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The Construction of Smooth Parseval Frames of Shearlets

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Abstract. The shearlet representation has gained increasing recognition in recent years as a framework for the efficient representation of multidimensional data. This representation consists of a countable collection of functions defined at various locations, scales and orientations, where the orientations are obtained through the use of shear matrices. While shear matrices offer the advantage of preserving the integer lattice and being more appropriate than rotations for digital implementations, the drawback is that the action of the shear matrices is restricted to cone-shaped regions in the frequency domain. Hence, in the standard construction, a Parseval frame of shearlets is obtained by combining different systems of cone-based shearlets which are projected onto certain subspaces of $L^2(\mathbb{R}^D)$ with the consequence that the elements of the shearlet system corresponding to the boundary of the cone regions lose their good spatial localization property. In this paper, we present a new construction yielding smooth Parseval frame of shearlets for $L^2(\mathbb{R}^D)$. Specifically, all elements of the shearlet systems obtained from this construction are compactly supported and C^∞ in the frequency domain, hence ensuring that the system has also excellent spatial localization.

Keywords and phrases: affine systems, nonlinear approximations, shearlets, sparse approximations, wavelets

Mathematics Subject Classification: 42C15, 42C40

1. Introduction

Several advanced multiscale representations were introduced during the last 10 years to overcome the limitations of wavelets and other traditional multiscale methods in dealing with multidimensional data efficiently. Indeed, wavelets are not very efficient in capturing the geometry of images with edges, so that they can only provide suboptimal approximations for these and other higher dimensional data. Starting with the introduction of wedgelets [8] and ridgelets [3], it was recognized that to achieve sparser representations of multidimensional data, one has to consider representation systems with ability to deal with anisotropic features much more efficiently than traditional wavelets. The construction of the system of curvelets [4] can be considered a breakthrough in this direction, since it was the first non-adaptive representation to be proved to provide (nearly) optimal approximations for a large class of images. Curvelets are a collection of analyzing functions defined not only at various scales and locations

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as wavelets do, but also at various orientations, with the number of orientations increasing at finer scales. Another fundamental property is that their supports are highly anisotropic, and this also plays a fundamental role in their approximation properties. However, the curvelet construction has a number of drawbacks. In particular, this system is not singly generated, i.e., it is not derived from the action of countably many operators applied to a single (or finite set) of generating functions. In addition, its construction involves rotations and these operators do not preserve the digital lattice, which prevents a direct transition from the continuum to the digital setting.

The *shearlets*, originally introduced by the authors and their collaborators in [13, 31], provide an alternative approach to the construction of sparse and efficient multidimensional representations, which overcomes the main drawbacks of the curvelet representation. In fact, shearlets were derived from the framework of wavelets with composite dilations, a method introduced to provide a truly multivariate extension of the wavelet framework through the use of affine transformations [20–22]. In this approach, the shearlet system is obtained by applying a countable collection of operators to a single or finite set of generators. In particular, the directional selectivity of the system is controlled through the use of shear matrices. This is another fundamental difference with respect to curvelets, since the shear matrices preserve the integer lattice and this allows the shearlet systems to provide a unified treatment of the continuum and digital setting. These properties and the special flexibility of the shearlet framework have made this approach very successful both as a theoretical and an applicable tool (see, for example, publications [6, 14, 16, 19, 28, 30] for the main theoretical results, and [5, 9–11, 30, 33–35] for applications).

In this paper, we introduce a new construction which allows us to obtain smooth Parseval frame of shearlets for $L^2(\mathbb{R}^D)$; specifically, all elements of these new shearlet systems are C^∞ and compactly supported in the frequency domain, hence ensuring that they are well localized. This new approach overcomes an outstanding problem concerning the construction of well localized Parseval frames of shearlets in $L^2(\mathbb{R}^D)$. In fact, the standard construction of reproducing shearlet systems produces smooth Parseval frames of shearlets defined only over certain proper subspaces of L^2; when shearlet systems associated to these subspaces are combined to obtain a reproducing system for the whole L^2 space, either the tightness of the system is lost (e.g., the enlarged shearlet system is a non-tight frame of $L^2(\mathbb{R}^D)$) or the system loses its spatial localization since the enlarged shearlet system is a Parseval frame of $L^2(\mathbb{R}^D)$, but not all of its elements are well localized. This new construction is especially relevant since the new smooth Parseval frames of shearlets for $L^2(\mathbb{R}^D)$ introduced in this paper offer the appropriate representation methods needed to provide highly sparse approximations for a large class of multidimensional data. In particular, for $D = 3$, this new smooth Parseval frame of shearlets provides nearly optimally sparse approximations piecewise C^2 functions in $L^2(\mathbb{R}^3)$ [17, 18].

1.1. Shearlet representations

In order to better explain the significance of the new construction presented in this paper, let us recall the main ideas of the traditional shearlet construction in dimension $D = 2$ (cf. [14, 31]). For an appropriate $\gamma \in L^2(\mathbb{R}^2)$, a system of shearlets is defined as a collection of functions of the form

$$\left\{\gamma_{j,\ell,k}(x) = |\det A_{(1)}|^{j/2}\, \gamma(B_{(1)}^{\ell} A_{(1)}^{j} x - k) : j, \ell \in \mathbb{Z}, k \in \mathbb{Z}^2\right\}, \tag{1.1}$$

where

$$A_{(1)} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \quad B_{(1)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{1.2}$$

Notice that the matrix $A_{(1)}$ is a dilation matrix whose integer powers produce anisotropic dilations and, more specifically, *parabolic scaling* dilations similar to those used in the curvelet construction; the *shear* matrix $B_{(1)}$ is non-expanding and its integer powers control the directional features of the shearlet system. The generator function γ is defined in the frequency domain as

$$\hat{\gamma}(\xi) = \hat{\gamma}(\xi_1, \xi_2) = w(\xi_1)\, v(\frac{\xi_2}{\xi_1}),$$

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and it is possible to choose functions w, $v \in C_c^\infty(\mathbb{R})$ so that the corresponding system (1.1) is a Parseval frame of $L^2(\mathbb{R}^2)$. As a result, the system (1.1) forms a Parseval frame of waveforms defined at various scales, orientations and locations, controlled by j, ℓ, and k, respectively.

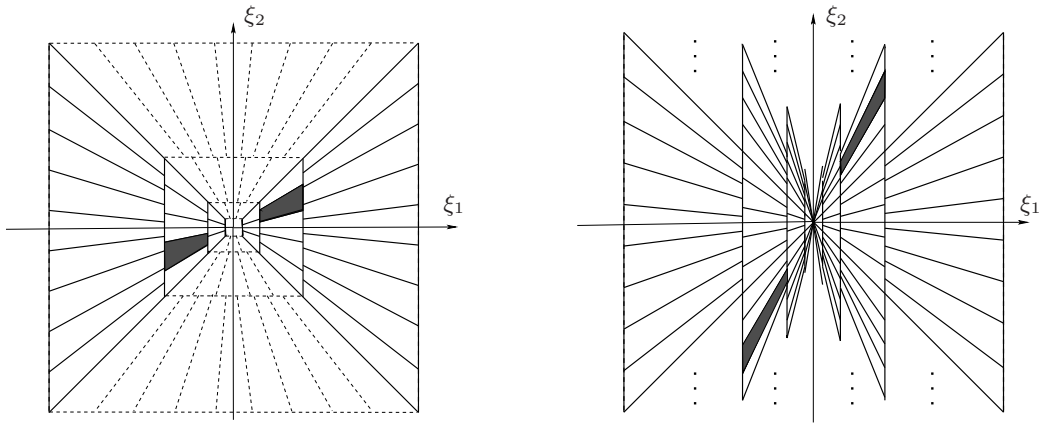


FIGURE 1. *The tiling of the frequency plane produced by: (a) a singly generated horizontal shearlet system; (b) two cone-based systems of horizontal and vertical shearlets. Note that the frequency supports of the elements of the shearlet system do overlap, so that the figures should be interpreted as descriptive of the essential frequency supports (i.e., the regions where 'most' of the L^2-norm is concentrated).*

Despite its simplicity, the shearlet system derived from this construction has a directional bias, due to the fact that the shear matrices $B_{(1)}^{\ell}$ produce elements whose support is increasingly more elongated along the vertical axis, in the Fourier domain, as $|\ell|$ increases (see illustration in Fig. 1(a)). This directional bias affects negatively the approximation properties of the system (1.1) and makes it impractical in most applications. Hence, to avoid this problem, the shearlet system is usually modified by: (i) restricting the shear parameter ℓ so that the frequency support of the shearlet system (1.1) is contained inside the cone $|\frac{\xi_2}{\xi_1}| \leq 1$, and (ii) adding a second shearlet system which is rotated by $\pi/2$ with respect to the first one. That is, for $d = 1, 2$, one defines the cone-based shearlet systems

$$\{\gamma_{j,\ell,k}^{(d)} = |\det A_{(d)}|^{-j/2}\, \gamma^{(d)}(B_{(d)}^{\ell} A_{(d)}^{j} x - k), \quad j \geq 0, |\ell| \leq 2^j, k \in \mathbb{Z}^2\}, \tag{1.3}$$

where $A_{(2)} = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 4 \end{smallmatrix}\right)$, $B_{(2)} = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, $\gamma^{(1)} = \gamma$, $\gamma^{(2)}$ is defined by $\hat{\gamma}^{(2)}(\xi_1, \xi_2) = w(\xi_2)\, v(\frac{\xi_1}{\xi_2})$, and the index $d = 1, 2$ is associated with the horizontal and vertical truncated cone regions given by

$$\widetilde{\mathcal{P}}_1 = \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\frac{\xi_2}{\xi_1}| \leq 1,\, |\xi_1| \geq \frac{1}{8}\right\}, \widetilde{\mathcal{P}}_2 = \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\frac{\xi_2}{\xi_1}| > 1,\, |\xi_1| \geq \frac{1}{8}\right\}, \tag{1.4}$$

respectively. In fact, for each value of index d, one can show that the shearlet systems (1.3) are smooth Parseval frames of the subspace of $L^2(\mathbb{R}^2)$ given by

$$L^2(\widetilde{\mathcal{P}}_d)^\vee = \{f \in L^2(\mathbb{R}^2) : \operatorname{supp} \hat{f} \subset \widetilde{\mathcal{P}}_d\}.$$

In order to obtain a Parseval frame of shearlets of $L^2(\mathbb{R}^2)$, each of these systems is projected onto the subspaces $L^2(\widetilde{\mathcal{P}}_d)^\vee$, $d = 1, 2$; in addition, a coarse scale system is included to take care of the low frequency

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region. That is, we obtain a *cone-based Parseval frame of shearlet* of $L^2(\mathbb{R}^2)$ of the form

$$\{P_{\widetilde{\mathcal{P}}_d}\gamma^{(d)}_{j,\ell,k} : j \geq 0, |\ell| \leq 2^j, k \in \mathbb{Z}^2, d = 1, 2\} \bigcup \{\Phi_k : k \in \mathbb{Z}^2\},$$

where $P_{\widetilde{\mathcal{P}}_d}$ is the orthogonal projection $(P_{\widetilde{\mathcal{P}}_d}h)^\wedge(\xi) = \hat{h}(\xi)\,\chi_{\widetilde{\mathcal{P}}_d}(\xi)$ and Φ is an appropriate band-limited function (cf. [13,14]). As illustrated in Fig. 1(b), the cone-based shearlet system produces a tiling of the Fourier plane whose directional selectivity is much more uniform than the previous construction, and this makes this approach highly preferable in most situations[1].

The outstanding issue in this cone-based construction is that the orthogonal projection operator has the effect of destroying the spatial localization properties of the shearlet elements whose frequency support overlaps the boundaries of the cone regions $\widetilde{\mathcal{P}}_1$ and $\widetilde{\mathcal{P}}_2$. This is due to the fact that the boundary shearlet elements are truncated in the frequency domain. While one can preserve the continuity in the frequency domain, it is not possible to match the boundary elements so that also the smoothness is preserved.

As indicated above, in this paper we introduce a new shearlet construction which allows us to obtain smooth Parseval frame of shearlets for $L^2(\mathbb{R}^D)$; specifically, all elements of these new shearlet systems are C^∞ and compactly supported in the frequency domain, hence ensuring that they are well localized. As will be discussed below in detail, this is obtained by slightly modifying the definition of the cone-based shearlets and by appropriately defining the shearlet elements whose frequency supports overlap the boundaries of the cone regions. This modified construction, which is rather saddle, becomes increasingly more complicated as the dimension D increases and is obtained at the expense of the affine structure of the system. That is, the new smooth Parseval frame of shearlets is not obtained from the action of a countable set of operators on a single or finite set of generators as in (1.3). Nevertheless, as will be clear below, the new system is indeed finitely generated in the frequency domain (unlike curvelets). In fact, the new construction corresponds very nicely to what is needed for some digital implementations such as the one considered in [10], where the shearlet decomposition is derived directly in the discrete Fourier domain. We also wish to mention that the philosophy of the shearlet construction presented in this paper is similar to the point of view of nonstationary tight framelets proposed by Bin Han [23, 24] and, more precisely, to the notion of directional tight framelets [24, Sec. 4].

Before presenting our new construction, it is useful to observe that an alternative solution for the construction of smooth reproducing shearlet systems for $L^2(\mathbb{R}^D)$ consists in relaxing the requirement that the shearlet system forms a Parseval frame and considering instead band-limited shearlet frames which are not tight, as it was recently proposed in [12]. The disadvantage of this approach is that it requires to carefully design the dual frame system. Even with a nice dual frame, this solution is less desirable than a Parseval frame for most applications. Finally, we recall that there is a very different approach to the construction of shearlet systems, recently introduced in [26, 29], which considers well localized *compactly supported* frames of shearlets, where the frames are not tight. It is an open question whether Parseval frames of such compactly supported systems of shearlets exist (cf.[25] for a related discussion).

The paper is organized as follows. In Sec. 2, we introduce the new systems of shearlets in dimension $D = 2$ and show that they produce smooth Parseval frames of $L^2(\mathbb{R}^2)$. In Sec. 3, we extend this construction to dimension $D = 3$. This construction is significantly more complicated, due to the need to define different types of boundary shearlet functions. In Sec. 4, we briefly describe how the new shearlet construction extends to higher dimensions D. In Sec. 5, we recall a notion of shearlet molecules previously introduced by the authors and show that the new shearlets are also shearlet molecules. This is useful to provide an additional insight into the close relationship between the new and old shearlet constructions.

[1]Note that the frequency tiling Fig. 1(b) is very similar to the one used by the contourlets [7], a purely discrete approach introduced as a discretization of curvelets

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2. A new shearlet construction ($D = 2$)

We start with the construction of a smooth Parseval frame of shearlets for $L^2(\mathbb{R}^2)$. This construction has some similarities with the construction of the so-called *digital curvelets* from [2]. Note, however, that no details (and no proof) for the smoothness of the construction are provided in the cited reference.

