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STABILITY OF MULTICLASS QUEUEING NETWORKS UNDER LONGEST-QUEUE AND LONGEST-DOMINATING-QUEUE SCHEDULING

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Abstract

We consider the stability of robust scheduling policies for multiclass queueing networks. These are open networks with arbitrary routeing matrix and several disjoint groups of queues in which at most one queue can be served at a time. The arrival and potential service processes and routeing decisions at the queues are independent, stationary, and ergodic. A scheduling policy is called *robust* if it does not depend on the arrival and service rates nor on the routeing probabilities. A policy is called *throughput-optimal* if it makes the system stable whenever the parameters are such that the system can be stable. We propose two robust policies: longest-queue scheduling and a new policy called longest-dominating-queue scheduling. We show that longest-queue scheduling is throughput-optimal for two groups of two queues. We also prove the throughput-optimality of longest-dominating-queue scheduling when the network topology is acyclic, for an arbitrary number of groups and queues.

Keywords: Stability; longest-queue scheduling; queueing network; fluid model

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Secondary 90B15; 60G17

1. Introduction

We consider the scheduling of multiclass queueing networks [4], [13]. These are queueing networks with disjoint groups of queues that cannot be scheduled simultaneously. The max-weight scheduling algorithm, proposed in [5] and [18], is known to be throughput-optimal for these networks. However, max-weight scheduling suffers from high complexity and also dependency on the knowledge of all of the service rates, queue lengths, and routeing probabilities. A natural low complexity scheduling algorithm is longest-queue (LQ) scheduling, studied in [8], [15], and [19], which we discuss in detail in this paper.

In general,we know that the utilization being less than 1 for each server is necessary but not sufficient for the stability of a queueing network. This condition specifies that work arrives at each server at rate less than 1. In Lu and Kumar [13] the authors provided an example of a network (Figure 1) with priority scheduling that is unstable despite satisfying the utilization condition. The priority is given to queue 2 in group 1 and to queue 3 in group 2. To see this, assume that $\mu_1 > \mu_3$ and $\mu_4 > \mu_2$ and that queue 3 is initially empty while queue 2 is not. Group 1 serves queue 2, so that queue 1 is not served and queue 3 remains empty. Eventually,

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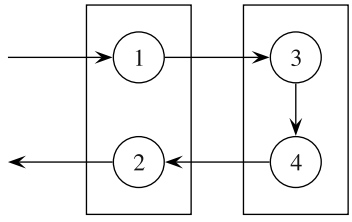


FIGURE 1: Lu–Kumar network.

queue 2 becomes empty and queues 1 and 3 are served until queue 3 becomes empty (because $\mu_1 > \mu_3$). At that time, queues 4 and 2 are served until queue 2 becomes empty (because $\mu_4 > \mu_2$). Thus, queues 2 and 3 are never served together. Consequently, they form a virtual group and the system cannot be stable unless the utilization of that virtual group is less than 1, which is an additional condition not implied by the original utilization condition. The notion of virtual group was introduced in [6]. For more details about the proof, see [4, Lemma 4.1.2].

If priority is given to the last buffer in each group (queues 2 and 4) or the first buffer in each group (queues 1 and 3), the network is stable. It was shown in [7], [10], and [11] that, for a re-entrant line, the last-buffer–first-serve and first-buffer–first-serve discipline with preemption is throughput-optimal.

We analyze the stability of the queueing networks under robust policies using their fluid model. The fluid model converts a stochastic system into a deterministic system, based on the functional strong law of large numbers [2], [3], [4], [14], [17]. Under weak assumptions, the stability of the fluid model implies the stability of the queueing network. To show the stability of the fluid model, we find a Lyapunov function for the differential equations of the system.

There has been relatively little work on the stability of LQ scheduling in the literature. We can mainly mention the following papers. In [12], Kumar *et al.* considered sufficient conditions for stability of stable marriage scheduling algorithms in input-queued switches. In Dimakis and Walrand [8] the authors identified new sufficient conditions for longest-queue-first (LQF) scheduling to be throughput-optimal. They combine properties of diffusion-scaled path functionals and local fluid limits into a sharper characterization of stability. See also [9] for a variation on the first-order sufficient condition of that paper (resource pooling). In Baharian and Tezcan [1] the authors considered LQF scheduling for parallel server systems. It was shown that the utilization condition is sufficient to prove stability if the underlying graph of the parallel server system is a tree. Furthermore, additional drift conditions were provided for the stability of a special parallel server system known as the X-model. The network model that we consider is different from all of these works on LQ scheduling. In a contemporaneous work Yang and Hu [20] showed the stability of Kumar–Seidman network under LQF policy. The network they considered is a special case of a multiclass queueing network with two groups of two queues considered in this paper.

In addition to LQ scheduling, we propose a new scheduling policy called longest-dominating-queue (LDQ). This policy is closely related to LQ scheduling. According to this policy, none of the queues that feed a larger maximum-length queue in the network is served. Among those queues that are not dominated by a maximum-length queue in the network, the longest one is served in each group. We use the fluid model and show that the maximum of the queue lengths is a Lyapunov function to prove the stability of LDQ scheduling in a general acyclic network.

