicated study on its own. We shall, nonetheless, identify those uses that might allow us to understand why mathematical inference does not appear to be semantically general. To examine why mathematical inference does not appear to be semantically general, we now consider three different senses in which a proposition can be said to belong to a subject matter in the context of ordinary mathematical discourse.

A first option is to think of the subject matter of a proposition as the *field of inquiry*[55] to which it belongs.[56] This sense of the notion turns out to be particularly hard to pin down precisely. It is true that some cases are quite unproblematic: when we speak for instance of the fundamental theorems of algebra and of arithmetic, this makes it clear that such theorems belong to the fields of algebra and arithmetic, respectively. Other cases are, however, much more intricate. Consider for instance the prime number theorem which states that the density of primes in the positive natural numbers is asymptotic to $1/\ln x$. At first sight, this is a theorem about the distribution of prime numbers, and so one would say that it belongs to the field of number theory. Yet, the original proof by Hadamard and de la Vallée Poussin makes heavy use of complex analysis, and so one might say

[53] Even though such a variation is natural in the case of logical rules of inference—e.g., it is natural to imagine modus ponens applied to propositions belonging to mathematics, physics, or any other field—it appears rather as counter-intuitive in the case of mathematical rules of inference. The reason is that a mathematical rule of inference comes with an *intended interpretation* of the expressions that occur in its inference schema. If we consider the mathematical rules of inference of our case study, say the rule associated to inference (I2), we can observe that the rule makes sense only insofar as the placeholders "X" and "Y" are replaced by expressions denoting natural numbers, and the expressions "×", "2" and "even" are interpreted in their intended way in the context of number theory. Yet, being counter-intuitive or not, we should consider such a variation in order to examine why mathematical rules of inference do not appear to be semantically formal.

[54] For an attempt to capture what it means for a proposition to be *about* a subject matter within a possible-worlds framework see (Lewis, 1988). For an extensive philosophical analysis of the notion of *aboutness* see (Yablo, 2014).

[55] Several different terms have been used synonymously to the one of 'field of inquiry' in the philosophical literature, for instance one might encounter the terms 'field of study', 'area of inquiry' or 'area of study', 'topic', 'discipline', 'domain', etc.

[56] This might be the sense of the notion that has been mostly used relatively to the discussions on the generality of logic, witness Goldfarb's quote at the beginning of this section and Barnes' quote in footnote 41.

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that the subject matter of the prime number theorem is complex analysis, since its proof is carried out in this mathematical field of inquiry. But Selberg and Erdös have provided in the middle of the past century an elementary proof of the theorem that does not make use of complex analysis, so does it still make sense to say that the subject matter of the prime number theorem is complex analysis? Similarly, should Fermat's last theorem be considered as a proposition belonging to number theory or to elliptic curve theory, or both? The reason why it is hard to determine precisely to which field of inquiry a given mathematical proposition belongs to is that the different branches of mathematics are, to a large extent, interconnected with each other, so that it is particularly hard to classify a given proposition into a specific or exclusive field of inquiry. From this perspective, a given mathematical proposition might happen to belong to a wide range of fields of inquiry. From another perspective, one might consider that the whole of mathematics can be derived from ZFC set theory, in which case one might simply say that there is only one field of inquiry, namely ZFC set theory, and that all the propositions of mathematics belong to this subject matter. These initial considerations show the difficulty of making precise what it means for a mathematical proposition to belong to a specific or exclusive field of inquiry,[57] and so already undermine the possibility of explaining why mathematical inference does not appear to be semantically general when the subject matter of a proposition is conceived as the field of inquiry to which it belongs. Yet, despite these difficulties, we can still consider the type of variation most commonly associated with this notion of subject matter. This variation consists simply in considering a proposition as belonging to several different fields of inquiry. This is precisely what we were doing when we said that the prime number theorem can be considered as belonging to number theory or complex analysis, and that Fermat's last theorem can be considered as belonging to number theory or to elliptic curve theory. The main observation to be made here is that varying the subject matter of a proposition in this sense does *not* change the intended interpretation of the considered proposition. It follows from this that, if a mathematical inference is truth-preserving when its premisses and conclusion are taken to belong to a certain field of inquiry, then it will still be truth-preserving when its premisses and conclusion are taken to belong to another field of inquiry. Thus, it is simply not the case that mathematical inference is not semantically general when the subject matter of a proposition is conceived as the field of inquiry to which it belongs, and so this notion cannot be used to explain why mathematical inference does not appear to be semantically general.

