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A FURTHER NOTE ON RAMANUJAN'S ARITHMETICAL FUNCTION $\tau(n)$

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1. This note is a sequel to two in earlier volumes of the *Proceedings*, the first by myself and the second by Wilton†.

Suppose that

$$f(z) = z\{(1-z)(1-z^2)(1-z^3)\ldots\}^{24} = \sum_1^\infty \tau(n) z^n \tag{1·1}$$

for $|z|<1$; that $x>0$; that

$$T_r(x) = \frac{1}{\Gamma(r+1)} \sum_{n\leqslant x} (x-n)^r \tau(n) \tag{1·2}$$

for $r \geqslant 0$; and that

$$T(x) = T_0(x) - \tfrac{1}{2}\tau(x), \tag{1·3}$$

where $\tau(x)$ is to mean 0 if x is not an integer. Thus

$$T(x) = \sum_{n\leqslant x}{}' \tau(n), \tag{1·4}$$

where the dash shows that the last term is to be halved when x is an integer;

$$\frac{dT_r(x)}{dx} = T_{r-1}(x) \tag{1·5}$$

for $r>1$; and

$$\int_0^x T(t)\,dt = \int_0^x T_0(t)\,dt = T_1(x). \tag{1·6}$$

Then $T_r(x)$ and $T(x)$ can be expressed by series of Bessel functions like those which occur in the standard lattice point problems‡. Thus Wilton proves that

$$T_r(x) = \left(\frac{1}{2\pi}\right)^r \sum_1^\infty \left(\frac{x}{n}\right)^{6+\frac{1}{2}r} \tau(n) J_{12+r}\{4\pi\sqrt{(nx)}\} \tag{1·7}$$

when $r>0$; but his method fails when $r=0$. This is the most interesting case, and it seems desirable that the gap should be filled.

† Hardy (2), Wilton (8).

‡ See Landau (5), pp. 183–308, where there is a very full account of Gauss's "circle" problem.

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Following the method used by Landau and myself in our discussion of the circle problem†, I prove

THEOREM A. *If* $x>0$ *then*

$$T(x)=\sum_1^\infty\left(\frac{x}{n}\right)^6\tau(n)\,J_{12}\{4\pi\sqrt{(nx)}\}. \tag{1·8}$$

The series is boundedly convergent in any finite interval $0<x_1\leqslant x\leqslant x_2$ *of values of* x, *and uniformly convergent in any such interval which includes no integral values.*

2. We need certain formulae in the theory of Bessel functions, which I quote from Watson's book‡. These are

$$\frac{d}{dx}\left\{\frac{J_\nu(c\sqrt{x})}{x^{\frac{1}{2}\nu}}\right\}=-\tfrac{1}{2}c\,\frac{J_{\nu+1}(c\sqrt{x})}{x^{\frac{1}{2}(\nu+1)}}; \tag{2·1}$$

$$|J_\nu(x)|<\frac{A_\nu}{x^{\frac{1}{2}}}; \tag{2·2}$$

$$\left|J_\nu(x)-\sqrt{\left(\frac{2}{\pi x}\right)}\cos(x-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)\right|<\frac{A_\nu}{x^{\frac{3}{2}}}; \tag{2·3}$$

$$(\tfrac{1}{2}x)^{-\nu}J_\nu(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}u^{-\nu-1}\exp\left(u-\frac{x^2}{4u}\right)du, \tag{2·4}$$

where $c>0$; and

$$\int_0^\infty J_\nu(ax)\,J_{\nu+1}(x)\,dx=\begin{cases}a^\nu & (0<a<1),\\ \frac{1}{2} & (a=1),\\ 0 & (a>1).\end{cases} \tag{2·5}$$

In (2·2) and (2·3) the A_ν are independent of x.

LEMMAS

3. LEMMA 1§. *If*

$$F(s)=f(e^{-s})=\sum_1^\infty\tau(n)\,e^{-ns}, \tag{3·1}$$

and $\sigma=\Re(s)>0$, *then*

$$F(s)=\left(\frac{2\pi}{s}\right)^{12}F\left(\frac{4\pi^2}{s}\right), \tag{3·2}$$

LEMMA 2||. *If* $r>0$ *then*

$$T_r(x)=\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}F(s)\,\frac{e^{xs}}{s^{r+1}}\,ds. \tag{3·3}$$

† Hardy and Landau (3). The argument is reproduced, with modifications, in Landau's book.

‡ Watson (7), pp. 16, 18, 45, 199, 406. Some of the formulae do not occur in the book in the exact form in which they are stated here, but there are equivalents of all of them.