Let ϕ be a C^∞ univariate function such that $0 \le \hat\phi \le 1$, $\hat\phi = 1$ on $[-\frac{1}{16}, \frac{1}{16}]$ and $\hat\phi = 0$ outside the interval $[-\frac{1}{8}, \frac{1}{8}]$. That is, ϕ is the scaling function of a Meyer wavelet, rescaled so that its frequency support is contained the interval $[-\frac{1}{8}, \frac{1}{8}]$. For $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, let

$$\widehat{\Phi}(\xi) = \widehat{\Phi}(\xi_1, \xi_2) = \hat\phi(\xi_1)\,\hat\phi(\xi_2) \tag{2.1}$$

and

$$W(\xi) = W(\xi_1, \xi_2) = \sqrt{\widehat{\Phi}^2(2^{-2}\xi_1, 2^{-2}\xi_2) - \widehat{\Phi}^2(\xi_1, \xi_2)}.$$

It follows that

$$\widehat{\Phi}^2(\xi_1, \xi_2) + \sum_{j\ge 0} W^2(2^{-2j}\xi_1, 2^{-2j}\xi_2) = 1 \quad \text{for } (\xi_1, \xi_2) \in \mathbb{R}^2. \tag{2.2}$$

Notice that each function $W_j^2 = W^2(2^{-2j}\cdot)$ has support into the Cartesian corona

$$C_j = [-2^{2j-1}, 2^{2j-1}]^2 \setminus [-2^{2j-4}, 2^{2j-4}]^2.$$

In particular, the functions W_j^2, $j \ge 0$, produce a smooth tiling of the frequency plane into Cartesian coronae:

$$\sum_{j\ge 0} W^2(2^{-2j}\xi) = 1 \quad \text{for } \xi \in \mathbb{R}^2 \setminus [-\frac{1}{8}, \frac{1}{8}]^2 \subset \mathbb{R}^2. \tag{2.3}$$

Next, let $v \in C^\infty(\mathbb{R})$ be chosen so that $\operatorname{supp} v \subset [-1, 1]$ and

$$|v(u-1)|^2 + |v(u)|^2 + |v(u+1)|^2 = 1 \quad \text{for } |u| \le 1. \tag{2.4}$$

In addition, we will assume that $v(0) = 1$ and that $v^{(n)}(0) = 0$ for all $n \ge 1$. It was shown in [14] that there are several examples of functions satisfying these properties. It follows from equation (2.4) that, for any $j \ge 0$,

$$\sum_{m=-2^j}^{2^j} |v(2^j\, u - m)|^2 = 1, \quad \text{for } |u| \le 1. \tag{2.5}$$

Using this notation we state the following definition.

Definition 2.1. For $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $V_{(1)}(\xi_1, \xi_2) = v(\frac{\xi_2}{\xi_1})$, the *horizontal shearlet system* for $L^2(\mathbb{R}^2)$ is defined as the countable collection of functions

$$\{\psi^{(1)}_{j,\ell,k} : j \ge 0, -2^j \le \ell \le 2^j, k \in \mathbb{Z}^2\}, \tag{2.6}$$

where

$$\hat\psi^{(1)}_{j,\ell,k}(\xi) = |\det A_{(1)}|^{-j/2}\, W(2^{-2j}\xi)\, V_{(1)}(\xi A^{-j}_{(1)} B^{-\ell}_{(1)})\, e^{2\pi i \xi A^{-j}_{(1)} B^{-\ell}_{(1)} k}, \tag{2.7}$$

and $A_{(1)}$, $B_{(1)}$ are given by (1.2).

The term *horizontal* refers to the fact that the elements of this system (except for $\ell = \pm 2^j$) have frequency support contained inside the horizontal cone $\widetilde{\mathcal{P}}_1$, given by (1.4) and that, as will be shown below, the system (2.6) forms a Parseval frame for the $L^2(\widetilde{\mathcal{P}}_1)^\vee$.

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Remark 2.1. The new horizontal shearlet system (2.6) has virtually the same space-frequency interpretation as the "standard" shearlet system (1.1) whose elements, in the Fourier domain, are

$$\hat{\gamma}_{j,\ell,k}(\xi) = |\det A_{(1)}|^{-j/2}\, w(2^{-2j}\xi_1)\, V_{(1)}(\xi A_{(1)}^{-j} B_{(1)}^{-\ell})\, e^{2\pi i \xi A_{(1)}^{-j} B_{(1)}^{-\ell} k}, \tag{2.8}$$

where w is a one-dimensional Meyer wavelet with frequency support in $[-\frac{1}{2}, \frac{1}{2}] \setminus [-\frac{1}{16}, \frac{1}{16}]$.

In fact, both the functions $\hat{\gamma}_{j,\ell,k}$, given by (2.8), and the elements $\hat{\psi}^{(1)}_{j,\ell,k}$, given by (2.6), are supported inside the trapezoidal regions

$$\{(\xi_1,\xi_2) : \xi_1 \in [-2^{2j-1}, -2^{2j-4}] \cup [2^{2j-4}, 2^{2j-1}], |\frac{\xi_2}{\xi_1} - \ell 2^{-j}| \le 2^{-j}\}$$

in the Fourier plane, so that they produce the same frequency tiling. However, due to the shear-invariance of the function w, the system of shearlets $\{\gamma_{j,\ell,k}\}$ is affine-like. In fact, we have that

$$\hat{\gamma}_{j,\ell,k}(\xi) = |\det A_{(1)}|^{-j/2} \hat{\gamma}(\xi A_{(1)}^{-j} B_{(1)}^{-\ell})\, e^{2\pi i \xi A_{(1)}^{-j} B_{(1)}^{-\ell} k},$$

where $\hat{\gamma}$ is the function $\hat{\gamma} = w\, V$, so that in space domain they produce the system (1.1). By contrast, the new system of shearlets (2.6) is not affine-like. In the space domain, its elements are of the form

$$\psi_{j,\ell,k}(x) = |\det A_{(1)}|^{j/2}\, \psi_{j,\ell}(B_{(1)}^{\ell} A_{(1)}^{j} x - k),$$

where

$$\hat{\psi}_{j,\ell}(\xi) = W(2^{-2j}\xi B_{(1)}^{\ell} A_{(1)}^{j})\, V(\xi).$$

As the expression above shows, the function W is not shear-invariant. On the other hand, as we will show below, the new definition will allow us to construct a smooth cone-based Parseval frame of shearlets not only for certain proper subspaces of $L^2(\mathbb{R}^2)$ but for the whole space $L^2(\mathbb{R}^2)$, whereas this is not possible using the affine-like shearlet construction.

The next observation shows that the system of shearlets (2.6) is a Parseval frame for $L^2(\widetilde{\mathcal{P}}_1)^\vee$. This result is similar to the one valid for the affine-like shearlet system (1.3).

Proposition 2.1. *The horizontal shearlet system* (2.6) *is a Parseval frame for* $L^2(\widetilde{\mathcal{P}}_1)^\vee$.

Proof. Notice that $(\xi_1,\xi_2)\, A_{(1)}^{-j} B_{(1)}^{-\ell} = (2^{-2j}\xi_1, 2^{-j}\xi_2 - \ell 2^{-2j}\xi_1)$. Hence, we can write the elements of the system of shearlets (2.6) as

$$\hat{\psi}^{(1)}_{j,\ell,k}(\xi_1,\xi_2) = 2^{-\frac{3}{2}j}\, W(2^{-2j}\xi_1, 2^{-2j}\xi_2)\, v\Big(2^j \frac{\xi_2}{\xi_1} - \ell\Big)\, e^{2\pi i \xi A_{(1)}^{-j} B_{(1)}^{-\ell} k}.$$

Let $f \in L^2(\widetilde{\mathcal{P}}_1)^\vee$. Using the change of variable $\eta = \xi A_{(1)}^{-j} B_{(1)}^{-\ell}$ and the notation $Q = [-\frac{1}{2}, \frac{1}{2}]^2$, we have:

$$\begin{aligned}
&\sum_{j\ge 0} \sum_{\ell=-2^j}^{2^j} \sum_{k\in\mathbb{Z}^2} |\langle \hat{f}, \hat{\psi}^{(1)}_{j,\ell,k}\rangle|^2 \\
&= \sum_{j\ge 0} \sum_{\ell=-2^j}^{2^j} \sum_{k\in\mathbb{Z}^2} \Big| \int_{\mathbb{R}^2} 2^{-\frac{3}{2}j}\, \hat{f}(\xi)\, W(2^{-2j}\xi_1, 2^{-2j}\xi_2)\, v\Big(2^j \frac{\xi_2}{\xi_1} - \ell\Big)\, e^{2\pi i \xi A_{(1)}^{-j} B_{(1)}^{-\ell} k}\, d\xi \Big|^2 \\
&= \sum_{j\ge 0} \sum_{\ell=-2^j}^{2^j} \sum_{k\in\mathbb{Z}^2} \Big| \int_{Q} 2^{\frac{3}{2}j}\, \hat{f}(\eta B_{(1)}^{\ell} A_{(1)}^{j})\, W(\eta_1, 2^{-j}(\eta_2 + \ell\eta_1))\, v\Big(\frac{\eta_2}{\eta_1}\Big)\, e^{2\pi i \eta k}\, d\eta \Big|^2
\end{aligned}$$

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$$= \sum_{j\geq 0}\sum_{\ell=-2^j}^{2^j}\int_Q 2^{3j}|\hat{f}(\eta B_{(1)}^{\ell}A_{(1)}^{j})|^2\,|W(\eta_1,2^{-j}(\eta_2+\ell\eta_1))|^2\,|v\left(\tfrac{\eta_2}{\eta_1}\right)|^2\,d\eta$$

$$= \sum_{j\geq 0}\sum_{\ell=-2^j}^{2^j}\int_{\mathbb{R}^2}|\hat{f}(\xi)|^2\,|W(2^{-2j}\xi_1,2^{-2j}\xi_2)|^2\,|v\Big(2^j\frac{\xi_2}{\xi_1}-\ell\Big)|^2\,d\xi$$

$$= \int_{\mathbb{R}^2}|\hat{f}(\xi)|^2\sum_{j\geq 0}\sum_{\ell=-2^j}^{2^j}|W(2^{-2j}\xi_1,2^{-2j}\xi_2)|^2\,|v\Big(2^j\frac{\xi_2}{\xi_1}-\ell\Big)|^2\,d\xi.$$

In the computation above, we have used the fact that the function

$$W(\eta_1,2^{-j}(\eta_2+\ell\eta_1))\,v\left(\tfrac{\eta_2}{\eta_1}\right)$$

is supported inside Q since $v(\frac{\eta_2}{\eta_1})$ is supported inside the cone $|\frac{\eta_2}{\eta_1}|\leq 1$ and $W(\eta_1,2^{-j}(\eta_2+\ell\eta_1))$ is supported inside the strip $|\eta_1|\leq\frac{1}{2}$.

Finally, using (2.3) and (2.5) if follows that

$$\sum_{j\geq 0}\sum_{\ell=-2^j}^{2^j}|W(2^{-2j}\xi_1,2^{-2j}\xi_2)|^2\,|v\Big(2^j\frac{\xi_2}{\xi_1}-\ell\Big)|^2=\sum_{j\geq 0}|W(2^{-2j}\xi_1,2^{-2j}\xi_2)|^2\sum_{\ell=-2^j}^{2^j}|v\Big(2^j\frac{\xi_2}{\xi_1}-\ell\Big)|^2$$
$$=\sum_{j\geq 0}|W(2^{-2j}\xi_1,2^{-2j}\xi_2)|^2=1 \text{ for } (\xi_1,\xi_2)\in\widetilde{\mathcal{P}}_1.$$

Thus, we conclude that, for each $f\in L^2(\widetilde{\mathcal{P}}_1)^\vee$, we have:

$$\sum_{j\geq 0}\sum_{\ell=-2^j}^{2^j}\sum_{k\in\mathbb{Z}^2}|\langle f,\hat{\psi}^{(1)}_{j,\ell,k}(\xi_1,\xi_2)\rangle|^2=\|f\|^2. \quad \square$$

A similar construction yields a Parseval frame for $L^2(\widetilde{\mathcal{P}}_2)^\vee$, where $\widetilde{\mathcal{P}}_2$ is given by (1.4). Namely, let us define a *vertical shearlet system* for $L^2(\mathbb{R}^2)$ as

$$\{\psi^{(2)}_{j,\ell,k}:\, j\geq 0,-2^j\leq\ell\leq 2^j,k\in\mathbb{Z}^2\}, \tag{2.9}$$

where

$$\hat{\psi}^{(2)}_{j,\ell,k}(\xi)=|\det A_{(2)}|^{-j/2}\,W(2^{-2j}\xi)\,V_{(2)}(\xi A_{(2)}^{-j}B_{(2)}^{-\ell})\,e^{2\pi i\xi A_{(2)}^{-j}B_{(2)}^{-\ell}k}, \tag{2.10}$$

$A_{(2)}=\left(\begin{smallmatrix}2&0\\0&4\end{smallmatrix}\right)$, $B_{(2)}=\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$, $V_{(2)}(\xi)=v(\frac{\xi_1}{\xi_2})$. Then a very similar argument to the one above gives:

Proposition 2.2. *The vertical shearlet system* (2.9) *is a Parseval frame for* $L^2(\widetilde{\mathcal{P}}_2)^\vee$.

2.1. A smooth Parseval frame of shearlets for $L^2(\mathbb{R}^2)$

We will derive a smooth Parseval frame of shearlets for $L^2(\mathbb{R}^2)$ using an appropriate combinations of the horizontal and vertical shearlet systems together with a coarse scale system, which will take care of the low frequency region. To ensure that all elements of this combined shearlet system are C_c^∞ in the frequency domain, those elements overlapping the boundaries of the cone regions in the frequency domain will be appropriately modified [2].

[2]As it will be clear from the calculations below, this modification of the boundary elements produces the desired smooth Parseval frame of shearlets; the same modification would not produce the same result using the systems (1.3).

