THE SINGLE SERVER QUEUE IN HEAVY TRAFFIC

By J. F. C. KINGMAN

Communicated by M. F. Atiyah

Received 23 November 1960

The process known as the single server queue has been intensively studied in recent years, and various solutions have been obtained. We here consider the case in which the inter-arrival times t of customers are independently and identically distributed, as are the service times u. For this problem Smith (1) has obtained a formal solution for the distribution of the queueing time v of a customer, when the system is in equilibrium.

When the traffic intensity $\rho = E(u)/E(t)$ is near its critical value unity, the queue will be long and the queueing time large. The purpose of the following analysis is to show that the variable

$$V = (1-\rho)v \tag{1}$$

has asymptotically, under certain conditions, an exponential distribution as $\rho \to 1$. A similar result for a certain storage process has been established by Kendall (2).

Suppose that we have a one-parameter family of queueing problems which we may assume, without loss of generality, to have

$$E(t) = 1, \quad E(u) = \rho = 1-\alpha. \tag{2}$$

Let the moment generating function of $x = u-t$ be $K_\alpha^*(s) = E(e^{-sx})$.

In order to use Smith's result we need to restrict the behaviour of $K_\alpha^*(s)$, and it is convenient to make the following assumptions:

(i) $\exists\, \lambda > 0$ such that $K_\alpha^*(s)$ is regular in $-\lambda\alpha+\epsilon \leqslant \Re(s) \leqslant \lambda\alpha-\epsilon$ for any $\epsilon > 0$.

(ii) $\exists\, \mu < \lambda$ such that $K_\alpha^*(s) \neq 1$ in $-\mu\alpha \leqslant \Re(s) \leqslant \mu\alpha$, $s \neq 0$.

(iii) $K_\alpha^{*\prime\prime}(0) = E(x^2)$ tends to a finite, non-zero limit 2β as $\alpha \to 0$.

(iv) $E(x^2 e^{\mu\alpha x})$ remains bounded as $\alpha \to 0$.

(v) If $\tau = \Im(s)$, $K_\alpha^*(s) = O(1/|\tau|)$ as $|\tau| \to \infty$.

Then, according to Smith ((1), § 4), the cumulant generating function of V is given by

$$\begin{aligned}
\phi_\alpha(s) &= \log E(e^{-sV}) = \log E(e^{-\alpha sv}) \\
&= \frac{\alpha s}{2\pi i}\int_{-\mu\alpha-i\infty}^{-\mu\alpha+i\infty} \log\left[\frac{\zeta-\lambda\alpha}{\zeta}\{1-K_\alpha^*(\zeta)\}\right]\frac{d\zeta}{\zeta(\zeta-\alpha s)} \quad (\Re(s) \geqslant 0) \\
&= \frac{s}{2\pi i}\int_{-\mu-i\infty}^{-\mu+i\infty} \log\left[\frac{z-\lambda}{z}\{1-K_\alpha^*(\alpha z)\}\right]\frac{dz}{z(z-s)} \\
&= \frac{s}{2\pi i}\int_{-\mu-i\infty}^{-\mu+i\infty} \log\left[\frac{z-\lambda}{z}\,\frac{K_\alpha^*(\alpha z)-1}{\alpha^2}\right]\frac{dz}{z(z-s)}.
\end{aligned} \tag{3}$$

https://doi.org/10.1017/S0305004100036094 Published online by Cambridge University Press

But
$$K_\alpha^*(\alpha z) = 1 + K_\alpha^{*\prime}(0)\,\alpha z + \tfrac{1}{2}K_\alpha^{*\prime\prime}(\alpha\theta z)\,\alpha^2 z^2 \quad (0 < \theta < 1)$$
$$= 1 + \alpha . \alpha z + \tfrac{1}{2}K_\alpha^{*\prime\prime}(\alpha\theta z)\,\alpha^2 z^2$$
$$= 1 + \alpha^2 z(1 + \tfrac{1}{2}zK_\alpha^{*\prime\prime}(\alpha\theta z))$$

so that
$$\frac{K_\alpha^*(\alpha z) - 1}{\alpha^2 z} \to 1 + \beta z \quad \text{as} \quad \alpha \to 0. \tag{4}$$

Also, the integrand in (3) has modulus equal to

$$|z(z-s)|^{-1}\,|\log(z-\lambda)\,(1 + \tfrac{1}{2}zK_\alpha^{*\prime\prime}(\alpha\theta z))|$$
$$\leqslant |z(z-s)|^{-1}\{|\log|z-\lambda|| + |\log|1 + \tfrac{1}{2}zK_\alpha^{*\prime\prime}(\alpha\theta z)|| + \pi\}$$

and
$$|K_\alpha^{*\prime\prime}(\alpha\theta z)| = |E(x^2 e^{-\alpha\theta z x})| \leqslant E(x^2 e^{\alpha\mu x})$$

which is bounded. Hence the integrand is less than a function of the form

$$|z(z-s)|^{-1}\{A \log|z| + B\}$$

which is integrable and independent of α. Hence we may apply the dominated convergence theorem to (3), using (4), to give

$$\phi_\alpha(s) \to \phi_0(s) = \frac{s}{2\pi i}\int_{-\mu-i\infty}^{-\mu+i\infty} \log(z-\lambda)\,(1+\beta z)\,\frac{dz}{z(z-s)}. \tag{5}$$