§ Wilton (8), p. 122. The formula is one of the equations of linear transformation of the modular functions.

|| See Hardy and Riesz (4), p. 50. The formula is a special case of a well-known general theorem in the theory of Dirichlet's series.

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LEMMA 3†. $$\tau^2(1)+\tau^2(2)+\ldots+\tau^2(n) = O(n^{12}).$$

LEMMA 4. $$\tau(n) = O(n^6).$$

LEMMA 5. *The series* $\Sigma n^{-27/4}|\tau(n)|$ *is convergent.*

For $\Sigma n^{-49/4}\tau^2(n)$ is convergent, by Lemma 3, and

$$(\Sigma n^{-27/4}|\tau(n)|)^2 \leqslant \Sigma n^{-5/4}\Sigma n^{-49/4}\tau^2(n).$$

LEMMA 6. *If* $x>0$ *then*

$$T_1(x) = \frac{1}{2\pi}\sum_1^\infty \left(\frac{x}{n}\right)^{13/2}\tau(n)J_{13}\{4\pi\sqrt{(nx)}\}. \tag{3·4}$$

The series is uniformly convergent in any finite interval $0 \leqslant x \leqslant X$.

This is (1·7) with $r=1$, and is proved in Wilton's note; but we can shorten his proof by the use of Lemma 3.

By Lemmas 1 and 2, we have

$$\begin{aligned} T_r(x) &= \frac{(2\pi)^{12}}{2\pi i}\int_{1-i\infty}^{1+i\infty}\frac{e^{xs}}{s^{13+r}}\sum_1^\infty \tau(n)\,e^{-4n\pi^2/s}\,ds \\ &= (2\pi)^{12}x^{12+r}\sum_1^\infty \tau(n)\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}u^{-13-r}\exp\left(u-\frac{4\pi^2nx}{u}\right)du \\ &= (2\pi)^{-r}\sum_1^\infty\left(\frac{x}{n}\right)^{6+\frac{1}{2}r}\tau(n)J_{12+r}\{4\pi\sqrt{(nx)}\}, \end{aligned}$$

provided that we may invert the order of integration and summation. This is permissible if

$$\frac{e^x}{2\pi}\int_{-\infty}^{\infty}\frac{1}{|1+it|^{13+r}}\sum_1^\infty|\tau(n)|\left|\exp\left(-\frac{4n\pi^2}{1+it}\right)\right|dt<\infty.$$

It is plainly sufficient to prove the integral from 1 to ∞ convergent. But, when $t\geqslant 1$,

$$\left|\exp\left(-\frac{4n\pi^2}{1+it}\right)\right| = \exp\left(-\frac{4n\pi^2}{1+t^2}\right)\leqslant e^{-2n\pi^2/t^2},$$

$$\{\Sigma|\tau(n)|e^{-2n\pi^2/t^2}\}^2 \leqslant \Sigma e^{-2n\pi^2/t^2}\Sigma\tau^2(n)e^{-2n\pi^2/t^2},$$

$$\Sigma e^{-2n\pi^2/t^2} = O(t^2),$$

and

$$\Sigma\tau^2(n)e^{-2n\pi^2/t^2} = O(t^{24}),$$

by Lemma 3. Hence

$$\sum_1^\infty|\tau(n)|\left|\exp\left(-\frac{4n\pi^2}{1+it}\right)\right| = O(t^{13}),$$

and the integral is convergent for $r>1$.

† Hardy (2), p. 676. Lemma 4 is a corollary of Lemma 3. It has been proved by Davenport and Salié, by a refinement of a method first used by Kloosterman, that

$$\tau(n) = O(n^{35/6+\epsilon})$$

for every positive ϵ; but we do not require this more delicate result. See Davenport (1) and Salié (6).

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This proves (1·7) for $r > 1$. In particular

$$T_2(x) = \tfrac{1}{2}\sum_{n\leqslant x}(x-n)^2\tau(n) = \left(\frac{1}{2\pi}\right)^2\sum_1^\infty\left(\frac{x}{n}\right)^7\tau(n)\,J_{14}\{4\pi\sqrt{(nx)}\}; \qquad (3\cdot5)$$

and (3·4) is the result of formal differentiation of (3·5). The general term of (3·4) is

$$O\{x^{13/2}n^{-13/2}\,|\,\tau(n)\,|\,n^{-1/4}x^{-1/4}\} = O\{n^{-27/4}\,|\,\tau(n)\,|\,\}$$

uniformly for $0 < x \leqslant X$; and $\Sigma n^{-27/4}\,|\,\tau(n)\,|$ is convergent, by Lemma 5. This justifies the differentiation and proves (3·4).