We use boldface letters to denote vectors and matrices. Matrix transposition operation is denoted by $[\cdot]^\top$ and $[\cdot]^{-\top}$ denotes the matrix inversion and transposition. Let $\boldsymbol{V}$ and $\boldsymbol{W}$ be

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matrices of size $L \times N$, then $\boldsymbol{V} \preceq \boldsymbol{W}$ means $v_{ij} \leq w_{ij}$ for all $i = 1, \ldots, L$ and $j = 1, \ldots, N$. Also, $\mathbf{1}^L$ and $\mathbf{0}^L$ stand for column vectors of 1s and 0s with length L, respectively, and $\mathbf{0}^{L\times N}$ stands for the $L \times N$ matrix of all 0 entries. The indicator function of set A is shown as $\mathbf{1}_A$.

The rest of the paper is organized as follows. In Section 2 we provide the precise network model and problem formulation. In Section 3 we focus on the LQ scheduling and prove that it is throughput-optimal for multiclass queueing networks with two groups of two queues. In Section 4 we propose the LDQ scheduling and prove that this new policy is throughput-optimal if the network topology is acyclic.

2. Problem definition and network model

In this section we first introduce the queueing network model that we consider. We discuss the fluid model used to analyze the stability of the queueing network. We also provide the utilization condition or stability condition of the network.

2.1. Network model

We consider a network with K queues and a routeing matrix $\boldsymbol{R}_{K\times K}$. The entry r_{ik} is the probability that a job goes to queue k upon leaving queue i. Therefore, a job leaves the network upon leaving queue i with probability $1 - \sum_k r_{ik}$. In each queue, jobs are served in their order of arrival. A queueing network is 'acyclic' if a job cannot visit any queue more than once; that is, if there exists no finite sequence of queues $(i_1, i_2, \ldots, i_\ell, i_1)$ such that $r_{i_1 i_2} r_{i_2 i_3} \cdots r_{i_\ell i_1} > 0$.

The random length of queue i at time $t \geq 0$ is $\bar{X}_i(t)$, the vector of service rates of the queues is $\boldsymbol{\mu} = [\mu_1, \mu_2, \ldots, \mu_K]^\top$, and the vector of exogenous arrival rates to the queues is $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \ldots, \lambda_K]^\top$. The exogenous arrival processes to the queues are independent stationary ergodic processes. In this network, not all the queues can be served at the same time. The queues are partitioned into J disjoint groups $\{\mathcal{G}_j\}_{j=1}^J$ and only one queue can be served in each group at a time. In Figure 2 we illustrate a multiclass queueing network with two groups of two queues: $\mathcal{G}_1 = \{1, 2\}$ and $\mathcal{G}_2 = \{3, 4\}$.

We assume that the network is open, i.e. all the jobs eventually leave the network. Since the network is open,

$$\boldsymbol{Q} = \boldsymbol{I} + \boldsymbol{R}^\top + (\boldsymbol{R}^\top)^2 + \cdots = (\boldsymbol{I} - \boldsymbol{R}^\top)^{-1}$$

is a finite positive matrix. So, the matrix $(\boldsymbol{I} - \boldsymbol{R}^\top)$ is invertible.

The service disciplines that we consider are independent of $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, and $\boldsymbol{R}$. The goal is to analyze the stability of this network when the utilization condition that will be stated in Section 2.2 holds. Let $\bar{X}_i(t)$ be the length of queue i at time t and $\bar{T}_i(t)$ be the random total amount of time that queue i has been scheduled up to time t. We use the fluid model to analyze the stability of the network. Let $X_i(t)$ be the fluid level of queue i at time t, and let $T_i(t)$ be the cumulative amount of time that queue i is served up to time t in the fluid model. Then the

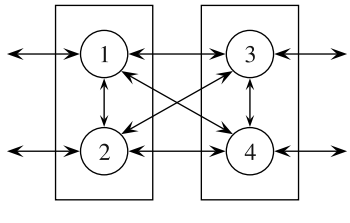


FIGURE 2: Multiclass queueing network with two groups of two queues.

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fluid model equations are

$$X_k(t) = X_k(0) + \lambda_k t - \mu_k T_k(t) + \sum_{i=1}^{K} r_{ik}\mu_i T_i(t), \qquad k = 1, \ldots, K. \tag{2.1}$$

The fluid model is *stable* if there exists some $\delta > 0$ such that, for each fluid solution with $\|\boldsymbol{X}(0)\| \leq 1$, we have $\boldsymbol{X}(t) = \boldsymbol{0}$ for $t > \delta$. Under weak conditions, the stability of the fluid model implies the stability of the queueing network (e.g. the Harris recurrence of a Markov model in the case of renewal arrival processes, independent and identically distributed service times and Markov routeing). See [3] and [4] for a discussion of such results.