A second option is to identify the subject matter of a proposition with its *meaning*, that is, with its intended interpretation consisting of a domain and an assignment for its nonlogical terms. As an illustration, consider the mathematical inference (I3) which has for only premiss "a^2 is even" and for conclusion "a is even". If we identify the subject matter of this mathematical inference with the meaning of its premiss and conclusion, we would then say that its subject matter consists of the natural numbers, the even predicate, and the square function, or equivalently that this mathematical inference is *about* the natural numbers, the even predicate, and the square function. Although this might not be the most widespread use of the notion of the subject matter of a proposition, such a use does occur in ordinary mathematical discourse. Interestingly, this sense of the notion makes

[57] Notice that this difficulty is even enhanced by the fact that the notion of field of inquiry itself depends on sociological and historical factors, and so is also susceptible of many contextual variations.

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it very easy to explain why mathematical inference does not appear to be semantically general: by identifying the subject matter of a proposition with its meaning, the notion of semantic generality boils down to the one of semantic formality, and so our explanation for why mathematical inference does not appear to be semantically formal holds *mutatis mutandis* for explaining why mathematical inference does not appear to be semantically general.

A third option is to think of the subject matter of a proposition as the *domain* of objects it talks about, that is, as the domain of its intended interpretation. In this sense, all the propositions in our case study would have for subject matter the natural numbers, while the subject matter of the propositions of real analysis would be the real numbers, the propositions of complex analysis the complex numbers, etc. This corresponds to a common usage in ordinary mathematical discourse, illustrated by such statements as "Fermat's last theorem is a proposition about the natural numbers". Interestingly, this notion admits a type of variation that is fairly natural from the point of view of ordinary mathematical discourse, and which consists in varying the domain of a proposition while keeping fixed, in a certain sense, the meaning of its nonlogical terms—call this type of variation *domain variation*.[58] To get a sense of what domain variation consists in, consider for instance the proposition "$x + y = y + x$" with $\mathbb{N}$ as the domain of its intended interpretation. In mathematical practice, this proposition can perfectly be (re)interpreted as being about $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$. When we consider such domain variations, there is a clear sense in which the meaning of "+" can be kept fixed. This sense is given, more specifically, by the *standard embedding* of $\mathbb{N}$ into $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, which preserves the initial meaning of the addition operation in $\mathbb{N}$ while extending it to these other domains.[59] Now, can mathematical rules of inference be nontruth-preserving under such domain variation? The answer is yes, and this can be seen from the mathematical rule of inference corresponding to mathematical induction. In §3, we formulated the inference schema of this mathematical rule of inference as follows:

$$\frac{\begin{array}{l} H(0) \\ H(X) \rightarrow H(X+1) \end{array}}{H(Y)},$$

where $H(_)$ is a placeholder for an expression involving an arbitrary variable ranging over $\mathbb{N}$, and X and Y are placeholders for arbitrary variables ranging over $\mathbb{N}$. It turns out that mathematical induction is not truth-preserving under domain variation. To see this, consider the instance of the above inference schema in which X and Y are replaced by the variables x and y ranging over $\mathbb{Z}$, and $H(_)$ is replaced by $0 \leq _$. In this case, we have that $H(0)$ is true since $0 \leq 0$, and we have that $H(x) \rightarrow H(x+1)$ is true

[58] The notion of domain variation bears a family resemblance with the issues discussed in sec. 12.9 of (Avigad, 2008) where Avigad describes a class of mathematical inferences involving reasoning about different domains. Avigad's examples show how this notion of domain variation is indeed present in mathematical practice, and points out the complexity of the mechanisms at work in mathematical practice to deal with such variations.

