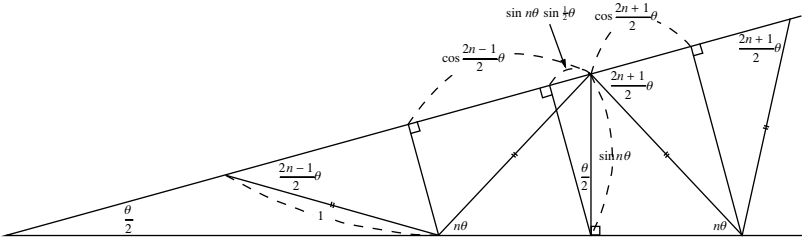
109.36 Proof without words: Lagrange's sine identity

$$\sum_{k=0}^{n} \sin k\theta = \frac{\cos\dfrac{\theta}{2} - \cos\dfrac{(2n+1)\theta}{2}}{2\sin\dfrac{\theta}{2}}$$

Lemma: $\sin n\theta \sin\frac{1}{2}\theta = \frac{1}{2}\big(\cos\frac{1}{2}((2n-1)\theta) - \cos\frac{1}{2}((2n+1)\theta)\big)$. That is,

$$\sin n\theta = \frac{\cos\frac{1}{2}((2n-1)\theta) - \cos\frac{1}{2}((2n+1)\theta)}{2\sin\frac{1}{2}\theta}.$$



Theorem (Lagrange's sine identity):

$$\sum_{k=0}^{n} \sin k\theta = \frac{\cos\frac{1}{2}\theta - \cos\frac{1}{2}((2n+1)\theta)}{2\sin\frac{1}{2}\theta}.$$

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109.37 A visual proof of a theorem on divergent series

In mathematics, divergent series have been widely studied in the relevant literature and are always of interest due to their applications in various sciences. For example, in probability theory, the well-known Borel-Cantelli lemma (second version) is expressed with assumptions involving divergent series; it states that if the series of probabilities of an infinite sequence of independent events is divergent, then the probability that each event occurs infinitely often is equal to 1 (see, e.g., [1]).

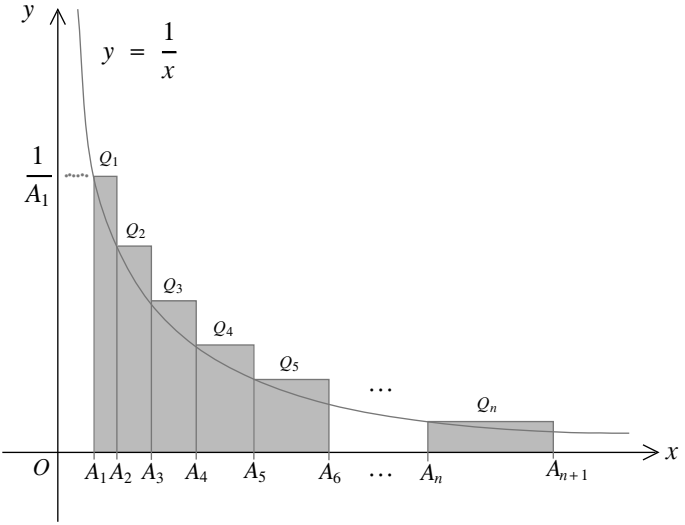
Among the divergent series, the series that diverge slowly are more interesting. Some of the most popular examples of slowly diverging series are $\sum_{k=1}^{\infty}\frac{1}{k}$, $\sum_{k=2}^{\infty}\frac{1}{k\ln k}$ and $\sum_{k=3}^{\infty}\frac{1}{k\ln k\ln\ln k}$. Usually, to show the divergence of such series, the integral test or Abel-Dini theorem is used (see, e.g., [2] and [3]). In a recent paper [4], the author established the following general theorem, which is useful for showing the divergence of slowly diverging series.

https://doi.org/10.1017/mag.2025.10138 Published online by Cambridge University Press

Theorem: Let $\{A_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of positive numbers tending to infinity. Then the series $\sum_{k=1}^{\infty} \frac{A_{k+1} - A_k}{A_k}$ is divergent.

Here, we offer a visual proof of the above theorem. Our visual proof is given below.

Proof: Note that the term above each rectangle (i.e. Q_i) is the area of that rectangle.



$$\overbrace{\frac{A_2 - A_1}{A_1}}^{Q_1} + \overbrace{\frac{A_3 - A_2}{A_2}}^{Q_2} + \overbrace{\frac{A_4 - A_3}{A_3}}^{Q_3} + \ldots + \overbrace{\frac{A_{n+1} - A_n}{A_n}}^{Q_n} > \int_{A_1}^{A_{n+1}} \frac{1}{x} dx = \ln\left(\frac{A_{n+1}}{A_1}\right).$$

Clearly, since $A_{n+1} \to \infty$ as $n \to \infty$, the right side of the above inequality tends to infinity as $n \to \infty$. Consequently the series on the left side diverges. This completes the proof.

As an example, let us consider the Theorem for $A_k = k$. In this case, the theorem implies that the series $\sum_{k=1}^{\infty} \frac{(k+1) - k}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$ (the harmonic series) diverges.

Acknowledgements

The authors would like to thank the referee for carefully reading the manuscript and providing constructive comments that substantially helped to improve the quality of the paper.

https://doi.org/10.1017/mag.2025.10138 Published online by Cambridge University Press

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109.38 A visual proof of convergence of a hyperpower sequence

In 2004, Azarpanah ([1]) wordlessly established that

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot}}}} = 2.$$

Recently, Mukherjee ([2]) gave a wordless proof of convergence of a hyperpower sequence:

$$\left(n^{\frac{1}{n}}\right)^{\left(n^{\frac{1}{n}}\right)^{\left(n^{\frac{1}{n}}\right)^{\cdot^{\cdot}}}} = -\frac{nW_0\left(-\frac{\ln n}{n}\right)}{\ln n} \text{ for } n \in \mathbb{N}\setminus\{1\},$$

where W_0 denotes the principal branch of the Lambert W function ([3]). Note that for $n = 2$, $-\frac{nW_0\left(-\frac{\ln n}{n}\right)}{\ln n} = 2$. In this Note, we visually prove a surprising result:

$$\left(a^{\frac{1}{a}}\right)^{\left(a^{\frac{1}{a}}\right)^{\left(a^{\frac{1}{a}}\right)^{\cdot^{\cdot}}}} = a, \text{ where } 1 < a \leqslant e.$$

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