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In the following, we will use the notation:

$$\mathcal{P}_1 = \left\{(\xi_1,\xi_2)\in\mathbb{R}^2 : |\frac{\xi_2}{\xi_1}|\le 1\right\}, \mathcal{P}_2 = \left\{(\xi_1,\xi_2)\in\mathbb{R}^2 : |\frac{\xi_2}{\xi_1}|> 1\right\}.$$

The new *shearlet system for* $L^2(\mathbb{R}^2)$ is given by

$$\left\{\widetilde{\psi}_{-1,k} : k\in\mathbb{Z}^2\right\}\bigcup\left\{\widetilde{\psi}_{j,\ell,k,d} : j\ge 0, |\ell|<2^j, k\in\mathbb{Z}^2, d=1,2\right\}\bigcup\left\{\widetilde{\psi}_{j,\ell,k} : j\ge 0, \ell=\pm 2^j, k\in\mathbb{Z}^2\right\}, \quad (2.11)$$

consisting of:

- the *coarse-scale shearlets* $\{\widetilde{\psi}_{-1,k} = \Phi(\cdot - k) : k\in\mathbb{Z}^2\}$, where Φ is given by (2.1);
- the *interior shearlets* $\{\widetilde{\psi}_{j,\ell,k,d} = \psi^{(d)}_{j,\ell,k} : j\ge 0, |\ell|<2^j, k\in\mathbb{Z}^2, d=1,2\}$, where the functions $\psi^{(d)}_{j,\ell,k}$, $d=1,2$ are given by (2.7) and (2.10), respectively;
- the *boundary shearlets* $\{\widetilde{\psi}_{j,\ell,k} : j\ge 0, \ell=\pm 2^j, k\in\mathbb{Z}^2\}$, obtained by joining together slightly modified versions of $\psi^{(1)}_{j,\ell,k}$ and $\psi^{(2)}_{j,\ell,k}$, for $\ell=\pm 2^j$, after that they have been restricted in the Fourier domain to the cones $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. The precise definition is given below:
  for $j\ge 1$, $\ell=\pm 2^j$, $k\in\mathbb{Z}^2$, we define

$$(\widetilde{\psi}_{j,\ell,k})^\wedge(\xi) = \begin{cases} 2^{-\frac{3}{2}j-\frac{1}{2}} W(2^{-2j}\xi_1, 2^{-2j}\xi_2)\, v\left(2^j\frac{\xi_2}{\xi_1}-\ell\right) e^{2\pi i \xi 2^{-1} A_{(1)}^{-j} B_{(1)}^{-\ell} k}, & \text{if } \xi\in\mathcal{P}_1 \\ 2^{-\frac{3}{2}j-\frac{1}{2}} W(2^{-2j}\xi_1, 2^{-2j}\xi_2)\, v\left(2^j\frac{\xi_1}{\xi_2}-\ell\right) e^{2\pi i \xi 2^{-1} A_{(1)}^{-j} B_{(1)}^{-\ell} k}, & \text{if } \xi\in\mathcal{P}_2; \end{cases}$$

  for $j=0$, $k\in\mathbb{Z}^2$, $\ell=\pm 1$, we define

$$(\widetilde{\psi}_{0,\ell,k})^\wedge(\xi) = \begin{cases} W(\xi_1,\xi_2)\, v\left(\frac{\xi_2}{\xi_1}-\ell\right) e^{2\pi i\xi k}, & \text{if } \xi\in\mathcal{P}_1 \\ W(\xi_1,\xi_2)\, v\left(\frac{\xi_1}{\xi_2}-\ell\right) e^{2\pi i\xi k}, & \text{if } \xi\in\mathcal{P}_2. \end{cases}$$

Note that the boundary shearlets $\{\widetilde{\psi}_{j,\ell,k} : j\ge 0, \ell=\pm 2^j, k\in\mathbb{Z}^2\}$ are compactly supported and C^∞ functions in the frequency domain. In fact, the support condition follows trivially from the definition. To show that $(\widetilde{\psi}_{j,\ell,k})^\wedge$ is continuous, it is easy to verify that the two terms of these piecewise defined functions are equal for $\xi_1=\pm\xi_2$. The smoothness is similarly verified by checking that the derivatives of these functions along the lines $\xi_1=\pm\xi_2$ are equal. In fact, the two terms of the functions $(\widetilde{\psi}_{j,2^j,k})^\wedge$ only differ by $v(2^j(\frac{\xi_2}{\xi_1}-1))$ and $v(2^j(\frac{\xi_2}{\xi_1}-1))$. Since $v^{(n)}(0)=0$ for all $n\ge 1$, it follows that all derivatives of these functions are equal when $\xi_1=\pm\xi_2$. This implies that the functions $(\widetilde{\psi}_{j,2^j,k})^\wedge$ are infinitely differentiable. The same observation holds for the functions $(\widetilde{\psi}_{j,-2^j,k})^\wedge$.

We can now prove this construction yields a smooth Parseval frame for $L^2(\mathbb{R}^2)$.

Theorem 2.1. *The shearlet system* (2.11) *is a Parseval frame for* $L^2(\mathbb{R}^2)$. *Furthermore, the elements of this system are* C^∞ *and compactly supported in the Fourier domain.*

Proof. The smoothness and compact support of the coarse-scale and interior shearlets in the Fourier domain follow directly from the definition. The smoothness and support conditions of the boundary shearlet elements have been discussed above. Hence, it only remains to show that the system of shearlets (2.11) is a Parseval frame of $L^2(\mathbb{R}^2)$.

We start by examining the tiling properties of the boundary elements for $\{\widetilde{\psi}_{j,2^j,k} : j\ge 1, k\in\mathbb{Z}^2\}$. Namely, for any $f\in L^2(\mathbb{R}^2)$, we observe that

$$\sum_{k\in\mathbb{Z}^2} |\langle \hat{f}, (\widetilde{\psi}_{j,2^j,k})^\wedge\rangle|^2$$

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$$= \sum_{k\in\mathbb{Z}^2} \left| \int_{\mathcal{P}_1} 2^{-\frac{3}{2}j-\frac{1}{2}} \hat{f}(\xi)\, W(2^{-2j}\xi_1, 2^{-2j}\xi_2)\, v\Big(2^j(\tfrac{\xi_2}{\xi_1}-1)\Big)\, e^{2\pi i \xi 2^{-1} A_{(1)}^{-j} B_{(1)}^{(-2^j)} k}\, d\xi \right|^2$$
$$+ \sum_{k\in\mathbb{Z}^2} \left| \int_{\mathcal{P}_2} 2^{-\frac{3}{2}j-\frac{1}{2}} \hat{f}(\xi)\, W(2^{-2j}\xi_1, 2^{-2j}\xi_2)\, v\Big(2^j(\tfrac{\xi_1}{\xi_2}-1)\Big)\, e^{2\pi i \xi 2^{-1} A_{(1)}^{-j} B_{(1)}^{(-2^j)} k}\, d\xi \right|^2. \tag{2.12}$$

We will use the change of variable $\eta = \xi 2^{-1} A_{(1)}^{-j} B_{(1)}^{-2^j}$. Hence, for $\eta = (\eta_1, \eta_2)$, we have that $\xi = 2\eta B_{(1)}^{2^j} A_{(1)}^{j} = (2^{2j+1}\eta_1, 2^{2j+1}\eta_1 + 2^{j+1}\eta_2)$. Using this variable change, we obtain the following:

$$v\left(2^j \tfrac{\xi_2}{\xi_1} - 2^j\right) = v\left(\tfrac{\eta_2}{\eta_1}\right);$$

$$v\left(2^j \tfrac{\xi_1}{\xi_2} - 2^j\right) = v\left(2^j \tfrac{2^{2j+1}\eta_1}{2^{2j+1}\eta_1 + 2^{j+1}\eta_2} - 2^j\right) = v\left(\tfrac{-\eta_2}{\eta_1 + 2^{-j}\eta_2}\right);$$

$$W(2^{-2j}\xi_1, 2^{-2j}\xi_2) = W(2\eta_1, 2^{-j+1}(\eta_2 + 2^j\eta_1)).$$

By the conditions on the support of v and W, it follows that $v(\frac{\eta_2}{\eta_1})$ is supported inside the region $|\frac{\eta_2}{\eta_1}| \le 1$ and $W(2\eta_1, 2^{-j+1}(\eta_2 + 2^j\eta_1))$ is supported inside the region $|\eta_1| \le \frac{1}{4}$. Thus, the function

$$\Gamma_{1,j}(\eta_1, \eta_2) = W\left(2\eta_1, 2^{-j+1}(\eta_2 + 2^j\eta_1)\right)\, v\left(\tfrac{\eta_2}{\eta_1}\right)$$

is supported inside Q. Next consider the function

$$\Gamma_{2,j}(\eta_1, \eta_2) = W\left(2\eta_1, 2^{-j+1}(\eta_2 + 2^j\eta_1)\right)\, v\left(\tfrac{-\eta_2}{\eta_1 + 2^{-j}\eta_2}\right).$$

Similar to the case above, the assumption on the support of W implies that $\Gamma_{2,j}$ is supported inside the region $|\eta_1| \le \frac{1}{4}$. In addition, the support condition of v implies that, for $j \ge 1$,

$$\left|\frac{\eta_2}{\eta_1 + 2^{-j}\eta_2}\right| \le 1.$$

It follows that

$$\left|\frac{\eta_2}{\eta_1}\right| \le \left|1 + 2^{-j}\frac{\eta_2}{\eta_1}\right| \le 1 + 2^{-j}\left|\frac{\eta_2}{\eta_1}\right|,$$

so that

$$(1 - 2^{-j})\left|\frac{\eta_2}{\eta_1}\right| \le 1,$$

and finally

$$\left|\frac{\eta_2}{\eta_1}\right| \le (1 - 2^{-j})^{-1} \le 2 \quad \text{since } j \ge 1.$$

This shows that, if $|\eta_1| \le \frac{1}{4}$, then $|\eta_2| \le 2|\eta_1| \le \frac{1}{2}$. Thus, $\Gamma_{2,j}(\eta)$ is also supported inside Q for $j \ge 1$. Using these observations, from (2.12) we have that, for $j \ge 1$:

$$\sum_{k\in\mathbb{Z}^2} |\langle \hat{f}, (\widetilde{\psi}_{j,2^j,k})^\wedge \rangle|^2$$
$$= \sum_{k\in\mathbb{Z}^2} \left| \int_{\mathcal{P}_1} 2^{-\frac{3}{2}j-\frac{1}{2}} \hat{f}(\xi)\, W(2^{-2j}\xi_1, 2^{-2j}\xi_2)\, v\left(2^j(\tfrac{\xi_2}{\xi_1}-1)\right)\, e^{2\pi i \xi 2^{-1} A_{(1)}^{-j} B_{(1)}^{(-2^j)} k}\, d\xi \right|^2$$
$$+ \sum_{k\in\mathbb{Z}^2} \left| \int_{\mathcal{P}_2} 2^{-\frac{3}{2}j-\frac{1}{2}} \hat{f}(\xi)\, W(2^{-2j}\xi_1, 2^{-2j}\xi_2)\, v\left(2^j(\tfrac{\xi_1}{\xi_2}-1)\right)\, e^{2\pi i \xi 2^{-1} A_{(1)}^{-j} B_{(1)}^{(-2^j)} k}\, d\xi \right|^2$$

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$$
\begin{aligned}
&= \sum_{k\in\mathbb{Z}^2} \left| \int_Q 2^{\frac{3}{2}j+\frac{1}{2}} \hat{f}(2\eta B_{(1)}^{2^j} A_{(1)}^j)\, W(2\eta_1, 2^{-j+1}(\eta_2+2^j\eta_1))\, v\left(\tfrac{\eta_2}{\eta_1}\right)\, e^{2\pi i\eta k}\, d\eta \right|^2 \\
&+ \sum_{k\in\mathbb{Z}^2} \left| \int_Q 2^{\frac{3}{2}j+\frac{1}{2}} \hat{f}(2\eta B_{(1)}^{2^j} A_{(1)}^j)\, W(2\eta_1, 2^{-j+1}(\eta_2+2^j\eta_1))\, v\left(\tfrac{-\eta_2}{\eta_1+2^{-j}\eta_2}\right)\, e^{2\pi i\eta k}\, d\eta \right|^2 \\
&= \int_{\mathcal{P}_1 2^{-1} A_{(1)}^{-j} B_{(1)}^{(-2^j)}} 2^{3j+1} |\hat{f}(2\eta B_{(1)}^{\ell} A_{(1)}^j)|^2\, |W(2\eta_1, 2^{-j+1}(\eta_2+2^j\eta_1))|^2\, |v\left(\tfrac{\eta_2}{\eta_1}\right)|^2\, d\eta \\
&+ \int_{\mathcal{P}_2 2^{-1} A_{(1)}^{-j} B_{(1)}^{(-2^j)}} 2^{3j+1} |\hat{f}(2\eta B_{(1)}^{\ell} A_{(1)}^j)|^2\, |W(2\eta_1, 2^{-j+1}(\eta_2+2^j\eta_1))|^2\, |v\left(\tfrac{-\eta_2}{\eta_1+2^{-j}\eta_2}\right)|^2\, d\eta \\
&= \int_{\mathcal{P}_1} |\hat{f}(\xi)|^2\, |W(2^{-2j}\xi_1, 2^{-2j}\xi_2)|^2\, |v\left(2^j(\tfrac{\xi_2}{\xi_1}-1)\right)|^2\, d\xi \\
&+ \int_{\mathcal{P}_2} |\hat{f}(\xi)|^2\, |W(2^{-2j}\xi_1, 2^{-2j}\xi_2)|^2\, |v\left(2^j(\tfrac{\xi_1}{\xi_2}-1)\right)|^2\, d\xi. \qquad (2.13)
\end{aligned}
$$

For $j=0$ and $\ell=1$, since $\mathrm{supp}\,(W)\subset Q$, we have the following:

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}^2} |\langle \hat{f}, (\widetilde{\psi}_{0,1,k})^\wedge\rangle|^2 \\
&= \sum_{k\in\mathbb{Z}^2} \left| \int_{\mathcal{P}_1} \hat{f}(\xi)\, W(\xi_1,\xi_2)\, v(\tfrac{\xi_2}{\xi_1}-1)\, e^{2\pi i\xi k}\, d\xi \right|^2 + \left| \int_{\mathcal{P}_2} \hat{f}(\xi)\, W(\xi_1,\xi_2)\, v(\tfrac{\xi_1}{\xi_2}-1)\, e^{2\pi i\xi k}\, d\xi \right|^2 \\
&= \int_{\mathcal{P}_1} |\hat{f}(\xi)|^2\, |W(\xi_1,\xi_2)|^2\, |v(\tfrac{\xi_2}{\xi_1}-1)|^2\, d\xi + \int_{\mathcal{P}_2} |\hat{f}(\xi)|^2\, |W(\xi_1,\xi_2)|^2\, |v(\tfrac{\xi_1}{\xi_2}-1)|^2\, d\xi. \qquad (2.14)
\end{aligned}
$$

Using (2.13) and (2.14) into (2.12), we conclude that for any $f\in L^2(\mathbb{R}^2)$

$$
\begin{aligned}
&\sum_{j\geq 0}\sum_{k\in\mathbb{Z}^2} |\langle \hat{f}, (\widetilde{\psi}_{j,2^j,k})^\wedge\rangle|^2 \\
&= \sum_{j\geq 0} \int_{\mathcal{P}_1} |\hat{f}(\xi)|^2\, |W(2^{-2j}\xi_1, 2^{-2j}\xi_2)|^2\, |v\left(2^j(\tfrac{\xi_2}{\xi_1}-1)\right)|^2\, d\xi \\
&+ \int_{\mathcal{P}_2} |\hat{f}(\xi)|^2\, |W(2^{-2j}\xi_1, 2^{-2j}\xi_2)|^2\, |v\left(2^j(\tfrac{\xi_1}{\xi_2}-1)\right)|^2\, d\xi.
\end{aligned}
$$