[59] It should be noted that varying the domain of a proposition in this way admits important restrictions depending on the nonlogical terms occurring in the considered proposition. For instance, even though the proposition "$x + y = y + x$" with domain $\mathbb{N}$ can be reinterpreted as a proposition about the complex numbers, the proposition "$x + y \leq y + x$" cannot since the relation "$\leq$" does not make sense on $\mathbb{C}$, although this proposition can perfectly be reinterpreted as a proposition with domain $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$.

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as well since if $0 \leq x$, then $0 \leq x + 1$. Yet, it is obviously false that $0 \leq y$ for any $y \in \mathbb{Z}$, and so we have here an instance of the above inference schema which is not truth-preserving. Consequently, even though mathematical induction is always truth-preserving when the domain of its premisses and conclusion is $\mathbb{N}$, it can be nontruth-preserving when its premisses and conclusion are taken from domain $\mathbb{Z}$. Mathematical induction is then a typical example of a mathematical rule of inference that is not truth-preserving under variation of the subject matter of its premisses and conclusion, when the subject matter of a proposition is taken as its domain and the variation thereof as domain variation.

Thus, mathematical inference does not appear to be semantically general because mathematical rules of inference can be nontruth-preserving under variation of the subject matter of the premisses and conclusion they are applied to when the subject matter of a proposition is understood either as its meaning, or as its domain. When the subject matter of a proposition is understood as the field of inquiry to which it belongs, mathematical rules of inference are indeed truth-preserving under variation of the subject matter of premisses and conclusion due to the fact that this type of variation keeps fixed the intended interpretation of the propositions involved.

Mathematical inference and syntactic generality. Why does mathematical inference not appear to be syntactically general? To claim that mathematical inference is not syntactically general amounts to claim that it is governed by rules of inference whose application is sensitive to the subject matter of the propositions they are applied to. Probably the most natural way to understand the notion of sensitivity is to say that any mathematical rule of inference possesses a restricted *range of application* which specifies to which propositions the rule can be applied in terms of their logical forms *and* their subject matter. This implies, in particular, that a mathematical rule of inference cannot be applied to some propositions if they belong to certain subject matters, even though such propositions might possess logical forms that are compatible with the inference schema of the considered rule. To understand why mathematical inference does not appear to be syntactically general, we shall first argue that the nonsyntactic generality of mathematical inference follows from its nonsemantic generality plus the requirement that the application of mathematical rules of inference be truth-preserving.

Recall that to say of mathematical inference that it is not semantically general is to say that it is governed by rules of inference which are not truth-preserving whatever the subject matter of the propositions they are applied to, i.e., which might lead from true premisses to false conclusions when premisses and conclusions belong to some specific subject matters. If one requires, in addition, that mathematical rules of inference be truth-preserving—which is the *sine qua none* requirement of any deductive rule of inference—then one should not apply a mathematical rule of inference to propositions belonging to subject matters for which the rule is not truth-preserving. In other words, this means that mathematical rules of inference should be sensitive to the subject matter of the propositions they are applied to. Thus, the nonsyntactic generality of mathematical inference follows from its nonsemantic generality plus the requirement that mathematical rules of inference be truth-preserving.

It should be noted that our characterization of syntactic generality also appeals to the notion of the subject matter of a proposition. This means that the implication we have just established between nonsyntactic generality and nonsemantic generality is parametrized, so to speak, to the notion of subject matter one is considering. This implication, together with the previous discussion on mathematical inference and semantic generality, generates

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the following conclusions about mathematical inference and syntactic generality: mathematical rules of inference are sensitive to the subject matter of the propositions they are applied to when the notion of the subject matter of a proposition is understood either as its meaning, or as its domain.