2.2. Utilization condition

Define the drift matrix $\boldsymbol{D}$ of the network by

$$\boldsymbol{D} = \boldsymbol{M}(\boldsymbol{R} - \boldsymbol{I}),$$

where $\boldsymbol{M} = \text{diag}\{\boldsymbol{\mu}\}$. Since $\boldsymbol{Q} := (\boldsymbol{I} - \boldsymbol{R}^\top)^{-1}$ is a positive matrix, we have

$$\boldsymbol{D}^{-\top} = -\boldsymbol{M}^{-1}\boldsymbol{Q} \preceq \boldsymbol{0}_{K\times K}. \tag{2.2}$$

Furthermore, let the vector of nominal traffic of the queues be $\boldsymbol{\nu} = [\nu_1, \nu_2, \ldots, \nu_K]^\top$. We have

$$\nu_i = \lambda_i + \sum_{k=1}^{K} r_{ki}\nu_k.$$

In matrix form

$$\boldsymbol{\nu} = (\boldsymbol{I} - \boldsymbol{R}^\top)^{-1}\boldsymbol{\lambda}. \tag{2.3}$$

By [4, Proposition 2.5.3], the necessary condition for the stability of the network (utilization condition) is

$$\sum_{i\in\mathcal{G}_j} \frac{\nu_i}{\mu_i} \leq 1 - \varepsilon \qquad \text{for all } j = 1, \ldots J, \tag{2.4}$$

for some $\varepsilon > 0$. The utilization condition expresses that work for each group arrives at the network at rate less than 1.

Using (2.2) and (2.3), we find an equivalent stability condition which is $\tilde{\boldsymbol{e}}_j^\top \boldsymbol{D}^{-\top}\boldsymbol{\lambda} + 1 \geq \varepsilon$ for all $j = 1, \ldots, J$, where $\tilde{\boldsymbol{e}}_j$ is a column vector of length K corresponding to group j such that component i of $\tilde{\boldsymbol{e}}_j$ is $\tilde{e}_j(i) = \mathbf{1}_{\{i\in\mathcal{G}_j\}}$.

3. LQ scheduling

In this section we define the LQ scheduling policy and analyze it using the fluid model. We show that LQ scheduling is throughput-optimal in the queueing network with two groups of two queues.

3.1. LQ scheduling for multiclass queueing networks

We provide the fluid model equations for LQ scheduling. Under this discipline, the longest queue in each group is served. To break ties, we can use a static priority scheduling among the maximum queues, or serve any of the maximum-length queues randomly.

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Let

$$S_j(t) = \left\{ i : i \in \mathcal{G}_j, X_i(t) = \max_{k \in \mathcal{G}_j} X_k(t) \right\}$$

be the set of queues with maximum fluid level in group j.

Fluid model equations are those defined by (2.1) together with

$$\sum_{i \in S_j} \dot{T}_i(t) = 1, \qquad \sum_{i \in \mathcal{G}_j \setminus S_j} \dot{T}_i(t) = 0,$$

which state that the server in each group spends all its time serving the longest queues. The main result of this paper is the following theorem.

Theorem 3.1. *The multiclass queueing network with two groups of two queues, $\mathcal{G}_1 = \{1, 2\}$ and $\mathcal{G}_2 = \{3, 4\}$, is stable under LQ scheduling if the utilization condition stated in (2.4) holds.*

The network is illustrated in Figure 2. In the rest of this section we establish the proof of Theorem 3.1. The key idea for the proof is to find a piecewise linear Lyapunov function. The stability of multiclass queueing networks under LQ scheduling for general networks remains an open problem.

Proof of Theorem 3.1. Let $\boldsymbol{W}(t) = \boldsymbol{M}^{-1}\boldsymbol{Q}\boldsymbol{X}(t) = -\boldsymbol{D}^{-\top}\boldsymbol{X}(t)$. Note that $\boldsymbol{Q}\boldsymbol{X}(t)$ denotes the vector of potential fluid level for each queue at time t, and $W_i(t)$ is the potential work to be done at queue i at time t. Let

$$a = \frac{\mu_1^{-1}(q_{13} + q_{14}) + \mu_2^{-1}(q_{23} + q_{24})}{\mu_3^{-1}(q_{33} + q_{34}) + \mu_4^{-1}(q_{43} + q_{44})}, \tag{3.1}$$

and

$$b = \frac{\mu_1^{-1}(q_{11} + q_{12}) + \mu_2^{-1}(q_{21} + q_{22})}{\mu_3^{-1}(q_{31} + q_{32}) + \mu_4^{-1}(q_{41} + q_{42})}, \tag{3.2}$$

where $\boldsymbol{Q} = [q_{ij}]$. In Lemma 3.1 we show that $a \le b$. Let $\beta \in [a, b]$. We show that

$$V(\boldsymbol{X}) = \max(f_1, f_2),$$

where $f_1 = W_1 + W_2 = \tilde{\boldsymbol{e}}_1^{\top}\boldsymbol{W}$ and $f_2 = \beta[W_3 + W_4] = \beta\tilde{\boldsymbol{e}}_2^{\top}\boldsymbol{W}$, is a Lyapunov function.

Lemma 3.1. *The following inequality holds: $a \le b$, where a is given in (3.1) and b is given in (3.2).*

Proof. It is enough to show the following four inequalities:

$$(q_{11} + q_{12})(q_{33} + q_{34}) \ge (q_{13} + q_{14})(q_{31} + q_{32}), \tag{3.3}$$
$$(q_{11} + q_{12})(q_{43} + q_{44}) \ge (q_{13} + q_{14})(q_{41} + q_{42}), \tag{3.4}$$
$$(q_{21} + q_{22})(q_{33} + q_{34}) \ge (q_{23} + q_{24})(q_{31} + q_{32}), \tag{3.5}$$
$$(q_{21} + q_{22})(q_{43} + q_{44}) \ge (q_{23} + q_{24})(q_{41} + q_{42}). \tag{3.6}$$

We prove (3.3). The proof of (3.4)–(3.6) is identical with a change of indices.