An analogous result holds for the boundary elements $(\widetilde{\psi}_{j,-2^j,k})^\wedge$. Hence, using the observations above and (2.5), we now have that for any $f\in L^2(\mathbb{R}^2)$:

$$
\begin{aligned}
&\sum_{d=1}^{2}\sum_{j\geq 0}\sum_{|\ell|<2^j}\sum_{k\in\mathbb{Z}^2} |\langle f, \widetilde{\psi}_{j,\ell,k,d}\rangle|^2 + \sum_{j\geq 0}\sum_{\ell=\pm 2^j}\sum_{k\in\mathbb{Z}^2} |\langle f, \widetilde{\psi}_{j,\ell,k}\rangle|^2 \\
&= \sum_{j\geq 0}\sum_{|\ell|<2^j}\sum_{k\in\mathbb{Z}^2} |\langle \hat{f}, \hat{\psi}^{(1)}_{j,\ell,k}\rangle|^2 + \sum_{j\geq 0}\sum_{|\ell|<2^j}\sum_{k\in\mathbb{Z}^2} |\langle \hat{f}, \hat{\psi}^{(2)}_{j,\ell,k}\rangle|^2 + \sum_{j\geq 0}\sum_{\ell=\pm 2^j}\sum_{k\in\mathbb{Z}^2} |\langle f, (\widetilde{\psi}_{j,\ell,k})^\wedge\rangle|^2 \\
&= \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \sum_{j\geq 0} |W(2^{-2j}\xi)|^2 \left(\sum_{|\ell|<2^j} |v\left(2^j\tfrac{\xi_2}{\xi_1}-\ell\right)|^2 + \sum_{|\ell|<2^j} |v\left(2^j\tfrac{\xi_1}{\xi_2}-\ell\right)|^2 \right) d\xi \\
&+ \int_{\mathcal{P}_1} |\hat{f}(\xi)|^2 \sum_{j\geq 0} |W(2^{-2j}\xi)|^2\, |v\left(2^j(\tfrac{\xi_1}{\xi_2}-1)\right)|^2\, d\xi \\
&+ \int_{\mathcal{P}_2} |\hat{f}(\xi)|^2 \sum_{j\geq 0} |W(2^{-2j}\xi)|^2\, |v\left(2^j(\tfrac{\xi_1}{\xi_2}-1)\right)|^2\, d\xi
\end{aligned}
$$

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$$
\begin{aligned}
&+\int_{\mathcal{P}_1} |\hat{f}(\xi)|^2 \sum_{j\geq 0} |W(2^{-2j}\xi)|^2 \, |v\left(2^j(\tfrac{\xi_1}{\xi_2}+1)\right)|^2 \, d\xi \\
&+\int_{\mathcal{P}_2} |\hat{f}(\xi)|^2 \sum_{j\geq 0} |W(2^{-2j}\xi)|^2 \, |v\left(2^j(\tfrac{\xi_1}{\xi_2}+1)\right)|^2 \, d\xi \\
&= \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \sum_{j\geq 0} |W(2^{-2j}\xi)|^2 \sum_{|\ell|\leq 2^j} \left(|v\left(2^j \tfrac{\xi_2}{\xi_1} - \ell\right)|^2 \chi_{\mathcal{P}_1}(\xi) + |v\left(2^j \tfrac{\xi_1}{\xi_2} - \ell\right)|^2 \chi_{\mathcal{P}_2}(\xi) \right) d\xi \\
&= \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \sum_{j\geq 0} |W(2^{-2j}\xi)|^2 d\xi
\end{aligned}
$$

Finally, using (2.2), we have that for any $f \in L^2(\mathbb{R}^2)$:

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}^2} |\langle f, \widetilde{\psi}_{-1,k}\rangle|^2 + \sum_{d=1}^{2} \sum_{j\geq 0} \sum_{|\ell|<2^j} \sum_{k\in\mathbb{Z}^2} |\langle f, \widetilde{\psi}_{j,\ell,k,d}\rangle|^2 + \sum_{j\geq 0} \sum_{\ell=\pm 2^j} \sum_{k\in\mathbb{Z}^2} |\langle f, \widetilde{\psi}_{j,\ell,k}\rangle|^2 \\
&= \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \, |\Phi(\xi)|^2 d\xi + \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \sum_{j\geq 0} |W(2^{-2j}\xi)|^2 d\xi \\
&= \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \left(|\Phi(\xi)|^2 + \sum_{j\geq 0} |W(2^{-2j}\xi)|^2 \right) d\xi = \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \, d\xi. \quad \square
\end{aligned}
$$

3. A new shearlet construction ($D = 3$)

In this section, we extend the construction presented above to $D = 3$. While the main ideas of the construction are similar to the 2D case, there are several technical additional difficulties, especially for what concerns the construction of the boundary shearlet elements.

As in the 2D case, let ϕ be a C^∞ univariate function such that $0 \leq \hat{\phi} \leq 1$, $\hat{\phi} = 1$ on $[-\frac{1}{16}, \frac{1}{16}]$ and $\hat{\phi} = 0$ outside the interval $[-\frac{1}{8}, \frac{1}{8}]$. For $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, define

$$
\widehat{\Phi}(\xi) = \widehat{\Phi}(\xi_1, \xi_2, \xi_3) = \hat{\phi}(\xi_1) \, \hat{\phi}(\xi_2) \, \hat{\phi}(\xi_3) \tag{3.1}
$$

and

$$
W(\xi) = \sqrt{\widehat{\Phi}^2(2^{-2}\xi) - \widehat{\Phi}^2(\xi)}.
$$

It follows that

$$
\widehat{\Phi}^2(\xi) + \sum_{j\geq 0} W^2(2^{-2j}\xi) = 1 \ \text{ for } \xi \in \mathbb{R}^3. \tag{3.2}
$$

Notice that each function $W_j^2 = W^2(2^{-2j}\cdot)$, $j \in \mathbb{Z}$, has support into the Cartesian corona

$$
C_j = [-2^{2j-1}, 2^{2j-1}]^3 \setminus [-2^{2j-4}, 2^{2j-4}]^3 \subset \mathbb{R}^3.
$$

The functions W_j^2, $j \geq 0$, produce a smooth tiling of the frequency plane into Cartesian coronae, where

$$
\sum_{j\geq 0} W^2(2^{-2j}\xi) = 1 \quad \text{for } \xi \in \mathbb{R}^3 \setminus [-\frac{1}{8}, \frac{1}{8}]^3. \tag{3.3}
$$

We will define 3 shearlet systems associated with the following pyramidal regions in $\mathbb{R}^3$:

$$
\mathcal{P}_1 = \left\{ (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : |\frac{\xi_2}{\xi_1}| \leq 1, |\frac{\xi_3}{\xi_1}| \leq 1 \right\},
$$

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$$\mathcal{P}_2 = \left\{(\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3 : |\frac{\xi_1}{\xi_2}| \le 1, |\frac{\xi_3}{\xi_2}| \le 1\right\},$$
$$\mathcal{P}_3 = \left\{(\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3 : |\frac{\xi_1}{\xi_3}| \le 1, |\frac{\xi_2}{\xi_3}| \le 1\right\}.$$

Let the anisotropic dilation matrices be given by

$$A_{(1)} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \; A_{(2)} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \; A_{(3)} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

and, for $\ell = (\ell_1, \ell_2) \in \mathbb{Z}^2$, let the *shear matrices* be defined by

$$B^{[\ell]}_{(1)} = \begin{pmatrix} 1 & \ell_1 & \ell_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; B^{[\ell]}_{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ \ell_1 & 1 & \ell_2 \\ 0 & 0 & 1 \end{pmatrix}, \; B^{[\ell]}_{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \ell_1 & \ell_2 & 1 \end{pmatrix}.$$

Hence, using these notation, we have the following definition.

Definition 3.1. For $\xi = (\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3$ and $d = 1, 2, 3$, the 3D *shearlet systems* associated with the pyramidal regions $\mathcal{P}_d$ are defined as the collections

$$\{\psi^{(d)}_{j,\ell,k} : j \ge 0, -2^j \le \ell_1, \ell_2 \le 2^j, k \in \mathbb{Z}^3\}, \tag{3.4}$$

where

$$\hat{\psi}^{(d)}_{j,\ell,k}(\xi) = |\det A_{(d)}|^{-j/2} \, W(2^{-2j}\xi) \, V_{(d)}(\xi A^{-j}_{(d)} B^{[-\ell]}_{(d)}) \, e^{2\pi i \xi A^{-j}_{(d)} B^{[-\ell]}_{(d)} k}, \tag{3.5}$$

$V_{(1)}(\xi_1,\xi_2,\xi_3) = v(\frac{\xi_2}{\xi_1})v(\frac{\xi_3}{\xi_1})$, $V_{(2)}(\xi_1,\xi_2,\xi_3) = v(\frac{\xi_1}{\xi_2})v(\frac{\xi_3}{\xi_2})$, $V_{(3)}(\xi_1,\xi_2,\xi_3) = v(\frac{\xi_1}{\xi_3})v(\frac{\xi_2}{\xi_3})$, and v is defined as in Sec. 2, that is, it is a C^∞ functions supported inside $[-1,1]$, with $v(0) = 0$, $v^{(n)}(0) = 1$ for all $n \ge 1$ and satisfying (2.4). Notice that $(B^{[\ell]}_{(d)})^{-1} = B^{[-\ell]}_{(d)}$.

Similar to the 2D case, each 3D shearlet system (3.4), for $d = 1, 2, 3$, is a Parseval frame for $L^2(\widetilde{\mathcal{P}}_d)^\vee$, the subspace of L^2 functions whose frequency support is contained inside the truncated pyramidal regions $\widetilde{\mathcal{P}}_d = \mathcal{P}_d \cap \{(\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3 : |\xi_d| \ge \frac{1}{8}\}$.

Proposition 3.1. *For each $d = 1, 2, 3$, the system of shearlets* (3.4) *is a Parseval frame for $L^2(\widetilde{\mathcal{P}}_d)^\vee$.*

Proof. We only present the argument for the pyramidal region $\mathcal{P}_1$ since the other cases are very similar. Notice first that

$$(\xi_1,\xi_2,\xi_3) \, A^{-j}_{(1)} B^{[-\ell]}_{(1)} = (2^{-2j}\xi_1, -\ell_1 2^{-2j}\xi_1 + 2^{-j}\xi_2, -\ell_2 2^{-2j}\xi_1 + 2^{-j}\xi_3).$$

Hence, we can write the elements of the system of shearlets (3.4), for $d = 1$, as

$$\hat{\psi}^{(1)}_{j,\ell,k}(\xi_1,\xi_2,\xi_3) = 2^{-2j} \, W(2^{-2j}\xi_1, 2^{-2j}\xi_2, 2^{-2j}\xi_3) \, v\left(2^j \tfrac{\xi_2}{\xi_1} - \ell_1\right) v\left(2^j \tfrac{\xi_3}{\xi_1} - \ell_2\right) e^{2\pi i \xi A^{-j}_{(1)} B^{[-\ell]}_{(1)} k}.$$

Let $f \in L^2(\widetilde{\mathcal{P}}_1)^\vee$. Using the change of variable $\eta = \xi A^{-j}_{(1)} B^{[-\ell]}_{(1)}$ and the notation $Q = [-\frac{1}{2}, \frac{1}{2}]^3$, we have:

$$\sum_{k\in\mathbb{Z}^3} |\langle \hat{f}, \hat{\psi}^{(1)}_{j,\ell,k}\rangle|^2$$
$$= \sum_{k\in\mathbb{Z}^3} \left| \int_{\mathbb{R}^3} 2^{-2j} \hat{f}(\xi) W(2^{-2j}\xi_1, 2^{-2j}\xi_2, 2^{-2j}\xi_3) \, v\left(2^j \tfrac{\xi_2}{\xi_1} - \ell_1\right) v\left(2^j \tfrac{\xi_3}{\xi_1} - \ell_2\right) e^{2\pi i \xi A^{-j}_{(1)} B^{[-\ell]}_{(1)} k} \, d\xi \right|^2$$

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$$= \sum_{k\in\mathbb{Z}^3} \left| \int_Q 2^{2j}\, \hat{f}(\eta B_{(1)}^{\ell} A_{(1)}^{j})\, W(\eta_1, 2^{-j}(\eta_2+\ell_1\eta_1), 2^{-j}(\eta_3+\ell_2\eta_1))\, v(\tfrac{\eta_2}{\eta_1})\, v(\tfrac{\eta_3}{\eta_1})\, e^{2\pi i \eta k}\, d\eta \right|^2$$

$$= \int_Q 2^{3j} |\hat{f}(\eta B_{(1)}^{\ell} A_{(1)}^{j})|^2\, |W(\eta_1, 2^{-j}(\eta_2+\ell\eta_1))|^2\, |v(\tfrac{\eta_2}{\eta_1})|^2\, |v(\tfrac{\eta_3}{\eta_1})|^2\, d\eta$$

$$= \int_{\mathbb{R}^3} |\hat{f}(\xi)|^2\, |W(2^{-2j}\xi_1, 2^{-2j}\xi_2, 2^{-2j}\xi_3)|^2\, |v\Big(2^j \frac{\xi_2}{\xi_1} - \ell\Big)|^2\, |v\Big(2^j \frac{\xi_3}{\xi_1} - \ell\Big)|^2\, d\xi. \tag{3.6}$$

In the computation above, we have used the fact that the function

$$W\left(\eta_1, 2^{-j}(\eta_2+\ell_1\eta_1), 2^{-j}(\eta_3+\ell\eta_2)\right)\; v(\tfrac{\eta_2}{\eta_1})\, v(\tfrac{\eta_3}{\eta_1})$$

is supported inside Q since $v(\frac{\eta_2}{\eta_1})$ is supported inside the region $|\frac{\eta_2}{\eta_1}| \le 1$, $v(\frac{\eta_3}{\eta_1})$ is supported inside the region $|\frac{\eta_3}{\eta_1}| \le 1$ and $W(\eta_1, 2^{-j}(\eta_2+\ell_1\eta_1), 2^{-j}(\eta_3+\ell\eta_2))$ is supported inside the region $|\eta_1| \le \frac{1}{2}$.

Using (3.3) and (2.5) if follows that

$$\sum_{j\ge 0} \sum_{\ell_1=-2^j}^{2^j} \sum_{\ell_2=-2^j}^{2^j} |W(2^{-2j}\xi_1, 2^{-2j}\xi_2, 2^{-2j}\xi_3)|^2\, |v\left(2^j \tfrac{\xi_2}{\xi_1} - \ell_1\right)|^2\, |v\left(2^j \tfrac{\xi_3}{\xi_1} - \ell_2\right)|^2$$

$$= \sum_{j\ge 0} |W(2^{-2j}\xi_1, 2^{-2j}\xi_2, 2^{-2j}\xi_3)|^2 \sum_{\ell_1=-2^j}^{2^j} |v\left(2^j \tfrac{\xi_2}{\xi_1} - \ell_1\right)|^2 \sum_{\ell_1=-2^j}^{2^j} |v\left(2^j \tfrac{\xi_3}{\xi_1} - \ell_2\right)|^2$$

$$= 1 \quad \text{for } (\xi_1,\xi_2,\xi_3) \in \widetilde{\mathcal{P}}_1. \tag{3.7}$$

Thus, using (3.6) and (3.7), we conclude that, for any $f \in L^2(\widetilde{\mathcal{P}}_1)^\vee$, we have:

$$\sum_{j\ge 0} \sum_{\ell_1=-2^j}^{2^j} \sum_{\ell_2=-2^j}^{2^j} \sum_{k\in\mathbb{Z}^2} |\langle f, \psi^{(1)}_{j,\ell_1,\ell_2,k}\rangle|^2 = \|f\|^2. \quad \square$$

3.1. A smooth Parseval frame of shearlets for $L^2(\mathbb{R}^3)$

Similar to the 2D case, we will obtain a Parseval frame of shearlets for $L^2(\mathbb{R}^3)$ by using an appropriate combination of the shearlet systems associated with the 3 pyramidal regions $\mathcal{P}_d$, $d = 1, 2, 3$, together with a coarse scale system, which will take care of the low frequency region. To ensure that all elements of this combined shearlet system are C_c^∞ in the Fourier domain, we will introduce appropriate boundary elements corresponding to the boundaries of the 3 pyramidal regions. The main difference with respect to the 2D case is that there are two classes of such boundary shearlets, corresponding to boundary planes (where 2 pyramidal regions intersect) and boundary lines (where 3 pyramidal regions intersect), respectively. As will be shown in Sec. 4, the same type of situation, requiring different types of boundary shearlets, will occur in higher dimensions.