Thus, mathematical inference does not appear to be syntactically general because the application of mathematical rules of inference is sensitive to the subject matter of the propositions they are applied to, when the subject matter of a proposition is understood either as its meaning, or as its domain. This follows from our previous observation that, for these two notions of the subject matter of a proposition, mathematical rules of inference are not truth-preserving whatever the subject matter of the propositions they are applied to, and therefore their application should be restricted, and thereby sensitive, to propositions belonging to those subject matters for which the rules are truth-preserving.

§7. Mechanicality. It has been claimed that mathematical inference departs from logical inference with respect to its *mechanicality*. For instance, Kreisel writes:

> Mathematical reasoning, except in the 'limiting' case of numerical computations, does not present itself to us as the execution of mechanical rules [...] The connection between reliability and the possibility of mechanical checking is usually, and somewhat uncritically, taken for granted. (Kreisel, 1970, p. 21)

In this section, we first identify two different senses in which logical inference is said to be mechanical in the context of formal deductive systems. We then examine why mathematical inference does not appear to be mechanical for each one of these two senses.

7.1. The mechanicality of logical inference. It is sometimes said that logic is *mechanical*—i.e., that it represents reasoning as a purely mechanical endeavor. Such a mechanical conception of reasoning can be traced back at least to Leibniz and his idea of a *characteristica universalis*—a universal language which would allow to express any proposition—and an associated *calculus ratiocinator*—a calculus of reasoning which would allow to mechanically decide the truth of any proposition expressed in the *characteristica*. The mechanical conception of reasoning has played a significant role in the subsequent development of logic posterior to Leibniz,[60] and has even taken central stage in the development of logic and the foundations of mathematics in the 1930s,[61] witness Gödel's opening paragraph in the incompleteness paper:

> The development of mathematics toward greater precision has led, as is well known, to the formalization of large tracts of it, so that one can prove any theorem using nothing but a few mechanical rules. (Gödel, 1931, p. 145)

Indeed, from the 1930s onwards, to be mechanical became an *essential* requirement of logical inference in the context of formal deductive systems. In this context, the mechanicality of logical inference amounts to the mechanicality of the rules of inference that govern it:

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[60] For an historical perspective of the mechanization of reasoning see (Marciszewski & Murawski, 1995). For a general overview of the mechanization of proof from a contemporary perspective see (MacKenzie, 2001).

[61] See (Sieg, 2009) for an historical perspective on the mechanical conception of reasoning during this period.

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Mechanicality: To say that logical inference is *mechanical* is to say that it is governed by rules of inference that are mechanical.[62]

We shall now distinguish between two senses in which the rules of inference of a formal deductive system have been said to be mechanical. This will yield two conceptions of the mechanicality of logical inference that we shall refer to as *mechanical applicability* and *mechanical checkability*.

Mechanical applicability. A rule of inference can be mechanical with respect to its *application*. It is in this sense that Gödel says of the rules of inference (of a formal deductive system) that they could be applied by a machine:

> [T]he outstanding feature of the rules of inference being that they are purely formal, i.e., refer only to the outward structure of the formulas, not to their meaning, so that they could be applied by someone who knew nothing about mathematics, or by a machine. (Gödel, 1995b, p. 45)

But what does the application of a logical rule of inference consist in? Here again, Gödel provides us with a precise description of what it means to apply a rule of inference:

> [A] rule of inference is nothing else but a mechanical procedure which allows [one] to determine of any given finite class of expressions whether anything can be inferred from them by means of the rule of inference under consideration, and if so to write down the conclusion. (Gödel, 1995a, p. 166)

Thus, to say that a rule of inference is mechanically applicable is to say that its application can be carried out by a mechanical procedure, where its application to a set of premisses consists in first determining whether something can be inferred from them, and if so to then write down the conclusion. By reformulating this idea in the terminology adopted in this paper, we obtain the following characterization of mechanical applicability:

Mechanical applicability: To say that logical inference is *mechanically applicable* is to say that it is governed by rules of inference whose application can be carried out by a mechanical procedure, where the application of a rule of inference consists in first determining whether the premisses to which it is applied are instances of the premiss schema of the inference rule, and if so to then write down the conclusion generated by the conclusion schema.

