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ISOLATED SUBGROUPS OF FINITE p-GROUPS

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Abstract

We say that a subgroup H is isolated in a group G if for each $x \in G$ either $x \in H$ or $\langle x \rangle \cap H = 1$. We determine the structure of finite p-groups with isolated minimal nonabelian subgroups and finite p-groups with an isolated metacyclic subgroup.

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1. Introduction

Let G be a finite group. We say that a subgroup H is isolated in G if for each $x \in G$ either $x \in H$ or $\langle x \rangle \cap H = 1$. In 2016, Janko [4] first studied isolated subgroups of finite p-groups and obtained many interesting results. Among them, he proved the following result for finite p-groups with isolated minimal nonabelian subgroups.

THEOREM 1.1 [4, Theorem 4]. *Let G be a nonabelian p-group of exponent greater than p which is not minimal nonabelian and all of whose minimal nonabelian subgroups are isolated in their normalisers. Then we have the following possibilities:*

(a) *$p > 2$ and G has an abelian maximal subgroup A of exponent $> p$ with $A = \mathrm{H}_p(G)$ and all minimal nonabelian subgroups are isomorphic to $\mathrm{S}(p^3)$ (a nonabelian group of order p^3 and exponent p) and, if S is any one of them, then $N_G(S)$ is of exponent p;*

(b) *$p = 3$ and G has a nonmetacyclic minimal nonabelian subgroup H of exponent greater than three and index three with $H = \mathrm{H}_3(G)$, where*

$$H = \langle a, b \mid a^{3^m} = b^{3^n} = c^3 = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle$$

with $m \geqslant 2$, $n \geqslant 1$ and either $m = n$ or $m = n + 1$, and each minimal nonabelian subgroup S distinct from H is isomorphic to $\mathrm{S}(27)$ and $N_G(S) = SZ(G)$, where $Z(G) \cong \mathrm{E}_9$ and $SZ(G) = S' = \mathrm{C}_3$.

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We will prove that $p \geqslant 5$ in the possibility (a) in Theorem 1.1, and the possibility (b) in Theorem 1.1 does not occur. Theorem 1.1 can therefore be refined as follows.

THEOREM 1.2. *Let G be a nonabelian p-group of exponent greater than p which is not minimal nonabelian. Then all minimal nonabelian subgroups of G are isolated in their normalisers if and only if all minimal nonabelian subgroups of G are isomorphic to $\mathrm{S}(p^3)$, where $p \geqslant 5$.*

We also determine the structure of finite p-groups which have an isolated metacyclic subgroup.

Let G be a finite p-group. We use $p^{e(G)}$ to denote the exponent of G. For any integer s with $0 \leq s \leq e = e(G)$, we define

$$\Omega_s(G) = \langle a \in G \mid a^{p^s} = 1 \rangle \quad \text{and} \quad \mho_s(G) = \langle a^{p^s} \mid a \in G \rangle.$$

We use E_{p^k} to denote an elementary abelian group of order p^k, $\mathrm{S}(p^3)$ $(p > 2)$ a nonabelian group of order p^3 and exponent p and D_{2^n} $(n \geqslant 3)$ a dihedral group of order 2^n. The Hughes subgroup $\mathrm{H}_p(G)$ is the subgroup of G generated by all elements of order not equal to p. We write 'H char G' to indicate that H is a characteristic subgroup of G. The other notation and terminology are standard, as in [2].

2. Finite p-groups with isolated minimal nonabelian subgroups

In this section, we prove Theorem 1.2. The first proposition is straightforward.

PROPOSITION 2.1. *Let G be a finite p-group.*

(1) *Assume that $K \leq G$ and $H \leq G$. If K is isolated in G, then $H \cap K$ is isolated in H. In particular, if $K \leq H$, then K is isolated in H.*
(2) *Assume that $N \leq H \leq G$ and $N \trianglelefteq G$. If H is isolated in G, then H/N is isolated in G/N.*

We also need the following lemmas.