The new shearlet system for $L^2(\mathbb{R}^3)$ is defined as the collection

$$\left\{\widetilde{\psi}_{-1,k} : k \in \mathbb{Z}^3\right\} \bigcup \left\{\widetilde{\psi}_{j,\ell,k,d} : j \ge 0, |\ell_1| \le 2^j, |\ell_2| < 2^j, k \in \mathbb{Z}^3, d = 1,2,3\right\}$$
$$\bigcup \left\{\widetilde{\psi}_{j,\ell,k} : j \ge 0, \ell_1, \ell_2 = \pm 2^j, k \in \mathbb{Z}^3\right\} \tag{3.8}$$

consisting of:

- the *coarse scale shearlets* $\{\widetilde{\psi}_{-1,k} = \Phi(\cdot - k) : k \in \mathbb{Z}^3\}$, where Φ is given by (3.1);
- the *interior shearlets* $\{\widetilde{\psi}_{j,\ell,k,d} = \psi^{(d)}_{j,\ell,k} : j \ge 0, |\ell_1||\ell_2| < 2^j, k \in \mathbb{Z}^3, d = 1,2,3\}$, where the functions $\psi^{(d)}_{j,\ell,k}$ are given by (3.5);

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– the *boundary shearlets* $\{\widetilde{\psi}_{j,\ell,k,d} : j \geq 0, \ell_1 = \pm 2^j, |\ell_2| < 2^j, k \in \mathbb{Z}^3, d = 1,2,3\}$ and $\{\widetilde{\psi}_{j,\ell,k} : j \geq 0, \ell_1, \ell_2 = \pm 2^j, k \in \mathbb{Z}^3\}$, obtained by joining together slightly modified versions of $\psi^{(1)}_{j,\ell,k}$, $\psi^{(2)}_{j,\ell,k}$ and $\psi^{(3)}_{j,\ell,k}$, for $\ell_1, \ell_2 = \pm 2^j$, after they have been restricted in the Fourier domain to the pyramidal regions $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$, respectively. Their precise definition is given below.

For $j \geq 1$, $\ell_1 = \pm 2^j$, $|\ell_2| < 2^j$, we define

$$(\widetilde{\psi}_{j,\ell_1,\ell_2,k,1})^\wedge(\xi) = \begin{cases} 2^{-2j-3} W(2^{-2j}\xi)\, v(2^j \frac{\xi_2}{\xi_1} - \ell_1)\, v(2^j \frac{\xi_3}{\xi_1} - \ell_2)\, e^{2\pi i \xi 2^{-2} A_{(1)}^{-j} B_{(1)}^{[-(\ell_1,\ell_2)]} k}, & \text{if } \xi \in \mathcal{P}_1 \\ 2^{-2j-3} W(2^{-2j}\xi)\, v(2^j \frac{\xi_1}{\xi_2} - \ell_1)\, v(2^j \frac{\xi_3}{\xi_2} - \ell_2)\, e^{2\pi i \xi 2^{-2} A_{(1)}^{-j} B_{(1)}^{[-(\ell_1,\ell_2)]} k}, & \text{if } \xi \in \mathcal{P}_2. \end{cases} \tag{3.9}$$

$$(\widetilde{\psi}_{j,\ell_1,\ell_2,k,2})^\wedge(\xi) = \begin{cases} 2^{-2j-3} W(2^{-2j}\xi)\, v(2^j \frac{\xi_1}{\xi_2} - \ell_2)\, v(2^j \frac{\xi_3}{\xi_2} - \ell_1)\, e^{2\pi i \xi 2^{-2} A_{(2)}^{-j} B_{(2)}^{[-(\ell_2,\ell_1)]} k}, & \text{if } \xi \in \mathcal{P}_2 \\ 2^{-2j-3} W(2^{-2j}\xi)\, v(2^j \frac{\xi_1}{\xi_3} - \ell_2)\, v(2^j \frac{\xi_2}{\xi_3} - \ell_1)\, e^{2\pi i \xi 2^{-2} A_{(2)}^{-j} B_{(2)}^{[-(\ell_2,\ell_1)]} k}, & \text{if } \xi \in \mathcal{P}_3. \end{cases}$$

$$(\widetilde{\psi}_{j,\ell_1,\ell_2,k,3})^\wedge(\xi) = \begin{cases} 2^{-2j-3} W(2^{-2j}\xi)\, v(2^j \frac{\xi_2}{\xi_1} - \ell_2)\, v(2^j \frac{\xi_3}{\xi_1} - \ell_1)\, e^{2\pi i \xi 2^{-2} A_{(3)}^{-j} B_{(3)}^{[-(\ell_1,\ell_2)]} k}, & \text{if } \xi \in \mathcal{P}_1 \\ 2^{-2j-3} W(2^{-2j}\xi)\, v(2^j \frac{\xi_1}{\xi_3} - \ell_1)\, v(2^j \frac{\xi_2}{\xi_3} - \ell_2)\, e^{2\pi i \xi 2^{-2} A_{(3)}^{-j} B_{(3)}^{[-(\ell_1,\ell_2)]} k}, & \text{if } \xi \in \mathcal{P}_3. \end{cases}$$

For $j \geq 1$, $\ell_1, \ell_2 = \pm 2^j$, we define

$$(\widetilde{\psi}_{j,\ell_1,\ell_2,k})^\wedge(\xi) = \begin{cases} 2^{-2j-3} W(2^{-2j}\xi)\, v(2^j \frac{\xi_2}{\xi_1} - \ell_1)\, v(2^j \frac{\xi_3}{\xi_1} - \ell_2)\, e^{2\pi i \xi 2^{-2} A_{(1)}^{-j} B_{(1)}^{[-(\ell_1,\ell_2)]} k}, & \text{if } \xi \in \mathcal{P}_1 \\ 2^{-2j-3} W(2^{-2j}\xi)\, v(2^j \frac{\xi_1}{\xi_2} - \ell_1)\, v(2^j \frac{\xi_3}{\xi_2} - \ell_2)\, e^{2\pi i \xi 2^{-2} A_{(1)}^{-j} B_{(1)}^{[-(\ell_1,\ell_2)]} k}, & \text{if } \xi \in \mathcal{P}_2, \\ 2^{-2j-3} W(2^{-2j}\xi)\, v(2^j \frac{\xi_1}{\xi_3} - \ell_1)\, v(2^j \frac{\xi_2}{\xi_3} - \ell_2)\, e^{2\pi i \xi 2^{-2} A_{(1)}^{-j} B_{(1)}^{[-(\ell_1,\ell_2)]} k}, & \text{if } \xi \in \mathcal{P}_3. \end{cases}$$

Similarly, for $j = 0$, $\ell_1 = \pm 1$, we define

$$(\widetilde{\psi}_{0,\ell_1,0,k,1})^\wedge(\xi) = \begin{cases} W(\xi)\, v(\frac{\xi_2}{\xi_1} - \ell_1)\, v(\frac{\xi_3}{\xi_1})\, e^{2\pi i \xi k}, & \text{if } \xi \in \mathcal{P}_1 \\ W(\xi)\, v(\frac{\xi_1}{\xi_2} - \ell_1)\, v(\frac{\xi_3}{\xi_2})\, e^{2\pi i \xi k}, & \text{if } \xi \in \mathcal{P}_2. \end{cases}$$

$$(\widetilde{\psi}_{0,\ell_1,0,k,2})^\wedge(\xi) = \begin{cases} W(\xi)\, v(\frac{\xi_1}{\xi_2})\, v(\frac{\xi_3}{\xi_2} - \ell_1)\, e^{2\pi i \xi k}, & \text{if } \xi \in \mathcal{P}_2 \\ W(\xi)\, v(\frac{\xi_1}{\xi_3})\, v(\frac{\xi_2}{\xi_3} - \ell_1)\, e^{2\pi i \xi k}, & \text{if } \xi \in \mathcal{P}_3. \end{cases}$$

$$(\widetilde{\psi}_{0,\ell_1,0,k,3})^\wedge(\xi) = \begin{cases} W(\xi)\, v(\frac{\xi_2}{\xi_1})\, v(\frac{\xi_3}{\xi_1} - \ell_1)\, e^{2\pi i \xi k}, & \text{if } \xi \in \mathcal{P}_1 \\ W(\xi)\, v(\frac{\xi_1}{\xi_3} - \ell_1)\, v(\frac{\xi_2}{\xi_3})\, e^{2\pi i \xi k}, & \text{if } \xi \in \mathcal{P}_3. \end{cases}$$

For $j = 0$, $\ell_1 = \pm 1$ and $\ell_2 = \pm 1$, we define

$$(\widetilde{\psi}_{0,\ell_1,\ell_2,k})^\wedge(\xi) = \begin{cases} W(\xi)\, v(\frac{\xi_2}{\xi_1} - \ell_1)\, v(\frac{\xi_3}{\xi_1} - \ell_2)\, e^{2\pi i \xi k}, & \text{if } \xi \in \mathcal{P}_1 \\ W(\xi)\, v(\frac{\xi_1}{\xi_2} - \ell_1)\, v(\frac{\xi_3}{\xi_2} - \ell_2)\, e^{2\pi i \xi k}, & \text{if } \xi \in \mathcal{P}_2 \\ W(\xi)\, v(\frac{\xi_1}{\xi_3} - \ell_1)\, v(\frac{\xi_2}{\xi_3} - \ell_2)\, e^{2\pi i \xi k}, & \text{if } \xi \in \mathcal{P}_3. \end{cases}$$

Note that the boundary shearlet functions are compactly supported in the Fourier domain by construction. In addition, it can be shown that they are C^∞ in the Fourier domain. In fact, let us consider the function $(\widetilde{\psi}_{j,2^j,\ell_2,k})^\wedge$, given by (3.9). To show that it is continuous, it is easy to verify that the two terms of the piecewise defined functions are equal when $\xi_1 = \xi_2$ and $\xi_1 = \xi_3$. The smoothness is verified by checking that the derivatives of these functions on the plane $\xi_1 = \xi_2 = \xi_3$ are continuous. Similarly to the 2D argument, we observe that the two terms of the piecewise defined functions $(\widetilde{\psi}_{j,2^j,\ell_2,k})^\wedge$ only differ by $v(2^j(\frac{\xi_2}{\xi_1} - 1))\, v(2^j \frac{\xi_3}{\xi_1} - \ell_2)$ and $v(2^j(\frac{\xi_1}{\xi_2} - 1))\, v(2^j \frac{\xi_3}{\xi_2} - \ell_2)$. Since $v^{(n)}(0) = 0$ for all $n \geq 1$, it follows that all derivatives of these functions are equal when $\xi_1 = \xi_2 = \xi_3$. This implies that the functions $(\widetilde{\psi}_{j,\ell_1,\ell_2,k})^\wedge(\xi)$, given by (3.9), are infinitely differentiable. A similar computation shows that all boundary shearlets are infinitely differentiable.

We have the following result.

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Theorem 3.1. *The 3D shearlets system* (3.8) *is a Parseval frame for* $L^2(\mathbb{R}^3)$. *Furthermore, the elements of this systems are* C^∞ *and compactly supported in the Fourier domain.*

Proof. The interior and coarse scale shearlets are clearly C^∞ and compactly supported in the Fourier domain by construction. As discussed above, the boundary shearlets also satisfy this property. Thus, it only remains to prove that the system of shearlets (3.8) is a Parseval frame.

We will first examine the tiling properties of the boundary shearlets starting with the functions $\{\widetilde{\psi}_{j,2^j,\ell_2,k,1} : j \geq 0, |\ell_2| < 2^j, k \in \mathbb{Z}^3\}$. For $f \in L^2(\mathbb{R}^3)$ we have:

$$\begin{aligned}
&\sum_{j\geq 0}\sum_{k\in\mathbb{Z}^3} |\langle \hat{f}, (\widetilde{\psi}_{j,2^j,\ell_2,k,1})^\wedge\rangle|^2 \\
&= \sum_{j\geq 0}\sum_{k\in\mathbb{Z}^3} \left| \int_{\mathcal{P}_1} 2^{-2j-3}\hat{f}(\xi)\, W(2^{-2j}\xi)\, v\left(2^j(\tfrac{\xi_2}{\xi_1}-1)\right) v\left(2^j\tfrac{\xi_3}{\xi_1}-\ell_2\right) e^{2\pi i\xi 2^{-2}A_{(1)}^{-j}B_{(1)}^{[-2^j,-\ell_2]}k}\, d\xi \right|^2 \\
&+ \sum_{j\geq 0}\sum_{k\in\mathbb{Z}^3} \left| \int_{\mathcal{P}_2} 2^{-2j-3}\hat{f}(\xi)\, W(2^{-2j}\xi)\, v\left(2^j(\tfrac{\xi_1}{\xi_2}-1)\right) v\left(2^j\tfrac{\xi_3}{\xi_2}-\ell_2\right) e^{2\pi i\xi 2^{-2}A_{(1)}^{-j}B_{(1)}^{[-2^j,-\ell_2]}k}\, d\xi \right|^2
\end{aligned} \tag{3.10}$$

We will use the change of variable $\eta = \xi 2^{-2}A_{(1)}^{-j}B_{(1)}^{[-2^j,-\ell_2]}$ and hence $\xi = 2^2\eta B_{(1)}^{[2^j,\ell_2]}A_{(1)}^{j} = (2^{2j+2}\eta_1, 2^{2j+2}\eta_1 + 2^{j+2}\eta_2, \ell_2 2^{j+2}\eta_1 + 2^{j+2}\eta_3)$. Using this change of variable we have

$$v\left(2^j(\tfrac{\xi_2}{\xi_1}-1)\right) v\left(2^j(\tfrac{\xi_3}{\xi_1}-1)\right) = v\left(\tfrac{\eta_2}{\eta_1}\right) v\left(\tfrac{\eta_3}{\eta_1}\right),$$

$$v\left(2^j(\tfrac{\xi_1}{\xi_2}-1)\right) = v\left(2^j\tfrac{2^{2j+2}\eta_1}{2^{2j+2}\eta_1+2^{j+2}\eta_2}-2^j\right) = v\left(\tfrac{-\eta_1}{\eta_1+2^{-j}\eta_2}\right),$$

$$v\left(2^j\tfrac{\xi_3}{\xi_2}-\ell_2\right) = v\left(2^j\tfrac{\ell_2 2^{j+2}\eta_1+2^{j+2}\eta_3}{2^{2j+2}\eta_1+2^{j+1}\eta_2}-\ell_2\right) = v\left(\tfrac{2^j\eta_3-\ell_2\eta_2}{2^j\eta_1+\eta_2}\right),$$

$$W(2^{-2j}\xi_1, 2^{-2j}\xi_2, 2^{-2j}\xi_3) = W(2^2\eta_1, 2^{-j+2}(\eta_2+2^j\eta_1), 2^{-j+2}(\eta_3+\ell_2\eta_1)).$$