LEMMA 7. *$T_1(x) = O(x^{25/4})$ for large x.*

For $$T_1(x) = O(x^{25/4}\Sigma n^{-27/4}\,|\,\tau(n)\,|) = O(x^{25/4}),$$

by Lemma 5.

LEMMA 8. *$T_0(x) = O(x^{49/8})$ for large x*†.

Suppose $0 < y < x$. Then

$$T_0(t) = T_0(x) + \sum_{x<n\leqslant t}\tau(n) = T_0(x) + O(yx^6)$$

uniformly for $x < t < x+y$. Hence

$$\begin{aligned} yT_0(x) &= \int_x^{x+y}\{T_0(t) + O(yx^6)\}\,dt \\ &= T_1(x+y) - T_1(x) + O(y^2x^6) = O(x^{25/4}) + O(y^2x^6). \end{aligned}$$

The result follows on taking $y = x^{1/8}$.

PROOF OF THEOREM A

4. We now follow the method used in the circle problem by Landau and myself. It is convenient to agree that $\tau(0)$ means 0, and that $z^{-\nu}J_\nu(z)$ is to be interpreted, when $z = 0$, as its limiting value.

We suppose that $$0 < x_1 \leqslant x \leqslant x_2, \quad N > 1.$$

Then

$$\begin{aligned} &x^6\sum_0^N\frac{\tau(n)}{x^6}J_{12}\{4\pi\sqrt{(nx)}\} - \frac{x^6}{N^6}T_0(N)\,J_{12}\{4\pi\sqrt{(Nx)}\} \\ &= -x^6\sum_0^N\tau(n)\left[\frac{J_{12}\{4\pi\sqrt{(Nx)}\}}{N^6} - \frac{J_{12}\{4\pi\sqrt{(nx)}\}}{n^6}\right] \\ &= -x^6\sum_0^N\tau(n)\int_n^N d\,\frac{J_{12}\{4\pi\sqrt{(xt)}\}}{t^6} \\ &= -x^6\int_0^N T_0(t)\,d\,\frac{J_{12}\{4\pi\sqrt{(xt)}\}}{t^6} = 2\pi x^{13/2}\int_0^N T_0(t)\frac{J_{13}\{4\pi\sqrt{(xt)}\}}{t^{13/2}}\,dt \\ &= 2\pi\left(\frac{x}{N}\right)^{13/2}T_1(N)\,J_{13}\{4\pi\sqrt{(Nx)}\} + 4\pi^2x^7\int_0^N T_1(t)\frac{J_{14}\{4\pi\sqrt{(xt)}\}}{t^7}\,dt. \end{aligned}$$

† A good deal more than this is known. Thus Wilton proves that $T_0(x) = O(x^6\log x)$, and we can prove more, in fact $T_0(x) = O(n^{127/18+\epsilon})$, by using the best known results about the order of $\tau(n)$.

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Hence, if
$$S = S(x, N) = x^6 \sum_0^N \frac{\tau(n)}{n^6} J_{12}\{4\pi\sqrt{(nx)}\}, \tag{4·1}$$

$$K = K(x, N) = 4\pi^2 x^7 \int_0^N T_1(t) \frac{J_{14}\{4\pi\sqrt{(xt)}\}}{t^7}\, dt, \tag{4·2}$$

we have

$$S = \left(\frac{x}{N}\right)^6 T_0(N) J_{12}\{4\pi\sqrt{(Nx)}\} + 2\pi\left(\frac{x}{N}\right)^{13/2} T_1(N) J_{13}\{4\pi\sqrt{(Nx)}\} + K. \tag{4·3}$$

5. We may substitute the series (3·4), with t for x, for $T_1(t)$ in K, and integrate term by term. We thus obtain

$$K = 2\pi x^7 \sum_1^\infty \frac{\tau(n)}{n^{13/2}} \int_0^N \frac{J_{13}\{4\pi\sqrt{(nt)}\} J_{14}\{4\pi\sqrt{(xt)}\}}{\sqrt{t}}\, dt \tag{5·1}$$

$$= x^{13/2} \sum_1^\infty \frac{\tau(n)}{n^{13/2}} \int_0^{4\pi\sqrt{(Nx)}} J_{14}(u) J_{13}\left\{u\sqrt{\left(\frac{n}{x}\right)}\right\} du,$$

on making the substitution $4\pi\sqrt{(xt)} = u$.