LEMMA 2.2 [6, Lemma 2.2]. *Suppose that G is a finite nonabelian p-group. Then the following conditions are equivalent:*

(1) *G is minimal nonabelian;*
(2) *$d(G) = 2$ and $|G'| = p$;*
(3) *$d(G) = 2$ and $\Phi(G) = Z(G)$.*

LEMMA 2.3. *Let G be a finite 2-group. If G has a proper subgroup H which is isolated in G, then H is abelian.*

PROOF. Let $T \leq G$ be such that $|T : H| = 2$. Since H is isolated in G, H is isolated in T and so $o(x) = 2$ for all $x \in T - H$. Let $y \in H$. Then $o(xy) = 2$. So, $y^x = y^{-1}$. Hence, $(ab)^x = (ab)^{-1}$ for all $a, b \in H$. On the other hand, $(ab)^x = a^x b^x = a^{-1}b^{-1}$. So, $ab = ba$, that is, H is abelian. □

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LEMMA 2.4. *Suppose that G is a finite 3-group, M is maximal in G and $g^3 = 1$ for all $g \in G - M$. If $[a, b] = 1$ for some $a, b \in M$, then $[a^x, b] = 1$ for all $x \in G - M$.*

PROOF. Let $x \in G - M$ and $y \in M$. Then $(xy)^3 = 1$. That is, $yxy = x^{-1}y^{-1}x^{-1}$. Now

$$\begin{aligned}[a^x, b] &= x^{-1}a^{-1}xb^{-1}x^{-1}axb \\ &= x^{-1}a^{-1}(xb^{-1}x)(xax)b \\ &= x^{-1}a^{-1}(bx^{-1}b)(a^{-1}x^{-1}a^{-1})b \\ &= x^{-1}a^{-1}bx^{-1}a^{-1}bx^{-1}a^{-1}b \\ &= (x^{-1}a^{-1}b)^3 = 1.\end{aligned}$$

□

LEMMA 2.5. *Suppose that G is a finite p-group and H is a proper minimal nonabelian subgroup of G. If H is isolated in G, then H is isomorphic to $\mathrm{S}(p^3)$.*

PROOF. By Lemma 2.3, we have $p > 2$. Let $T \leq G$ be such that $|T : H| = p$. Since H is isolated in G, H is isolated in T and so all elements in $T - H$ are of order p.

If $p \geqslant 5$, then, by [5, Corollary 3.4], T has at least two minimal nonabelian subgroups of index p. Moreover, by [1, Theorem 2.7], we have $\mathrm{cl}(G) \leqslant 3$. It follows from [2, Theorem 7.1] that T is regular. Notice that $T = \langle T - H\rangle$ and all elements in $T - H$ are of order p. We have $\exp(T) = p$. Moreover, $\exp(H) = p$. Hence, $H \cong \mathrm{S}(p^3)$.

In the following, assume that $p = 3$.

First, we claim that $T' \leq \mathrm{Z}(H) = \Phi(H)$. In fact, since H is minimal nonabelian, $\mathrm{Z}(H) = \Phi(H)$ and $|H : \mathrm{Z}(H)| = 3^2$ by Lemma 2.2. Let $h \in H - \mathrm{Z}(H)$. It follows from $|H : \mathrm{Z}(H)| = 3^2$ that $\mathrm{C}_H(h) = \langle h\rangle\mathrm{Z}(H)$ and $\mathrm{C}_H(h)$ is maximal in H. Notice that $o(t) = 3$ for all $t \in T - H$. By Lemma 2.4, $[h^t, h] = 1$. So, $h^t \in \mathrm{C}_H(h)$. It follows that $\mathrm{C}_H(h) \trianglelefteq T$. Since $|T/\mathrm{C}_H(h)| = 3^2$, we have $T' \leq \mathrm{C}_H(h)$. Since h is arbitrary, $T' \leq \Phi(H) = \mathrm{Z}(H)$.

Next, we claim that $H \cong \mathrm{S}(27)$. Otherwise, $\exp(H) > \exp(\mathrm{Z}(H))$. Hence, $a \notin \mathrm{Z}(H)$ for all $a \in H$ with $o(a) = \exp(H)$. Let $x \in T - H$. Then $o(x) = o(xa^{-1}) = 3$. Since $T' \leq \mathrm{Z}(H)$,

$$1 = (xa^{-1})^3 = x^3[x, a]^{\binom{3}{2}}[x, a, x][x, a, a]a^{-3} = [x, a]^3[x, a, x]a^{-3}.$$