By the conditions on the support of v and W, it follows that $v(\frac{\eta_2}{\eta_1})\, v(\frac{\eta_3}{\eta_1})$ is supported inside the cone $\mathcal{P}_1$ and $W(2^2\eta_1, 2^{-j+2}(\eta_2+2^j\eta_1), 2^{-j+2}(\eta_3+\ell_2\eta_1))$ is supported inside the region $|\eta_1| \leq \frac{1}{8}$. Thus, the function

$$\Gamma_{1,j}(\eta) = W(2^2\eta_1, 2^{-j+2}(\eta_2+2^j\eta_1), 2^{-j+2}(\eta_3+\ell_2\eta_1))\, v\left(\tfrac{\eta_2}{\eta_1}\right) v\left(\tfrac{\eta_3}{\eta_1}\right)$$

is supported inside Q. Next, consider the function

$$\Gamma_{2,j}(\eta) = W(2^2\eta_1, 2^{-j+2}(\eta_2+2^j\eta_1), 2^{-j+2}(\eta_3+\ell_2\eta_1))\, v\left(\tfrac{-\eta_2}{\eta_1+2^{-j}\eta_2}\right) v\left(\tfrac{2^j\eta_3-\ell_2\eta_2}{2^j\eta_1+\eta_2}\right).$$

We will show that also the support of $\Gamma_{2,j}$ is contained inside Q. In fact, the assumption on the support of W implies that $|\eta_1| \leq \frac{1}{8}$. In addition, the support condition of v implies that

$$\left|\frac{\eta_2}{\eta_1+2^{-j}\eta_2}\right| \leq 1.$$

It follows that

$$\left|\frac{\eta_2}{\eta_1}\right| \leq \left|1+2^{-j}\frac{\eta_2}{\eta_1}\right| \leq 1+2^{-j}\left|\frac{\eta_2}{\eta_1}\right|,$$

so that

$$(1-2^{-j})\left|\frac{\eta_2}{\eta_1}\right| \leq 1,$$

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and finally

$$\left|\frac{\eta_2}{\eta_1}\right| \le (1-2^{-j})^{-1} \le 2 \quad \text{for } j \ge 1.$$

This shows that, if $|\eta_1| \le \frac{1}{8}$, then $|\eta_2| \le 2|\eta_1| \le \frac{1}{2}$. Again, by the support condition on v, we have that

$$\left|\frac{2^j\eta_3 - \ell_2\eta_2}{2^j\eta_1+\eta_2}\right| \le 1.$$

It follows that

$$\left|\frac{\eta_3}{\eta_1} - 2^{-j}\ell_2\frac{\eta_2}{\eta_1}\right| \le \left|1+2^{-j}\frac{\eta_2}{\eta_1}\right| \le 1+2^{-j}\left|\frac{\eta_2}{\eta_1}\right|.$$

Hence

$$\left|\frac{\eta_3}{\eta_1}\right| \le 1+2^{-j}\left|\frac{\eta_2}{\eta_1}\right| + 2^{-j}|\ell_2|\left|\frac{\eta_2}{\eta_1}\right| \le 4 \quad \text{since } \ j\ge 1, \ \ |\frac{\eta_2}{\eta_1}| \le 2.$$

Combined with the observation above, this shows that, if $|\eta_1| \le \frac{1}{8}$, then $|\eta_3| \le \frac{1}{2}$. Thus, also $\Gamma_{2,j}(\eta)$ is supported inside Q, for each $j \ge 1$. Using these observations, from (3.10) we have that, for $j \ge 1$, $|\ell_2| < 2^j$:

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^3} |\langle \hat{f}, (\widetilde{\psi}_{j,2^j,\ell_2,k,1})^\wedge\rangle|^2 \\
&= \sum_{k\in\mathbb{Z}^3}\left|\int_{\mathcal{P}_1} 2^{-2j-3}\hat{f}(\xi)\,W(2^{-2j}\xi)\,v\left(2^j(\tfrac{\xi_2}{\xi_1}-1)\right)\,v\left(2^j\tfrac{\xi_3}{\xi_1}-\ell_2\right)\,e^{2\pi i\xi 2^{-2}A_{(1)}^{-j}B_{(1)}^{[-2^j,-\ell_2]}k}\,d\xi\right|^2 \\
&+ \sum_{k\in\mathbb{Z}^3}\left|\int_{\mathcal{P}_2} 2^{-2j-3}\hat{f}(\xi)\,W(2^{-2j}\xi)\,v\left(2^j(\tfrac{\xi_1}{\xi_2}-1)\right)\,v\left(2^j\tfrac{\xi_3}{\xi_2}-\ell_2\right)\,e^{2\pi i\xi 2^{-2}A_{(1)}^{-j}B_{(1)}^{[-2^j,-\ell_2]}k}\,d\xi\right|^2 \\
&= \sum_{k\in\mathbb{Z}^3}\left|\int_Q 2^{2j+3}\hat{f}(2^2\eta B_{(1)}^{2^j}A_{(1)}^j)W(2^2\eta_1, 2^{2-j}(\eta_2+2^j\eta_1), 2^{2-j}(\eta_3+\ell_2\eta_1))v(\tfrac{\eta_2}{\eta_1})v(\tfrac{\eta_3}{\eta_1})e^{2\pi i\eta k}d\eta\right|^2 \\
&+ \sum_{k\in\mathbb{Z}^3}\left|\int_Q 2^{2j+3}\hat{f}(2^2\eta B_{(1)}^{2^j}A_{(1)}^j)\,W(2^2\eta_1, 2^{2-j}(\eta_2+2^j\eta_1), 2^{2-j}(\eta_3+\ell_2\eta_1))\,v\left(\tfrac{-\eta_2}{\eta_1+2^{-j}\eta_2}\right)\right. \\
&\left.\times\, v\left(\tfrac{2^j\eta_3-\ell_2\eta_2}{2^j\eta_1+\eta_2}\right)\,e^{2\pi i\eta k}\,d\eta\right|^2 \\
&= \int_{\mathcal{P}_1 2^{-2}A_{(1)}^{-j}(B_{(1)}^{[2^j,\ell_2]})^{-1}} 2^{4j+6}|\hat{f}(2^2\eta B_{(1)}^{\ell}A_{(1)}^j)|^2\,|W(2^2\eta_1, 2^{2-j}(\eta_2+2^j\eta_1), 2^{2-j}(\eta_3+\ell_2\eta_1))|^2 \\
&\times|v\left(\tfrac{\eta_2}{\eta_1}\right)|^2\,|v\left(\tfrac{\eta_3}{\eta_1}\right)|^2\,d\eta \\
&+ \int_{\mathcal{P}_2 2^{-2}A_{(1)}^{-j}(B_{(1)}^{[2^j,\ell_2]})^{-1}} 2^{4j+6}|\hat{f}(2^2\eta B_{(1)}^{\ell}A_{(1)}^j)|^2\,|W(2^2\eta_1, 2^{2-j}(\eta_2+2^j\eta_1), 2^{2-j}(\eta_3+\ell_2\eta_1))|^2 \\
&\times|v\left(\tfrac{-\eta_2}{\eta_1+2^{-j}\eta_2}\right)|^2\,|v\left(\tfrac{2^j\eta_3-\ell_2\eta_2}{2^j\eta_1+\eta_2}\right)|^2\,d\eta \\
&= \int_{\mathcal{P}_1} |\hat{f}(\xi)|^2\,|W(2^{-2j}\xi)|^2\,|v\left(2^j(\tfrac{\xi_2}{\xi_1}-1)\right)|^2\,|v\left(2^j\tfrac{\xi_3}{\xi_1}-\ell_2\right)|^2\,d\xi \\
&+ \int_{\mathcal{P}_2} |\hat{f}(\xi)|^2\,|W(2^{-2j}\xi)|^2\,|v\left(2^j(\tfrac{\xi_1}{\xi_2}-1)\right)|^2\,|v\left(2^j\tfrac{\xi_3}{\xi_1}-\ell_2\right)|^2\,d\xi. \qquad (3.11)
\end{aligned}$$

For $j=0$, since $\mathrm{supp}\,(W) \subset Q$ we have that

$$\sum_{k\in\mathbb{Z}^3} |\langle \hat{f}, (\widetilde{\psi}_{0,1,0,k,1})^\wedge\rangle|^2 \ = \sum_{k\in\mathbb{Z}^3}\left|\int_{\mathcal{P}_1} \hat{f}(\xi)\,W(\xi)\,v\left(\tfrac{\xi_2}{\xi_1}-1\right)\,v\left(\tfrac{\xi_3}{\xi_1}\right)\,e^{2\pi i\xi k}\,d\xi\right|^2$$

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$$+\sum_{k\in\mathbb{Z}^3}\left|\int_{\mathcal{P}_2}\hat{f}(\xi)\,W(\xi_1)\,v\left(\tfrac{\xi_1}{\xi_2}-1\right)\,v\left(\tfrac{\xi_3}{\xi_2}\right)\,e^{2\pi i\xi k}\,d\xi\right|^2$$
$$=\int_{\mathcal{P}_1}|\hat{f}(\xi)|^2\,|W(\xi)|^2\,|v\left(\tfrac{\xi_2}{\xi_1}-1\right)|^2\,|v\left(\tfrac{\xi_3}{\xi_1}\right)|^2\,d\xi$$
$$+\int_{\mathcal{P}_2}|\hat{f}(\xi)|^2\,|W(\xi)|^2\,|v\left(\tfrac{\xi_1}{\xi_2}-1\right)|^2\,|v\left(\tfrac{\xi_3}{\xi_2}\right)|^2\,d\xi, \tag{3.12}$$

for any $f\in L^2(\mathbb{R}^3)$. Using (3.11) and (3.12) into (3.10), we conclude that, for $|\ell_2|<2^j$,

$$\sum_{j\geq 0}\sum_{k\in\mathbb{Z}^3}|\langle\hat{f},(\widetilde{\psi}_{j,2^j,\ell_2,k,1})^\wedge\rangle|^2$$
$$=\sum_{j\geq 0}\int_{\mathcal{P}_1}|\hat{f}(\xi)|^2\,|W(2^{-2j}\xi)|^2\,|v\left(2^j(\tfrac{\xi_2}{\xi_1}-1)\right)|^2\,|v\left(2^j\tfrac{\xi_3}{\xi_1}-\ell_2\right)|^2\,d\xi$$
$$+\sum_{j\geq 0}\int_{\mathcal{P}_2}|\hat{f}(\xi)|^2\,|W(2^{-2j}\xi)|^2\,|v\left(2^j(\tfrac{\xi_1}{\xi_2}-1)\right)|^2\,|v\left(2^j\tfrac{\xi_3}{\xi_2}-\ell_2\right)|^2\,d\xi, \tag{3.13}$$

with a similar result holding for $\widetilde{\psi}_{j,-2^j,\ell_2,k}$.

Similar computations to the one above show that, for $\ell_1=\pm 1$, $|\ell_2|<2^j$,

$$\sum_{j\geq 0}\sum_{k\in\mathbb{Z}^3}|\langle\hat{f},(\widetilde{\psi}_{j,\ell_1,\ell_2,k,2})^\wedge\rangle|^2$$
$$=\sum_{j\geq 0}\int_{\mathcal{P}_2}|\hat{f}(\xi)|^2\,|W(2^{-2j}\xi)|^2\,|v\left(\tfrac{\xi_1}{\xi_2}-\ell_2\right)|^2\,|v\left(2^j\tfrac{\xi_3}{\xi_2}-\ell_1\right)|^2\,d\xi$$
$$+\sum_{j\geq 0}\int_{\mathcal{P}_3}|\hat{f}(\xi)|^2\,|W(2^{-2j}\xi)|^2\,|v\left(2^j\tfrac{\xi_1}{\xi_3}-\ell_2\right)|^2\,|v\left(2^j\tfrac{\xi_2}{\xi_3}-\ell_1\right)|^2\,d\xi \tag{3.14}$$

and that

$$\sum_{j\geq 0}\sum_{k\in\mathbb{Z}^3}|\langle\hat{f},(\widetilde{\psi}_{j,\ell_1,\ell_2,k,3})^\wedge\rangle|^2$$
$$=\sum_{j\geq 0}\int_{\mathcal{P}_1}|\hat{f}(\xi)|^2\,|W(2^{-2j}\xi)|^2\,|v\left(\tfrac{\xi_2}{\xi_1}-\ell_2\right)|^2\,|v\left(2^j\tfrac{\xi_3}{\xi_1}-\ell_1\right)|^2\,d\xi$$
$$+\sum_{j\geq 0}\int_{\mathcal{P}_3}|\hat{f}(\xi)|^2\,|W(2^{-2j}\xi)|^2\,|v\left(2^j\tfrac{\xi_1}{\xi_3}-\ell_1\right)|^2\,|v\left(2^j\tfrac{\xi_2}{\xi_3}-\ell_2\right)|^2\,d\xi. \tag{3.15}$$

A similar computation, for $j\geq 1$, $\ell_1,\ell_2=\pm 2^j$ yields:

$$\sum_{k\in\mathbb{Z}^3}|\langle\hat{f},(\widetilde{\psi}_{j,\ell_1,\ell_2,k})^\wedge\rangle|^2$$
$$=\int_{\mathcal{P}_1}|\hat{f}(\xi)|^2\,|W(2^{-2j}\xi)|^2\,|v\left(2^j\tfrac{\xi_2}{\xi_1}-\ell_1\right)|^2\,|v\left(2^j\tfrac{\xi_3}{\xi_1}-\ell_2\right)|^2\,d\xi$$
$$+\int_{\mathcal{P}_2}|\hat{f}(\xi)|^2\,|W(2^{-2j}\xi)|^2\,|v\left(2^j\tfrac{\xi_1}{\xi_2}-\ell_1\right)|^2\,|v\left(2^j\tfrac{\xi_3}{\xi_1}-\ell_2\right)|^2\,d\xi$$
$$+\int_{\mathcal{P}_3}|\hat{f}(\xi)|^2\,|W(2^{-2j}\xi)|^2\,|v\left(2^j\tfrac{\xi_1}{\xi_3}-\ell_1\right)|^2\,|v\left(2^j\tfrac{\xi_2}{\xi_3}-\ell_2\right)|^2\,d\xi. \tag{3.16}$$

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Similarly, for $j = 0$, $\ell_1, \ell_2 = \pm 1$,

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^3} |\langle \hat{f}, (\widetilde{\psi}_{0,\ell_1,\ell_2,k})^\wedge\rangle|^2 \\
&= \int_{\mathcal{P}_1} |\hat{f}(\xi)|^2\, |W(\xi)|^2\, |v\left(\tfrac{\xi_2}{\xi_1} - \ell_1\right)|^2\, |v\left(\tfrac{\xi_3}{\xi_1} - \ell_2\right)|^2\, d\xi \\
&+ \int_{\mathcal{P}_2} |\hat{f}(\xi)|^2\, |W(\xi)|^2\, |v\left(\tfrac{\xi_1}{\xi_2} - \ell_1\right)|^2\, |v\left(\tfrac{\xi_3}{\xi_2} - \ell_2\right)|^2\, d\xi \\
&+ \int_{\mathcal{P}_3} |\hat{f}(\xi)|^2\, |W(\xi)|^2\, |v\left(\tfrac{\xi_1}{\xi_3} - \ell_1\right)|^2\, |v\left(\tfrac{\xi_2}{\xi_3} - \ell_2\right)|^2\, d\xi. \qquad (3.17)
\end{aligned}$$