Recall that $\boldsymbol{Q} = (\boldsymbol{I} - \boldsymbol{R}^{\top})^{-1}$. Denote by $\boldsymbol{I} - \boldsymbol{R}$ an M-matrix [16]. That is, $\boldsymbol{I} - \boldsymbol{R}$ has nonnegative diagonal entries, nonpositive off-diagonal entries, and the sum of entries of each

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row is nonnegative. M-matrices have positive principal minors [16]; thus, $\det(\boldsymbol{I}-\boldsymbol{R})$ is strictly positive, as the network is open. The following equality can be computed:

$$\det\left(\sqrt{\det(\boldsymbol{I}-\boldsymbol{R})}\begin{bmatrix} q_{11}+q_{12} & q_{13}+q_{14} \\ q_{31}+q_{32} & q_{33}+q_{34}\end{bmatrix}\right) = (1+r_{21}-r_{22})(1+r_{43}-r_{44})+(r_{23}-r_{24})(r_{42}-r_{41}).$$

It is now sufficient to show that the right-hand side of the above equality is nonnegative. This can be shown as follows:

$$\begin{aligned}(1+r_{21}-r_{22})(1+r_{43}-r_{44})+(r_{23}-r_{24})(r_{42}-r_{41}) &\geq (1-r_{22})(1-r_{44})-(r_{23}+r_{24})(r_{42}+r_{41}) \\ &\geq 0.\end{aligned}$$

The last inequality is due to the fact $1-r_{22}\geq r_{23}+r_{24}$ and $1-r_{44}\geq r_{42}+r_{41}$. This completes the proof of the lemma. □

Now we show that $V(\boldsymbol{X})$ is a Lyapunov function. First, it is clear that $V(\boldsymbol{X})=\boldsymbol{0}$ if and only if $\boldsymbol{X}=0$. We show that at a regular point t, $\dot{V}(t)\leq -\delta$ if $\boldsymbol{X}(t)\neq\boldsymbol{0}$ for some $\delta>0$ by considering three different cases. A time t is regular if each component of $\boldsymbol{X}(t)$ is differentiable at time t, and function $V(\boldsymbol{X}(t))$ is differentiable at time t.

Case 1. Suppose that $f_1(t)>0$ and $f_2(t)>0$. The fluid model equation in (2.1) can be written in vector form as

$$\boldsymbol{X}(t)=\boldsymbol{X}(0)+\boldsymbol{\lambda}t+\boldsymbol{D}^{\top}\boldsymbol{T}(t).$$

Then,

$$\dot{f}_1(t)=\tilde{\boldsymbol{e}}_1^{\top}\boldsymbol{M}^{-1}\boldsymbol{Q}(\boldsymbol{\lambda}+\boldsymbol{D}^{\top}\dot{\boldsymbol{T}}(t))=-\tilde{\boldsymbol{e}}_1^{\top}\boldsymbol{D}^{-\top}\boldsymbol{\lambda}-1\leq-\varepsilon,$$

by the utilization condition and the fact that $\dot{T}_1+\dot{T}_2=1$ since LQ scheduling is work-conserving. Similarly,

$$\dot{f}_2(t)=-\tilde{\boldsymbol{e}}_2^{\top}\boldsymbol{D}^{-\top}\boldsymbol{\lambda}-1\leq-\varepsilon.$$

Thus, $\dot{V}(t)\leq-\varepsilon$ in this case.

Case 2. Suppose that group 1 is empty, and group 2 is nonempty. That is, $X_1(t)=X_2(t)=0$ and $\boldsymbol{X}(t)\neq\boldsymbol{0}$. Then, similar to case 1, we can show that $\dot{f}_2(t)\leq-\varepsilon$. However, $\dot{f}_1(t)$ is not necessarily strictly negative. Thus, we need to show that at all regular points t in case 2, either $\dot{f}_1(t)<0$ or $f_2(t)\geq f_1(t)$ (or both). Since $X_1(t)=X_2(t)=0$, we can write $f_1(t)=c_1X_3(t)+c_2X_4(t)$ and $f_2(t)=c_3X_3(t)+c_4X_4(t)$ for some positive constants c_i, $i=1,2,3,4$. We write f_1 in the following two expressions:

$$f_1=\frac{c_1}{c_3}f_2+\left(c_2-\frac{c_1c_4}{c_3}\right)X_4, \tag{3.7}$$

$$f_1=\frac{c_2}{c_4}f_2+\left(c_1-\frac{c_2c_3}{c_4}\right)X_3. \tag{3.8}$$

Recall that under LQ scheduling $\dot{T}_3=1$ and $\dot{T}_4=0$ if $X_3>X_4$, which implies that $\dot{X}_3<0$ and $\dot{X}_4\geq0$ in case 2, since $\dot{f}_2=c_3\dot{X}_3+c_4\dot{X}_4<0$. Similarly, by LQ scheduling $X_4>X_3$ implies that $\dot{T}_4=1$, $\dot{T}_3=0$, $\dot{X}_4<0$, and $\dot{X}_3\geq0$ in case 2. Now suppose that $c_1c_4<c_2c_3$. Then, we expand f_1 according to (3.7). Then, $X_4>X_3$ implies that $\dot{f}_1<-(c_1/c_3)\varepsilon$. We show