Now, we consider $[x, a, x]$. Without loss of generality, we may assume that $H = \langle a, b\rangle$ and $H' = \langle c\rangle$. Notice that $[x, a] \in T' \leq \mathrm{Z}(H) = \Phi(H) = \langle a^3, b^3, c\rangle$. We may assume that $[x, a] = a^{3i}b^{3j}c^k$. Since $|H'| = 3$, we have $H' \leq \mathrm{Z}(T)$. So, $[c^k, x] = 1$. Hence,

$$[x, a, x] = [a^{3i}b^{3j}c^k, x] = [a^{3i}b^{3j}, x] = [a^{3i}, x][b^{3j}, x] = [a^i, x]^3[b^j, x]^3.$$

Since $o(a) = \exp(H) > \exp(\mathrm{Z}(H))$, $o([x, a]^3) < o(a^{-3})$ and $o([x, b]^3) < o(a^{-3})$. So, $o([x, a, x]) < o(a^{-3})$. It follows that $[x, a]^3[x, a, x]a^{-3} \neq 1$. This is a contradiction. □

LEMMA 2.6 [7, Theorem 4.5]. *Assume that G is a finite nonabelian p-group and p an odd prime. Then all minimal nonabelian subgroups of G are isomorphic to $\mathrm{S}(p^3)$ if and only if $\exp(G) = p$ or $G = \mathrm{H}_p(G) \rtimes \langle a\rangle$, a semidirect product of $\mathrm{H}_p(G)$ by $\langle a\rangle$, where the Hughes subgroup $\mathrm{H}_p(G) = B_1 \times B_2 \times \cdots \times B_n$ is an abelian subgroup of index p*

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and $o(a) = p$. Moreover, for all i with $1 \le i \le n$, $B_i\langle a\rangle$ is a group of maximal class with an abelian subgroup B_i of index p, and all elements of $B_i\langle a\rangle - B_i$ are of order p.

LEMMA 2.7. *Let G be a nonabelian p-group of exponent greater than p, where $p \geqslant 5$. If all minimal nonabelian subgroups of G are isomorphic to $\mathrm{S}(p^3)$, then $\mathrm{N}_G(S)$ is of exponent p for any minimal nonabelian subgroup S of G.*

PROOF. Assume the contrary and $\exp(\mathrm{N}_G(S)) > p$ for some minimal nonabelian subgroup S of G. Then there exists $a \in \mathrm{N}_G(S) - S$ such that $o(a) = p^2$. Let $L = S\langle a\rangle$. Then $|L| \leqslant p^5$ and hence $\mathrm{cl}(L) \leqslant 4$. So, L is regular. By [2, Proposition 10.28], L is generated by minimal nonabelian subgroups. So, $\exp(L) = p$, which is a contradiction. □

PROOF OF THEOREM 1.2. ($\Leftarrow$:) By Lemma 2.7, S is isolated in $\mathrm{N}_G(S)$ for any minimal nonabelian subgroup S of G.

($\Rightarrow$:) By Lemma 2.3, $p > 2$. By Lemma 2.5, all minimal nonabelian subgroups of G are isomorphic to $\mathrm{S}(p^3)$. By Lemma 2.6, $G = \mathrm{H}_p(G) \rtimes \langle a\rangle$ has the structure described as Lemma 2.6. Assume that $p = 3$. We will prove that this is a contradiction.

If $\mathrm{cl}(G) = 2$, then G is regular. Notice that all elements of $G - \mathrm{H}_3(G)$ are of order three and $G = \langle G - \mathrm{H}_3(G)\rangle$. We have $\exp(G) = 3$. This contradicts the hypothesis of $\exp(G) > 3$.

If $\mathrm{cl}(G) \geqslant 3$, then there exists some j such that $\mathrm{cl}(B_j\langle a\rangle) \geqslant 3$. Let $M = \mathrm{Z}_3(B_j\langle a\rangle)\langle a\rangle$, where $\mathrm{Z}_3(B_j\langle a\rangle)$ is the third term of the upper central series of $B_j\langle a\rangle$. Then M is of maximal class and $|M| = 3^4$. By checking the list of groups of order 3^4, there is a $c \in M$ such that $c^3 \neq 1$. Let S be a minimal nonabelian subgroup of M. Then $M \leq \mathrm{N}_G(S)$. Since $c^3 \cap S \neq 1$, S is not isolated in its normaliser. This is a contradiction. □

REMARK 2.8. If G is the group in Theorem 1.2 and all minimal nonabelian subgroups of G are isomorphic to $\mathrm{S}(p^3)$, then, by Lemma 2.6, G has an abelian maximal subgroup A of exponent greater than p with $A = \mathrm{H}_p(G)$. By Lemma 2.7, $\mathrm{N}_G(S)$ is of exponent p for any minimal nonabelian subgroup S of G. This means that Theorem 1.2 contains all conclusions in Theorem 1.1.