Using (3.16) and (3.17), we conclude that, for $\ell_1, \ell_2 = \pm 2^j$,

$$\begin{aligned}
&\sum_{j\geq 0}\sum_{k\in\mathbb{Z}^3} |\langle \hat{f}, (\widetilde{\psi}_{j,\ell_1,\ell_2,k})^\wedge\rangle|^2 \\
&= \sum_{j\geq 0}\int_{\mathcal{P}_1} |\hat{f}(\xi)|^2\, |W(2^{-2j}\xi)|^2\, |v\left(2^j\tfrac{\xi_2}{\xi_1} - \ell_1\right)|^2\, |v\left(2^j\tfrac{\xi_3}{\xi_1} - \ell_2\right)|^2\, d\xi \\
&+ \sum_{j\geq 0}\int_{\mathcal{P}_2} |\hat{f}(\xi)|^2\, |W(2^{-2j}\xi)|^2\, |v\left(2^j\tfrac{\xi_1}{\xi_2} - \ell_1\right)|^2\, |v\left(2^j\tfrac{\xi_3}{\xi_1} - \ell_2\right)|^2\, d\xi \\
&+ \sum_{j\geq 0}\int_{\mathcal{P}_3} |\hat{f}(\xi)|^2\, |W(2^{-2j}\xi)|^2\, |v\left(2^j\tfrac{\xi_1}{\xi_3} - \ell_1\right)|^2\, |v\left(2^j\tfrac{\xi_2}{\xi_3} - \ell_2\right)|^2\, d\xi. \qquad (3.18)
\end{aligned}$$

Finally, using the same argument as in the proof of Proposition 3.1 it is easy to verify that for any $f \in L^2(\mathbb{R}^3)$:

$$\begin{aligned}
&\sum_{j\geq 0}\sum_{|\ell_1|,|\ell_2|<2^j}\sum_{k\in\mathbb{Z}^3} |\langle f, \psi^{(1)}_{j,\ell_1,\ell_2,k}\rangle|^2 \\
&= \int_{\mathbb{R}^3} |\hat{f}(\xi)|^2 \sum_{j\geq 0} |W(2^{-2j}\xi)|^2 \sum_{|\ell_1|,|\ell_2|<2^j} |v\left(2^j\tfrac{\xi_2}{\xi_1} - \ell_1\right)|^2 |v\left(2^j\tfrac{\xi_3}{\xi_1} - \ell_2\right)|^2\, d\xi,
\end{aligned}$$

with a similar result holding for the interior shearlets $\{\psi^{(d)}_{j,\ell_1,\ell_2,k}\}$, $d = 2, 3$. Thus, using the last observation together with equalities (3.13), (3.14), (3.15) and (3.18), we now have that for any $f \in L^2(\mathbb{R}^3)$:

$$\begin{aligned}
&\sum_{d=1}^{3}\sum_{j\geq 0}\sum_{|\ell_1|\leq 2^j}\sum_{|\ell_2|<2^j}\sum_{k\in\mathbb{Z}^3} |\langle f, \widetilde{\psi}_{j,\ell_1,\ell_2,k,d}\rangle|^2 + \sum_{j\geq 0}\sum_{\ell_1,\ell_2=\pm 2^j}\sum_{k\in\mathbb{Z}^3} |\langle f, \widetilde{\psi}_{j,\ell_1,\ell_2,k}\rangle|^2 \\
&= \sum_{d=1}^{3}\sum_{j\geq 0}\sum_{|\ell_1|,|\ell_2|<2^j}\sum_{k\in\mathbb{Z}^3} |\langle f, \psi^{(d)}_{j,\ell_1,\ell_2,k}\rangle|^2 + \sum_{d=1}^{3}\sum_{j\geq 0}\sum_{\ell_1=\pm 2^j}\sum_{|\ell_2|<2^j}\sum_{k\in\mathbb{Z}^3} |\langle f, \widetilde{\psi}_{j,\ell_1,\ell_2,k,d}\rangle|^2 \\
&+ \sum_{j\geq 0}\sum_{\ell_1,\ell_2=\pm 2^j}\sum_{k\in\mathbb{Z}^3} |\langle f, \widetilde{\psi}_{j,\ell_1,\ell_2,k}\rangle|^2 \\
&= \int_{\mathbb{R}^3} |\hat{f}(\xi)|^2 \sum_{j\geq 0} |W(2^{-2j}\xi)|^2 \left(\sum_{|\ell_1|,|\ell_2|<2^j} |v\Big(2^j\tfrac{\xi_2}{\xi_1} - \ell_1\Big)\, v\Big(2^j\tfrac{\xi_3}{\xi_1} - \ell_2\Big)|^2 + \right. \\
&\left. + \sum_{|\ell_1|,|\ell_2|<2^j} |v\Big(2^j\tfrac{\xi_1}{\xi_2} - \ell_1\Big)\, v\Big(2^j\tfrac{\xi_3}{\xi_2} - \ell_2\Big)|^2 + \sum_{|\ell_1|,|\ell_2|<2^j} |v\Big(2^j\tfrac{\xi_1}{\xi_3} - \ell_1\Big)\, v\Big(2^j\tfrac{\xi_2}{\xi_3} - \ell_2\Big)|^2 \right) d\xi
\end{aligned}$$

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$$
\begin{aligned}
&+\int_{\mathcal{P}_1}|\hat{f}(\xi)|^2\sum_{j\geq 0}|W(2^{-2j}\xi)|^2\sum_{\ell_1=\pm 2^j}\sum_{|\ell_2|<2^j}|v\left(2^j\tfrac{\xi_2}{\xi_1}-\ell_1\right)|^2|v\left(2^j\tfrac{\xi_3}{\xi_1}-\ell_2\right)|^2d\xi\\
&+\int_{\mathcal{P}_2}|\hat{f}(\xi)|^2\sum_{j\geq 0}|W(2^{-2j}\xi)|^2\sum_{\ell_1=\pm 2^j}\sum_{|\ell_2|<2^j}|v\left(2^j\tfrac{\xi_1}{\xi_2}-\ell_1\right)|^2|v\left(2^j\tfrac{\xi_3}{\xi_2}-\ell_2\right)|^2d\xi\\
&+\int_{\mathcal{P}_2}|\hat{f}(\xi)|^2\sum_{j\geq 0}|W(2^{-2j}\xi)|^2\sum_{\ell_1=\pm 2^j}\sum_{|\ell_2|<2^j}|v\left(2^j\tfrac{\xi_1}{\xi_2}-\ell_2\right)|^2|v\left(2^j\tfrac{\xi_3}{\xi_2}-\ell_1\right)|^2d\xi\\
&+\int_{\mathcal{P}_3}|\hat{f}(\xi)|^2\sum_{j\geq 0}|W(2^{-2j}\xi)|^2\sum_{\ell_1=\pm 2^j}\sum_{|\ell_2|<2^j}|v\left(2^j\tfrac{\xi_1}{\xi_3}-\ell_2\right)|^2|v\left(2^j\tfrac{\xi_2}{\xi_3}-\ell_1\right)|^2d\xi\\
&+\int_{\mathcal{P}_1}|\hat{f}(\xi)|^2\sum_{j\geq 0}|W(2^{-2j}\xi)|^2\sum_{\ell_1=\pm 2^j}\sum_{|\ell_2|<2^j}|v\left(2^j\tfrac{\xi_2}{\xi_1}-\ell_2\right)|^2|v\left(2^j\tfrac{\xi_3}{\xi_1}-\ell_1\right)|^2d\xi\\
&+\int_{\mathcal{P}_3}|\hat{f}(\xi)|^2\sum_{j\geq 0}|W(2^{-2j}\xi)|^2\sum_{\ell_1=\pm 2^j}\sum_{|\ell_2|<2^j}|v\left(2^j\tfrac{\xi_1}{\xi_3}-\ell_1\right)|^2|v\left(2^j\tfrac{\xi_2}{\xi_3}-\ell_2\right)|^2d\xi\\
&+\int_{\mathcal{P}_1}|\hat{f}(\xi)|^2\sum_{j\geq 0}|W(2^{-2j}\xi)|^2\sum_{\ell_1,\ell_2=\pm 2^j}|v\left(2^j\tfrac{\xi_2}{\xi_1}-\ell_1\right)|^2|v\left(2^j\tfrac{\xi_3}{\xi_1}-\ell_2\right)|^2\,d\xi\\
&+\int_{\mathcal{P}_2}|\hat{f}(\xi)|^2\sum_{j\geq 0}|W(2^{-2j}\xi)|^2\sum_{\ell_1,\ell_2=\pm 2^j}|v\left(2^j\tfrac{\xi_1}{\xi_2}-\ell_1\right)|^2|v\left(2^j\tfrac{\xi_3}{\xi_2}-\ell_2\right)|^2\,d\xi\\
&+\int_{\mathcal{P}_3}|\hat{f}(\xi)|^2\sum_{j\geq 0}|W(2^{-2j}\xi)|^2\sum_{\ell_1,\ell_2=\pm 2^j}|v\left(2^j\tfrac{\xi_1}{\xi_3}-\ell_1\right)|^2|v\left(2^j\tfrac{\xi_2}{\xi_3}-\ell_2\right)|^2\,d\xi\\
&=\int_{\mathbb{R}^3}|\hat{f}(\xi)|^2\sum_{j\geq 0}|W(2^{-2j}\xi)|^2\Bigg(\sum_{|\ell_1|,|\ell_2|\leq 2^j}|v\left(2^j\tfrac{\xi_2}{\xi_1}-\ell_1\right)v\left(2^j\tfrac{\xi_3}{\xi_1}-\ell_2\right)|^2\chi_{\mathcal{P}_1}(\xi)\\
&+\sum_{|\ell_1||\ell_2|\leq 2^j}|v\Big(2^j\tfrac{\xi_1}{\xi_2}-\ell_1\Big)v\Big(2^j\tfrac{\xi_3}{\xi_2}-\ell_2\Big)|^2\chi_{\mathcal{P}_2}(\xi)+\sum_{|\ell|\leq 2^j}|v\Big(2^j\tfrac{\xi_1}{\xi_3}-\ell_1\Big)v\Big(2^j\tfrac{\xi_2}{\xi_3}-\ell_2\Big)|^2\chi_{\mathcal{P}_3}(\xi)\Bigg)d\xi\\
&=\int_{\mathbb{R}^3}|\hat{f}(\xi)|^2\sum_{j\geq 0}|W(2^{-2j}\xi)|^2d\xi.
\end{aligned}
$$

In the last step we used (2.5). From the calculations above, using (3.2) it follows that for any $f\in L^2(\mathbb{R}^3)$:

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}^3}|\langle f,\widetilde{\psi}_{-1,k}\rangle|^2+\sum_{d=1}^{3}\sum_{j\geq 0}\sum_{|\ell_1|\leq 2^j}\sum_{|\ell_2|<2^j}\sum_{k\in\mathbb{Z}^3}|\langle f,\widetilde{\psi}_{j,\ell_1,\ell_2,k,d}\rangle|^2+\sum_{j\geq 0}\sum_{\ell_1,\ell_2=\pm 2^j}\sum_{k\in\mathbb{Z}^3}|\langle f,\widetilde{\psi}_{j,\ell_1,\ell_2,k}\rangle|^2\\
&=\int_{\mathbb{R}^3}|\hat{f}(\xi)|^2\,|\Phi(\xi)|^2d\xi+\int_{\mathbb{R}^3}|\hat{f}(\xi)|^2\sum_{j\geq 0}|W(2^{-2j}\xi)|^2d\xi\\
&=\int_{\mathbb{R}^3}|\hat{f}(\xi)|^2\left(|\Phi(\xi)|^2+\sum_{j\geq 0}|W(2^{-2j}\xi)|^2\right)d\xi=\int_{\mathbb{R}^3}|\hat{f}(\xi)|^2\,d\xi.\quad\square
\end{aligned}
$$

4. Higher dimensions

The construction of smooth Parseval frames of shearlets can be naturally generalized to higher dimensions. In this section, we briefly outline the main ideas needed to construct such shearlet systems in the D-dimensional setting.

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For $\xi = (\xi_1, \dots, \xi_D) \in \mathbb{R}^D$, one can define a smooth low pass function $\widehat{\Phi}(\xi)$ supported in $[-\frac{1}{8}, \frac{1}{8}]^D$ as done in the previous sections, so that the function

$$W(\xi) = \sqrt{\widehat{\Phi}^2(2^{-2}\xi) - \widehat{\Phi}^2(\xi)}$$

satisfies

$$\widehat{\Phi}^2(\xi) + \sum_{j\geq 0} W^2(2^{-2j}\xi) = 1 \quad \text{for } \xi \in \mathbb{R}^D.$$

Next, let us define the anisotropic dilations matrices

$$A_{(1)} = \begin{pmatrix} 4 & 0 & \dots & 0 \\ 0 & 2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 2 \end{pmatrix}, A_{(2)} = \begin{pmatrix} 2 & 0 & \dots & 0 \\ 0 & 4 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 2 \end{pmatrix}, \dots, A_{(D)} = \begin{pmatrix} 2 & 0 & \dots & 0 \\ 0 & 2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 4 \end{pmatrix},$$

and, for $\ell = (\ell_1, \dots, \ell_{D-1})$, the shear matrices

$$B^{[\ell]}_{(1)} = \begin{pmatrix} 1 & \ell_1 & \dots & \ell_{D-1} \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}, B^{[\ell]}_{(2)} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ \ell_1 & 1 & \dots & \ell_{D-1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}, \dots, B^{[\ell]}_{(D)} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \ell_1 & \ell_2 & \dots & 1 \end{pmatrix}.$$

To obtain a smooth Parseval frame of shearlets for $L^2(\mathbb{R}^D)$, one has to combine a set of coarse-scale shearlets, interior shearlets and boundary shearlets, similar to the constructions given above for $D = 2, 3$. In particular, the coarse-scale shearlets are similarly defined as the elements of the form $\{\Phi(\cdot - k) : k \in \mathbb{Z}^D\}$. The interior shearlets are the elements

$$\{\psi^{(d)}_{j,\ell,k} : j \geq 0, -2^j \leq \ell_1, \dots, \ell_{D-1} \leq 2^j, k \in \mathbb{Z}^D, d = 1, \cdots, D\}, \tag{4.1}$$

where

$$\hat{\psi}^{(d)}_{j,\ell,k}(\xi) = |\det A_{(d)}|^{-j/2}\, W(2^{-2j}\xi)\, V_{(d)}(\xi A^{-j}_{(d)} B^{[-\ell]}_{(d)})\, e^{2\pi i \xi A^{-j}_{(d)} B^{[-\ell]}_{(d)} k},$$

and

$$V_{(d)}(\xi_1, \cdots, \xi_D) = \prod_{\substack{m=1,\cdots,D \\ m\neq d}} v\left(\frac{\xi_m}{\xi_d}\right).$$