Two remarks are in order with respect to this characterization of mechanical applicability. Firstly, the notion of mechanical procedure can be made fully precise using the notion of a Turing machine. Secondly, and as we can see from Gödel's quote above, the notion of mechanical applicability is connected to the formality of logical inference or, more precisely, to what we have called the syntactic formality of logical inference: it is the fact that logical rules of inference operate on the logico-syntactic structures of formulas that makes it possible for their applications to be carried out mechanically. Yet, it should be noted that the syntactic formality of logical inference does *not* imply its mechanical applicability. To see this, it suffices to consider a rule of inference that is syntactically

[62] This notion is discussed by Dutilh Novaes (2011, sec. 3.1) under the heading of 'formal as computable'.

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formal but not mechanical—or not *effective* to use contemporary terminology—such as, for instance, the nonrecursive version of the ω-rule.

Mechanical checkability. The rules of inference of a formal deductive system can be mechanical in the sense that there is a mechanical procedure to *check* that any given inference is made according to one of the rules of inference. This is indeed a necessary requirement of any formal deductive system, as Church put it:

> [T]he rules of inference, taken together, shall be effective in the strong sense that there is a method by which, whenever a proposed immediate inference is given of one well-formed formula as conclusion from others as premisses, it can always be determined effectively whether or not this proposed immediate inference is in accordance with the rules of inference. (Church, 1956, p. 51)

Importantly, the notion of mechanical checkability does not apply to a single rule of inference taken in isolation, but rather to *all* the rules of inference of a formal deductive system taken *together*. Such a notion of mechanical applicability can be characterized as follows:

Mechanical checkability: To say that logical inference is *mechanically checkable* is to say that it is governed by rules of inference for which there is a mechanical procedure to check whether any given logical inference is in accordance with one of the rules of inference of a given formal deductive system, i.e., whether any given logical inference is an instance of the inference schema of one of the rules of inference.

Three remarks are in order with respect to our characterization of mechanical checkability. Firstly, the notion of mechanical procedure can here again be made fully precise using the notion of a Turing machine. Secondly, mechanical checkability is concerned with *checking* that an inference has been made in accordance with one of the rules of inference, while mechanical applicability is concerned with *applying* such rules of inference. Thirdly, and as we have just mentioned, the notion of mechanical checkability applies to all the rules of inference of a formal deductive system taken together, while the notion of mechanical applicability applies to a single rule of inference taken in isolation.

7.2. Why does mathematical inference not appear to be mechanical? It has often been claimed that, contrary to logical inference, mathematical inference is not mechanical. Thanks to the two conceptual characterizations of the mechanicality of logical inference just identified, we are now in a position to examine precisely why mathematical inference does not appear to be mechanical. In the following, we shall see that if mathematical inference does not appear to be mechanical, it is not because mathematical rules of inference are not mechanically applicable or not mechanically checkable. Rather, it is because mathematical rules of inference do not meet two essential preconditions, respectively, to the mechanical applicability and the mechanical checkability of logical inference, that is, mathematical rules of inference are not made *explicit* and are not *fixed* at the outset.

Mathematical inference and mechanical applicability. Why does mathematical inference not appear to be mechanically applicable? To claim that mathematical inference is not mechanically applicable would amount to claim that the application of mathematical rules of inference cannot be carried out by a mechanical procedure. In order to examine this claim, recall, as was stressed in §3, that we are not saying that drawing a mathematical