We write
$$\xi = \sqrt{\left(\frac{n}{x}\right)}, \tag{5·2}$$

$$I = I(n, N, x) = \int_0^{4\pi\sqrt{(Nx)}} J_{14}(u) J_{13}(\xi u)\, du = \int_0^\infty - \int_{4\pi\sqrt{(Nx)}}^\infty = I_1 - I_2, \tag{5·3}$$

$$K = x^{13/2} \sum_1^\infty \frac{\tau(n)}{n^{13/2}} I = x^{13/2} \sum_1^\infty \frac{\tau(n)}{n^{13/2}} I_1 - x^{13/2} \sum_1^\infty \frac{\tau(n)}{n^{13/2}} I_2 = K_1 - K_2. \tag{5·4}$$

Then, by (2·5),
$$I_1 = \left(\frac{n}{x}\right)^{13/2}, \ \tfrac{1}{2}, \ 0$$

according as $n < x$, $n = x$, $n > x$; and

$$K_1 = \sum_{n \leqslant x}{}' \tau(n) = T(x);$$

so that (4·3) takes the form

$$S - T(x) = \left(\frac{x}{N}\right)^6 T_0(N) J_{12}\{4\pi\sqrt{(Nx)}\} + 2\pi\left(\frac{x}{N}\right)^{13/2} T_1(N) J_{13}\{4\pi\sqrt{(Nx)}\} - K_2. \tag{5·5}$$

6. We have to show that the three terms on the right in (5·4) tend to 0 when $N \to \infty$. The first two are

$$O(N^{-6} N^{49/8} N^{-1/4}) = O(N^{-1/8}), \quad O(N^{-13/2} N^{25/4} N^{-1/4}) = O(N^{-1/2}),$$

by Lemmas 8 and 7, and tend to 0 uniformly for x of (x_1, x_2). It remains to dispose of K_2.

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We have

$$J_{14}(u) = -\sqrt{\left(\frac{2}{\pi}\right)} u^{-1/2} \cos(u - \tfrac{1}{4}\pi) + O(n^{-3/2}),$$

$$J_{13}(\xi u) = \sqrt{\left(\frac{2}{\pi}\right)} x^{1/4} n^{-1/4} u^{-1/2} \cos(\xi u - \tfrac{3}{4}\pi) + O(x^{3/4} n^{-3/4} u^{-3/2}),$$

$$J_{14}(u) J_{13}(\xi u) = -\frac{2}{\pi} x^{1/4} n^{-1/4} u^{-1} \cos(u - \tfrac{1}{4}\pi) \cos(\xi u - \tfrac{3}{4}\pi) + R(u) \quad (6{\cdot}1)$$

$$= Q(u) + R(u),$$

say, where

$$R(u) = R(u, n, x) = O(x^{3/4} n^{-3/4} u^{-2}) + O(x^{1/4} n^{-1/4} u^{-2}) + O(x^{3/4} n^{-3/4} u^{-3}),$$

with O's uniform in all the variables. It follows that

$$R(u) = O(n^{-1/4} u^{-2})$$

uniformly for $n \geqslant 1$, $x_1 \leqslant x \leqslant x_2$ and $u \geqslant 4\pi\sqrt{x}$; and that the contribution of $R(u)$ to K_2 is

$$O\left\{\sum_1^\infty \frac{|\tau(n)|}{n^{27/4}} \int_{4\pi\sqrt{(Nx)}}^\infty \frac{du}{u^2}\right\} = O(N^{-1/2}).$$

and tends uniformly to 0.

Finally $$Q(u) = -\frac{1}{\pi} x^{1/4} n^{-1/4} u^{-1} \{\sin(\xi - 1) u - \cos(\xi + 1) u\},$$

and the contribution of $Q(u)$ to K_2 is

$$K^* = -\frac{1}{\pi} x^{27/4} \sum_1^\infty \frac{\tau(n)}{n^{27/4}} L(n, N, x), \quad (6{\cdot}2)$$

where

$$L(n, N, x) = \int_{4\pi\sqrt{(Nx)}}^\infty \frac{\sin(\xi - 1) u}{u} du - \int_{4\pi\sqrt{(Nx)}}^\infty \frac{\cos(\xi + 1) u}{u} du$$

$$= \operatorname{sgn}(\xi - 1) \int_{4\pi|\xi-1|\sqrt{(Nx)}}^\infty \frac{\sin w}{w} dw - \int_{4\pi(\xi+1)\sqrt{(Nx)}}^\infty \frac{\cos w}{w} dw,$$

and $\operatorname{sgn}(\xi - 1)$ means 0 if $\xi = 1$. The integrals are uniformly bounded for $n \geqslant 1$, $N \geqslant 1$, $x_1 \leqslant x \leqslant x_2$, so that the series (6·2) is uniformly convergent; and $L(n, N, x) \to 0$ when $N \to \infty$, for every n. Hence

$$K^* \to 0.$$

The convergence is uniform in (x_1, x_2) if $\xi \neq 1$ for any n and any x of the interval, i.e. if the interval contains no integral x.

This completes the proof of the theorem.

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