Notice that the index d is associated to the D-dimensional pyramid

$$\mathcal{P}_d = \left\{\xi \in \mathbb{R}^D : |\frac{\xi_1}{\xi_d}| \leq 1, \dots, |\frac{\xi_D}{\xi_d}| \leq 1\right\},$$

and each set of interior shearlets $\{\psi^{(d)}_{j,\ell,k}\}$ is supported inside $\mathcal{P}_d$ in the Fourier domain. To define the boundary shearlets, it is useful to recall that in the case $D = 3$ there are 3 pyramidal regions and we defined 2 sets of boundary shearlets: one set corresponding to the 6 planes where 2 different pyramidal regions intersect; another set corresponding to the 4 lines where 3 different pyramidal regions intersect. For the D-dimensional shearlet system, the situation is similar. In fact, there are D D-dimensional pyramidal regions. Hence we introduce several sets of boundary shearlets, each one associated with the L-hyperplanes where $D - L$ pyramidal regions intersect, for $L = 1, \dots, D-1$. One can verify that there are $2^{D-L}C(D, D-L)$ L-hyperplanes where $D - L$ pyramidal regions intersect, where $C(n,k) = \frac{n!}{(n-k)!k!}$. Specifically, the boundary shearlets corresponding to the $D-1$-hyperplanes (there are $2^{D-1}C(D, D-1)$ of them) are defined for $\ell_1 = \pm 2^j$, $|\ell_2| \dots |\ell_{D-1}| < 2^j$ and have the form

$$(\widetilde{\psi}_{j,\ell,k,d})^\wedge(\xi) = \begin{cases} 2^{-(D-1)\frac{D}{2}}\, 2^{-\frac{D+1}{2}}\, W(2^{-2j}\xi)\, V_{(q)}(\xi A^{-j}_{(q)} B^{[-\ell]}_{(q)})\, e^{2\pi i \xi 2^{-(D-1)} A^{-j}_{(q)} B^{[-\ell]}_{(q)} k}, & \text{if } \xi \in \mathcal{P}_q \\ 2^{-(D-1)\frac{D}{2}}\, 2^{-\frac{D+1}{2}}\, W(2^{-2j}\xi)\, V_{(q')}(\xi A^{-j}_{(q')} B^{[-\ell]}_{(q')})\, e^{2\pi i \xi 2^{-(D-1)} A^{-j}_{(q)} B^{[-\ell]}_{(q)} k}, & \text{if } \xi \in \mathcal{P}_{q'}, \end{cases}$$

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for all combinations of pyramidal regions $\mathcal{P}_q$ and $\mathcal{P}_{q'}$. The boundary shearlets corresponding to the $(D-2)$-hyperplanes (there are $2^{D-2}C(D,D-2)$ of them) are defined for $\ell_1,\ell_2=\pm 2^j$, $|\ell_3|,\dots|\ell_{D-1}|<2^j$ and have the form

$$(\widetilde{\psi}_{j,\ell,k,d})^\wedge(\xi)=\begin{cases}2^{-(D-1)\frac{D}{2}}\,2^{-\frac{D+1}{2}}\,W(2^{-2j}\xi)\,V_{(q)}(\xi A_{(q)}^{-j}B_{(q)}^{[-\ell]})\,e^{2\pi i\xi 2^{-(D-1)}A_{(q)}^{-j}B_{(q)}^{[-\ell]}k}, & \text{if } \xi\in\mathcal{P}_q\\ 2^{-(D-1)\frac{D}{2}}\,2^{-\frac{D+1}{2}}\,W(2^{-2j}\xi)\,V_{(q')}(\xi A_{(q')}^{-j}B_{(q')}^{[-\ell]})\,e^{2\pi i\xi 2^{-(D-1)}A_{(q)}^{-j}B_{(q)}^{[-\ell]}k}, & \text{if } \xi\in\mathcal{P}_{q'},\\ 2^{-(D-1)\frac{D}{2}}\,2^{-\frac{D+1}{2}}\,W(2^{-2j}\xi)\,V_{(q')}(\xi A_{(q'')}^{-j}B_{(q'')}^{[-\ell]})\,e^{2\pi i\xi 2^{-(D-1)}A_{(q)}^{-j}B_{(q)}^{[-\ell]}k}, & \text{if } \xi\in\mathcal{P}_{q''},\end{cases}$$

for all combinations of pyramidal regions $\mathcal{P}_q$, $\mathcal{P}_{q'}$ and $\mathcal{P}_{q''}$. Similarly one proceeds for the boundary shearlets corresponding to the L-hyperplanes, where $L=(D-3),(D-4),\dots,1$.

5. Shearlet Molecules

The new systems of shearlets discussed above form Parseval frames of band-limited, well localized functions ranging at various scales and orientations under the action of anisotropic matrices and shear transformations. The sparsity properties of these function systems are closely related to their localization properties, their directionality and their ability to deal with anisotropic features through the parabolic scaling which is produced by the dilation matrices. Indeed, one can try to define abstractly the type conditions which are required by a generic shearlet-like system in order to form a sparse representation system and this leads to a notion of *shearlet molecule*.

This type of notion was originally introduced by the authors in [15], in connection with the study of the shearlet representation of Fourier Integral Operators, where the following definition was introduced (in dimension $D=2$).

Definition 5.1. Let $\mathcal{M}=\{(j,\ell,k): j\geq 0, |\ell|\leq 2^j, k\in\mathbb{Z}^2\}$ and $\mathcal{M}_j=\{(j,\ell,k): |\ell|\leq 2^j, k\in\mathbb{Z}^2\}$. For $\mu=(j,\ell,k)\in\mathcal{M}$, the function $m_\mu(x)=2^{3j/2}\,a_\mu(B_{(1)}^{\ell}A_{(1)}^{j}x-k)$ is an horizontal shearlet molecule with regularity R if the function α_μ satisfies the following properties:

(i) for each $\gamma=(\gamma_1,\gamma_2)\in\mathbb{N}\times\mathbb{N}$ and each $N\geq 0$ there is a constant $C_N>0$ such that

$$|\partial_x^\gamma a_\mu(x)|\leq C_{\gamma,N}\,(1+|x|)^{-N}; \tag{5.1}$$

(ii) for each $M\leq R$ and each $N\geq 0$, there is a constant $C_{M,N}>0$ such that

$$|\hat{a}_\mu(\xi)|\leq C_{M,N}\,(1+|\xi|)^{-2N}\,(2^{-2j}+|\xi_1|)^M. \tag{5.2}$$

For $\mu=(j,\ell,k)\in\mathcal{M}$, the function $m_\mu^{(2)}(x)=2^{3j/2}\,a_\mu(B_{(2)}^{\ell}A_{(2)}^{j}x-k)$ is a vertical shearlet molecule with regularity R if the $\alpha_\mu^{(2)}$ satisfies (5.1) and for each $M\leq R$ and each $N\geq 0$ there is a constant $C_{N,M}>0$ such that

$$\left|\hat{a}_\mu^{(2)}(\xi)\right|\leq C_{N,M}\,(1+|\xi|)^{-2N}\,(2^{-2j}+|\xi_2|)^M.$$

The constants $C_{\gamma,N}$ and $C_{N,M}$ are independent of μ.

Notice that this definition of shearlet molecule is similar to the curvelet molecules introduced by Candès and Demanet [1]. Both definitions adapt the notion of vaguelettes of Coifman and Meyer [32].

It is a very simple exercise to show that the affine-like horizontal shearlets (1.1) are also horizontal shearlet molecules, and the same holds for the corresponding vertical systems.

It turns out that also the new shearlet systems introduced in the sections above are shearlet molecules.

Proposition 5.1. *The horizontal shearlet system* (2.6) *is a system of horizontal shearlet molecules.*

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Proof. Recall that the elements of the system (2.6) are of the form

$$\psi^{(1)}_{j,\ell,k}(x) = |\det A_{(1)}|^{j/2}\, \psi_{j,\ell}(B^{\ell}_{(1)}A^{j}_{(1)}x - k), \quad j \geq 0, |\ell| \leq 2^j, k \in \mathbb{Z}^2,$$

where

$$\hat{\psi}_{j,\ell}(\xi) = W_{j,\ell}(\xi)\, V_{(1)}(\xi)$$

and

$$W_{j,\ell}(\xi) = W(2^{-2j}\xi B^{\ell}_{(1)}A^{j}_{(1)}) = W(\xi_1, 2^{-j}(\xi_2 - \ell\xi_1)).$$

Hence, we need to show that $\psi_{j,\ell}$ satisfies (5.1) and (5.2).

Notice that $\hat{\psi}_{j,\ell}$ is compactly supported, uniformly for each j, ℓ. In fact, the support conditions on W and $V_{(1)}$ impose that $\operatorname{supp}\hat{\psi}_{j,\ell} \subset \{(\xi_1,\xi_2) \in \mathbb{R}^2 : |\frac{\xi_2}{\xi_1}| \leq 1, \xi_1 \in [-\frac{1}{2},\frac{1}{2}] \setminus [-\frac{1}{16},\frac{1}{16}]$. From this observation, it is easy to verify that for all $\beta = (\beta_1,\beta_2)$ such that $|\beta| \leq 2N$, $i = 1, 2$, there is a constant C_N such that

$$\left|\partial^{\beta}_{\xi_i}\left(|\xi|^{2N}\hat{\psi}_{j,\ell}(\xi)\right)\right| \leq C_N.$$

This implies that (5.1) is satisfied.

To show that (5.2) is satisfied, notice first that support condition of V implies that there is a constant C_N such that

$$|V_{(1)}(\xi)| \leq C_N\,(1 + |\xi|)^{-2N}.$$

Next, a direct calculation show that, for all $M \in \mathbb{N}$, there is a constant C_M such that

$$\left|\partial^{M}_{\xi_1} W_{j,\ell}(\xi)\right| \leq C_M,$$

uniformly with respect to j, ℓ, and ξ, and that

$$\partial^{n}_{\xi_1} W_{j,\ell}(\xi)\big|_{\xi_1=0} = 0, \quad n = 0, 1, \ldots, M-1,$$

due to the support assumptions on W. Hence (5.2) is satisfied. □

A very similar argument shows that vertical shearlet system (2.9) is a system of vertical shearlet molecules. In addition, taking again advantage of the assumptions on W and v, one can show by direct computation that also the boundary shearlets $\{\tilde{\psi}_{j,\ell,k} : j \geq 0, \ell = \pm 2^j, k \in \mathbb{Z}^2\}$ defined in Sec. 2.1 are horizontal shearlet molecules.

Let us briefly examine a few implications of Definition 5.1. If $m_\mu(x)$ is an horizontal shearlet molecule with regularity R, then it follows from (5.1) that

$$|(2\pi i\xi)^{\gamma}\,\hat{a}_\mu(\xi)| \leq \|\partial^{\gamma} a_\mu\|_{L^1} \leq C_\gamma,$$

and, thus, for all $N \geq 0$ there is a constant C_N such that

$$|\hat{a}_\mu(\xi)| \leq C_N\,(1 + |\xi|)^{-N}.$$

It follows that for all $N \geq 0$ there is a constant C_N such that

$$|\hat{m}_\mu(\xi)| \leq C_N\, 2^{-3j/2}\,(1 + |\xi A^{-j} B^{-\ell})^{-N}. \tag{5.3}$$

On the other hand, from (5.2) it follows that for each $M \leq R$ and each $N \geq 0$ there is a constant $C_{N,M} > 0$ such that

$$|\hat{m}_\mu(\xi)| = |\hat{a}_\mu(\xi\, A^{-j}\, B^{-\ell})| \leq C_{N,M}\, 2^{-3j/2}\, \left\{2^{-2j}\,(1 + |\xi_1|)\right\}^{M}\, \left(1 + |\xi\, A^{-j}\, B^{-\ell}|\right)^{-2N} \tag{5.4}$$

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Thus, combining (5.3) and (5.4), it follows that for each $M \leq R$ and each $N \geq 0$ there is a constant $C_{N,M} > 0$ such that

$$|\hat{m}_\mu(\xi)| \leq C_{N,M}\, 2^{-3j/2}\, \min\left\{1, 2^{-2j}\,(1+|\xi_1|)\right\}^M\, \left(1+|\xi\, A^{-j}\, B^{-\ell}|\right)^{-2N}. \tag{5.5}$$

Similarly if $m_\mu^{(v)}(x)$ is a vertical shearlet molecule with regularity R, then for all $N \geq 0$ and all $M \leq R$ there is a constant $C_{N,M}$ independent of μ such that:

$$|\hat{m}_\mu^{(v)}(\xi)| \leq C_{N,M}\, 2^{-3j/2}\, \min\left\{1, 2^{-2j}\,(1+|\xi_2|)\right\}^M\, \left(1+|\xi\, A_{(v)}^{-j}\, B_{(v)}^{-\ell}|\right)^{-2N}. \tag{5.6}$$

These properties are particularly useful, as discussed in [15], where it is shown that there is a notion of almost orthogonality associated with the shearlet molecules. Related to this, it is useful to recall that shearlet molecules were recently used to develop a notion of *sparsity equivalence* in [27], implying, essentially, that all systems satisfying the conditions given in Definition 5.1 share the same sparsity properties.

This point of view makes more precise the observation that there is a close affinity between the affine-like shearlets (1.3) and the new shearlets (2.6). Not only both systems form a Parseval frame for the same subspace $L^2(\mathcal{P}_1)^\vee$. They are also both horizontal shearlet molecules, implying that they have essentially the same geometrical and sparsity properties (when considered as systems in $L^2(\mathcal{P}_1)^\vee$).

Finally, it is clear that the definition of shearlet molecules can be extended to higher dimensions. In particular, in dimension $D = 3$ we have the following definition.

Definition 5.2. Let $\mathcal{M} = \{(j,\ell,k) : j \geq 0, \ell = (\ell_1,\ell_2), |\ell_1|, |\ell_2| \leq 2^j, k \in \mathbb{Z}^3\}$. For $\mu = (j,\ell_1,\ell_2,k) \in \mathcal{M}$ and $d = 1,2,3$, the function $m_\mu^{(d)}(x) = |\det A_{(d)}|^{j/2}\, a_\mu(B_{(d)}^{[\ell]} A_{(d)}^j x - k)$ is a shearlet molecule associated with the pyramidal region $\mathcal{P}_d$ of regularity R if the function α_μ satisfies the following properties:

(i) for each $\gamma = (\gamma_1,\gamma_2,\gamma_3) \in \mathbb{N}^3$ and each $N \geq 0$ there is a constant $C_N > 0$ such that

$$|\partial_x^\gamma a_\mu(x)| \leq C_{\gamma,N}\,(1+|x|)^{-N}; \tag{5.7}$$

(ii) for each $M \leq R$ and each $N \geq 0$ there is a constant $C_{N,M} > 0$ such that

$$|\hat{a}_\mu(\xi)| \leq C_{N,M}\,(1+|\xi|)^{-2N}\,(2^{-2j}+|\xi_d|)^M. \tag{5.8}$$

The constants $C_{\gamma,N}$ and $C_{N,M}$ are independent of μ.

A calculation similar to one above shows that the new cone-based shearlets (3.5) are shearlet molecules associated with the pyramidal regions $\mathcal{P}_d$.

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