THEOREM 2.9. *Let G be a nonabelian p-group which is not minimal nonabelian. Then all minimal nonabelian subgroups of G are isolated in G if and only if $\exp(G) = p$.*

PROOF. ($\Leftarrow$:) It is obvious.

($\Rightarrow$:) Since all minimal nonabelian subgroups of G are isolated in G, all minimal nonabelian subgroups of G are isolated in their normalisers. By Lemma 2.5, all minimal nonabelian subgroups of G are isomorphic to $\mathrm{S}(p^3)$.

Assume that $\exp(G) > p$. Then, by Lemma 2.6, $G = \mathrm{H}_p(G) \rtimes \langle a\rangle$ has the structure described in Lemma 2.6. Since $\exp(G) > p$, there exists some j such that $\exp(B_j\langle a\rangle) > p$. So, $\exp(B_j) > p$ and hence $\mathfrak{V}_1(B_j) \neq 1$. Since $B_j\langle a\rangle$ is a group of maximal class, $|\mathrm{Z}(B_j\langle a\rangle)| = p$. Hence, $\mathrm{Z}(B_j\langle a\rangle) \leq \mathfrak{V}_1(B_j)$. Notice that B_j is abelian. Then there exists $b \in B_j$ such that $\mathrm{Z}(B_j\langle a\rangle) = \langle b^p\rangle$. Let $D = \mathrm{Z}_2(B_j\langle a\rangle)\langle a\rangle$, where $\mathrm{Z}_2(B_j\langle a\rangle)$ is the second term of the upper central series of $B_j\langle a\rangle$. Then D is a nonabelian

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group of order p^3. So, $D \cong \mathrm{S}(p^3)$. Since $1 \neq \langle b^p \rangle = \mathrm{Z}(B_j\langle a \rangle) \leq D$, D is not isolated in G. This is a contradiction. □

REMARK 2.10. By Theorem 2.9, the p-groups satisfying the hypothesis of [4, Corollary 5] do not exist.

3. Finite p-groups with an isolated metacyclic subgroup

Firstly, we prove that if H is an isolated metacyclic subgroup, then H is abelian.

THEOREM 3.1. *Let G be a nonabelian p-group and H a proper metacyclic subgroup of G. If H is isolated in $N_G(H)$, then H is abelian. In particular, if H is isolated in G, then H is abelian.*

PROOF. Assume the contrary and that H is nonabelian. By Lemma 2.3, $p > 2$. Let $N = N_G(H)$ and $K = \Phi(H')$. Since K is a characteristic subgroup of H, $K \trianglelefteq N$. By Proposition 2.1, H/K is isolated in N/K. By Lemma 2.2, it is easy to see that H/K is minimal nonabelian. By Lemma 2.5, H/K is isomorphic to $\mathrm{S}(p^3)$, which contradicts the assumption that H is metacyclic. So, H is abelian. □

In the following, we consider finite p-groups with an isolated abelian subgroup.

LEMMA 3.2 [3]. *Let G be a finite p-group with $\mathrm{H}_p(G) \neq 1$. Then $|G : \mathrm{H}_p(G)| \leqslant p$ if G satisfies one of the following conditions:*

(a) *G_i/G_{i+1} is cyclic for all $i \geqslant 2$;*
(b) *G is metabelian.*

THEOREM 3.3. *Let G be a nonabelian p-group and H an abelian subgroup of G with $\exp(H) > p$. If H is isolated in $N_G(H)$, then the following assertions hold:*

(1) *$N_G(H)/H$ has no subgroup isomorphic to E_{p^2};*
(2) *$N_G(H) = G$;*
(3) *$C_G(H) = H$.*

PROOF. (1) Assume the contrary and let $L/H \leq N_G(H)/H$ be such that $L/H \cong \mathrm{E}_{p^2}$. Notice that H is isolated in $N_G(H)$. So, all elements in $L - H$ are of order p. Hence, $\mathrm{H}_p(L) \leq H$. Since $\exp(H) > p$, we have $H \leq \mathrm{H}_p(L)$. So, $H = \mathrm{H}_p(L)$. It follows that $|L : \mathrm{H}_p(L)| = p^2$. On the other hand, since $L/H \cong \mathrm{E}_{p^2}$, we have $L' \leq H$. Hence, L is metabelian. So, by Lemma 3.2, $|L : \mathrm{H}_p(L)| \leqslant p$, which is a contradiction.