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inference consists in applying a mathematical rule of inference[63]—since this would be a claim on the nature of mathematical inference at the algorithmic level—and that mathematical rules of inference have been introduced as descriptions of the inference processes underlying mathematical inferences at the functional level. Insofar as the mechanical applicability of logical inference appeals precisely to the fact that drawing a logical inference consists in applying a logical rule of inference, it does not make sense to ask directly whether mathematical rules of inference, *as rules*, are mechanically applicable. We can, nonetheless, reformulate the above claim and ask if the *inference processes* underlying mathematical inferences are mechanically applicable. Now, if we accept that (1) agents in mathematical practice are always able to decide whether an inference process can be applied to some premisses, (2) the intuitive notion of a mechanical procedure is entirely captured by the formal one of a Turing machine, and (3) the computational power of the human mind does not exceed the one of a Turing machine, then we are led to the conclusion that mathematical rules of inference must be mechanically applicable. Thus, if mathematical inference does not appear to be mechanically applicable, this is not because mathematical rules of inference are not mechanically applicable.

To understand why mathematical inference does not appear to be mechanically applicable, one should first realize that an essential precondition to the mechanical applicability of logical inference is that the rules of inference that govern it be made *explicit*: for a given agent—either a human or a machine—to mechanically apply a logical rule of inference to a set of premisses, it must have the rule at its disposal and, for this purpose, the rule must have been stated explicitly. This corresponds to an essential feature of formal deductive systems, as Curry put it:

> In the older conception a mathematical theory consisted of a set of postulates and their logical consequences. The trouble with that idea is that no one knows exactly what a logical consequence is, and the paradoxes have shown that our intuitive feelings on the subject are not reliable. In the modern conception this vague and subjective notion is replaced by the objective one of derivability according to explicitly stated rules. (Curry, 1950, p. 1)

The requirement of mechanical applicability ensures, in turn, that the process of applying such explicitly stated rules of inference is 'practically' feasible, in the sense that there is a mechanical procedure for doing so. For these reasons, the requirement that the logical rules of inference of a formal deductive system be made explicit constitutes a precondition to the mechanical applicability of logical inference in the context of formal deductive systems.

Mathematical inference does not appear to be mechanically applicable because mathematical rules of inference do not meet this precondition. In mathematical practice—e.g., in mathematical textbooks, monographs, or research articles—one would almost never encounter a set of rules of inference being stated explicitly. Indeed, this feature of mathematical inference and mathematical proof was considered to be a weakness that formal

[63] Notice that drawing a logical inference, in the context of formal deductive systems, does consist in applying a logical rule of inference. For this to be taken as a differentiating property between mathematical inference and logical inference, one should argue that drawing a mathematical inference does not consist in applying some rules of inference, which would be a claim on the nature of mathematical inference at the algorithmic level. We are not making such a claim since, as we said in §3, we remain agnostic on the nature of mathematical inference at the algorithmic level.

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deductive systems precisely aimed to overcome. This is exactly the point made by Curry in the above quote, when he says that the "vague and subjective" notion of logical consequence in ordinary mathematical practice should be "replaced by the objective one of derivability according to explicitly stated rules". This point finds its origin in Frege where the specification of the rules of inference in the *Begriffsschrift* is presented as a main step over traditional mathematical axiomatics:

> I demand—and in this I go beyond Euclid—that all methods of inference employed be specified in advance [. . .]. (Frege, 1964, p. 2)

Thus, mathematical inference does not appear to be mechanically applicable not because mathematical rules of inference are not mechanically applicable, but because mathematical rules of inference are not made *explicit*, which constitutes an essential precondition to the mechanical applicability of logical inference. This is one of the main differences between mathematical and logical rules of inference, and was seen as a weakness of mathematical inference that the development of formal deductive systems precisely aimed to overcome.

Mathematical inference and mechanical checkability. Why does mathematical inference not appear to be mechanically checkable? To claim that mathematical inference is not mechanically checkable would amount to claim that checking whether a given mathematical inference is in accordance with one of the (admitted) mathematical rules of inference in mathematical practice cannot be achieved by a mechanical procedure. Equivalently, this would amount to claim that there does not exist a mechanical procedure for checking whether any given mathematical inference in mathematical practice is *correct* or not. To examine this claim, we shall first observe that an important feature of mathematical practice is that the participants engaged in it can very quickly reach an agreement on whether a mathematical proof is correct or not, and so on whether the mathematical inferences that comprise it are correct or not.[64] So, if we accept that (1) agents engaged in mathematical practice have the ability to decide whether any given mathematical inference is correct or not, (2) the intuitive notion of a mechanical procedure is entirely captured by the formal one of a Turing machine, and (3) the computational power of the human mind does not exceed the one of a Turing machine, then we are led to the conclusion that mathematical rules of inference must be mechanically checkable. Thus, if mathematical inference does not appear to be mechanically checkable, this is not because mathematical rules of inference are not mechanically checkable.