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that if $X_3 \geq X_4$ then $f_2 \geq f_1$ in this case as follows. Fixing $X_4 \geq 0$, the function

$$\frac{f_2}{f_1} = \frac{c_3 X_3 + c_4 X_4}{c_1 X_3 + c_2 X_4}$$

is increasing in X_3 since $c_1 c_4 < c_2 c_3$ by assumption. Thus,

$$\inf_{X_3 :\, X_4 \leq X_3} \frac{f_2}{f_1}$$

is achieved when $X_3 = X_4$. Therefore, it is enough to show that $f_2 \geq f_1$ when $X_3 = X_4$. When $X_1 = X_2 = 0$ and $X_3 = X_4$, we have

$$\frac{f_2}{f_1} = \frac{\beta[\mu_3^{-1}(q_{33} + q_{34}) + \mu_4^{-1}(q_{43} + q_{44})]}{\mu_1^{-1}(q_{13} + q_{14}) + \mu_2^{-1}(q_{23} + q_{24})} = \frac{\beta}{a} \geq 1. \tag{3.9}$$

In the above derivation, we used (3.1) and the fact that $\beta \in [a, b]$.

Similarly, suppose that $c_1 c_4 > c_2 c_3$. Then, we expand f_1 according to (3.8). Then, $X_3 > X_4$ implies that $\dot{f}_1 < -(c_2/c_4)\varepsilon$. We show that if $X_4 \geq X_3$ then $f_2 \geq f_1$ in this case as follows. Fixing $X_3 \geq 0$, the function

$$\frac{f_2}{f_1} = \frac{c_3 X_3 + c_4 X_4}{c_1 X_3 + c_2 X_4}$$

is increasing in X_4 since $c_1 c_4 > c_2 c_3$ by assumption. Then,

$$\inf_{X_4 :\, X_3 \leq X_4} \frac{f_2}{f_1}$$

is achieved when $X_4 = X_3$. Therefore, it is enough to show that $f_2 \geq f_1$ when $X_3 = X_4$, which is already shown in (3.9).

Finally, supposing that $c_1 c_4 = c_2 c_3$, $f_1/f_2 = c_1/c_3$. Thus,

$$\dot{f}_1 = \frac{c_1}{c_3}\dot{f}_2 \leq -\frac{c_1}{c_3}\varepsilon.$$

Case 3. Suppose that group 2 is empty, and group 1 is nonempty. Then, $\dot{f}_1(t) \leq -\varepsilon$. Similar to case 2, we need to show that either $\dot{f}_2(t) < 0$ or $f_1(t) \geq f_2(t)$ (or both). Since in case 3, $X_3 = X_4 = 0$, we can write $f_1 = c_1' X_1 + c_2' X_2$ and $f_2 = c_3' X_1 + c_4' X_2$ for some positive constants c_i', $i = 1, 2, 3, 4$. We have

$$f_2 = \frac{c_3'}{c_1'} f_1 + \left(c_4' - \frac{c_3' c_2'}{c_1'}\right) X_2, \tag{3.10}$$

$$f_2 = \frac{c_4'}{c_2'} f_1 + \left(c_3' - \frac{c_4' c_1'}{c_2'}\right) X_1. \tag{3.11}$$

Similar to case 2, by LQ scheduling, $X_1 > X_2$ implies that $\dot{X}_1 < 0$ and $\dot{X}_2 \geq 0$. Also, $X_2 > X_1$ implies that $\dot{X}_2 < 0$ and $\dot{X}_1 \geq 0$. Now suppose that $c_1' c_4' > c_2' c_3'$. Then, $X_2 > X_1$ implies that $\dot{f}_2 < -(c_3'/c_1')\varepsilon$ due to (3.10). We show that if $X_1 \geq X_2$ then $f_1 \geq f_2$. Fixing $X_2 \geq 0$, the function

$$\frac{f_1}{f_2} = \frac{c_1' X_1 + c_2' X_2}{c_3' X_1 + c_4' X_2}$$

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is increasing in X_1 since $c_1'c_4' > c_2'c_3'$. Thus,

$$\inf_{X_1:\, X_1 \geq X_2} \frac{f_1}{f_2}$$

is achieved when $X_1 = X_2$. Therefore, it is enough to show that $f_1 \geq f_2$ when $X_1 = X_2$. When $X_3 = X_4 = 0$ and $X_1 = X_2$, we have

$$\frac{f_1}{f_2} = \frac{\mu_1^{-1}(q_{11} + q_{12}) + \mu_2^{-1}(q_{21} + q_{22})}{\beta[\mu_3^{-1}(q_{33} + q_{34}) + \mu_4^{-1}(q_{41} + q_{42})]} = \frac{b}{\beta} \geq 1. \tag{3.12}$$

In the above derivation, we used (3.2) and the fact that $\beta \in [a, b]$.