(2) By (1), $N_G(H)/H$ has a unique subgroup of order p. Set $M/H \leq N_G(H)/H$ be such that $|M/H| = p$. Then M char $N_G(H)$. Notice that all elements in $M - H$ are of order p. We have $\mathrm{H}_p(M) = H$. It follows that H char M and hence H char $N_G(H)$. So, $N_G(N_G(H)) = N_G(H)$. Thus, $N_G(H) = G$.

(3) Assume the contrary and let $A \leq C_G(H)$ be such that $|A : H| = p$. Since H is isolated in $N_G(H)$, H is isolated in A. Hence, all elements in $A - H$ are of order p. Since A is abelian and $A = \langle A - H \rangle$, we have $\exp(A) = p$, contradicting $\exp(H) > p$. □

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THEOREM 3.4. *Let G be a nonabelian p-group and H an abelian subgroup of G with $\exp(H) > p$. If H is isolated in G, then $p \leqslant d(H) + 1$.*

PROOF. If $p = 2$, then the conclusion is obvious. In the following, we may therefore assume that $p > 2$. Let $T \leq G$ be such that $|T : H| = p$. Notice that H is isolated in T. So, all elements in $T - H$ are of order p. Hence, $\mho_1(T) = \mho_1(H)$. So, $|T : \mho_1(T)| = |T : \mho_1(H)| = p|H : \mho_1(H)| = p|H : \Phi(H)| = p^{d(H)+1}$.

Assume that $p > d(H) + 1$. Then $|T/\mho_1(T)| \leqslant p^{p-1}$. Hence, T is absolutely regular. So, by [2, Theorem 9.8], T is regular. Since $T = \langle T - H \rangle$, we have $\exp(T) = p$, contradicting $\exp(H) > p$. Hence, $p \leqslant d(H) + 1$. □

COROLLARY 3.5. *Let G be a nonabelian p-group and H a proper metacyclic subgroup of G with $\exp(H) > p$. If H is isolated in G, then $p = 2$ or 3.*

COROLLARY 3.6. *Let G be a nonabelian p-group and H a cyclic subgroup of G with $\exp(H) > p$. If H is isolated in G, then $G \cong \mathrm{D}_{2^{e+1}}$, where $e \geqslant 2$.*

PROOF. Let $H = \langle a \rangle$ and $|H| = p^e$, where $e \geqslant 2$. By Theorem 3.4, $p = 2$. By Theorem 3.3, $H \trianglelefteq G$ and there is a unique subgroup M such that $|M/H| = 2$. Let $M = \langle a, b \rangle$. Since H is isolated in G, we have $b^2 = (ba)^2 = 1$. Hence, $a^b = a^{-1}$. So, $M \cong \mathrm{D}_{2^{e+1}}$. We claim that $M = G$. Otherwise, there is a $c \in G$ such that $c^2 = b$. Let $a^c = a^i$. Then $a^b = a^{c^2} = a^{i^2}$. It follows that $i^2 \equiv -1 \pmod{2^e}$, which is a contradiction. Hence, $G = M \cong \mathrm{D}_{2^{e+1}}$. □

COROLLARY 3.7. *Let G be a nonabelian p-group and H an isolated abelian subgroup of G. Assume that H is noncyclic with type invariants*

$$(e_1, e_2, \ldots, e_d), \quad \text{where } e_1 \geqslant e_2 \geqslant \cdots \geqslant e_d \text{ and } e_1 \geqslant 2.$$

If $p > 2$ or $p = 2$ and $|G : H| \geqslant 4$, then $e_1 - e_2 \leqslant 1$.

