Note on the Lorentz group. By Mr J. Brill, St John's College.

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The Lorentz group is defined by the equations

$$X_r = a_{1r}x_1 + a_{2r}x_2 + a_{3r}x_3 + a_{4r}x_4 \quad (r = 1, 2, 3, 4),$$

where the coefficients are subject to the conditions

$$a_{r1}^2 + a_{r2}^2 + a_{r3}^2 - a_{r4}^2 = 1 \,(r = 1, 2, 3), \quad a_{41}^2 + a_{42}^2 + a_{43}^2 - a_{44}^2 = -1,$$

$$a_{r1}a_{s1} + a_{r2}a_{s2} + a_{r3}a_{s3} - a_{r4}a_{s4} = 0 \,(r, s = 1, 2, 3, 4; r \neq s).$$

It is an intransitive group possessing the single invariant

$$x_4^2 - x_1^2 - x_2^2 - x_3^2.$$

By an adaptation of the work of Hermite and Cayley the said coefficients may be realized on writing $a_{rs} = u_{rs}/v$, where

$$\begin{aligned}
v &= 1 + a^2 + b^2 + c^2 - f^2 - g^2 - h^2 - k^2,\\
u_{11} &= 1 + a^2 - b^2 - c^2 + f^2 - g^2 - h^2 + k^2,\\
u_{21} &= 2c + 2(ab + fg) - 2hk,\\
u_{31} &= -2b + 2(ca + hf) + 2gk,\\
u_{41} &= -2f + 2(bh - cg) - 2ak,\\
u_{12} &= -2c + 2(ab + fg) + 2hk,\\
u_{22} &= 1 + b^2 - c^2 - a^2 + g^2 - h^2 - f^2 + k^2,\\
u_{32} &= 2a + 2(bc + gh) - 2fk,\\
u_{42} &= -2g + 2(cf - ah) - 2bk,\\
u_{13} &= 2b + 2(ca + hf) - 2gk,\\
u_{23} &= -2a + 2(bc + gh) + 2fk,\\
u_{33} &= 1 + c^2 - a^2 - b^2 + h^2 - f^2 - g^2 + k^2,\\
u_{43} &= -2h + 2(ag - bf) - 2ck,\\
u_{14} &= -2f + 2(cg - bh) - 2ak,\\
u_{24} &= -2g + 2(ah - cf) - 2bk,\\
u_{34} &= -2h + 2(bf - ag) - 2ck,\\
u_{44} &= 1 + a^2 + b^2 + c^2 + f^2 + g^2 + h^2 + k^2,
\end{aligned}$$

where, for brevity, we have written

$$k = af + bg + ch.$$

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The group has thus six parameters a, b, c, f, g, h. If we apply the transformation, whose parameters are A, B, C, F, G, H, to that whose parameters are a, b, c, f, g, h, and the resulting parameters are a, b, c, f, g, h, we have

$$\begin{aligned}
\mathrm{a} &= (A + a + Cb - Bc + Gh - Hg + Fk + Kf)/U,\\
\mathrm{b} &= (B + b + Ac - Ca + Hf - Fh + Gk + Kg)/U,\\
\mathrm{c} &= (C + c + Ba - Ab + Fg - Gf + Hk + Kh)/U,\\
\mathrm{f} &= (F + f + Cg - Gc + Hb - Bh - Ak - Ka)/U,\\
\mathrm{g} &= (G + g + Ah - Ha + Fc - Cf - Bk - Kb)/U,\\
\mathrm{h} &= (H + h + Bf - Fb + Ga - Ag - Ck - Kc)/U,
\end{aligned}$$

where $U = 1 - Aa - Bb - Cc + Ff + Gg + Hh - Kk,$

and $$K = AF + BG + CH.$$

Further, we find that

$$\mathrm{af} + \mathrm{bg} + \mathrm{ch} = (K + k + Af + Fa + Bg + Gb + Ch + Hc)/U.$$

From the above formulae we can obtain the equations of each of the parameter groups, having first substituted for K and k their values.

The six operators

$$y_3\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial y_3} + y_6\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial y_6}, \quad y_1\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial y_1} + y_4\frac{\partial}{\partial y_6} - y_6\frac{\partial}{\partial y_4},$$

$$y_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial y_2} + y_5\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial y_5}, \quad y_3\frac{\partial}{\partial y_5} + y_5\frac{\partial}{\partial y_3} - y_2\frac{\partial}{\partial y_6} - y_6\frac{\partial}{\partial y_2},$$

$$y_1\frac{\partial}{\partial y_6} + y_6\frac{\partial}{\partial y_1} - y_3\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial y_3}, \quad y_2\frac{\partial}{\partial y_4} + y_4\frac{\partial}{\partial y_2} - y_1\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial y_1},$$

define a group having the same structure as the Lorentz group. This group possesses the two invariants

$$y_1y_4 + y_2y_5 + y_3y_6 \text{ and } y_1^2 + y_2^2 + y_3^2 - y_4^2 - y_5^2 - y_6^2,$$

and the covariant

$$\begin{aligned}
&(x_1y_1 + x_2y_2 + x_3y_3)^2 - (x_3y_5 - x_2y_6 + x_4y_1)^2\\
&\quad - (x_1y_6 - x_3y_4 + x_4y_2)^2 - (x_2y_4 - x_1y_5 + x_4y_3)^2.
\end{aligned}$$

Taking into account the second of the above invariants together with the invariant of the Lorentz group, viz.

$$x_4^2 - x_1^2 - x_2^2 - x_3^2,$$

we deduce another covariant

$$\begin{aligned}
&(x_1y_4 + x_2y_5 + x_3y_6)^2 - (x_2y_3 - x_3y_2 + x_4y_4)^2\\
&\quad - (x_3y_1 - x_1y_3 + x_4y_5)^2 - (x_1y_2 - x_2y_1 + x_4y_6)^2
\end{aligned}$$

from the above.

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If we take the two expressions

$$(x_1y_1 + x_2y_2 + x_3y_3)^2 - (x_3y_5 - x_2y_6 + x_4y_1)^2 - (x_1y_6 - x_3y_4 + x_4y_2)^2 - (x_2y_4 - x_1y_5 + x_4y_3)^2$$

and

$$(x_1z_1 + x_2z_2 + x_3z_3)^2 - (x_3z_5 - x_2z_6 + x_4z_1)^2 - (x_1z_6 - x_3z_4 + x_4z_2)^2 - (x_2z_4 - x_1z_5 + x_4z_3)^2$$

conjointly; then, in addition to the invariants involving the y's, and those involving the z's, we obtain the two mixed invariants

$$y_1z_4 + y_4z_1 + y_2z_5 + y_5z_2 + y_3z_6 + y_6z_3$$

and

$$y_1z_1 + y_2z_2 + y_3z_3 - y_4z_4 - y_5z_5 - y_6z_6.$$

We have also the covariant

$$(x_1y_1 + x_2y_2 + x_3y_3)(x_1z_1 + x_2z_2 + x_3z_3) - (x_3y_5 - x_2y_6 + x_4y_1)(x_3z_5 - x_2z_6 + x_4z_1) - (x_1y_6 - x_3y_4 + x_4y_2)(x_1z_6 - x_3z_4 + x_4z_2) - (x_2y_4 - x_1y_5 + x_4y_3)(x_2z_4 - x_1z_5 + x_4z_3).$$

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