Now
$$\phi_0(s) = \frac{s}{2\pi i}\left\{\int_{-\mu-i\infty}^{-\mu+i\infty} \log(z-\lambda)\,\frac{dz}{z(z-s)} + \int_{-\mu-i\infty}^{-\mu+i\infty} \log(1+\beta z)\,\frac{dz}{z(z-s)}\right\}$$
$$= \frac{s}{2\pi i}(I_1 + I_2) \quad \text{(say)}.$$

Let $C_1(R)$, $C_2(R)$ be the contours consisting of the line segment $(-\mu - iR, -\mu + iR)$, together with the semi-circles standing on it, to the left and right respectively. Then

$$I_1 = \lim_{R\to\infty}\int_{C_1(R)} \log(z-\lambda)\,\frac{dz}{z(z-s)} = 0$$

since $C_1(R)$ contains neither $z = 0$, s or λ. We may evidently choose μ such that $\beta\mu < 1$, and then

$$I_2 = -\lim_{R\to\infty}\int_{C_2(R)} \log(1+\beta z)\,\frac{dz}{z(z-s)}$$
$$= -2\pi i[\log(1+\beta . 0)/(-s) + \log(1+\beta s)/s]$$
$$= -\frac{2\pi i}{s}\log(1+\beta s).$$

Hence
$$\phi_\alpha(s) \to -\log(1+\beta s) \quad \text{as} \quad \alpha \to 0 \quad (\Re(s) \geqslant 0). \tag{6}$$

By the continuity theorem for characteristic functions, this implies that V has, asymptotically, an exponential distribution with mean β. If we write σ_1^2 and σ_2^2 for the variances of t and u, we have

$$\beta = \tfrac{1}{2}\lim_{\alpha\to 0}\frac{\sigma_1^2 + \sigma_2^2}{E(t)}. \tag{7}$$

We may note that the same conclusion also holds for the waiting time $w = u + v$, since $\alpha u \to 0$ in probability.

https://doi.org/10.1017/S0305004100036094 Published online by Cambridge University Press

That fact that an exponential distribution arises in this case is a reflexion of the fact that the queue discipline is of a 'first come, first served' type. For example, if customers are chosen at random for service, then (at least for Poisson arrivals), the asymptotic distribution function is of the form

$$1-2\sqrt{(2V)}\,K_1(2\sqrt{(2V)}),$$

where K_1 is the usual Bessel function. The proof of this result will be published elsewhere.

REFERENCES

(1) Smith, W. L. *Proc. Cambridge Philos. Soc.* 49 (1953), 449.
(2) Kendall, D. G. *J. Roy. Statist. Soc. Ser.* B, 19 (1957), 209.

Statistical Laboratory
Cambridge

THE MATHEMATICAL EQUIVALENCE OF THE BORN APPROXIMATION AND THE METHOD OF IMPACT PARAMETERS

By A. M. ARTHURS

Communicated by C. A. Coulson

Received 19 November 1960

1. *Introduction.* It has been found by Frame (2) that the impact parameter approximation tends to the Born approximation at high energies and weak interactions, but although the result itself is correct for collisions involving *s-s* transitions, there is an error in the analysis which invalidates the entire derivation. We shall replace this by a simple proof which depends on the properties of Hankel transforms. For other transitions the result does not hold (Bates (1)).

2. *The problem.* For *s-s* excitation the problem reduces to proving that

$$\lim_{\kappa\to\infty}\kappa^2\int_0^\pi\theta\,d\theta\left|\int_0^\infty rU(r)\,J_0(\kappa\theta r)\,dr\right|^2=\int_0^\infty|U(r)|^2\,r\,dr, \tag{1}$$

where J_0 is the usual Bessel function and $U(r)$ is an effective potential defined by Frame (2).

To follow Frame's analysis we write the left-hand side of (1) as

$$L=\lim_{\kappa\to\infty}\kappa^2\int_0^\infty\int_0^\infty rU(r)\,r'U(r')\,dr\,dr'\int_0^\pi J_0(\kappa r\theta)\,J_0(\kappa r'\theta)\,\theta\,d\theta.$$

Performing the integration over θ we have

$$L=\lim_{\kappa\to\infty}\int_0^\infty\int_0^\infty rr'U(r)\,U(r')\,dr\,dr'\,\frac{\kappa}{r^2-r'^2}\,[r'J_0(\kappa r)\,J_0'(\kappa r')-rJ_0(\kappa r')\,J_0'(\kappa r)], \tag{2}$$

https://doi.org/10.1017/S0305004100036094 Published online by Cambridge University Press