PROOF. Assume that $e_1 - e_2 \geqslant 2$. By Theorem 3.3, $H \trianglelefteq G$. Hence, $\Omega_{e_2}(H) \trianglelefteq G$. Consider the quotient group $\overline{G} = G/\Omega_{e_2}(H)$. Obviously, $\overline{H} \cong \mathrm{C}_{p^{e_1-e_2}}$. By Corollary 3.6, $\overline{G} \cong \mathrm{D}_{2^{e_1-e_2+1}}$. Hence, $|\overline{G} : \overline{H}| = 2$, contradicting $|G : H| \geqslant 4$. Thus, $e_1 - e_2 \leqslant 1$. □

PROPOSITION 3.8. *Suppose that $G \cong \mathrm{C}_{p^2} \times \mathrm{C}_p$ and $P \in \mathrm{Syl}_p(\mathrm{Aut}(G))$. If $p = 2$, then $P \cong \mathrm{D}_8$. If $p > 2$, then $P \cong \mathrm{S}(p^3)$.*

PROOF. Let $G = \langle x \rangle \times \langle y \rangle \cong \mathrm{C}_{p^2} \times \mathrm{C}_p$, where $o(x) = p^2, o(y) = p$. By calculation, $|\mathrm{Aut}(G)| = p^3(p-1)^2$. Hence, $|P| = p^3$. Let

$$\alpha : x \to x, y \to yx^p,$$
$$\beta : x \to xy, y \to y,$$
$$\gamma : x \to x^{1-p}, y \to y.$$

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By straightforward calculation, $\alpha, \beta, \gamma \in \mathrm{Aut}(G)$ and

$$o(\alpha) = o(\beta) = o(\gamma) = p, \quad [\alpha, \beta] = \gamma, \quad [\gamma, \alpha] = [\gamma, \beta] = 1.$$

Hence, $\langle \alpha, \beta, \gamma \rangle \in \mathrm{Syl}_p(\mathrm{Aut}(G))$ and the conclusions in the theorem hold. □

In the following result, we determine finite p-groups with an isolated metacyclic subgroup.

THEOREM 3.9. *Let G be a nonabelian p-group. Then G has a metacyclic subgroup with exponent greater than p which is isolated in its normaliser if and only if G is isomorphic to one of the following nonisomorphic groups:*

(I) $\mathrm{D}_{2^{e+1}}$, *where* $e \geqslant 2$;
(II) $\langle a, b, t \mid a^{2^e} = b^{2^{e'}} = t^2 = 1, [a, b] = 1, a^t = a^{-1}, b^t = b^{-1} \rangle$, *where* $e \geqslant e'$ *and* $e \geqslant 2$;
(III) $\langle a, x, b \mid a^{2^e} = b^{2^{e'}} = x^4 = 1, [a, b] = 1, a^x = ab, b^x = b^{-1}a^{-2} \rangle$, *where* $e \geqslant e' \geqslant e - 1 \geqslant 1$;
(IV) $\langle a, t, b \mid a^{3^e} = b^{3^{e'}} = t^3 = 1, [a, b] = 1, [a, t] = b, [b, t] = a^{-3}b^{-3} \rangle$, *where* $e \geqslant e' \geqslant e - 1 \geqslant 1$.

PROOF. ($\Rightarrow$:) Let H be a metacyclic subgroup of G with exponent greater than p and H be isolated in $N_G(H)$. By Theorem 3.1, H is abelian. Moreover, by Theorem 3.3, $N_G(H) = G$. So, H is isolated in G and $H \trianglelefteq G$.

If H is cyclic, then, by Corollary 3.6, $G \cong \mathrm{D}_{2^{e+1}}$, giving the group (I) in the theorem.

In the following, we assume that $d(H) = 2$. By Corollary 3.5, $p = 2$ or 3.

Case 1: $p = 2$. Assume that $H = \langle a \rangle \times \langle b \rangle \cong \mathrm{C}_{2^e} \times \mathrm{C}_{2^{e'}}$, where $e \geqslant e'$, $e \geqslant 2$. By Theorem 3.3, $H \trianglelefteq G$ and there is a unique subgroup T such that $|T/H| = 2$. Since H is isolated in G, H is isolated in T and so all elements in $T - H$ are of order two.

Let $t \in T - H$. Then $(ht)^2 = 1$ for all $h \in H$. Hence, $t^2 = 1$ and $h^t = h^{-1}$ for all $h \in H$. If $T = G$, then we get the group (II) in the theorem.