To understand why mathematical inference does not appear to be mechanically checkable, one should first realize that an essential precondition to the mechanical checkability of logical inference is that the logical rules of inference be *fixed* at the outset, i.e., before one can produce or check any formal proof within the considered formal deductive system.[65] This feature was mentioned explicitly by Frege in the above quote when he demands of his formal deductive system that "all methods of inference employed be specified *in advance*" (my emphases). It is easy to see why this is an obvious requirement of any formal deductive system: formal deductive systems have been developed in order to provide an objective criterion for decisively deciding whether a given chain of inferences constitute or not a

[64] This observation has been made several times in the literature, for instance by Azzouni (2004) and Leitgeb (2009).

[65] Another precondition to the mechanical checkability of logical inference is again that the rules of inference that govern it be made *explicit*. This aspect has already been discussed in the case of mechanical applicability.

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correct proof, and this is achieved precisely by fixing at the outset a set of logical rules of inference governing the logical inferences to be admitted in proof. If the logical rules of inference were not fixed at the outset, then an invalid proof with respect to a given set of rules of inference might become valid later on if new rules of inference are admitted, and so it would be impossible to decisively decide at any given point in time whether a chain of inferences constitutes an *in*correct proof. The criterion of mechanical checkability ensures, in turn, that the process of decisively deciding whether a chain of inferences is or is not a correct proof, i.e, of deciding for each inference in the proof whether it is in accordance with one of the rules of inference, is 'practically' feasible, in the sense that there is a mechanical procedure for doing so. For this reason, the requirement that the rules of inference of a formal deductive system be fixed at the outset constitutes a precondition to the mechanical checkability of logical inference in the context of formal deductive systems.

Mathematical inference does not appear to be mechanically checkable because it fails to meet this precondition. Why does mathematical inference not present itself as governed by a *fixed* set of rules of inference? To answer this question, imagine that we undertake the task of recording the rules that govern mathematical inference in mathematical practice. If we do so for the mathematical proof of our case study, we will then be faced with mathematical inferences that do not correspond to the application of a mathematical rule of inference that would have been previously introduced or used. The mathematical inferences (I3) and (I8) are typical examples of such cases. In these two cases, we would be led to record two new mathematical rules of inference. This recording enterprise appears as a virtually open-ended process in which one might potentially always find new rules of inference that have not been previously used or introduced. Interestingly, Frege makes the very same observation in the following passage:

> If we try to list all the laws governing the inferences that occur when arguments are conducted in the usual way, we find an almost unsurveyable multitude which apparently has no precise limits. The reason for this, obviously, is that these inferences are composed of simpler ones. (Frege, 1984, p. 235 (362))

Frege tells us, in addition, why the recording of rules of inference in mathematical practice is potentially an open-ended process, and this is simply because one can always combine together two or more rules of inference that have been previously used or introduced in order to obtain a new one. This is exactly what happens in the case of the mathematical inference (I7), which can be seen as the result of combining the mathematical rules of inference (I2) and (I3).

Thus, mathematical inference does not appear to be mechanically checkable not because mathematical rules of inference are not mechanically checkable, but because mathematical rules of inference are not *fixed* at the outset, which constitutes a precondition to the mechanical checkability of logical inference. This is due to the fact that mathematical rules of inference can always be combined to obtain new ones, which has for consequence to render the set of mathematical rules of inference admitted in mathematical practice virtually open-ended.