Similarly, suppose that $c_1'c_4' < c_2'c_3'$. Then, $X_1 > X_2$ implies that $\dot{f}_2 < -(c_4'/c_2')\varepsilon$ due to (3.11). We show that if $X_2 \geq X_1$ then $f_1 \geq f_2$. Fixing $X_1 \geq 0$, the function f_1/f_2 is increasing in X_2 since $c_2'c_3' > c_1'c_4'$. Then,

$$\inf_{X_2:\, X_1 \leq X_2} \frac{f_1}{f_2}$$

is achieved when $X_2 = X_1$. Therefore, it is enough to show that $f_1 \geq f_2$ when $X_1 = X_2$, which is already shown in (3.12).

Finally, supposing that $c_1'c_4' = c_2'c_3'$, $f_1/f_2 = c_1'/c_3'$. Thus,

$$\dot{f}_2 = \frac{c_3'}{c_1'}\dot{f}_1 \leq -\frac{c_3'}{c_1'}\varepsilon.$$

Now taking

$$\delta = \min\left(\varepsilon, \frac{c_1}{c_3}\varepsilon, \frac{c_2}{c_4}\varepsilon, \frac{c_3'}{c_1'}\varepsilon, \frac{c_4'}{c_2'}\varepsilon\right) > 0,$$

we have $\dot{V}(t) \leq -\delta$. This completes the proof of Theorem 3.1. □

We show the simulation result for the Lu–Kumar network (Figure 1) in the fluid model. In the simulation, service rates are $\boldsymbol{\mu} = [3, 1, 1, 1]^\top$, and arrival rate is $\lambda = 0.4$. Initial fluid levels are $\boldsymbol{X}(0) = [40, 30, 20, 10]^\top$. In Figure 3 we show the trajectories of the fluid in different queues versus time. As we can see, the fluid level at all queues becomes 0 after some finite time.

4. LDQ scheduling

4.1. Policy

Let $\bar{S}$ be the set of global maxima in the network

$$\bar{S} = \left\{i :\, \bar{X}_i = \max_k \bar{X}_k\right\}.$$

We define a queue to be 'dominating' if it belongs to the set of global maxima in the network, or if it does not feed another globally maximum-length queue in the network. More formally, for each group j, the dominating set of queues $\bar{\mathcal{D}}_j$ is defined to be

$$\bar{\mathcal{D}}_j = \{i \in \mathcal{G}_j : r_{is} = 0 \text{ if } \bar{X}_i < \bar{X}_s \text{ for all } s \in \bar{S}\}.$$

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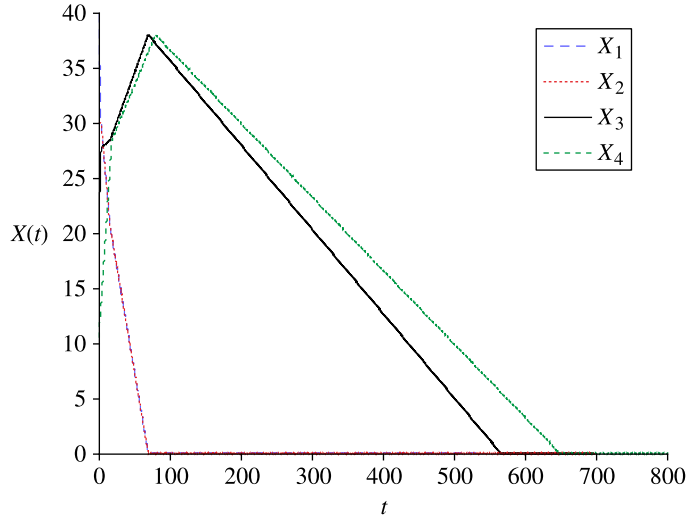


FIGURE 3: Simulation result. The trajectories of the fluid levels in the queues for the Lu–Kumar network under LQ scheduling.

The scheduling policy is to serve the longest queue in $\bar{\mathcal{D}}_j$. If $\bar{\mathcal{D}}_j = \varnothing$, it does not serve any queues from group j. As an example, see Figure 4, where queue 3 is the maximum with length 30. Since jobs leaving both queues 1 and 2 can be destined to queue 3, $\bar{\mathcal{D}}_1 = \varnothing$ and $\bar{\mathcal{D}}_2 = \{3, 4\}$, so no queues in group 1 are served, and queue 3 in group 2 is served. As we can see, while the policy is not work-conserving, it always serves the queue with maximum-length in the network. The policy is robust to the knowledge of service rates and exact values of routeing probabilities. The drawback in comparison with LQ scheduling is that it needs global knowledge of queue lengths and the topology of the network.

4.2. Main result for LDQ scheduling

We analyze the fluid model of the system under the LDQ scheduling policy. Let S be the set of queues in the network with maximum fluid level; thus

$$S = \left\{ i : X_i = \max_k X_k \right\}.$$

Let

$$\mathcal{D}_j = \{ i \in \mathcal{G}_j : r_{is} = 0 \text{ if } X_i < X_s \text{ for all } s \in S \}$$

be the set of dominating queues in group j in the fluid model. Let

$$\tilde{S}_j = \left\{ i : i \in \mathcal{D}_j, X_i = \max_{k \in \mathcal{D}_j} X_k \right\}$$

be the set of longest dominating queues in group j. Define a subset of groups $\mathcal{J} \subseteq \{1, 2, \ldots, J\}$ as follows: $\mathcal{J} = \{ j \mid \text{there exists } i \in \mathcal{G}_j \cap S \text{ such that } r_{is} = 0 \text{ for all } s \in S \}$. In words, $\mathcal{J}$ is the set of groups that have a globally maximum-length queue which does not feed another