Next, assume that $T < G$. By Corollary 3.7, $e \geqslant e' \geqslant e - 1 \geqslant 1$. Let $M \leq G$ be such that $|M : T| = 2$. By Theorem 3.3, $M/H \cong \mathrm{C}_4$. Let $x \in M - T$. Since $H \trianglelefteq M$, we may assume that $a^x = a^i b^j, b^x = a^k b^l$, where $2 \nmid il - jk$. Notice that $x^2 \in T - H$.

If $2 \mid i$, then $e = e'$ and $2 \nmid j$ since $o(a^x) = o(a)$. Replacing b with $a^{i-1}b^j$, we have $a^x = ab$.

If $2 \nmid i$ and $2 \nmid j$, then, replacing b with $a^{i-1}b^j$, we have $a^x = ab$.

If $2 \nmid i$ and $2 \mid j$, then $2 \nmid l$ since $2 \nmid il - jk$. We claim that $2 \nmid k$. Otherwise, $4 \mid kj$. By calculation, $a^{-1} = a^{x^2} = a^{i^2+kj}b^{ij+lj}$. It follows that $-1 \equiv i^2 \pmod 4$, which is a contradiction. Hence, $2 \nmid k$. Moreover, $e = e'$. Replacing a and b with b and $a^k b^{l-1}$, we also have $a^x = ab$.

Hence, we may assume that $a^x = ab$. By calculation, $a^{-1} = a^{x^2} = (ab)^x = abb^x$. So, $b^x = b^{-1}a^{-2}$.

We claim that $M = G$. If not, we have $|G/H| \geqslant 8$. Without loss of generality, we may assume that $|G/H| = 8$. Notice that $\mathfrak{V}_2(H)$ char H and $H \trianglelefteq G$. We have $\mathfrak{V}_2(H) \trianglelefteq G$. Let

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$\overline{G} = G/\mho_2(H)$ and $\overline{H} = H/\mho_2(H)$. By Proposition 2.1, $\overline{H}$ is isolated in $\overline{G}$. Obviously, $|\overline{G} : \overline{H}| = 8$ and $\exp(\overline{H}) = 4$. So, we always assume that $\exp(H) = 4$. Notice $d(H) = 2$ and H is abelian. We have $H \cong C_4 \times C_2$ or $H \cong C_4^2$. For the latter, let $H_1 \leq H$ be such that $|H_1| = 2$ and $H_1 \trianglelefteq G$. Consider the quotient group G/H_1. It is easy to see that H/H_1 is isolated in G/H_1, $|G/H_1 : H/H_1| = 8$ and $H/H_1 \cong C_4 \times C_2$. So, we always assume that $H \cong C_4 \times C_2$. Notice that H is isolated in G. By Theorem 3.3, $C_G(H) = H$. Hence,

$$G/H = N_G(H)/C_G(H) \lesssim \mathrm{Aut}(H).$$

Notice that $|G/H| = 8$. By Proposition 3.8, $G/H \cong D_8$. So, G/H has a subgroup isomorphic to E_4. This contradicts Theorem 3.3.

Summarising, we have $M = G$. So, we get the group (III) in the theorem.

Case 2: $p = 3$. Assume that $H \cong C_{3^e} \times C_{3^{e'}}$, where $e \geqslant e'$, $e \geqslant 2$. By Corollary 3.7, $e \geqslant e' \geqslant e - 1 \geqslant 1$. By Theorem 3.3, $H \trianglelefteq G$ and there is a unique subgroup T such that $|T/H| = 3$. Since H is isolated in G, H is isolated in T and so all elements in $T - H$ are of order three.

Let $M < H$ be such that $|M/\mho_1(H)| = 3$ and $M \trianglelefteq G$. Take $t \in T - H$, $a \in H - M$ and $b \in M - \mho_1(H)$. Then $t^3 = 1$ and $H = \langle a \rangle \times \langle b \rangle$. Notice that $\langle b^3, a^9 \rangle = \mho_1(M)$ is a normal subgroup of T and $|\mho_1(H)/\mho_1(M)| = 3$.