§8. Conclusion. The objective of this work was to contribute to the building of a clear and organized picture of the deviation of mathematical proof from formal proof. Our approach has consisted in addressing the issue at the level of the elementary steps of deduction of the two types of proofs, and in examining the various claims put forward in the literature to differentiate between them. To this end, we have proceeded in two steps:

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first, we have analyzed the meaning of the differentiating claims by characterizing the properties that occur in them, resulting in the taxonomy given in Table 7; second, we have investigated the various reasons supporting the differentiating claims, and this for all the various acceptations of the differentiating claims previously identified and characterized.

Table 7. *Taxonomy of the differentiating properties*

Formality	Syntactic formality Semantic formality
Generality	Schematic generality Syntactic generality Semantic generality
Mechanicality	Mechanical applicability Mechanical checkability

Our study results thereby in a precise conceptual apparatus for expressing and discussing the various properties differentiating mathematical inference and logical inference, and provides a first inventory of the various reasons supporting the observations of those differences. In this way, our study offers a clear and organized picture of the deviation of mathematical inference from logical inference, and insofar as those constitute the building blocks of mathematical proof and formal proof, respectively, our study provides an essential element—if not *the* essential element—of a clear and organized picture of the deviation of mathematical proof from formal proof.

As mentioned in the introduction, the importance of understanding the nature of the deviation of mathematical proof from formal proof for the philosophy of mathematics lies in the fact that the notion of formal proof has often been considered to provide an accurate descriptive account of the mathematical proofs to be found in ordinary mathematical practice. From this perspective, any deviation of mathematical proof from formal proof constitutes an epistemological challenge to the validity and accuracy of this descriptive account, and thus demands a philosophical response. Two main strategies have been adopted in the literature to respond to this challenge, the essential difference between them lying in the role they intend to attribute to the notion of formal proof in the new philosophical account of mathematical proof to be proposed.

The first strategy recognizes that the mathematical proofs to be found in mathematical practice do deviate from the ideal of formal proof, but yet considers that one can still *retain* the notion of formal proof as the core component of our philosophical account of mathematical proof. To this end, the main idea is to argue that, although mathematical proofs are not formal proofs *per se*, a mathematical proof can still be conceived as a *sketch* of a formal proof, a view famously proposed by Bourbaki (1970) and Mac Lane (1986), and more recently endorsed by Azzouni (2004), Avigad (2006), Burgess (2015), and Weir (2016), among others. The main task facing this strategy is to specify what it means for a mathematical proof to be a sketch of a formal proof, and then to explain how a typical mathematical agent involved in mathematical practice can recognize and evaluate whether or not a given ordinary mathematical proof does constitute a sketch of a formal proof.

The second strategy considers that the deviation of mathematical proof from formal proof constitutes a *sufficient* reason to abandon *altogether* the notion of formal proof in our philosophical account of mathematical proof. The main task facing this strategy is, of course, to come up and develop a completely new alternative account of mathematical proof. Several philosophers of mathematics have recently engaged in this philosophical

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project. We can mention, in particular, Aberdein (2006) who explored the possibility of exploiting resources from the fields of informal logic and argumentation theory to model mathematical proofs; Leitgeb (2009) who proposed a Gödelian-inspired view in which mathematical proofs are conceived as possessing both semantic and intuitive components, and in which those components are considered to be epistemologically interdependent; or Larvor (2012) who put forward a view according to which the validity of a mathematical proof depends on its logical form as well as its content, and in which mathematical inferences can also act on nonpropositional representations such as diagrams and notational expressions.

Yet, whatever strategy one wishes to adopt, develop, and defend, evaluating its merits with respect to the objective of providing a philosophical account of mathematical proof that purports to be more faithful to mathematical practice requires a set of data against which any such account can be confronted and tested. If one accepts the analysis developed in this work, then the differentiating claims already constitute a significant set of data that any philosophical account of mathematical inference and proof purporting to be more faithful to mathematical practice ought to be able to accommodate and explain.

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