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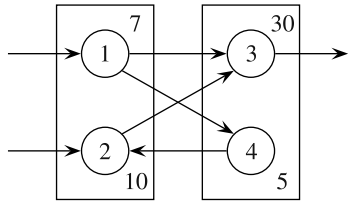


FIGURE 4: Illustration of LDQ scheduling.

globally maximum-length queue. The scheduling policy implies that

$$\sum_{i\in\tilde{S}_j} \dot{T}_i(t) = 1 \quad \text{if } j \in \mathcal{J}, \tag{4.1}$$

$$\sum_{i\in\tilde{S}_j} \dot{T}_i(t) \le 1 \quad \text{if } j \notin \mathcal{J}, \qquad \sum_{i\in\mathcal{G}_j\setminus\tilde{S}_j} \dot{T}_i(t) = 0. \tag{4.2}$$

The fluid model equations for this policy are those stated in (2.1) together with (4.1) and (4.2).

Theorem 4.1. *A multiclass queueing networks are stable under LDQ scheduling if the utilization condition stated in (2.4) holds and the network is acyclic.*

Proof. We prove Theorem 4.1 using the Lyapunov function $V(\boldsymbol{X}) = \max_i\{X_i\}$. Let $|S(t)| = L'(t)$. Let $S'_j = \mathcal{G}_j \cap S$ be the set of maximum-length (maximum fluid level) queues in group j. At time t, if $L' = 1$, that is, the queue with maximum fluid level is unique, $\dot{V}(\boldsymbol{X}(t))$ is clearly negative. Here is a proof. Let $S = \{i\}$. Then $\dot{T}_i = 1$. Moreover, no jobs will be scheduled destined to the queue with maximum length in the network. Thus, $\dot{V}(\boldsymbol{X}(t)) = \lambda_i - \mu_i < 0$. If the maximum is not unique, at a regular point t, the drifts of the queues in the set of maxima are equal (see [4, Lemma 2.8.6]). That is, $\dot{X}_i(t) = \dot{X}_k(t)$ if $i, k \in S(t)$.

By LDQ scheduling, there is no internal flow to the sub-network of queues in S coming from other queues. That is, if $i \in S$, $\sum_{k\notin S}\mu_k r_{ki}\dot{T}_k = 0$. Thus, we can only consider the sub-network of queues constructed by S to analyze the drift of the Lyapunov function. Let the corresponding drift matrix and exogenous arrival vector to set S be $\boldsymbol{D}_{L'}$ and $\boldsymbol{\lambda}_{L'}$. Suppose that J' groups ($J' \le J$) have queues in S, so $S = \bigcup_{j=1}^{J'} S'_j$ (with some abuse of notation due to relabelling the J' groups by 1 to J'). Let $L'_j = |S'_j|$. At a regular point t, the drifts of the queues in $S(t)$ are all equal to $\dot{V}(t)$. In order to complete the proof we need the following lemma.

Lemma 4.1. *If $\boldsymbol{X}(t) \neq \boldsymbol{0}$, $\dot{V}(t) < 0$.*

Proof. Define matrix $\boldsymbol{E}_{L'\times J'}$ as

$$\boldsymbol{E} = [\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{J'}],$$

where $\boldsymbol{e}_j$ is a column vector of length L', given by

$$\boldsymbol{e}_j = [\boldsymbol{0}^{\sum_{k=1}^{j-1} L'_k}, \boldsymbol{1}^{L'_j}, \boldsymbol{0}^{\sum_{k=j+1}^{J'} L'_k}]^\top.$$

Writing (the derivative of) (2.1) in matrix form together with the fact that at a regular point t, if $i \in S$, $\dot{V}(t) = \dot{X}_i(t)$, we have

$$\begin{bmatrix} -(\boldsymbol{D}_{L'})^\top & \boldsymbol{E} \\ \boldsymbol{E}^\top & \boldsymbol{0}^{J'\times J'} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{T}}_{L'} \\ \dot{V}\boldsymbol{1}^{J'} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_{L'} \\ \boldsymbol{t}_{J'}, \end{bmatrix}, \tag{4.3}$$

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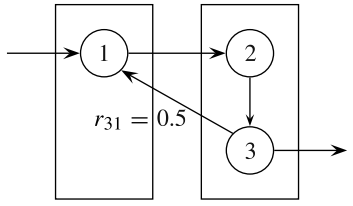


FIGURE 5: A simple cyclic network. An illustration that in cyclic networks there may be no groups in $\mathcal{J}$. Note that in this example, if all the queues have equal fluid level, they all belong to the set of global maxima in the network. Now since the network is cyclic, there is no output queue that does not feed any other maximum-length queue.