We claim that $[a, t] \in M - \mho_1(H)$. Otherwise, $[a, t] \in \mho_1(H)$. It follows that $[a, t, t] \in \mho_1(M) = \langle b^3, a^9 \rangle$. By calculation, $1 = (at^{-1})^3 = a^3[a, t]^{\binom{3}{2}}[a, t, t]$. Hence, $a^3 = [a, t]^{-3}[a, t, t]^{-1} \in \langle b^3, a^9 \rangle$, which is a contradiction. So, $[a, t] \in M - \mho_1(H)$. Replacing b with $[a, t]$, we have $[a, t] = b$ and $[b, t] = a^{-3}b^{-3}$.

Now, we claim that $T = G$. Otherwise, we have $|G/H| \geqslant 9$. Without loss of generality, we may assume that $|G/H| = 9$. By Theorem 3.3, $G/H \cong C_9$. By a similar argument to that for $p = 2$, we always assume that $H \cong C_9 \times C_3$. By Theorem 3.3, we have $C_G(H) = H$. Hence,

$$G/H = N_G(H)/C_G(H) \lesssim \mathrm{Aut}(H).$$

By Proposition 3.8, $G/H \lesssim S(27)$. Hence, $\exp(G/H) = 3$. This contradicts $G/H \cong C_9$. Hence, $T = G$ and we get the group (IV) in the theorem.

($\Leftarrow$:) For the group (I), it is obvious that the unique cyclic maximal subgroup of G is isolated in G.

For the group (II), it is easy to see that $\langle b, t \rangle$ is a dihedral group. Hence, $o(b^i t) = 2$. Since $\langle a, b^i t \rangle$ is also a dihedral group, $o(a^i b^i t) = 2$. Hence, $H = \langle a, b \rangle$ is isolated in G.

For the group (III), since $\langle a, b, x^2 \rangle$ is isomorphic to the group (II), we have $o(a^i b^i x^2) = 2$ and $(a^i b^i x^{2k} x)^2 \notin H$. Hence, $H = \langle a, b \rangle$ is isolated in G.

The group (IV) is a 3-group of maximal class. Its uniform elements are all of order three. Hence, $H = \langle a, b \rangle$ is isolated in G. □

The following corollary is a refinement of [4, Theorem 13].

COROLLARY 3.10. *Let G be a nonmetacyclic p-group of exponent greater than p. Then all maximal metacyclic subgroups are isolated in their normalisers in G if and only*

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if $G = \langle a, t, b \mid a^{3^e} = b^{3^{e'}} = t^3 = 1, [a,b] = 1, [a,t] = b, [b,t] = a^{-3}b^{-3}\rangle$, *where* $e \geqslant e' \geqslant e-1 \geqslant 1$, $H = \mathrm{H}_3(G) = \langle a\rangle \times \langle b\rangle$ *is noncyclic abelian, all other maximal metacyclic subgroups are isomorphic to* E_9 *and* G *is of maximal class.*

PROOF. By Theorem 3.1, metacyclic subgroups of G are of exponent p or abelian. Hence, for $p = 2$, there is no nonabelian metacyclic subgroup in G. By Theorem 3.9, G is isomorphic to one of the groups (II)–(IV). For the group (II), $\langle a^{2^{e-2}}, t\rangle$ is a nonabelian metacyclic subgroup. For the group (III), $\langle a^{2^{e-2}}, x^2\rangle$ is a nonabelian metacyclic subgroup. Hence, G is isomorphic to the group (IV).

Now, it is easy to see that $H = \mathrm{H}_3(G) = \langle a\rangle \times \langle b\rangle$ is noncyclic abelian and G is of maximal class. Let H_1 be a maximal metacyclic subgroup of G such that $H_1 \neq H$. Since $G = HH_1$, we have $|H_1 : H_1 \cap H| = 3$ and $H \cap H_1 \leq Z(G)$. Hence, $H_1 = \langle x, Z(G)\rangle$, where $x \in G - H$. Since $x^3 = 1$, we have $H_1 \cong \mathrm{E}_9$. It is easy to see that $N_G(H_1) = \langle x, Z_2(G)\rangle$, where $Z_2(G) = \Omega_1(H)$ is the unique normal subgroup of order nine in G. Hence, $N_G(H_1) \cong \mathrm{S}(27)$ and H_1 is isolated in $N_G(H_1)$. Summarising, all maximal metacyclic subgroups are isolated in their normalisers in G. □

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