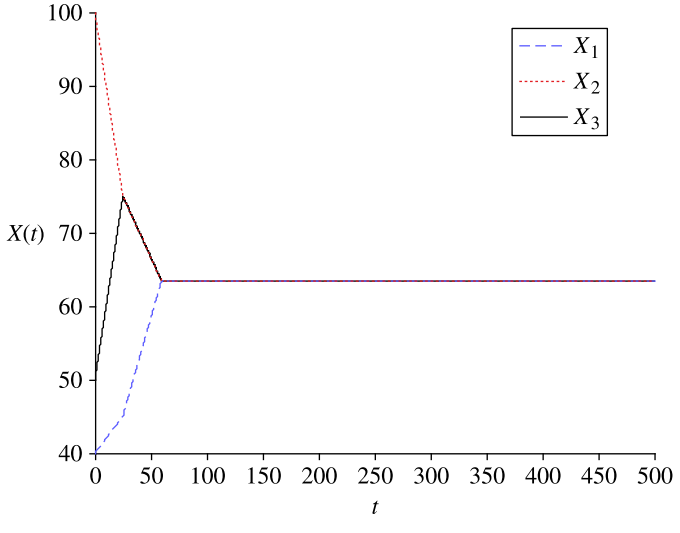


FIGURE 6: Simulation result. The trajectories of the fluid levels in the queues for the cyclic network shown in Figure 5 under LDQ scheduling.

where vector $\boldsymbol{t}_{J'} = [t_j]$, $t_j = \sum_{i \in S'_j} \dot{T}_i \leq 1$. By (4.1), at least one of the components of $\boldsymbol{t}_{J'}$ is equal to 1, if the network topology is acyclic. The reason is that the sub-network certainly has an 'output' queue, which does not feed any other queues in the sub-network. Note that if there is a cycle in the network, such a queue may not exist (see Figure 5). The group which contains the 'output' queue, say group j^*, has the property that $j^* \in \mathcal{J}$; thus, $\sum_{i \in S'_{j^*}} \dot{T}_i = 1$.

Solving (4.3) using the block inverse formula, we have

$$\dot{V}\mathbf{1}^{J'} = (\boldsymbol{E}^\top {\boldsymbol{D}_{L'}}^{-\top} \boldsymbol{E})^{-1} (\boldsymbol{E}^\top {\boldsymbol{D}_{L'}}^{-\top} \boldsymbol{\lambda}_{L'} + \boldsymbol{t}_{J'}).$$

The vector $(\boldsymbol{E}^\top {\boldsymbol{D}_{L'}}^{-\top} \boldsymbol{\lambda}_{L'} + \boldsymbol{t}_{J'})$ is strictly positive in at least one component corresponding to group j^* by the utilization condition of this sub-network (which is a weaker condition than the utilization condition of the whole network). Since $\boldsymbol{E}^\top {\boldsymbol{D}_{L'}}^{-\top} \boldsymbol{E}$ is a negative matrix and $\boldsymbol{E}^\top {\boldsymbol{D}_{L'}}^{-\top} \boldsymbol{E}(\dot{V}\mathbf{1}^{J'})$ is not a negative vector, $\dot{V}$ cannot be nonnegative. □

As a consequence, we see that $V(\boldsymbol{X}(t))$ is a Lyapunov function and the proof of Theorem 4.1 is complete. □

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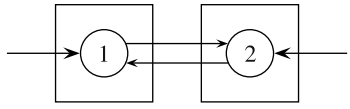


FIGURE 7: A simple cycle.

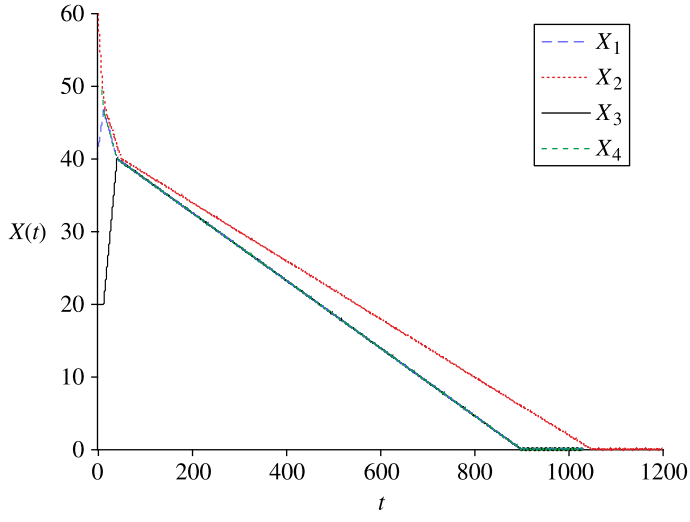


FIGURE 8: Simulation result. The trajectories of the fluid levels in the queues for Lu-Kumar network under LDQ scheduling.

The simulation results show that the network shown in Figure 5 is indeed unstable under LDQ scheduling. In this simulation all the service rates are 1, and the arrival rate is 0.2. The result of the simulation is shown in Figure 6. As we can see, the fluid levels of the queues do not reach 0.

The basic intuition behind this fact is the following. Consider the simple case of Figure 7. We have one queue in each group so no scheduling is needed. However, in the stochastic system, LDQ serves only one of the queues at a time if the queue-lengths are not equal. This illustrates that LDQ may not be throughput-optimal if we have a cycle in the network.

Finally, we see the simulation result for the Lu–Kumar network under LDQ scheduling. The network topology is acyclic, so we expect LDQ to be stable. The simulation result shown in Figure 8 verifies this. In this simulation, all the service rates are 1, and the arrival rate is 0.4. The conclusion from Figure 7 is that the fluid level at all queues becomes 0 after some finite time.

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